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Dynamics of a Diffusive Model with Spatial Memory and Nonlinear Boundary Condition*

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Abstract In this paper, we investigate the existence and stability of steady-state and periodic solutions for a heterogeneous diffusive model with spatial memory and nonlinear boundary conditions, employing Lyapunov-Schmidt reduction and eigenvalue theory. Our findings reveal that when the interior reaction term is weaker than the boundary reaction term, no Hopf bifurcation occurs regardless of time delay. Conversely, when the interior reaction term is stronger than the boundary reaction term, the presence of Hopf bifurcation is determined by the spatial memory delay.

Keywords Spatial memory, stability, Hopf bifurcation, nonlinear boundary condition

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1. Inroduction

Reaction-diffusion systems play a crucial role in both natural sciences and engineering, utilized in biological population dynamics, chemical reactor design, physical studies of material defects, medical disease modeling, and environmental pollutant diffusion. These models enhance our understanding of complex system behaviors and provide a foundation for improving technologies and devising effective strategies. In recent years, extensive research has been conducted on delayed reaction-diffusion equations, particularly focusing on the existence, uniqueness, monotonicity, stability, and bifurcation of steady-state solutions (for example, [2], [5], [8], [9], [11]). Reaction-diffusion models with spatial memory, maturation time, and linear boundary conditions have been extensively explored by Ji and Wu [20], Wang, Fan and Wang [26] and so on. However, research on models containing nonlinear boundary conditions remains limited. Our research includes the impact of external factors such as environmental conditions and resource distribution, which change the dynamical behavior of populations. These factors are crucial in designing more realistic and applicable models, thereby reflecting the complexity of ecological systems. The introduction of nonlinear boundary conditions reflects the complexity of real-world boundary interactions, allowing for a more accurate depiction of system boundary effects and revealing their impact on stability and

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bifurcation phenomena. In this study, we investigate how memory and maturation delays influence the dynamical behavior of nonlinear boundary problems. For convenience, we explore the following system with the memory delay equal to the mature delay:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u + d\nabla\cdot\left(u\nabla u_\tau\right) + \lambda u\left(m(x) - u_\tau\right), & x \in \Omega, \\ \dfrac{\partial u}{\partial \mathbf{n}} = \lambda h(x,u), & x \in \partial\Omega, \end{cases} \tag{1.1}$$

for $t > 0$, where $\tau > 0$, Δ is the Laplace operator, Ω is a connected bounded open domain in $\mathbb{R}^N$ $(N \geq 1)$ with smooth boundary $\partial\Omega$, $\mathbf{n}$ is the unit outer normal to $\partial\Omega$, $u = u(x,t) \in \mathbb{R}$, $h \in C^{1+\epsilon}(\partial\Omega \times \mathbb{R}, \mathbb{R})$ for some $0 < \epsilon < 1$, $h(x,\cdot) \in C^3(\mathbb{R},\mathbb{R})$ and $h(x,0) = 0$ for all $x \in \partial\Omega$, $u_\tau = u(x,t-\tau)$, in the equation (1.1), the current rate of change of the population $\frac{\partial u}{\partial t}$ depends on the population quantity at the past time point $t - \tau$. In system (1.1), a single bifurcation parameter, λ, controls both the internal and boundary reactions. When $\lambda = 0$, the equation becomes a flux-free diffusion equation with spatial memory.

In biology, the time delay $\tau \geq 0$ describes the averaged memory period, $u(x,t)$ represents the population density of a species at time t and location x, $m(x)$ is the intrinsic growth rate or the carrying capacity which can represent the situation of resource at x, in $d\nabla\cdot\left(u\nabla u_\tau\right)$ delay τ represents the averaged memory period, while in $\lambda u\left(m(x) - u_\tau\right)$ delay τ corresponds to the maturation time. In this paper, we focus on the case where memory and maturation delays are identical. The boundary conditions indicate that individuals reaching boundary $\partial\Omega$ are removed from the habitat at a rate determined by the current population density at that location.

Some scholars have studied the dynamical behaviors near the steady-state solutions of diffusion systems with Dirichlet boundary condition or Neumann boundary condition (for example, [7], [12], [18], [22]). However, many phenomena and processes exhibit nonlinear characteristics, such as the turbulence in fluids, nonlinear elasticity in materials, and variable rates in chemical reactions. Therefore, we incorporate more general nonlinear boundaries into the typical memory-inclusive diffusion systems. A lot of literature employs the center manifold method to investigate Hopf bifurcation (for example, [23], [24], [10]). However, the reduced equations through this method often remain high-dimensional, posing significant challenges for studying high-dimensional or even infinite-dimensional equations. In this paper, we utilize the Lyapunov-Schmidt procedure, aiming for a more precise and efficient characterization of system (1.1). Although studies such as An, Wang and Wang [1], Chen, Lou and Wei [6], and Ji and Wu [19] have investigated the impact of spatial memory on population dynamics, and Cantrell and Cosner [3], Cantrell, Cosner and Martínez [4], and Guo [14,15] have explored the effects of nonlinear boundary conditions, few studies are devoted to their combined influences. Model (1.1) enables a thorough analysis, providing deeper insights into the dynamical behavior of models with both nonlinear boundaries and spatial memory.

The objective of this paper is to determine the set λ for which steady-state solutions exist, and to ascertain the uniqueness, stability, and Hopf bifurcation of these positive steady-state solutions based on the values of τ. Ji and Wu [20] explored the stability of steady-state solutions and Hopf bifurcations in model (1.1) with Neumann boundary condition. By incorporating nonlinear boundary condition, this paper facilitates a deeper understanding of the intrinsic mechanisms of complex systems, thereby offering more precise theoretical support for practical applications. Additionally, we adopt a multifaceted theoretical frameworks and approaches to address the complexities of our research subject. Initially, we employ eigenvalue theory and bifurcation theory to establish the existence and stability of steady-state solutions. Following this, we use the Lyapunov-Schmidt reduction, which enables us to investigate the balance between the influences of the interior and boundary reaction terms on the occurrence of bifurcations, particularly the Hopf bifurcation.

The paper is organized as follows: In Section 2, we establish the existence and bifurcation of nontrivial steady-state solutions by treating λ as a bifurcation parameter and employing the Lyapunov-Schmidt reduction. For the linearized system at u_λ^* in Section 3, we analyze the eigenvalue distribution of its infinitesimal generator $\mathcal{A}_{\tau,\lambda}$. The findings show that if the internal reaction term is weaker than the boundary reaction term, then regardless of variations in time delay τ, system (1.1) will not experience a Hopf bifurcation. If the internal reaction term is stronger than the boundary reaction term, the occurrence of a Hopf bifurcation in system (1.1) is governed by the internal reaction delay τ. In Section 4, the Lyapunov-Schmidt reduction is employed to elucidate the one-to-one correspondence between periodic solutions near u_λ^* and u_s^* in system (1.1) and zeros of the simplified bifurcation map. This method enables the establishment of criteria for the existence and direction of periodic solution bifurcation branches, bypassing the center manifold reduction method.

For convenience, we introduce the following notations. Denote by $L^p(\Omega)$ $(p \in \mathbb{N})$ the Lebesgue space of integrable functions defined on Ω, and let $W^{k,p}(\Omega)$ $(k \geq 0)$ be the Sobolev space of the L^p-functions $f(x)$ defined on Ω whose derivatives $\frac{\mathrm{d}^n}{\mathrm{d}x^n}f$ $(n = 1, \ldots, k)$ also belong to $L^p(\Omega)$. Denote the spaces $\mathbb{X} = W^{2,p}(\Omega)$ and $\mathbb{Y} = L^p(\Omega) \times W^{(p-1)/p,p}(\partial\Omega)$. For a space Z, we also define the complexification of Z to be $Z_{\mathbb{C}} := Z \oplus \mathrm{i}Z = \{x_1 + \mathrm{i}x_2 : x_1, x_2 \in Z\}$. Denote by $\mathcal{C}_\tau^k = C^k([-\tau, 0], \mathbb{X}_{\mathbb{C}})$ the Banach space of k-times continuously differentiable mappings from $[-\tau, 0]$ into $\mathbb{X}_{\mathbb{C}}$ equipped with the supremum norm $\|\phi\| = \sup\{\|\phi^{(j)}(\theta)\|_{\mathbb{X}_{\mathbb{C}}} : \theta \in [-\tau, 0], j = 0, 1, ..., k\}$ for $\phi \in \mathcal{C}_\tau^k$. For a linear operator L: $Z_1 \to Z_2$, we denote the domain of L by $\mathrm{Dom}(L)$ and the range of L by $\mathrm{Range}L$.

2. Existence of steady-state solutions

In this section, the existence of steady-state solutions is studied by the following equation:

$$\begin{cases} 0 = \Delta u + d\nabla \cdot (u \nabla u) + \lambda u\,(m(x) - u)\,, & x \in \Omega, \\ \dfrac{\partial u}{\partial \mathbf{n}} = \lambda h(x, u), & x \in \partial\Omega. \end{cases} \tag{2.1}$$

Solving (2.1) can be reduced to seek nontrivial zero points of the following operator:

$$F(u, \lambda) = \begin{pmatrix} F_1(u, \lambda) \\ F_2(u, \lambda) \end{pmatrix} = \begin{pmatrix} \Delta u + d\nabla \cdot (u \nabla u) + \lambda u\,(m(x) - u) \\ \frac{\partial u}{\partial \mathbf{n}} - \lambda h(x, u) \end{pmatrix}.$$

2.1. Bifurcation from $(0, \lambda)$ for some $\lambda \in \mathbb{R}$

First of all, it is easy to see that, for every fixed parameter value $\lambda \in \mathbb{R}$, $F(u, \lambda) = 0$ always has a trivial solution $u = 0$. Namely, $F(u, \lambda) = 0$ for all values of the parameter λ. If we want to prove the uniqueness of these solutions by the implicit function theorem, we need to compute the derivative of F with respect to u evaluated at $(0, \lambda)$, which is given by

$$\mathcal{L}_\lambda u = \begin{pmatrix} \Delta u + \lambda m(x) u \\ \frac{\partial u}{\partial \mathbf{n}} - \lambda h_u(x, 0) u \end{pmatrix}.$$

For the sake of completeness, we review some results for the existence of principal eigenvalues from Umezu [25].

Lemma 2.1 (Umezu [25]). *Consider the eigenvalue problem*

$$\begin{cases} 0 = \Delta u + \lambda m(x) u, & x \in \Omega, \\ \dfrac{\partial u}{\partial \boldsymbol{n}} = \lambda h_u(0, u) u, & x \in \partial\Omega. \end{cases} \tag{2.2}$$

Assume that

$$\textit{either } \; m(x) \not\leq 0 \; \textit{ in } \; \Omega \; \textit{ or } \; h_u(0, u) u \not\leq 0 \; \textit{ on } \; \partial\Omega. \tag{2.3}$$

The problem (2.2) has a unique positive principal eigenvalue λ_1 if and only if

$$\int_\Omega m(x) dx + \int_{\partial\Omega} h_u(0, u) d\sigma < 0 \tag{2.4}$$

and it is characterized by the formula

$$\lambda_1 = \inf \left\{ \mathcal{Q} u : u \in W^{1,2}(\Omega), \; \int_\Omega m(x) u^2(x) dx + \int_{\partial\Omega} h_u(0, u) u^2(x) d\sigma > 0 \right\},$$

where

$$\mathcal{Q} u = \frac{\int_\Omega |\nabla u(x)|^2 dx}{\int_\Omega m(x) u^2(x) dx + \int_{\partial\Omega} h_u(0, u) u^2(x) d\sigma} \quad \text{for} \;\; u \in W^{1,2}(\Omega).$$

As long as (2.3) and (2.4) hold, λ_1 is a unique principal eigenvalue of eigenvalue problem (2.2), with an associated eigenfunction φ_1 satisfying $\int_\Omega \varphi_1^2(x) dx = 1$. Therefore, $\mathrm{Ker}\mathcal{L}_{\lambda_1} = \mathrm{span}\,\{\varphi_1\}$. It is easy to see that $(y_1, y_2) \in \mathrm{Range}\mathcal{L}_{\lambda_1}$ if and only if

$$\int_\Omega \varphi_1(x) y_1(x) dx = \int_{\partial\Omega} \varphi_1(x) y_2(x) d\sigma.$$

Thus, $\mathcal{L}_{\lambda_1}$ is a Fredholm operator of index zero. Denote the spaces $\mathbb{X} = W^{2,p}(\Omega)$ and $\mathbb{Y} = L^p(\Omega) \times W^{(p-1)/p,p}(\partial\Omega)$. Then we decompose the spaces $\mathbb{X}$ and $\mathbb{Y}$ as

$$\mathbb{X} = \mathrm{Ker}\mathcal{L}_{\lambda_1} \oplus \mathbb{X}_1, \qquad \mathbb{Y} = \mathrm{Range}\mathcal{L}_{\lambda_1} \oplus \mathbb{Y}_1,$$

Next, we apply the Lyapunov-Schmidt reduction. Define projection Q: $\mathbb{Y} \to \mathbb{Y}_1$ and the equation $F(u, \lambda) = 0$ is equivalent to

$$QF(\xi + \eta, \lambda) = 0, \quad (I - Q) F(\xi + \eta, \lambda) = 0, \tag{2.5}$$

where $\xi \in \mathrm{Ker}\mathcal{L}_{\lambda_1}$ and $\eta \in \mathbb{X}_1$. Note that

$$(I-Q)F(0,\lambda)=0, \quad (I-Q)F_\xi(0,\lambda_1)=\mathcal{L}_{\lambda_1}.$$

So we can apply the implicit function theorem and obtain a continuously differentiable map $h: \ \mathcal{U} \to \mathbb{X}_1$ such that $h(0,\lambda)=0$ and

$$(I-Q)F(\xi+h(\xi,\lambda),\lambda)=0, \tag{2.6}$$

where $\mathcal{U}$ is an open neighborhood of $(0,\lambda_1)$ in $\mathrm{Ker}\mathcal{L}_{\lambda_1}\times\mathbb{R}$. Thus, the first equation in (2.5) can be written as

$$\mathcal{F}(\xi,\lambda) \triangleq QF(\xi+h(\xi,\lambda),\lambda)=0. \tag{2.7}$$

In view of $\mathcal{F}(0,\lambda_1)=0$ and $\mathcal{F}_\xi(0,\lambda_1)=0$, each solution to $\mathcal{F}(\xi,\lambda)=0$ in $\mathcal{U}$ one-to-one corresponds to some solution to $F(u,\lambda)=0$.

