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Analysis of a Contact Problem Modeled by Hemivariational Inequalities in Thermo-Piezoelectricity

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Abstract We study a quasistatic contact problem from both variational and numerical perspectives, focusing on a thermo-piezoelectric body interacting with an electrically and thermally rigid foundation. The contact is modeled with a normal damped response and unilateral constraint for the velocity field, associated with a total slip-dependent version of Coulomb's law of dry friction. The electrical and thermal conditions on the contact surface are described by Clarke's subdifferential boundary conditions. We formulate the problem's weak form as a system combining a variational-hemivariational inequality with two hemivariational inequalities. Utilizing recent results in the theory of hemivariational inequalities, along with the fixed point method, we demonstrate the existence and uniqueness of the weak solution. Furthermore, we examine a fully discrete scheme for the problem employing the finite element method, and we establish error estimates for the approximate solutions.

Keywords Thermo-piezoelectric materials, friction, hemivariational inequality, fixed point argument, finite element method

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1. Introduction

Recently, the study of contact problems between thermo-piezoelectric bodies, has garnered significant attention in both industrial and real-world scenarios, and remains an active area of research. These problems arise from the coupling of mechanical, electrical and thermal properties, In the literature, several mathematical results address thermo-piezoelectric contact problems. Some findings on mathematical modeling and variational analysis can be found in [1, 2, 8, 9, 13, 14]. Additionally, numerical schemes and their error estimates are discussed in [2, 7, 8, 13]. We extend these results to a quasistatic case by incorporating nonmonotone boundary conditions defined by Clarke's subdifferential and employing the principles of hemivariational inequalities.

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The theory of hemivariational inequalities was introduced in the early 1980s by Panagiotopoulos in [28, 29]. This theory is grounded in the properties of Clarke's subdifferential for locally Lipschitz functions, which may be nonconvex. These inequalities have been instrumental in describing and analyzing various problems in Mechanics, Physics, and Engineering Sciences, particularly in Contact Mechanics [10, 20].

The present paper introduces a new mathematical model for a quasistatic frictional contact between a thermo-piezoelectric body and an electrically and thermally conducting rigid foundation. The novelty of this model lies in the application of the normal damped response and unilateral constraint for the velocity field. The damper coefficient depends on the normal displacement, associated with a version of Coulomb's law of dry friction, in such a way that the friction bound depends on the total slip, and in modeling the electrical and thermal conditions on the contact surface using subdifferential boundary conditions involving nonconvex functionals. From a mathematical perspective, we demonstrate the well-posedness of the resulting model. To approximate the solution, we propose a fully discrete scheme and estimate the error between the numerical solution and the exact solution, achieving optimal order accuracy for the linear finite element method under additional regularity assumptions.

The rest of the paper is organized as follows. In Section 2, we present the model of a thermo-piezoelectric body in a quasistatic frictional contact with a conductive rigid foundation. In Section 3, we introduce the notation and assumptions for the problem's data and derive the variational formulation of the problem. Section 4 contains the existence and uniqueness proof for a weak solution to the problem. Finally, in Section 5, we propose a fully discrete scheme for the numerical solution, along with related error estimates and convergence results.

2. Problem statement

In the current section we present a classic formulation of the contact problem of a thermo-piezoelectric body with a thermally and electrically conducting rigid foundation in a quasistatic process.

We consider a thermo-piezoelectric body which initially occupies a bounded domain $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ with a smooth boundary $\Gamma = \partial\Omega$. The body is acted upon by body forces of density f_0, volume electric charges of density q_0, and volume heat source term q_{th} on Ω. It is also subject to mechanical, electrical and thermal constraints at its boundary. To formulate these constraints we divide Γ into three measurable and disjoint parts Γ_1, Γ_2 and Γ_3 on one hand, such that $|\Gamma_1| > 0$, and we also consider a partition of $\Gamma_1 \cup \Gamma_2$ into two measurable and disjoint parts Γ_a and Γ_b on the other hand, such that $|\Gamma_a| > 0$. We assume that the body is clamped on Γ_1, the electrical potential vanishes on Γ_a and the temperature is zero on $\Gamma_1 \cup \Gamma_2$. We also assume that surface tractions of density f_2 act on Γ_2 and a surface electrical charge of density q_b is prescribed on Γ_b. Over the contact surface Γ_3, the body may come frictional contact with a conductive obstacle, the so called foundation, whose potential and temperature are assumed to be maintained at φ_F and θ_F, respectively.