In order to obtain the coefficients of the terms in the Taylor expansion of the reduced equation, because of $\dim\mathbb{Y}_1=1$, we can find $\phi\in\mathbb{Y}$ satisfying $\|\phi\|_{\mathbb{Y}}=1$ such that $\mathbb{Y}_1=\mathrm{span}\{\phi\}$. From the Hahn-Banach theorem, there exists a vector ζ in the dual space $\mathbb{Y}^*$ of $\mathbb{Y}$ such that $\langle\zeta,\phi\rangle=1$ and $\langle\zeta,y\rangle=0$ for all $y\in\mathrm{Range}\mathcal{L}_{\lambda_1}$, where $\langle\cdot,\cdot\rangle:\mathbb{Y}^*\times\mathbb{Y}\to\mathbb{R}$ denotes the duality between$\mathbb{Y}^*$ and Y and is defined as

$$\langle v,u\rangle=\int_\Omega v(x)u_1(x)dx-\int_{\partial\Omega} v(x)u_2(x)d\sigma$$

for all $v\in\mathbb{Y}^*$ and $u=(u_1,u_2)\in\mathbb{Y}$. Obviously, there exists $\psi\in\mathbb{Y}^*$ such that

$$\langle\psi,u\rangle=\int_\Omega \varphi_1 u_1(x)dx-\int_{\partial\Omega}\varphi_1 u_2(x)d\sigma,$$

and hence $\mathrm{Ker}\psi=\mathrm{Range}\mathcal{L}_{\lambda_1}$. Thus, the projection Q is given by $Qy=\langle\psi,y\rangle\,\phi$ for $y\in\mathbb{Y}$. In view of $\mathrm{dimKer}\mathcal{L}_{\lambda_1}=1$ and equation (2.7), for all $v\varphi_1\in\mathrm{Ker}\mathcal{L}_{\lambda_1}$ with $v\in\mathbb{R}$, we can define $G:\mathbb{R}^2\to\mathbb{R}$ and

$$G(v,\lambda)=\langle\psi,\mathcal{F}(v\varphi_1,\lambda)\rangle=\langle\psi,F(v\varphi_1+h(v\varphi_1,\lambda),\lambda)\rangle\,.$$

It is easy to verify $G(0,\lambda_1)=0$ and

$$G(v,\lambda)=v\left[\varrho(\lambda-\lambda_1)+\frac{\kappa}{2}v+\frac{\vartheta}{6}v^2+o(v^2)\right],$$

where

$$\begin{cases}\varrho=\langle\psi,F_{\lambda u}\left[\varphi_1\right]\rangle\,,\\ \kappa=\langle\psi,F_{uu}\left[\varphi_1,\varphi_1\right]\rangle\,,\\ \vartheta=\langle\psi,F_{uuu}\left[\varphi_1,\varphi_1,\varphi_1\right]\rangle+3\,\langle\psi,F_{uu}\left[\varphi_1,h_{\xi\xi}\left[\varphi_1,\varphi_1\right]\right]\rangle\,.\end{cases}$$

Here, the bilinear form $F_{uu}\left[\cdot,\cdot\right]$ and tri-linear form $F_{uuu}\left[\cdot,\cdot,\cdot\right]$ denote the second- and third-order Fréchet derivatives of F with respect to u, evaluated at $(u,\lambda)=(0,\lambda_1)$, respectively. Let h_ξ and $h_{\xi\xi}$ be the first- and second-order Fréchet derivatives of h with respect to ξ, evaluated at $(\xi,\lambda)=(0,\lambda_1)$, respectively.

The following analysis will examine $\kappa\neq0$ and $\kappa=0$ individually. We start with the case where $\kappa\neq0$. Notice that $G(0,\lambda_1)=0$ and $G_v(0,\lambda_1)=\kappa\neq0$, then by applying the implicit function theorem we know that there exists a constant

$\delta > 0$ and a continuously differentiable mapping $v : (\lambda_1 - \delta, \lambda_1 + \delta) \to \mathbb{R}$ such that $G(v_\lambda, \lambda) = v_\lambda \left[\varrho(\lambda - \lambda_1) + \frac{\kappa}{2}v + o(v)\right] = 0$ and $v_{\lambda_1} = v$ for $\lambda \in (\lambda_1 - \delta, \lambda_1 + \delta)$, that is to say,

$$v_{\lambda_1} = \frac{2\varrho(\lambda_1 - \lambda)}{\kappa} + o(|\lambda - \lambda_1|).$$

Thus, for all $\lambda \in (\lambda_1 - \delta, \lambda_1 + \delta)$, (2.1) has a nontrivial solution $u_\lambda = v_\lambda\varphi_1 + h(v_\lambda\varphi_1, \lambda)$, and $\lim_{\lambda\to\lambda_1} u_\lambda = 0$.

Next consider the case where $\kappa = 0$ and $\vartheta \neq 0$. Then we get $G(v_\lambda, \lambda) = v_\lambda \left[\varrho(\lambda - \lambda_1) + \frac{\vartheta}{6}v^2 + o(v^2)\right] = 0$, so

$$v_{\lambda_1}^2 = \frac{6\varrho(\lambda_1 - \lambda)}{\vartheta} + o(|\lambda - \lambda_1|).$$

We can see that equation (2.1) has a nontrivial solution $v_\lambda^\pm$ when $\varrho\vartheta(\lambda - \lambda_1) > 0$.

It follows from (2.6) that $\mathcal{L}_{\lambda_1} h_{\xi\xi}[\varphi_1, \varphi_1] + (I - Q)F_{uu}[\varphi_1, \varphi_1] = 0$ and hence the three quantities ϱ, κ, and ϑ can be obtained as follows:

$$\begin{aligned}
\varrho &= \int_\Omega \varphi_1^2(x)m(x)dx - \int_{\partial\Omega} \varphi_1^2(x)h_u(x,0)d\sigma, \\
\kappa &= \int_\Omega \varphi_1(x)\left[2d\nabla\cdot(\varphi_1(x)\nabla\varphi_1(x)) - 2\lambda_1\varphi_1^2(x)\right]dx + \int_{\partial\Omega} \lambda_1\varphi_1^3(x)h_{uu}(x,0)d\sigma, \\
\vartheta &= 3\int_\Omega \varphi_1(x)\left[d\nabla\cdot(\zeta_0(x)\nabla\varphi_1(x)) + d\nabla\cdot(\varphi_1(x)\nabla\zeta_0(x)) - 2\lambda_1\zeta_0(x)\varphi_1(x)\right]dx \\
&\quad + \int_{\partial\Omega} \lambda_1\zeta_0(x)\varphi_1^2(x)h_{uu}(x,0) + \int_{\partial\Omega} \lambda_1\varphi_1^4(x)h_{uuu}(x,0)d\sigma,
\end{aligned} \tag{2.8}$$

where $\zeta_0(x) = h_{\xi\xi}[\varphi_1, \varphi_1]$. To conclude, the following result is established.

Theorem 2.1. **(i)** *Under the assumptions (2.3) and (2.4), if $\kappa \neq 0$, there exists a constant $\delta > 0$ and a continuously differentiable mapping $v : \lambda \to v_\lambda$ from $(\lambda_1 - \delta, \lambda_1 + \delta)$ to $\mathbb{R}$ such that (2.1) has a nontrivial solution $u_\lambda = v_\lambda\varphi_1 + h(v_\lambda\varphi_1, \lambda)$, which satisfies $\lim_{\lambda\to\lambda_1} u_\lambda = 0$.*

(ii) *Under the assumptions (2.3) and (2.4), if $\kappa = 0$ and $\varrho\vartheta < 0$ (respectively,> 0) then there exist a constant $\lambda^* > \lambda_1$ (respectively, $\lambda^* < \lambda_1$) and two continuously differentiable mappings $\lambda \to v_\lambda^\pm$ from (λ_1, λ^*) to $\mathbb{R}$ (respectively, from (λ^*, λ_1) to $\mathbb{R}$) such that (2.1) has two nontrivial solutions $u_\lambda^\pm = v_\lambda^\pm\varphi_1 + h(v_\lambda^\pm\varphi_1, \lambda)$, which satisfies $\lim_{\lambda\to\lambda_1} u_\lambda^\pm = 0$.*

2.2. Bifurcation from $(u_0, 0)$ for some $u_0 \in \mathbb{R}$

In this section, we consider bifurcation from $(u_0, 0)$ for some $u_0 \in \mathbb{R}$. Note that $F_u(\cdot\,, 0)$ satisfies

$$F_u(\cdot\,, 0) = \begin{pmatrix} (1 + du_0)\Delta \\ \frac{\partial}{\partial\mathbf{n}} \end{pmatrix}. \tag{2.9}$$

Then we have $\text{Ker}F_u(\cdot\,,0) = \text{span}\,\{1\}$. Subsequently, we consider the range space of $F_u(\cdot\,,0)$. Let $(y_1, y_2) \in \text{Range}F_u(\cdot\,,0)$ and $w(x) \in \mathbb{X}$ satisfy

$$\begin{cases}(1+du_0)\Delta w = y_1, & x \in \Omega,\\ \dfrac{\partial w}{\partial \mathbf{n}} = y_2, & x \in \partial\Omega.\end{cases}$$

It is easy to see that $(y_1, y_2) \in \text{Range}F_u(\cdot\,,0)$ if and only if

$$\int_\Omega y_1(x)dx = \int_{\partial\Omega}(1+du_0)y_2(x)d\sigma.$$

Therefore, we can infer that $\text{codimRange}F_u(\cdot\,,0) = 1$ and $F_u(\cdot\,,0)$ is a Fredholm operator of index zero. We decompose the spaces $\mathbb{X}$ and $\mathbb{Y}$ as

$$\mathbb{X} = \text{Ker}F_u(\cdot\,,0) \oplus \mathbb{X}_2, \qquad \mathbb{Y} = \text{Range}F_u(\cdot\,,0) \oplus \mathbb{Y}_2,$$

where $\mathbb{X}_2$ is the complement space of $\text{Ker}F_u(\cdot\,,0)$ in $\mathbb{X}$, $\mathbb{Y}_2$ is the complement space of $\text{Range}F_u(\cdot\,,0)$ in $\mathbb{Y}$. Using a similar argument from the previous subsection, we can argue that original bifurcation problem can be reduced to the problem of finding zeros of a map $\widetilde{G} : \mathbb{R}^2 \to \mathbb{R}$ given by

$$\widetilde{G}(v,\lambda) = \int_\Omega F_1(u_0 + v + \widetilde{h}(v,\lambda))dx - \int_{\partial\Omega}(1+du_0)F_2(u_0 + v + \widetilde{h}(v,\lambda))d\sigma,$$

where $\widetilde{h} : \mathbb{R} \to \mathbb{X}_2$ is a continuously differentiable map satisfying $h(0,0) = 0$ and

$$(I - \widetilde{Q})F(u_0 + v + \widetilde{h}(v,\lambda),\lambda) = 0,$$

and $\widetilde{Q}$ denotes the projection operator from $\mathbb{Y}$ onto $\mathbb{Y}_2$ along $\text{Range}F_u(\cdot\,,0)$. Notice that $\widetilde{G}(0,0) = 0$, $\widetilde{G}_v(0,0) = 0$ and

$$\widetilde{G}_\lambda(0,0) = \int_\Omega u_0(m(x) - u_0)dx + \int_{\partial\Omega}(1+du_0)h(x,u_0)d\sigma.$$

Due to the implicit function theorem, if $\widetilde{G}_\lambda(0,0) \neq 0$, then $(u_0, 0)$ is not a bifurcating point of the map F. So we arrive at the following result.

Theorem 2.2. *If $\widetilde{G}_\lambda(0,0) \neq 0$, then no steady-state bifurcation occurs in the vicinity of $(u,\lambda) = (u_0, 0)$, that is to say, the steady-state solution set of (1.1) near $(u_0, 0)$ consists precisely of the trivial curve $\{(u,0) : u \in \mathbb{R}\}$.*

In the following, we shall focus on the specific case where $\widetilde{G}_\lambda(0,0) = 0$, i.e.,

$$\int_\Omega u_0(m(x) - u_0)dx + \int_{\partial\Omega}(1+du_0)h(x,u_0)d\sigma = 0. \tag{2.10}$$

Given that $\nabla\widetilde{G}(0,0) = 0$, we proceed to compute the Hessian matrix of $\widetilde{G}$ evaluated at $(0,0)$, which is given by

$$\text{Hess}(\widetilde{G}) = \begin{pmatrix}\widetilde{G}_{vv}(0,0) & \widetilde{G}_{v\lambda}(0,0)\\ \widetilde{G}_{\lambda v}(0,0) & \widetilde{G}_{\lambda\lambda}(0,0)\end{pmatrix}.$$

In view of Lemma 2.5 of Liu, Shi and Wang [21], we have the following results:

(i) If $\det\mathrm{Hess}(\widetilde{G}) > 0$, then $(0,0)$ is the unique zero of $\widetilde{G}$ near $(0,0)$;

(ii) If $\det\mathrm{Hess}(\widetilde{G}) < 0$, for $s \in (-\sigma, \sigma)$, there exist two C^{k-1} curves $(v_1(s), \lambda_1(s))$ and $(v_2(s), \lambda_2(s))$ satisfying $(v_1(0), \lambda_1(0)) = (0,0) = (v_2(0), \lambda_2(0))$, such that the solution set of $\widetilde{G}(v, \lambda) = 0$ consists of exactly these two curves near $(0,0)$. Moreover, s can be rescaled so that $(v_1'(0), \lambda_1'(0))$ and $(v_2'(0), \lambda_2'(0))$ are the two linear independent solutions of

$$\widetilde{G}_{vv}(0,0)x^2 + 2\widetilde{G}_{v\lambda}(0,0)xy + \widetilde{G}_{\lambda\lambda}(0,0)y^2 = 0.$$

Note that $\widetilde{G}_{vv}(0,0) = \widetilde{G}_{\lambda\lambda}(0,0) = 0$ and

$$\widetilde{G}_{v\lambda}(0,0) = \int_\Omega (m(x) - 2u_0)dx + \int_{\partial\Omega} (1 + du_0)h_u(x, u_0)d\sigma.$$

Thus, if

$$\Xi \triangleq \int_\Omega (m(x) - 2u_0)dx + \int_{\partial\Omega} (1 + du_0)h_u(x, u_0)d\sigma \neq 0, \tag{2.11}$$

then $\det\mathrm{Hess}(\widetilde{G}) = -\widetilde{G}^2_{v\lambda}(0,0) < 0$ and there exist two C^{k-1} curves $(v_1(s), \lambda_1(s))$ and $(v_2(s), \lambda_2(s))$ for $s \in (-\sigma, \sigma)$ satisfying $(v_1(0), \lambda_1(0)) = (0,0) = (v_2(0), \lambda_2(0))$, such that the solution set of $\widetilde{G}(v, \lambda) = 0$ consists of exactly these two curves near $(0,0)$ and that $(v_1'(0), \lambda_1'(0)) = (0,1)$ and $(v_2'(0), \lambda_2'(0)) = (1,0)$. In particular, the solution curve $(v_2(s), \lambda_2(s)) = (1,0)s + o(s)$ is identical to the trivial branch $\{(u,0) : u > 0\}$. Consequently, we establish the following theorem regarding the existence of two solution curves that are tangent to each other at the bifurcation point.

Theorem 2.3. *Assume that there exists $u_0 \in \mathbb{R}$ such that both (2.10) and (2.11) hold. Then the set of solutions of (2.1) near $(u, \lambda) = (u_0, 0)$ consists precisely of the trivial curve $\{(u,0) : u \in \mathbb{R}\}$ and the curve $\{(u_s^*, \lambda(s)) : s \in (-\sigma, \sigma)$ for some $\sigma > 0\}$, where $u_1(s)$ takes the form*

$$u_1(s) = u_0 + v_1(s) + \widetilde{h}(v_1(s), \lambda_1(s))$$

with $(v_1(0), \lambda_1(0)) = (0,0)$ and $(v_1'(0), \lambda_1'(0)) = (0,1)$.

3. Stability of steady-state solutions

In this section, we investigate the eigenvalue problem of model (1.1) at the nontrivial steady-state solution. Then we carry out a stability analysis and determine the conditions under which a Hopf bifurcation occurs at this nontrivial steady-state solution, guided by the distribution of eigenvalues.

3.1. Steady-state solutions established in Theorem 2.1

Theorem 2.1 guarantees the existence of an open set $\Lambda \subseteq \mathbb{R}$ containing λ_1 on its boundary, for which system (1.1) has a spatially nonhomogeneous steady-state solution u_λ^* when $\lambda \in \Lambda$. Furthermore, the form of u_λ^* is given by $u_\lambda^* = v_\lambda\varphi_1 + h(v_\lambda\varphi_1, \lambda)$, with v_λ satisfying $G(v_\lambda, \lambda) = 0$ as defined in Section 2.1. To explore the local dynamical behavior near the steady-state solution $u = u_\lambda^*$ of system (1.1), we linearize system (1.1) at that point:

$$\begin{cases} \dfrac{\partial v}{\partial t} = \Delta v + d\nabla\cdot(v\nabla u_\lambda^*) + d\nabla\cdot(u_\lambda^*\nabla v_\tau) + \lambda v(m(x) - u_\lambda^*) - \lambda u_\lambda^* v_\tau, & x \in \Omega, \\ \dfrac{\partial v}{\partial \mathbf{n}} = \lambda h_u(x, u_\lambda^*)v, & x \in \partial\Omega. \end{cases} \tag{3.1}$$

The characteristic equation for this linear system is obtained by considering solutions of the form $v(x,t) = u(x)\exp\{\mu t\}$ with $u \in \mathbb{X}$. Such solutions are nontrivial if and only if $m(\lambda, \mu, \tau)u = 0$ has a nontrivial solution u, where for $(\lambda, \mu, \tau) \in \Lambda \times \mathbb{C} \times \mathbb{R}^+$, $m(\lambda, \mu, \tau) : \mathbb{X} \to \mathbb{Y}$ and $m(\lambda, \mu, \tau)u$ is defined as

$$\begin{pmatrix} \Delta u + d\nabla\cdot(u\nabla u_\lambda^*) + d\nabla\cdot(u_\lambda^*\nabla u)e^{-\mu\tau} + \lambda u(m(x) - u_\lambda^*) - \lambda u_\lambda^* u e^{-\mu\tau} - \mu u \\ \frac{\partial u}{\partial \mathbf{n}} - \lambda h_u(x, u_\lambda^*)u \end{pmatrix}.$$

Denote by $\mathcal{A}_{\tau,\lambda}$ the infinitesimal generator of the semigroup generated by the linearized system (3.1). In fact, for all $\psi \in C_\tau^1$ satisfying $\frac{\partial}{\partial \mathbf{n}}\psi(0) = \lambda h_u(x, u_\lambda^*)\psi(0)$, we have $(\mathcal{A}_{\tau,\lambda}\psi)(\theta) = \psi'(\theta)$ for all $\theta \in [-\tau, 0)$, and

$$\begin{aligned} \mathcal{A}_{\tau,\lambda}\psi(0) =& \Delta\psi(0) + d\nabla\cdot(\psi(0)\nabla u_\lambda^*) + d\nabla\cdot(u_\lambda^*\nabla\psi(-\tau)) \\ &+ \lambda\psi(0)(m(x) - u_\lambda^*) - \lambda u_\lambda^*\psi(-\tau). \end{aligned}$$

Thus, the spectrum of $\mathcal{A}_{\tau,\lambda}$ is

$$\sigma(\mathcal{A}_{\tau,\lambda}) = \{\mu \in \mathbb{C} : m(\lambda, \mu, \tau)u = 0 \text{ for some } u \in \mathbb{X}_\mathbb{C} \setminus \{0\}\}.$$

In order to define the adjoint operator $\mathcal{A}_{\tau,\lambda}^*$ of $\mathcal{A}_{\tau,\lambda}$, we need to define the bilinear form and consider the formal adjoint equation of system (3.1):

$$\begin{cases} \dfrac{\partial u}{\partial t} = -\Delta u - du\Delta u_\lambda^* - \lambda u(m(x) - u_\lambda^*) + \lambda u_\lambda^* u(t+\tau), & x \in \Omega, \\ \dfrac{\partial u}{\partial \mathbf{n}} = -\lambda h_u(x, u_\lambda^*)u, & x \in \partial\Omega. \end{cases} \tag{3.2}$$

Assuming that v and u are solutions to equations (3.1) and (3.2), respectively, we obtain

$$\frac{d}{dt}\int_\Omega uv dx = \int_\Omega du_\lambda^*(v(t-\tau)\Delta u - v\Delta u(t+\tau))dx + \int_\Omega d\nabla u_\lambda^*(u\nabla v(t-\tau) - u(t+\tau)\nabla v)dx,$$

which implies that

$$\int_\Omega uv dx + \int_{t-\tau}^t \int_\Omega du_\lambda^*(v(\theta)\Delta u(\tau+\theta)) + d\nabla u_\lambda^*(u(\tau+\theta)\nabla v(\theta))dx d\theta$$

is a constant for all $t \geq -\tau$. Thus, one can set $t = 0$ in the above function to define the bilinear form

$$\begin{aligned} \langle\psi, \varphi\rangle_1 =& \int_{-\tau}^0 \int_\Omega \left[du_\lambda^*(\varphi(x,\theta)\Delta\overline{\psi(x,\tau+\theta)}) + d\nabla u_\lambda^*(\overline{\psi(x,\tau+\theta)}\nabla\varphi(x,\theta)\right] dx d\theta \\ &+ \int_\Omega \overline{\psi(x,0)}\varphi(x,0)dx. \end{aligned}$$

Then one has $\langle \mathcal{A}^*_{\tau,\lambda}\psi, \varphi\rangle_1 = \langle \psi, \mathcal{A}_{\tau,\lambda}\varphi\rangle_1$ for all $\varphi \in C^1_\tau$ and $\psi \in C^1([0,\tau], \mathbb{X}^*_{\mathbb{C}})$, where $\mathcal{A}^*_{\tau,\lambda}$ denotes the adjoint operator of $\mathcal{A}_{\tau,\lambda}$, that is, for all $\psi \in C^1([0,\tau], \mathbb{X}^*_{\mathbb{C}})$ satisfying $\frac{\partial}{\partial \mathbf{n}}\psi(0) = \lambda h_u(x, u^*_\lambda)\psi(0)$, we have $(\mathcal{A}^*_{\tau,\lambda}\psi)(\theta) = -\psi'(\theta)$ for $\theta \in (0,\tau]$, and

$$\begin{aligned}(\mathcal{A}^*_{\tau,\lambda}\psi)(0) =& \Delta\psi(0) + d\nabla\cdot(\psi(0)\nabla u^*_\lambda) + d\nabla\cdot(u^*_\lambda\nabla\psi(\tau)) \\ &+ \lambda\psi(0)(m(x) - u^*_\lambda) - \lambda u^*_\lambda\psi(\tau).\end{aligned}$$

Thus, $\mu \in \sigma(\mathcal{A}_{\tau,\lambda})$ if and only if there exists $v \in \mathbb{X}^*_{\mathbb{C}}\setminus\{0\}$ such that $m^*(\lambda,\mu,\tau)v = 0$, where $m^*(\lambda,\mu,\tau)$: $\mathbb{X}^* \to \mathbb{Y}^*$ and $m^*(\lambda,\mu,\tau)u$ is defined as

$$\begin{pmatrix} \Delta u + d\nabla\cdot(u\nabla u^*_\lambda) + d\nabla\cdot(u^*_\lambda\nabla u)e^{-\mu\tau} + \lambda u(m(x) - u^*_\lambda) - \lambda u^*_\lambda u e^{-\mu\tau} - \mu u \\ \frac{\partial u}{\partial \mathbf{n}} - \lambda h_u(x, u^*_\lambda)u \end{pmatrix}.$$

Thus, we conclude that $\mu \in \sigma(\mathcal{A}_{\tau,\lambda})$ if and only if there exists $u \in \mathbb{X}_{\mathbb{C}} \setminus \{0\}$ such that $m(\lambda,\mu,\tau)u = 0$, which is also equivalent to that there exists $v \in \mathbb{X}^*_{\mathbb{C}} \setminus \{0\}$ such that $m^*(\lambda,\mu,\tau)u = 0$.

We now consider the existence of zero eigenvalues of $\mathcal{A}_{\tau,\lambda}$.