We denote by $[0, T]$ the time interval of interest, where $T > 0$, and by $x \in \Omega \cup \Gamma$ and $t \in [0, T]$ the spatial and the time variable, respectively. Sometimes, we omit the explicit dependence of various functions on x and t. Moreover, we use Div and div to represent the divergence operators for tensor and vector fields, respectively, that is

$$\mathrm{Div}\tau = (\tau_{ij,j}), \quad \forall \tau \in \mathbb{S}^d,$$
$$\mathrm{div} v = v_{i,i}, \quad \forall v \in \mathbb{R},$$

where the index that follows a comma indicates a partial derivative with respect to the corresponding component of the spatial variable. We denote the space of second order symmetric tensors on $\mathbb{R}^d$ by $\mathbb{S}^d$. Additionally, we define the inner product and its associated norm on $\mathbb{R}^d$ and $\mathbb{S}^d$ by

$$u \cdot v = u_i v_i, \quad \|v\| = \sqrt{v \cdot v}, \quad \forall u, v \in \mathbb{R}^d,$$
$$\sigma \cdot \tau = \sigma_{ij}\tau_{ij}, \quad \|\tau\| = \sqrt{\tau \cdot \tau}, \quad \forall \sigma, \tau \in \mathbb{S}^d.$$

We denote by ν the unit outward normal on boundary Γ and we shall adopt the usual notation for normal and tangential components of vectors and tensors

$$u = u_\nu \nu + u_\tau, \quad u_\nu = u \cdot \nu \quad \text{and} \quad \sigma\nu = \sigma_\nu \nu + \sigma_\tau, \quad \sigma_\nu = (\sigma\nu) \cdot \nu.$$

We will use the standard notation for Lebesgue and Sobolev spaces associated with Ω and Γ. For a real Banach space $(B, \|\cdot\|_B)$, we denote by B^* the dual space of B and we use the notation $(\cdot,\cdot)_{B^* \times B}$ to represent the duality pairing between B^* and B. For $1 \le p \le \infty$, we use the usual notation for the space $L^p(0,T;B)$. We denote by $C(0,T;B)$ the space of continuous functions from $[0,T]$ to B.

Let $\lambda : B \to \mathbb{R}$ be a locally Lipschitz function. The (Clarke) generalized directional derivative of λ at $x \in B$ in the direction $\nu \in B$, denoted by $\lambda^0(x;\nu)$, is defined by

$$\lambda^0(x;\nu) = \limsup_{y \to x, \omega \downarrow 0} \frac{\lambda(y + \omega\nu) - \lambda(y)}{\omega},$$

and the (Clarke) generalized gradient of λ at x, denoted by $\partial\lambda(x)$, is a subset of B^* given by

$$\partial\lambda(x) = \{\zeta \in B^* \mid \lambda^0(x;\nu) \ge (\zeta, \nu)_{B^* \times B} \text{ for all } \nu \in B\}.$$

A locally Lipschitz function λ is called (Clarke) regular at $x \in B$ if for all $\nu \in B$ the one-sided directional derivative $\lambda^0(x;\nu)$ exists and satisfies $\lambda^0(x;\nu) = \lambda'(x;\nu)$ for all $\nu \in B$.

Finally, to present our problem, we denote by $u : \Omega \times (0,T) \to \mathbb{R}^d$ the displacement field, $\theta : \Omega \times (0,T) \to \mathbb{R}$ the temperature field, $\varphi : \Omega \times (0,T) \to \mathbb{R}$ the electric potential, $\sigma : \Omega \times (0,T) \to \mathbb{S}^d$ the stress tensor, $q : \Omega \times (0,T) \to \mathbb{R}^d$ the heat flux vector and $D : \Omega \times (0,T) \to \mathbb{R}^d$ the electric displacement field. Moreover, let $\varepsilon(u) = (\nabla u + (\nabla u)^T)/2$ denote the linearized strain tensor, where $(\nabla u)^T$ denotes the transpose of ∇u.

Now, we present the classical model for the quasistatic Coulomb's frictional contact problem for thermo-piezoelectric materials.

Problem 2.1. *Find a displacement field $u : \Omega \times (0,T) \to \mathbb{R}^d$, an electric potential $\varphi : \Omega \times (0,T) \to \mathbb{R}$ and a temperature field $\theta : \Omega \times (0,T) \to \mathbb{R}$ such that for all $t \in (0,T)$*

$$\sigma(t) = \mathcal{A}\varepsilon(\dot{u}(t)) + \mathcal{F}\varepsilon(u(t)) - \mathcal{E}^T E(\varphi(t)) - \theta(t)\mathcal{M} \qquad \text{in } \Omega, \tag{2.1}$$

$$D(t) = \mathcal{B}E(\varphi(t)) + \mathcal{E}\varepsilon(u(t)) - \theta(t)\mathcal{P} \qquad in\ \Omega, \tag{2.2}$$

$$q(t) = -\mathcal{K}\nabla\theta(t) \qquad in\ \Omega, \tag{2.3}$$

$$-\operatorname{Div}\sigma(t) = f_0(t) \qquad in\ \Omega, \tag{2.4}$$

$$\operatorname{div}D(t) = q_0(t) \qquad in\ \Omega, \tag{2.5}$$

$$\dot{\theta}(t) + \operatorname{div}q(t) = \mathcal{N}(\dot{u}(t)) + q_{th}(t) \qquad in\ \Omega, \tag{2.6}$$