Lemma 3.1. **(i)** *For each $(\tau,\lambda) \in \mathbb{R}^+ \times \Lambda$, if $0 \in \sigma(\mathcal{A}_{\tau,\lambda})$, then $\kappa = 0$.*

(ii) *For each $(\tau,\lambda) \in \mathbb{R}^+ \times \Lambda$, if $\kappa = 0$ and $F_{uu}[\varphi_1,\varphi_1] \neq 0$, then $0 \in \sigma(\mathcal{A}_{\tau,\lambda})$.*

Proof. If $0 \in \sigma(\mathcal{A}_{\tau,\lambda})$, then there exists $u \in \mathbb{X}_{\mathbb{C}} \setminus \{0\}$ such that $m(\lambda, 0, \tau)u = 0$. Note that $m(\lambda_1, 0, \tau)u = \mathcal{L}_{\lambda_1}$ and $\mathrm{Ker}\mathcal{L}_{\lambda_1} = \{\varphi_1\}$, then u takes the form

$$u = a_\lambda\varphi_1 + v_\lambda b_\lambda,$$

where $a_\lambda \in \mathbb{R}$, $b_\lambda \in \mathbb{X}_1$ and $a^2_\lambda + b^2_\lambda \neq 0$, then we have

$$a_\lambda m(\lambda, 0, \tau)\varphi_1 + v_\lambda m(\lambda, 0, \tau)b_\lambda = 0.$$

It is easy to see that

$$\widetilde{\varphi}_1 \triangleq \lim_{\lambda\to\lambda_1} \frac{m(\lambda,0,\tau)\varphi_1}{v_\lambda} = F_{uu}[\varphi_1,\varphi_1] + F_{\lambda u}[\varphi_1]\lim_{\lambda\to\lambda_1}\frac{\lambda-\lambda_1}{v_\lambda},$$

then

$$\langle\psi, \widetilde{\varphi}_1\rangle = \langle\psi, F_{uu}[\varphi_1,\varphi_1]\rangle + \left\langle \psi, F_{\lambda u}[\varphi_1](-\frac{\kappa}{2\varrho})\right\rangle = \frac{\kappa}{2}.$$

Thus, we have $a_{\lambda_1}\widetilde{\varphi}_1 + \mathcal{L}_{\lambda_1}b_{\lambda_1} = 0$. If $\kappa \neq 0$, then $\widetilde{\varphi}_1 \notin \mathrm{Range}\mathcal{L}_{\lambda_1}$, then $a_{\lambda_1} = 0$. This, together with the fact that $b_{\lambda_1} \in \mathbb{X}_1$ and $\mathcal{L}_{\lambda_1}$ is invertible if it is restricted in $\mathbb{X}_1$, implies that $b_{\lambda_1} = 0$, which contradicts $a^2_\lambda + b^2_\lambda \neq 0$. Therefore, $\kappa = 0$.

If $\kappa = 0$ and $F_{uu}[\varphi_1,\varphi_1] \neq 0$, let $u = \varphi_1 + v_\lambda b$, where $b \in \mathbb{X}_1$. Substituting u into $m(\lambda, 0, \tau)u = 0$, we have

$$H(b,\lambda) \triangleq \frac{m(\lambda,0,\tau)\varphi_1}{v_\lambda} + m(\lambda,0,\tau)b = 0$$

and

$$\lim_{\lambda\to\lambda_1} H(b,\lambda) = F_{uu}[\varphi_1,\varphi_1] + \mathcal{L}_{\lambda_1}b.$$

It follows from $\kappa = 0$ that $F_{uu}[\varphi_1,\varphi_1] \in \mathrm{Range}\mathcal{L}_{\lambda_1}$, and hence there exists $b^* \in \mathbb{X}_{\mathbb{C}} \setminus \{0\}$ such that $F_{uu}[\varphi_1,\varphi_1] + \mathcal{L}_{\lambda_1}b^* = 0$, that is to say, $H(b^*,\lambda_1) = 0$. Note that $F_{uu}[\varphi_1, \varphi_1] \neq 0$ and $H_b(b^*, \lambda_1) = \mathcal{L}_{\lambda_1}$. Then applying the implicit function theorem, there exists a continuously differentiable mapping $\lambda \to b_\lambda$ from Λ to $\mathbb{X}_1$ such that $b_{\lambda_1} = b^*$ and $H(b_\lambda, \lambda) = 0$, i.e., $m(\lambda, 0, \tau)u = 0$ has nontrivial solution $u = \varphi_1 + v_\lambda b_\lambda$. Therefore $0 \in \sigma(\mathcal{A}_{\tau,\lambda})$. The proof is completed. □

Considering Lemma 3.1 and Theorem 2.1, the following results are obtained.

Theorem 3.1. **(i)** *Under the assumptions (2.3) and (2.4), if $\kappa \neq 0$, then there exists $\delta > 0$ such that for each $\lambda \in \Lambda = (\lambda_1 - \delta) \cup (\lambda_1 + \delta)$, system (1.1) has exactly one spatially nonhomogeneous steady state solution, whose associated infinitesimal generator $\mathcal{A}_{\tau,\lambda}$ has no zero eigenvalue.*

(ii) *Under the assumptions (2.3) and (2.4), if $\kappa = 0$ and $F_{uu}[\varphi_1, \varphi_1] \neq 0$ and $\varrho\vartheta < 0$ (respectively, > 0), then there exists a constant $\delta > 0$ such that for each $\lambda \in \Lambda = (\lambda_1, \lambda_1 + \delta)$ (respectively, $\lambda \in \Lambda = (\lambda_1 - \delta, \lambda_1)$), system (1.1) has exactly two spatially nonhomogeneous steady state solutions, each of whose associated infinitesimal generator has a zero eigenvalue.*

In what follows, we investigate the existence of purely imaginary eigenvalues of $\mathcal{A}_{\tau,\lambda}$. For each $(\tau, \lambda) \in \mathbb{R}^+ \times \Lambda$, $i\omega \in \sigma(\mathcal{A}_{\tau,\lambda})$ if and only if there exists $u \in \mathbb{X}_{\mathbb{C}} \setminus \{0\}$ such that

$$m(\lambda, i\omega, \tau)u = 0. \tag{3.3}$$

Lemma 3.2. *If $(\omega, \tau, u) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{X}_{\mathbb{C}} \setminus \{0\}$ solves (3.3), then ω can be regarded as the function of λ and $\frac{\omega(\lambda)}{v_\lambda}$ is bounded.*

Proof. From (3.3), it is deduced that

$$\begin{aligned} 0 &= \langle \overline{u}, m(\lambda, i\omega, \tau)u \rangle \\ &= \int_{\partial\Omega} \lambda h_u(x, u_\lambda^*)|u|^2 d\sigma + \int_\Omega \overline{u}[\Delta u + d\nabla\cdot(u\nabla u_\lambda^*) + d\nabla\cdot(u_\lambda^*\nabla u)e^{-i\omega\tau}]dx \\ &\quad + \int_\Omega \lambda|u|^2(m(x) - u_\lambda^*) - \lambda u_\lambda^*|u|^2 e^{-i\omega\tau} - i\omega|u|^2 - |\nabla u|^2 dx. \end{aligned}$$

Separating the real and imaginary parts, we deduce that

$$\hbar(\omega, \lambda) \triangleq \omega \int_\Omega |u|^2 dx + d\sin(\omega\tau) \int_\Omega \overline{u}\cdot(u_\lambda^*\nabla u)dx - \sin(\omega\tau)\lambda \int_\Omega u_\lambda^*|u|^2 dx.$$

Note that $\hbar(0, \lambda_1) = 0$ and $\hbar_\omega(0, \lambda_1) = |u|^2|\Omega| \neq 0$. Then applying the implicit function theorem, there exists $\omega(\lambda)$ such that $\omega(\lambda_1) = 0$ and $\hbar(\omega(\lambda), \lambda) = 0$, i.e.,

$$\omega(\lambda) \int_\Omega |u|^2 dx + d\sin(\omega(\lambda)\tau) \int_\Omega \overline{u}\cdot(u_\lambda^*\nabla u)dx - \sin(\omega(\lambda)\tau)\lambda \int_\Omega u_\lambda^*|u|^2 dx.$$

Furthermore, the boundedness of $\frac{\omega(\lambda)}{v_\lambda}$ is readily apparent, thereby completing the proof. □

We can reformulate the target equation $m(\lambda, i\omega, \tau)u = 0$ as equation $\widetilde{m}(\lambda, \gamma, \theta)u = 0$, where $\widetilde{m}(\lambda, \gamma, \theta) : \mathbb{X}_{\mathbb{C}} \to \mathbb{Y}_{\mathbb{C}}$ and $\widetilde{m}(\lambda, \gamma, \theta)u$ is given by

$$\begin{pmatrix} \Delta u + d\nabla\cdot(u\nabla u_\lambda^*) + d\nabla\cdot(u_\lambda^*\nabla u)e^{-i\theta} + \lambda u(m(x) - u_\lambda^*) - \lambda u_\lambda^* u e^{-i\theta} - iv_\lambda\gamma u \\ \frac{\partial u}{\partial \mathbf{n}} - \lambda h_u(x, u_\lambda^*)u \end{pmatrix},$$

where $\omega = v_\lambda\gamma$ and $\omega\tau = \theta + 2j\pi$ $(j \in \mathbb{N})$. Note that $\widetilde{m}(\lambda_1, \gamma, \theta) = \mathcal{L}_{\lambda_1}$ and $\mathrm{Ker}\mathcal{L}_{\lambda_1} = \mathrm{span}\,\{\varphi_1\}$. Let $u = \varphi_1 + v_\lambda b$, where $b \in \mathbb{X}_{1\mathbb{C}}$. Consequently, $\mathcal{H}(b, \gamma, \theta, \lambda) = 0$ holds, with $\mathcal{H} : \mathbb{X}_{1\mathbb{C}} \times \mathbb{R} \times [0, 2\pi) \times \Lambda \to \mathbb{Y}$ defined by

$$\mathcal{H}(b, \gamma, \theta, \lambda) = \frac{\widetilde{m}(\lambda, \gamma, \theta)\varphi_1}{v_\lambda} + \widetilde{m}(\lambda, \gamma, \theta)b.$$

Obviously, we have $\mathcal{H}(b, \gamma, \theta, \lambda_1) = \Phi(\gamma, \theta) + \mathcal{L}_{\lambda_1} b$, where

$$\begin{aligned}\Phi(\gamma, \theta) =& F_{\lambda u}(\varphi_1, \varphi_1)\frac{\lambda - \lambda_1}{v_\lambda} \\ &+ \begin{pmatrix} d\nabla\cdot(\varphi_1\nabla\varphi_1) + d\nabla\cdot(\varphi_1\nabla\varphi_1)e^{-i\theta} - \lambda_1\varphi_1^2 - \lambda_1\varphi_1^2 e^{-i\theta} - i\gamma\varphi_1 \\ -\lambda_1 h_{uu}(x, 0)\varphi_1^2 \end{pmatrix}.\end{aligned}$$

For convenience, let

$$\kappa_1 = \int_\Omega \varphi_1\left[d\nabla\cdot(\varphi_1\nabla\varphi_1) - \lambda_1\varphi_1^2\right]dx, \quad \kappa_2 = \int_{\partial\Omega} \lambda_1 h_{uu}(x, 0)\varphi_1^3 d\sigma. \tag{3.4}$$

Then we have $\kappa = 2\kappa_2 + \kappa_1$. Define $\mathcal{R} : \mathbb{R} \times [0, 2\pi) \to \mathbb{R}$ as $\mathcal{R}(\gamma, \theta) = \langle\psi, \Phi(\gamma, \theta)\rangle$. Consequently, it follows that

$$\mathcal{R}(\gamma, \theta) = \kappa_1 e^{-i\theta} + \frac{\kappa_2}{2} - i\gamma,$$

where

$$\gamma_0 = \frac{\sqrt{4\kappa_1^2 - \kappa_2^2}}{2}, \quad \theta_0 = Arg\frac{-k_2 - 2\gamma i}{2\kappa_1}.$$

Therefore

$$\kappa \neq 0, \quad 4\kappa_1^2 \geq \kappa_2^2 \tag{3.5}$$

is sufficient for $\mathcal{R}(\gamma, \theta) = 0$ to hold. It follows that $\Phi(\gamma, \theta) \in \mathrm{Range}\mathcal{L}_{\lambda_1}$, which implies that there exists $b^* \in \mathbb{X}_{1\mathbb{C}} \setminus \{0\}$ such that $\Phi(\gamma, \theta) + \mathcal{L}_{\lambda_1} b^* = 0$, i.e., $\mathcal{H}(b^*, \gamma, \theta, \lambda_1) = 0$. This, combined with the implicit function theorem and the isomorphic property of $\mathcal{H}_{(b,\gamma,\theta)}(b, \gamma, \theta, \lambda_1)$, implies there exists a continuously differentiable mapping $\lambda \to (b(\lambda), \gamma(\lambda), \theta(\lambda))$ from Λ to $\mathbb{X}_{1\mathbb{C}} \times \mathbb{R} \times [0, 2\pi)$ such that $b(\lambda_1) = b^*$, $\gamma(\lambda_1) = \gamma_0$, $\theta(\lambda_1) = \theta_0$ and $\mathcal{H}(b(\lambda), \gamma(\lambda), \theta(\lambda), \lambda) = 0$ for all $\lambda \in \Lambda$. Thus, $\widetilde{m}(\lambda, \gamma(\lambda), \theta(\lambda))u = 0$ has a nontrivial solution $u = u_\lambda \in \mathbb{X}_\mathbb{C}$, where $u_\lambda = \varphi_1 + v_\lambda b(\lambda)$. This implies that the eigenfunction of $i\omega_\lambda \in \sigma(\mathcal{A}_{\tau_{j,\lambda},\lambda})$ is u_λ, where

$$\omega_\lambda = v_\lambda\gamma(\lambda), \;\; \tau_{j,\lambda} = \frac{\theta(\lambda) + 2j\pi}{\omega(\lambda)} \;\; (j \in \mathbb{N}). \tag{3.6}$$

In conclusion, the following result is obtained.

Lemma 3.3. *Under the condition (3.5), for each* $\lambda \in \Lambda$, $\omega(\lambda)$ *and* $\tau_{j,\lambda}$ *defined as in (3.6),* $\mathcal{A}_{\tau,\lambda}$ *has a pair of simple purely imaginary eigenvalues at* $\tau = \tau_{j,\lambda}$*. Moreover,*

(i) *These purely imaginary eigenvalues are* $\pm i\omega_\lambda$*.*

(ii) *The eigenspace associated with eigenvalue* $i\omega_\lambda$ *is only spanned by* ζ_0*, where* $\zeta_0(x, t) = u_\lambda(x)\exp\{i\omega_\lambda t\}$*.*

(iii) *There exist $\delta > 0$ and continuously differentiable mapping: $\mu : (\tau_{j,\lambda} - \delta, \tau_{j,\lambda} + \delta) \to \mathbb{C}$ such that $\mu(\tau_{j,\lambda}) = i\omega_\lambda$ and $\mu(\tau) \in \mathcal{A}_{\tau,\lambda}$ for all $\tau \in (\tau_{j,\lambda} - \delta, \tau_{j,\lambda} + \delta)$. Furthermore, $\frac{d}{dt} Re\{\mu(\tau)\}\,|_{\tau=\tau_{j,\lambda}} > 0$ for λ close enough to λ_1.*

(iv) *There exists $\varrho_\lambda = \varphi_1 + v_\lambda d(\lambda) \in \mathbb{Y}^*_{\mathbb{C}} \setminus \{0\}$ such that $m^*(\lambda, -i\omega_\lambda, \tau_\lambda)\varrho_\lambda = 0$. Moreover,*

$$\lim_{\lambda \to \lambda_1} \Pi_{j,\lambda} = 1,$$

where $\Pi_{j,\lambda} = \left\langle \varrho_\lambda e^{i\omega\theta}, \zeta_0(x,t) \right\rangle_1$, that is,

$$\Pi_{j,\lambda} = \int_\Omega \overline{\varrho_\lambda} u_\lambda dx + d\tau e^{-i\omega\tau} \int_\Omega u^*_\lambda (u_\lambda \Delta \overline{\varrho_\lambda}) + \nabla u^*_\lambda (\overline{\varrho_\lambda} \nabla u_\lambda) dx.$$

Proof. From the preceding analysis, it is evident that assumptions (i), (ii) and (iv) are satisfied. Consequently, it remains to confirm (iii) and demonstrate that $i\omega_\lambda$ is a simple eigenvalue of $\mathcal{A}_{\tau,\lambda}$. Conversely, assume that $i\omega_\lambda$ is not simple. Then there exists $\psi \in C^1([-\tau, 0], \mathbb{X}_{\mathbb{C}})$ such that $(\mathcal{A}_{\tau,\lambda} - i\omega_\lambda)\psi = ue^{i\omega_\lambda(\cdot)}$, that is to say,

$$\begin{cases} \psi'(\theta) = - i\omega_\lambda \psi(\theta) + ue^{i\omega_\lambda \theta}, \\ \psi'(0) = \Delta\psi(0) + d\nabla\cdot(\psi(0)\nabla u^*_\lambda) + d\nabla\cdot(u^*_\lambda \nabla\psi(-\tau)) \\ \qquad\qquad + \lambda\psi(0)(m(x) - u^*_\lambda) - \lambda u^*_\lambda \psi(-\tau). \end{cases} \tag{3.7}$$

From the first equation of (3.7), we deduce that $\psi(\theta) = (\theta u + p)e^{i\omega_\lambda \theta}$, with p and u satisfying

$$\frac{\partial u}{\partial \mathbf{n}} = \lambda h_u(x, u^*_\lambda)u, \quad \frac{\partial p}{\partial \mathbf{n}} = \lambda h_u(x, u^*_\lambda)u \quad on\ \partial\Omega.$$

Substituting $\psi(\theta) = (\theta u + p)e^{i\omega_\lambda \theta}$ into the second equation of (3.7) results in

$$u = \Delta p + d\nabla\cdot(p\nabla u^*_\lambda) + d\nabla\cdot(u^*_\lambda \nabla p)e^{-i\omega_\lambda \tau} + \lambda p(m(x) - u^*_\lambda) - \lambda u^*_\lambda p e^{-i\omega_\lambda \tau} - i\omega_\lambda p.$$

Thus, we have

$$\begin{pmatrix} u \\ \frac{\partial p}{\partial \mathbf{n}} - \lambda h_u(x, u^*_\lambda)p \end{pmatrix} = \begin{pmatrix} u \\ 0 \end{pmatrix} = m(\lambda, i\omega_\lambda, \tau_\lambda)p.$$

Computing the inner product of the aforementioned equation with $\overline{\varrho_\lambda}$ gives $-\Pi_{j,\lambda} = 0$, leading to a contradiction. Therefore, $i\omega_\lambda$ is simple.

Note that $u_\lambda \notin \text{Range}m(\lambda, i\omega_\lambda, \tau_\lambda)$ and $m(\lambda, i\omega_\lambda, \tau_\lambda)u_\lambda = 0$. Then 0 is a simple eigenvalue of $m(\lambda, i\omega_\lambda, \tau_\lambda)$. Furthermore, it is evident that $m(\lambda, i\omega_\lambda, \tau_\lambda)$ is a Fredholm operator with index zero. Therefore, we have the following direct sum decomposition:

$$\mathbb{Y}_{\mathbb{C}} = \text{Ker}m(\lambda, i\omega_\lambda, \tau_\lambda) \oplus \text{Range}m^*(\lambda, -i\omega_\lambda, \tau_\lambda),$$

which induces a decomposition

$$\text{Ker}m(\lambda, i\omega_\lambda, \tau_\lambda) \oplus \mathbb{X}_{0\mathbb{C}},$$

with $\mathbb{X}_{0\mathbb{C}} = \mathbb{X}_{\mathbb{C}} \cap \text{Range}m^*(\lambda, -i\omega_\lambda, \tau_\lambda)$. Define a mapping $\mathcal{P} : \mathbb{C} \times \mathbb{R}^+ \times \mathbb{X}_{0\mathbb{C}} \to \mathbb{Y}_{\mathbb{C}}$ by

$$\mathcal{P}(\mu, \tau, u) = m(\lambda, \mu, \tau)(u_\lambda + u).$$

Clearly, $\mathcal{P}(i\omega_\lambda, \tau_{j,\lambda}, 0) = 0$ and $\mathcal{P}_{(u,\mu)}(i\omega_\lambda, \tau_{j,\lambda}, 0)$ is an isomorphism. By means of the implicit function theorem, there exists a constant $\delta > 0$ and a continuously differentiable mapping $\tau \to (\mu, u)$ from $(\tau_{j,\lambda} - \delta, \tau_{j,\lambda} + \delta)$ to $\mathbb{C} \times \mathbb{X}_{\mathbb{C}}$ such that $\mu(\tau_{j,\lambda}) = i\omega_\lambda$ and $u(\tau_{j,\lambda}) = u_\lambda$ and that

$$\mathcal{P}(\mu(\tau), \tau, u(\tau)) = 0 \text{ for all } \tau \in (\tau_{j,\lambda} - \delta, \tau_{j,\lambda} + \delta).$$

Calculating the inner product of the above equation with $\overline{\varrho_\lambda}$ yields

$$\mu'(\tau_{j,\lambda}) = -\omega_\lambda e^{-i\theta(\lambda)} \int_\Omega [d\nabla \cdot (u_\lambda^* \nabla u_\lambda) + u_\lambda^* u_\lambda] \overline{\varrho_\lambda} dx,$$

then we have

$$\lim_{\lambda \to \lambda_1} \mu'(\tau_{j,\lambda}) = \lim_{\lambda \to \lambda_1} \frac{\mu'(\tau_{j,\lambda})}{v_\lambda} = 0$$

and

$$\lim_{\lambda \to \lambda_1} \frac{\mu'(\tau_{j,\lambda})}{v_\lambda^2} = -i\gamma e^{-i\theta(\lambda)} k_1 = (\gamma \sin\theta(\lambda) + i\gamma\cos\theta(\lambda))k_1.$$

So

$$\operatorname{sgnRe}\left\{\lim_{\lambda \to \lambda_1} \frac{\mu'(\tau_{j,\lambda})}{v_\lambda^2}\right\} = \operatorname{sgnRe}\left\{\frac{\sin^2\theta(\lambda)\left[\int_\Omega \varphi_1 \left[d\nabla\cdot(\varphi_1\nabla\varphi_1) - \lambda_1\varphi_1^2\right] dx\right]^2}{\int_\Omega |u|^2 dx}\right\} = 1$$

and hence $\mathrm{Re}\{\mu'(\tau_{j,\lambda})\} > 0$ for λ close enough to λ_1. □

Remark 3.1. Lemma 3.3 indicates that for the steady-state solution u_λ^* specified in Theorem 2.1, the corresponding infinitesimal generator $\mathcal{A}_{\tau,\lambda}$ meets the transversality condition at $\tau = \tau_{j,\lambda}$ for $\lambda \in \Lambda$ when λ is sufficiently close to λ_1. Consequently, a Hopf bifurcation occurs at $\tau = \tau_{j,\lambda}$, resulting in the emergence of a branch of periodic orbits of (1.1) from $(\tau, u) = (\tau_{j,\lambda}, u_\lambda^*)$. Condition (3.5) means that regardless of changes in the time delay τ, Hopf bifurcation cannot occur in system (1.1) when the interior reaction term is weaker than the boundary reaction term.

We next consider the existence of purely imaginary eigenvalues of $\mathcal{A}_{0,\lambda}$.

Lemma 3.4. *Assume that $\kappa \neq 0$. Then for each $\lambda \in \Lambda$ satisfying $\varrho(\lambda - \lambda_1) > 0$, $\mathcal{A}_{0,\lambda}$ has only eigenvalues of negative real parts. Conversely, when $\varrho(\lambda - \lambda_1) < 0$, $\mathcal{A}_{0,\lambda}$ possesses at least one eigenvalue with a positive real part.*

Proof. We first demonstrate that $\mathcal{A}_{0,\lambda}$ has no purely imaginary eigenvalue when $\kappa \neq 0$. Assuming that $\mathcal{A}_{0,\lambda}$ possesses purely imaginary eigenvalues, and applying the reasoning from Lemma 3.3, we find $\mathcal{R}(\gamma, 0) = \kappa_1 + \frac{\kappa_2}{2} - i\gamma$, leading to a contradiction. Thus, $\mathcal{A}_{0,\lambda}$ indeed has no purely imaginary eigenvalue.

Let $\{\lambda_k^*\}_{k=1}^\infty \in \Lambda$ satisfy $\lim_{k\to\infty} \lambda_k^* = \lambda_1$ such that $\mu_k \in \sigma(\mathcal{A}_{0,\lambda})$. Then there exists $\{u_k\}_{k=1}^\infty \in \mathbb{X}_{\mathbb{C}}$ satisfying $\lim_{k\to\infty} u_k = \varphi_1$ such that for each $k \in \mathbb{N}$, $m(\lambda_k^*, \mu_k, 0)u_k = 0$, i.e.,

$$\begin{pmatrix} \Delta u_k + d\nabla\cdot(u_k\nabla u_\lambda^*) + d\nabla\cdot(u_\lambda^*\nabla u_k) + \lambda_k^* u_k(m(x) - u_\lambda^*) - \lambda_k^* u_\lambda^* u_k - \mu_k u_k \\ \frac{\partial u}{\partial \mathbf{n}} - \lambda_k^* h_u(x, u_\lambda^*) u_k \end{pmatrix} = 0.$$

Note that

$$\lim_{k\to\infty} \frac{u_{\lambda_k^*}^*}{\lambda_k^* - \lambda_1} = \varphi_1 \lim_{k\to\infty} \frac{v_{\lambda_k^*}^*}{\lambda_k^* - \lambda_1} = \frac{-2\varrho\varphi_1}{\kappa}.$$

It follows from $\langle \psi, m(\lambda_k^*, \mu_k, 0)u_k \rangle = 0$ that

$$\lim_{k\to\infty} \mu_k = 0, \quad \lim_{k\to\infty} \frac{\mu_k}{2(\lambda_1 - \lambda_k^*)} = \varrho.$$

This implies that for sufficiently large k, $\operatorname{Re}\{\mu_k\}$ has the same sign as $\varrho(\lambda_1 - \lambda_k^*)$. This completes the proof. □

Remark 3.2. Lemma 3.4 indicates that when $\varrho(\lambda - \lambda_1) > 0$, all eigenvalues of $\mathcal{A}_{0,\lambda}$ have negative real parts, implying the stability of the steady-state solution u_λ^* of (1.1) for $\tau = 0$. However, Lemma 3.3 suggests that large delays τ might induce nonlinear oscillations, affecting computational performance significantly. Therefore, the time delay τ can be viewed as a source of instability and oscillatory behavior in system (1.1).

Considering Lemmas 3.3 and 3.4, the following conclusions about the stability of the steady-state solution u_λ^*, as established in Theorem 2.1, are derived.

Theorem 3.2. **(i)** *Assume that $\kappa \neq 0$. Then for each $(\tau, \lambda) \in \mathbb{R}^+ \times \Lambda$ satisfying $\varrho(\lambda - \lambda_1) < 0$ and $|\lambda - \lambda_1| \ll 1$, $A_{\tau,\lambda}$ has at least one eigenvalue with a positive real part. Consequently, the steady-state solution u_λ^* of system (1.1), as defined in Theorem 2.1, is unstable.*

(ii) *Assume that $k \neq 0$ and $4\kappa_1^2 < \kappa_2^2$. Then for each $(\tau, \lambda) \in \mathbb{R}^+ \times \Lambda$ satisfying $\varrho(\lambda - \lambda_1) > 0$ and $|\lambda - \lambda_1| \ll 1$, all eigenvalues of $A_{\tau,\lambda}$ have negative real parts, so the steady-state solution u_λ^* of system (1.1) is locally asymptotically stable.*

(iii) *Under assumption (3.5), for each $\lambda \in \Lambda$ satisfying $\varrho(\lambda - \lambda_1) > 0$ and $|\lambda - \lambda_1| \ll 1$, if $\tau \in [0, \tau_{0,\lambda})$, all eigenvalues of $A_{\tau,\lambda}$ have negative real parts. In contrast, for $(\tau_{n,\lambda}, \tau_{n+1,\lambda}]$ with $n \in \mathbb{N}$, $A_{\tau,\lambda}$ has exactly $2(n+1)$ solutions with positive real parts. Thus, the steady-state solution u_λ^* of system (1.1) remains locally asymptotically stable for $\tau \in [0, \tau_{0,\lambda})$ and becomes unstable for $\tau > \tau_{0,\lambda}$, as established in Theorem 2.1.*

3.2. Steady-state solutions established in Theorem 2.3

Theorem 2.3 means that for any $u_0 \in \mathbb{R}$ satisfying (2.10) and (2.11), in the vicinity of $(u, \lambda) = (u_0, 0)$, system (1.1) has two branches of steady-state solutions: one is the trivial solution $(u, 0)$ with $u > 0$, and the other is the bifurcating solution (u_s^*, λ_s^*), $s \in (-\sigma, \sigma)$, where $\sigma > 0$, and u_s^* is defined as

$$u_s^* = u_0 + v(s) + \widetilde{h}(v(s), \lambda(s))$$

with

$$(v(0), \lambda(0)) = (0, 0), \quad (v'(0), \lambda'(0)) = (0, 1)$$

and $v(s), \lambda(s)$ satisfying

$$\int_\Omega F_1(u_0 + v(s) + \widetilde{h}(v(s), \lambda(s)), \lambda)\, dx = (1 + du_0) \int_{\partial\Omega} F_2(u_0 + v(s) + \widetilde{h}(v(s), \lambda(s)), \lambda)\, d\sigma.$$

In the subsequent analysis, we consider the linearization of system (1.1) at $u = u_s^*$ with $\lambda = \lambda(s)$:

$$\begin{cases} \dfrac{\partial v}{\partial t} =\Delta v + d\nabla\cdot(v\nabla u_s^*) + d\nabla\cdot(u_s^*\nabla v_\tau) \\ \qquad\quad + \lambda(s)v(m(x) - u_s^*) - \lambda(s)u_s^* v_\tau, \qquad x \in \Omega, \\ \dfrac{\partial v}{\partial \mathbf{n}} =\lambda h_u(x, u_s^*)v, \qquad\qquad\qquad\qquad\quad x \in \partial\Omega. \end{cases} \tag{3.8}$$

Similar to Section 3, let $\widetilde{A}_{\tau,s}$ denote the infinitesimal generator of the semigroup generated by system (3.8). Then $\mu \in \mathbb{C}$ is in the spectral set of $\sigma(\widetilde{A}_{\tau,s})$ of $\widetilde{A}_{\tau,s}$ if and only if $\hat{m}(s, \mu, \tau)u = 0$ for some $u \in X_{\mathbb{C}} \setminus \{0\}$, which is also equivalent to that there exists $v \in X_{\mathbb{C}}^* \setminus \{0\}$ such that $\hat{m}^*(\lambda, \bar{\mu}, \tau)v = 0$. Here, $\hat{m}(s, \mu, \tau)u = \hat{m}^*(s, \mu, \tau)u$, and $\hat{m}(s, \mu, \tau)u$ is given by

$$\begin{pmatrix} \Delta u + d\nabla\cdot(u\nabla u_s^*) + d\nabla\cdot(u_s^*\nabla u)e^{-\mu\tau} + \lambda(s)u(m(x) - u_s^*) - \lambda(s)u_s^* u e^{-\mu\tau} - \mu u \\ \frac{\partial u}{\partial \mathbf{n}} - \lambda(s)h_u(x, u_s^*)u \end{pmatrix}.$$

Lemma 3.5. *Under the assumptions of Theorem 2.3, we obtain* $0 \notin \sigma(\widetilde{A}_{\tau,s})$.

Proof. Suppose that $0 \in \sigma(\widetilde{A}_{\tau,s})$. Consequently, there exists a $u \in \mathbb{X} \setminus \{0\}$ such that $\hat{m}(0, u, \tau) = 0$. Observations from $\hat{m}(0, 0, \tau) = F_u(u_0, 0)$ and $\mathrm{Ker}F_u(u_0, 0) = \mathrm{span}\{1\}$ imply that u can be expressed as $u = a_s + sb_s$, with $a_s \in \mathbb{R}$ and $b_s \in \mathbb{X}_2$ satisfying $a_s^2 + b_s^2 \neq 0$. Hence $a_s\hat{m}(s, 0, \tau) + s\hat{m}(s, 0, \tau)b_s = 0$.

It is straightforward to see that

$$\lim_{s\to 0} \frac{\hat{m}(s, 0, \tau)\cdot 1}{s} = \begin{pmatrix} m(x) - 2u_0 \\ -h_u(x, u_0) \end{pmatrix} = F_{\lambda u}(u_0, 0).$$

From condition (2.11), since $F_{\lambda u}(\cdot, u_0) \notin \mathrm{Range}F_u(u_0, 0)$, we conclude that $a_0 = 0$. This together with the invertibility of $F_u(u_0, 0)$ restricted in $\mathbb{X}_2$, indicates $b_0 = 0$, a contradiction. Therefore, $0 \notin \sigma(\widetilde{A}_{\tau,s})$. The proof is completed. $\square$

Due to Lemma 3.2, the condition $i\omega \notin \sigma(\widetilde{A}_{\tau,s})$ for $\omega \neq 0$ indicates that ω can be regarded as the function of s and $\omega(s)/s$ is bounded. Consequently, by setting $\omega(s) = s\gamma$, we rewrite $\hat{m}(s, \mathrm{i}s\gamma, \tau, \rho)$ as $\overline{m}(s, \gamma, \theta, \rho)$, with $\overline{m}(s, \gamma, \theta, \rho)$: $\mathbb{X}_{\mathbb{C}} \to \mathbb{Y}_{\mathbb{C}}$ and $\overline{m}(s, \gamma, \theta)u$ is given by

$$\begin{pmatrix} \Delta u + d\nabla\cdot(u\nabla u_s^*) + d\nabla\cdot(u_s^*\nabla u)e^{-i\theta} + \lambda(s)u(m(x) - u_s^*) - \lambda(s)u_s^* u e^{-i\theta} - is\gamma u \\ \frac{\partial u}{\partial \mathbf{n}} - \lambda h_u(x, u_s^*)u \end{pmatrix},$$

where $\omega = s\gamma$ and $\omega\tau = \theta + 2j\pi$ $(j \in \mathbb{N})$. Deriving from $\hat{m}(0, 0, \tau) = F_u(u_0, 0)$ and $\mathrm{Ker}F_u(u_0, 0) = \mathrm{span}\{1\}$, it is established that u has the form $u = 1 + sb$ for some $b \in \mathbb{X}_{2\mathbb{C}}$. Consequently, this results in $\widetilde{\mathcal{H}}(b, \gamma, \theta, s) = 0$, with $\widetilde{\mathcal{H}}$ mapping from $X_{1\mathbb{C}} \times \mathbb{R} \times [0, 2\pi) \times (-\sigma, \sigma)$ to Y, defined by:

$$\widetilde{\mathcal{H}}(b, \gamma, \theta, s) = \frac{\hat{m}(s, \gamma, \theta)\cdot 1}{s} + \hat{m}(s, \gamma, \theta)b.$$

At $s = 0$, $\widetilde{\mathcal{H}}(b, \gamma, \theta, 0) = \widetilde{\Phi}(\gamma, \theta) + F_u(u_0, 0)b$, where

$$\widetilde{\Phi}(\gamma, \theta) = \begin{pmatrix} m(x) - (1 + e^{-i\theta})u_0 - i\gamma \\ -h_u(x, u_0) \end{pmatrix}.$$

Define $\widetilde{\mathcal{R}} : \mathbb{R} \times [0, 2\pi) \to \mathbb{R}$ as

$$\widetilde{\mathcal{R}}(\gamma, \theta) = \int\limits_{\Omega} m(x)dx + (1 + du_0) \int\limits_{\partial\Omega} h_u(x, u_0)d\sigma - (1 + e^{-i\theta})u_0|\Omega| - i\gamma|\Omega|. \quad (3.9)$$

Throughout the remaining part of this section, we always assume that

$$\left| \int\limits_{\Omega} m(x)dx + (1 + du_0) \int\limits_{\partial\Omega} h_u(x, u_0)d\sigma - u_0|\Omega| \right| \leq |u_0|\Omega||. \quad (3.10)$$

Then there exists $(\theta_*, \gamma_*) \in [0, 2\pi) \times \mathbb{R}$ such that $\widetilde{R}(\gamma_*, \theta_*) = 0$, which implies that $\widetilde{\Phi}(\gamma_*, \theta_*) \in \text{Range}F_u(u_0, 0)$. Additionally, there exists $b_* \in X_{2\mathbb{C}} \setminus \{0\}$ for which $\widetilde{H}(b_*, \gamma_*, \theta_*, 0) = 0$. Applying the implicit function theorem and the isomorphism of $D_{(b,\gamma,\theta)}\widetilde{H}(b, \gamma, \theta, 0)$, a continuously differentiable mapping $s \to (b(s), \gamma(s), \theta(s))$ is defined from $(-\sigma, \sigma)$ to $\mathbb{X}_{2\mathbb{C}} \times \mathbb{R} \times [0, 2\pi)$ such that $b(0) = b_*$, $\gamma(0) = \gamma_*$, $\theta(0) = \theta_*$ and $\widetilde{H}(b(s), \gamma(s), \theta(s), s) = 0$ for all $s \in \Lambda$. Consequently, $\hat{m}(s, \gamma(s), \theta(s))u = 0$ admits a nontrivial solution $u = \widetilde{u}_s \in \mathbb{X}_{\mathbb{C}}$, where $\widetilde{u}_s = 1 + sb(s)$. This configuration confirms $i\widetilde{\omega}_s \in \sigma(\widetilde{A}_{\tau_{j,s},s})$ with the associated eigenfunction $\widetilde{u}_s = 1 + sb(s)$, where

$$\widetilde{\omega}_s = s\gamma(s), \quad \widetilde{\tau}_{j,s} = \frac{\theta(s) + 2j\pi}{\omega_s}. \quad (3.11)$$

In summary, we have the following result.

Lemma 3.6. *Under the condition (3.10) and the assumptions of Theorem 2.3, for each $s \in (-\sigma, \sigma)$, define $\widetilde{\omega}_s$ and $\widetilde{\tau}_{j,s}$ as in (3.11). Then for each $s \in (-\sigma, \sigma)$, $\mathcal{A}_{\tau,s}$ has a pair of purely imaginary eigenvalues at $\tau = \widetilde{\tau}_{j,s}$. Moreover,*

(i) *These purely imaginary eigenvalues are $\pm i\widetilde{\omega}_s$.*

(ii) *The eigenspace associated with eigenvalue $i\widetilde{\omega}_s$ is only spanned by ζ_0, where $\zeta_0(x, t) = \widetilde{u}_s \exp\{i\widetilde{\omega}_s t\}$.*

(iii) *There exists $\delta > 0$ and C^1-mapping $\mu : (\widetilde{\tau}_{j,s} - \delta, \widetilde{\tau}_{j,s} + \delta) \to \mathbb{C}$ such that $\mu(\widetilde{\tau}_{j,s}) = i\widetilde{\omega}_s$ and $\mu(\tau) \in \sigma(\mathcal{A}_{\tau,s})$ for all $\tau \in (\widetilde{\tau}_{j,\rho,s} - \delta, \widetilde{\tau}_{j,s} + \delta)$. Furthermore, $\frac{d}{dt} Re\{\mu(\tau)\} |_{\tau=\widetilde{\tau}_{j,s}} > 0$ for s close enough to 0.*

(iv) *There exists $\widetilde{\varrho}_s = 1 + sd(s) \in \mathbb{Y}^*_{\mathbb{C}} \setminus \{0\}$ such that $\hat{m}^*(s, -i\widetilde{\omega}_s, \widetilde{\tau}_{j,s})\widetilde{\varrho}_s = 0$. Moreover,*

$$\lim_{s\to 0} \widetilde{\Pi}_{j,s} = |\Omega|,$$

where

$$\Pi_{j,s} = \int_{\Omega} \overline{\widetilde{\varrho}_s}\widetilde{u}_s dx + d\tau e^{-i\omega\tau} \int\limits_{\Omega} \widetilde{u}^*_s(\widetilde{u}_s \Delta\overline{\widetilde{\varrho}_s}) + \nabla\widetilde{u}^*_s(\overline{\widetilde{\varrho}_s}\nabla\widetilde{u}_s)dx.$$

Omit the proof of Lemma 3.6 due to its similarity to Lemma 3.3. Lemma 3.6 implies that for the steady-state solution u_s^*, the associated infinitesimal generator $\widetilde{A}_{\tau,s}$ satisfies the transversality condition at $\tau = \widetilde{\tau}_{j,s}$ for $s \in (-\sigma, \sigma)$. Consequently, a Hopf bifurcation is observed at $\tau = \widetilde{\tau}_{j,s}$, resulting in the emergence of a branch of periodic orbits of (1.1) starting from $(\widetilde{\tau}_{j,s}, u_s^*)$.

According to condition (3.10), no matter how the time delay τ changes, Hopf bifurcation is precluded near the steady-state solution u_s^* in system (1.1) if the internal reaction is small enough, that is:

$$\left| \int_\Omega m(x)\, dx + (1 + du_0) \int_{\partial\Omega} h_u(x, u_0)\, d\sigma - u_0|\Omega| \right| > |u_0||\Omega|.$$

Note that system (1.1) with $\tau = 0$ has no Hopf bifurcation near the steady-state solution u_s^*. Therefore, it is the interior reaction delay τ that determines the existence of Hopf bifurcation in system (1.1) near the steady-state solution u_s^* under the condition (3.10).

In what follows, we consider the stability of the steady-state solution u_s^*. For this purpose, we start with the case where $\tau = 0$.

Note that system (1.1) with $\tau = 0$ has no Hopf bifurcation near the steady-state solution u_s^*. Consequently, it is the internal reaction delay τ that primarily influences the occurrence of Hopf bifurcation in system (1.1) around the steady-state solution u_s^*, as specified under condition (3.10).

Lemma 3.7. *Under the assumptions of Theorem 2.3, for each $s \in (-\sigma, \sigma)$ satisfying $\Xi\lambda(s) < 0$ (respectively, > 0), $\widetilde{\mathcal{A}}_{0,s}$ has only eigenvalue with negative real part (respectively, at least one eigenvalue with a positive real part).*

Proof. Let $\{s_k\}_{k=1}^\infty \subseteq (-\sigma, \sigma)$ be a sequence convergent to 0 as $k \to \infty$ such that for each $k \in \mathbb{N}$. $\widetilde{\mathcal{A}}_{0,s}$ has only eigenvalue μ_k. There exists $\{u_k\}_{k=1}^\infty \in \mathbb{X}_\mathbb{C}$ convergent to 1 as $k \to \infty$ such that for each $k \in \mathbb{N}$, $\widetilde{m}(s_k, \mu_k, 0)u_k = 0$, and hence

$$\int_\Omega \left[d\nabla\cdot (u_k \nabla u_{s_k}^*) + d\nabla\cdot (u_{s_k}^* \nabla u_k) + \lambda(s_k)u_k(m(x) - u_{s_k}^*) - \lambda(s_k)u_{s_k}^* u_k - \mu_k u_k \right] dx$$
$$+ (1 + du_0) \int_{\partial\Omega} \lambda(s_k) h_u(x, u_{s_k}^*) u_k d\sigma = 0.$$

This implies that

$$\lim_{k\to\infty} \mu_k = 0, \quad |\Omega| \lim_{k\to\infty} \frac{u_k}{\lambda(s_k)} = \Xi.$$

This implies that for sufficiently large k, $\mathrm{Re}\{\mu_k\}$ has the same sign as $\Xi\lambda(s_k)$. □

Considering Lemmas 3.6 and 3.7, we establish the stability of the steady state solution u_s^*.

Theorem 3.3. *Assume that there exists $u_0 \in \mathbb{R}$ such that both (2.10) and (2.11) hold, we have*

(i) *For each $(\tau, s) \in \mathbb{R}_+ \times (-\sigma, \sigma)$ satisfying $\Xi\lambda(s_k) > 0$, the steady-state solution u_s^* is unstable.*

(ii) *Assume that condition (3.10) does not hold. Then for each $(\tau, s) \in \mathbb{R}_+ \times (-\sigma, \sigma)$ satisfying $\Xi\lambda(s_k) < 0$, the steady-state solution u_s^* is locally asymptotically stable.*

(iii) *Assume that condition (3.10) hold. Then for each* $(\tau, s) \in \mathbb{R}_+ \times (-\sigma, \sigma)$ *satisfying* $\Xi\lambda(s_k) < 0$*, the steady-state solution* u_s^* *is locally asymptotically stable when* $\tau \in [0, \widetilde{\tau}_{0,s})$*, and is unstable when* $\tau \in [\widetilde{\tau}_{n,s}, \widetilde{\tau}_{n+1,s})$ *,* $n \in \mathbb{N}_0$*.*

4. Hopf bifurcation

In this section, we study the Hopf bifurcation at the steady-state solution u_λ^* and u_s^*, as established in Theorems 2.1 and 2.3, respectively. This phenomenon occurs when τ passes through the critical thresholds $\tau_{j,\lambda}$ and $\widetilde{\tau}_{j,s}$. For convenience, we call a Hopf bifurcation *forward* if there exist periodic solutions when parameter value $\tau > \tau_{j,\lambda}$ (or $\tau > \widetilde{\tau}_{j,s}$), and *backward* if $\tau < \tau_{j,\lambda}$ (or $\tau < \widetilde{\tau}_{j,s}$). We shall investigate the bifurcation direction and monotonicity of the period of the bifurcating closed invariant curve, and determine the conditions for the occurrence of Hopf bifurcation and identify both supercritical and subcritical scenarios. Additionally, we shall investigate the monotonicity of the period of the bifurcating closed invariant curve.

In the context of a Banach space W, define $C_T(W)$ and $C_T^1(W)$ as the sets of continuous and differentiable T-periodic mappings from $\mathbb{R}$ to W, respectively, where T is defined as $2\pi/\omega_\lambda$. Consider the norms:

$$\|x\|_{0,W} = \max_{t\in[0,T]} \{\|u(t)\|_W\}$$

for any $u \in C_T(W)$, and

$$\|u\|_{1,W} = \max\{\|u\|_{0,W}, \|u'\|_{0,W}\}$$

for $u \in C_T^1(W)$. Equipped with these norms, $C_T(W)$ and $C_T^1(W)$ are Banach spaces. Moreover, $C_T(W)$ represents a Banach representation of the group $\mathbb{S}^1$ with the operation defined by

$$\theta \cdot u(t) = u(t+\theta) \quad \text{for all} \quad \theta \in \mathbb{S}^1.$$

The inner product for $C_T(\mathbb{Y}_\mathbb{C}^*)$ and $C_T(\mathbb{Y}_\mathbb{C})$ is introduced as:

$$(v, u) = \frac{1}{T}\int_0^T \langle v(t), u(t)\rangle \, dt$$

for $u, v \in C_T(\mathbb{Y}_\mathbb{C}^*) \times C_T(\mathbb{Y}_\mathbb{C})$.

We start with Hopf bifurcation near the steady-state solutions established in Theorem 2.1 under the condition (3.5). According to Lemma 3.3, for each fixed $\lambda \in \Lambda$, $\mathcal{A}_{\tau_{j,\lambda},\lambda}$ has a pair of simple purely imaginary eigenvalues $\pm i\omega_\lambda$. Moreover, there exists $\mu_\lambda \in \mathbb{X}_\mathbb{C} \setminus \{0\}$ such that

$$m(\lambda, i\omega_\lambda, \tau_{j,\lambda})u_\lambda = 0, \qquad \lim_{\lambda\to\lambda_1} u_{j,\lambda} = \varphi_1.$$

It follows that there exists $\varrho_\lambda \in \mathbb{X}_\mathbb{C}^* \setminus \{0\}$ such that

$$m^*(\lambda, -i\omega_\lambda, \tau_{j,\lambda})\varrho_\lambda = 0, \qquad \lim_{\lambda\to\lambda_1} \varrho_{j,\lambda} = \varphi_1, \qquad \Pi_{j,\lambda} \neq 0.$$

For any $\beta \in (-1, 1)$, setting $u(t) = u((1+\beta)t)$ and letting $v(t,\beta,\tau) = v(t-(1+\beta)\tau)$, then equation (1.1) can be rewritten as

$$\begin{pmatrix} (1+\beta)\frac{d}{dt} - \Delta \\ -\frac{\partial}{\partial \mathbf{n}} \end{pmatrix} v(t) = \begin{pmatrix} d\nabla\cdot[v(t)\nabla v(t,\beta,\tau)] + \lambda v(t)\,[m(x) - v(t,\beta,\tau)] \\ -\lambda h(x,u) \end{pmatrix}.$$

Let $v_{t,\beta}(t) = v(t+(1+\beta)\theta)$ for $\theta \in [-\tau, 0]$. The operator $\mathcal{F} : C_T^1(\mathbb{X})\times\mathbb{R}_+\times(-1,1) \to C_T(\mathbb{Y})$ and $\mathcal{F}(v,\tau,\beta)$ is defined as follows:

$$\begin{pmatrix} -(1+\beta)\frac{dv(t)}{dt} + \Delta v(t) + \nabla\cdot\left[v(t)\nabla v_{t,\beta}(t)\right] + \lambda v(t)\left[m(x) - v_{t,\beta}(t)\right] \\ \frac{\partial}{\partial \mathbf{n}} - \lambda h(x,u) \end{pmatrix}.$$

By adjusting the parameter β, we explore not only the solutions of (1.1) that adhere to the period T but also those that approximate it. Specifically, solutions to $\mathcal{F}(v,\tau,\beta) = 0$ represent $\frac{T}{1+\beta}$-periodic solutions of (1.1). This characterizes $\mathcal{F}$ as $\mathbb{S}^1$-equivariant:

$$\theta \cdot \mathcal{F}(v,\tau,\beta) = \mathcal{F}(\theta\cdot v,\tau,\beta)$$

for $\theta \in \mathbb{S}^1$. Let $\mathcal{L}_{\tau_{j,\lambda}}$ be the first derivative of $\mathcal{F}$ with respect to v at $(v,\tau,\beta) = (0,\tau_{j,\lambda},0)$. Then the elements of $\mathrm{Ker}\mathcal{L}_{\tau_{j,\lambda}}$ correspond to solutions of the linear system $\mathcal{L}_{\tau_{j,\lambda}}u = 0$ satisfying $u(T) = u(t+T)$. This kernel is spanned by $\{\zeta_{0\lambda}, \overline{\zeta}_{0\lambda}\}$, with $\zeta_{0\lambda}$ in $C_T(\mathbb{X}_{\mathbb{C}})$ characterized by $\zeta_{0\lambda}(t) = u_\lambda e^{i\omega_\lambda t}$ for all $t \in \mathbb{R}$. With respect to the inner product on $C_T(\mathbb{Y}^*_{\mathbb{C}}) \times C_T(\mathbb{Y}_{\mathbb{C}})$, the adjoint operator $\mathcal{L}^*_{\tau_{j,\lambda}}$ is defined as follows:

$$(\mathcal{L}^*_{\tau_{j,\lambda}}v)(t) = \begin{pmatrix} \dfrac{\partial v}{\partial t} + \Delta v + dv\Delta u^*_\lambda + \lambda v(m(x) - u^*_\lambda) - \lambda u^*_\lambda v(t+\tau), \\ \dfrac{\partial v}{\partial \mathbf{n}} - \lambda h_u(x,u^*_\lambda)v \end{pmatrix}.$$

The kernel of $\mathcal{L}^*_{\tau_{j,\lambda}}$, $\mathrm{Ker}\mathcal{L}^*_{\tau_{j,\lambda}}$, is spanned by $\{\zeta^*_{0\lambda}, \overline{\zeta}^*_{0\lambda}\}$, where $\zeta^*_{0\lambda}$ in $C_T(\mathbb{X}^*_{\mathbb{C}})$ is expressed as $\zeta^*_{0\lambda}(t) = \varrho_\lambda e^{i\omega_\lambda t}$. The spaces $C_T(\mathbb{Y}_{\mathbb{C}})$ and $C^1_T(\mathbb{X}_{\mathbb{C}})$ are respectively decomposed into

$$C_T(\mathbb{Y}_{\mathbb{C}}) = \mathrm{Ker}\mathcal{L}^*_{\tau_{j,\lambda}} \oplus \mathrm{Range}\mathcal{L}_{\tau_{j,\lambda}}, \quad C^1_T(\mathbb{X}_{\mathbb{C}}) = \mathrm{Ker}\mathcal{L}_{\tau_{j,\lambda}} \oplus \mathrm{Range}\mathcal{L}^*_{\tau_{j,\lambda}}.$$

Inspired by the works of Guo and Li [16], let P denote the projection operator from $C_T(\mathbb{Y}_{\mathbb{C}})$ to $\mathrm{Range}\mathcal{L}_{\tau_{j,\lambda}}$ along $\mathrm{Ker}\mathcal{L}^*_{\tau_{j,\lambda}}$, retaining $\mathbb{S}^1$-equivariance. By employing the Lyapunov-Schmidt reduction, we simplify the Hopf bifurcation problem to finding zeros of the following $\mathbb{S}^1$-equivariant map

$$\mathcal{G}(z,\tau,\beta) = (\zeta^*_{0\lambda}, \mathcal{F}(z\zeta_{0\lambda} + \overline{z}\overline{\zeta}_{0\lambda} + W(z\zeta_{0\lambda} + \overline{z}\overline{\zeta}_{0\lambda},\tau,\beta),\beta)),$$

where $W : \mathrm{Ker}\mathcal{L}_{\tau_{j,\lambda}} \times \mathbb{R}_+ \times (-1,1) \to \mathrm{Range}\mathcal{L}^*_{\tau_{j,\lambda}}$ is a continuously differentiable $\mathbb{S}^1$-equivariant map such that $W(0,\tau_{j,\lambda},0) = 0$ and $P\mathcal{F}(v + W(v,\tau,\beta),\tau,\beta) = 0$ for all $(v,\tau,\beta) \in \mathrm{Ker}\mathcal{L}_{\tau_{j,\lambda}} \times \mathbb{R}_+ \times (-1,1)$. It is easy to see that $\mathcal{G}_z(0,\tau,0) = 0$ and $\mathcal{G}_{\overline{z}}(0,\tau,0) = 0$. Using similar arguments as in Golubitsky and Schaeffer [13], we can find two functions $\Re$ and $\Im$: $\mathbb{R}^2_+ \times (-1,1) \to \mathbb{R}$ such that

$$\mathcal{G}(z,\tau,\beta) = \Re(|z|^2,\tau,\beta)z + \Im(|z|^2,\tau,\beta)iz,$$

and hence that seeking the zeros of $\mathcal{G}$ is equivalent to solving either $r = 0$ or $\Re(r^2,\tau,\beta) = 0$ and $\Im(r^2,\tau,\beta) = 0$. It is easy to obtain that

$$\mathcal{G}_\tau(z,\tau_{j,\lambda},0) = \mu'(\tau)z + O(|z|^2),$$

$$\mathcal{G}_\beta(z,\tau_{j,\lambda},0) = -i\omega_\lambda z + O(|z|^2).$$

Then

$$\begin{pmatrix} \Re_\tau(0,\tau_{j,\lambda},0) & \Re_\beta(0,\tau_{j,\lambda},0) \\ \Im_\tau(0,\tau_{j,\lambda},0) & \Im_\beta(0,\tau_{j,\lambda},0) \end{pmatrix} = \omega_\lambda Re\left\{\mu'(\tau_{j,\lambda})\right\} \neq 0.$$

We can apply the implicit function theorem, and obtain a unique function $\tau = \tau(r^2)$ and $\beta = \beta(r^2)$ satisfying $\tau(0) = \tau_{j,\lambda}$ and $\beta(0) = 0$ such that

$$\Re(r^2, \tau(r^2), \beta(r^2)) = 0, \quad \Im(r^2, \tau(r^2), \beta(r^2)) = 0 \tag{4.1}$$

for all sufficiently small r. That is to say, $\mathcal{G}(z, \tau(r^2), \beta(r^2)) = 0$ for z sufficiently near 0. Therefore, system (1.1) has a bifurcation of periodic solutions. Namely, we have the following result.

Theorem 4.1. *Assume that (3.5) holds. Then for each $\lambda \in \Lambda$ satisfying $|\lambda - \lambda_1| \ll 1$, a Hopf bifurcation for (1.1) occurs at $(u,\tau) = (u_\lambda^*, \tau_{j,\lambda})$. Namely, in every neighborhood of $(u,\tau) = (u_\lambda^*, \tau_{j,\lambda})$ there is a branch of periodic solutions $u_{j,\tau}(x,t) \to u_\lambda^*$ as $\tau \to \tau_{j,\lambda}$. The period T_τ of $u_{j,\tau}(x,t)$ satisfies that $T_\tau \to \frac{2\pi}{\omega_\lambda}$ as $\tau \to \tau_{j,\lambda}$.*

From Guo and Wu [17], Faria, Teresa and Huang [10], and Wu [27], the bifurcation direction is determined by sign $\tau'(0)$, and the monotonicity of the period of bifurcating closed invariant curve depends on sign $\beta'(0)$. We know from (4.1) that

$$\tau'(0) = \frac{\mathrm{Re}\left\{\Gamma_{j,\lambda}\right\}}{Re\left\{\mu'(\tau_{j,\lambda})\right\}}, \quad \beta'(0) = \frac{\mathrm{Im}\left\{\mu'(\tau_{j,\lambda})\overline{\Gamma_{j,\lambda}}\right\}}{\omega_\lambda Re\left\{\mu'(\tau_{j,\lambda})\right\}},$$

where

$$\begin{aligned}&\Gamma_{j,\lambda} \\ =&(\zeta_0^*, \mathcal{F}_3(\tau_{j,\lambda}, u_\lambda^*)(\zeta_{0\lambda}, \zeta_{0\lambda}, \overline{\zeta}_{0\lambda}) + \mathcal{F}_2(\tau_{j,\lambda}, u_\lambda^*)(\overline{\zeta}_{0\lambda}, W_{20}) + 2\mathcal{F}_2(\tau_{j,\lambda}, u_\lambda^*)(\zeta_{0\lambda}, W_{11}))\end{aligned}$$

and

$$\begin{aligned} W_{20} =& -\mathcal{L}_{\tau_{j,\lambda}}^{-1} P\mathcal{F}_2(\tau_{j,\lambda}, u_\lambda^*)(\zeta_{0\lambda}, \zeta_{0\lambda}), \\ W_{11} =& -\mathcal{L}_{\tau_{j,\lambda}}^{-1} P\mathcal{F}_2(\tau_{j,\lambda}, u_\lambda^*)(\zeta_{0\lambda}, \overline{\zeta}_{0\lambda}). \end{aligned} \tag{4.2}$$

Theorem 4.2. *The Hopf bifurcation at $\tau = \tau_{j,\lambda}$ is supercritical (respectively, subcritical) if $Re\{\Gamma_{j,\lambda}\} < 0$ (respectively, > 0). The period is greater than (respectively, smaller than) $\frac{2\pi}{\omega_\lambda}$ if $Im\left\{\mu'(\tau_{j,\lambda})\overline{\Gamma_{j,\lambda}}\right\} > 0$ (respectively, < 0).*

The remaining part of this section is devoted to the Hopf bifurcation at the steady-state solution u_s^* established in Theorem 2.3, occurring when τ crosses the critical value $\widetilde{\tau}_{j,s}$. For $\widetilde{T} = 2\pi/\omega_s$, the inner product for $C_{\widetilde{T}}(\mathbb{Y}_{\mathbb{C}}^*)$ and $C_{\widetilde{T}}(\mathbb{Y}_{\mathbb{C}})$ is introduced as:

$$(v, u) = \frac{1}{\widetilde{T}} \int_0^{\widetilde{T}} \langle v(t), u(t) \rangle \, dt$$

for $(u, v) \in C_{\widetilde{T}}(\mathbb{Y}_{\mathbb{C}}^*) \times C_{\widetilde{T}}(\mathbb{Y}_{\mathbb{C}})$. Let $v_{t,\beta}(t) = v(t + (1+\beta)\theta)$ for $\theta \in [-\tau, 0]$. The operator $\widetilde{\mathcal{F}} : C_{\widetilde{T}}(\mathbb{X}) \times \mathbb{R}_+ \times (-1, 1) \to C_{\widetilde{T}}(\mathbb{Y})$ and $\widetilde{\mathcal{F}}(v, \tau, \beta)$ is defined as follows:

$$\begin{pmatrix} -(1+\beta)\frac{dv(t)}{dt} + \Delta v(t) + \nabla\cdot \left[v(t)\nabla v_{t,\beta}(t)\right] + \lambda v(t)\left[m(x) - v_{t,\beta}(t)\right] \\ \frac{\partial}{\partial \mathbf{n}} - \lambda h(x, u) \end{pmatrix}.$$

Let $\mathcal{L}_{\widetilde{\tau}_{j,s}}$ be the first derivative of $\widetilde{\mathcal{F}}$ with respect to v at $(v,\tau,\beta)=(0,\widetilde{\tau}_{j,s},0)$. Then the elements of $\mathrm{Ker}\mathcal{L}_{\widetilde{\tau}_{j,s}}$ correspond to solutions of the linear system $\mathcal{L}_{\widetilde{\tau}_{j,s}}u=0$ satisfying $u(T)=u(t+T)$. It follows that

$$\mathrm{Ker}\mathcal{L}_{\widetilde{\tau}_{j,s}}=\mathrm{span}\left\{\eta_{0s},\overline{\eta}_{0s}\right\},\quad \mathrm{Ker}\mathcal{L}^*_{\widetilde{\tau}_{j,s}}=\mathrm{span}\left\{\eta^*_{0s},\overline{\eta}^*_{0s}\right\},$$

where

$$\eta_{0s}=\widetilde{u}_s e^{i\widetilde{\omega}_s t},\quad \eta^*_{0s}=\widetilde{\varrho}_s e^{i\widetilde{\omega}_s t}\quad (\eta_{0s}\in C_{\widetilde{T}}(\mathbb{X}),\eta^*_{0s}\in C_{\widetilde{T}}(\mathbb{X}^*)).$$

Using a similar argument to the previous, let Q denote the projection operator from $C_{\widetilde{T}}(\mathbb{Y}_{\mathbb{C}})$ to $\mathrm{Range}\mathcal{L}_{\widetilde{\tau}_{j,s}}$ along $\mathrm{Ker}\mathcal{L}^*_{\widetilde{\tau}_{j,s}}$. It follows that

$$\tau'(0)=\frac{\mathrm{Re}\left\{\widetilde{\Gamma}_{j,s}\right\}}{|\Omega|Re\left\{\mu'(\widetilde{\tau}_{j,s})\right\}},\quad \beta'(0)=\frac{\mathrm{Im}\left\{\mu'(\widetilde{\tau}_{j,s})\overline{\widetilde{\Gamma}_{j,s}}\right\}}{|\Omega|\widetilde{\omega}_s Re\left\{\mu'(\widetilde{\tau}_{j,s})\right\}},$$

where

$$\widetilde{\Gamma}_{j,s}=(\eta^*_{0s},\widetilde{\mathcal{F}}_3(\widetilde{\tau}_{j,s},u^*_s)(\eta_{0s},\eta_{0s},\overline{\eta}_{0s})+\widetilde{\mathcal{F}}_2(\widetilde{\tau}_{j,s},u^*_s)(\overline{\eta}_{0s},W_{20})+2\widetilde{\mathcal{F}}_2(\widetilde{\tau}_{j,s},u^*_s)(\eta_{0s},W_{11}))$$

and

$$W_{20}=-\mathcal{L}^{-1}_{\widetilde{\tau}_{j,s}}Q\widetilde{\mathcal{F}}_2(\widetilde{\tau}_{j,s},u^*_s)(\eta_{0s},\eta_{0s}),\quad W_{11}=-\mathcal{L}^{-1}_{\widetilde{\tau}_{j,s}}Q\widetilde{\mathcal{F}}_2(\widetilde{\tau}_{j,s},u^*_s)(\eta_{0s},\overline{\eta}_{0s}).\quad (4.3)$$

We can obtain the following result.

Theorem 4.3. *Assume that there exists $u_o\in\mathbb{R}$ such that (2.10), (2.11) and (3.10) hold. Then for each $s\in(\sigma,\sigma)$, a Hopf bifurcation for equation(1.1) occurs at $(u,\tau)=(u^*_s,\widetilde{\tau}_{j,s})$. Namely, in every neighborhood of $(u,\tau)=(u^*_s,\widetilde{\tau}_{j,s})$ there is a branch of periodic solutions $u_{j,\tau}(x,t)$ satisfying $u_{j,\tau}(x,t)\to u^*_s$ as $\tau\to\widetilde{\tau}_{j,s}$. The period T_τ of $u_{j,\tau}(x,t)$ satisfies that $T_\tau\to\frac{2\pi}{\widetilde{\omega}_s}$ as $\tau\to\widetilde{\tau}_{j,s}$. Moreover, the Hopf bifurcation at $\tau=\widetilde{\tau}_{j,s}$ is supercritical (respectively, subcritical) if $Re\left\{\widetilde{\Gamma}_{j,s}\right\}<0$ (respectively, >0). The period is greater than (respectively, smaller than) $\frac{2\pi}{\widetilde{\omega}_s}$ if $Im\left\{\mu'(\widetilde{\tau}_{j,s})\overline{\widetilde{\Gamma}_{j,s}}\right\}>0$ (respectively, <0).*

5. Conclusions

In this paper, we investigate the dynamics of a heterogeneous diffusive model with spatial memory and nonlinear boundary conditions. Firstly, the existence of steady-state solutions is investigated by using the Lyapunov-Schmidt reduction and regarding λ as a bifurcation parameter. Next, we discuss the eigenvalues of infinitesimal generators of a linearized system semigroup at the steady-state solutions of equation (1.1) by employing eigenvalue theory. Moreover, we determine the bifurcation direction for each branch of steady-state and periodic solutions using Lyapunov-Schmidt reduction.

This study contributes to the theoretical understanding of reaction-diffusion models with spatial memory and nonlinear boundary conditions by providing a detailed analysis of the existence and stability of both steady-state and periodic solutions. Using the Lyapunov-Schmidt reduction and eigenvalue theory, our results offer new insights into how the balance between the interior and boundary reaction terms influences the occurrence of bifurcations, particularly the Hopf bifurcation. These findings deepen our understanding of the interplay between internal dynamics and boundary effects, enriching the theory of bifurcations in spatially heterogeneous systems. Moreover, our results suggest potential avenues for further exploration of memory-dependent phenomena in reaction-diffusion systems, which could have broader applications to the study of biological, ecological, and physical systems where spatial and temporal delays play a critical role.

Future investigations could significantly benefit from exploring several aspects of reaction-diffusion systems. One promising area involves examining the effects of more complex boundary conditions, such as those incorporating time delays, which could profoundly affect dynamics. Another prospective could include distinguishing between memory delay and maturation delay, to better understand their roles and interactions in influencing system behavior.

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