$$u(t) = 0 \qquad on\ \Gamma_1, \tag{2.7}$$

$$\sigma(t)\nu = f_2(t) \qquad on\ \Gamma_2, \tag{2.8}$$

$$\theta(t) = 0 \qquad on\ (\Gamma_1 \cup \Gamma_2), \tag{2.9}$$

$$\varphi(t) = 0 \qquad on\ \Gamma_a, \tag{2.10}$$

$$D(t)\cdot\nu = q_b(t) \qquad on\ \Gamma_b, \tag{2.11}$$

$$\left.\begin{aligned} &\sigma_\nu(t) + \varsigma(t) \le 0, \quad \dot{u}_\nu(t) - g \le 0,\\ &(\sigma_\nu(t) + \varsigma(t))(\dot{u}_\nu(t) - g) = 0,\\ &\varsigma(t) \in h_\nu(t, u_\nu(t))\partial j_\nu(\dot{u}_\nu(t)), \end{aligned}\right\} \qquad on\ \Gamma_3, \tag{2.12}$$

$$\left.\begin{aligned} &\|\sigma_\tau(t)\| \le F_b\left(t, \int_0^t \|u_\tau(s)\|\, ds\right),\\ &-\sigma_\tau(t) = F_b\left(t, \int_0^t \|u_\tau(s)\|\, ds\right)\frac{\dot{u}_\tau(t)}{\|\dot{u}_\tau(t)\|}, \ \ if\ \dot{u}_\tau(t) \neq 0, \end{aligned}\right\} \qquad on\ \Gamma_3, \tag{2.13}$$

$$D(t)\cdot\nu \in \partial j_e(\varphi(t) - \varphi_F) \qquad on\ \Gamma_3, \tag{2.14}$$

$$q(t)\cdot\nu \in \partial j_c(\theta(t) - \theta_F) \qquad on\ \Gamma_3, \tag{2.15}$$

$$u(0) = u_0,\ \ \theta(0) = \theta_0 \qquad in\ \Omega. \tag{2.16}$$

We provide a brief commentary on the equations and boundary conditions (2.1) to (2.16). Equations (2.1)-(2.3) represent the thermo-electro-viscoelastic constitutive law in which $\mathcal{A}$ is the viscosity tensor, $\mathcal{F}$ is the elasticity tensor, $\mathcal{E}$ is the piezoelectric tensor, $\mathcal{M}$ is the thermal expansion tensor, $\mathcal{B}$ is the electric permittivity tensor, $\mathcal{P}$ is the thermal expansion tensor, $\mathcal{K}$ is the thermal conductivity tensor, and $E(\varphi) = -\nabla\varphi$ is the electric field. Equations (2.4)-(2.6) denote the equilibrium conditions for the stress, electric displacement, and heat flux fields, respectively. Here, the function $\mathcal{N}$ describes the impact of the velocity field on temperature. In [3], the function $\mathcal{N}$ was specified as a linear function $\mathcal{N}(\zeta) = -\mathcal{M}\cdot\varepsilon(\zeta)$. Conditions (2.7)-(2.8), (2.9) and (2.10)-(2.11) represent the mechanical, thermal and electrical boundary conditions, respectively, whose physical interpretation was discussed earlier in the second paragraph of this section. Condition (2.12) represents the normal damped response condition in such a way that the normal velocity is limited, in which $g > 0$ represents a given bound. j_ν is a prescribed function and the condition $\varsigma(t) \in h_\nu(t, u_\nu(t))\partial j_\nu(\dot{u}_\nu(t))$ on Γ_3 represents a generalization of the normal damped response condition where h_ν is a given damper coefficient depending on the normal displacement [23]. An example of the function j_ν is presented in [18]

$$j_\nu(r) = \begin{cases} 0 & \text{if } r < 0, \\ \dfrac{a-b}{2a} r^2 + br & \text{if } 0 \le r \le a, \\ ar + \dfrac{a(b-a)}{2} & \text{if } r > a, \end{cases}$$

with $0 < a < b$. We refer to [22, 26] for more examples. (2.13) represents a version of the Coulomb law of dry friction, where F_b denotes the friction bound. Details on such a frictional contact condition is found in [26] and some references therein. Condition (2.14) represents the electrical conductivity requirement over Γ_3 where j_e is a prescribed function. We can take for example

$$j_e(r) = k_e \int_0^r p(s)ds \quad \text{for all} \quad r \in \mathbb{R},$$

where p is a prescribed real-valued function and k_e is the electric conductivity coefficient. For the choice $p(r) = r$, the condition (2.14) rewrites to the following form

$$D(t) \cdot \nu = k_e(\varphi(t) - \varphi_F) \qquad \text{on} \quad \Gamma_3.$$

For more examples of subdifferential boundary conditions similar to (2.14), we refer to [24, 25]. Relation (2.15) describes the heat exchange between Γ_3 and the foundation, where j_c is a prescribed function given by

$$j_c(r) = \frac{1}{2} k_c r^2, \quad \forall r \in \mathbb{R},$$

such that k_c is the heat exchange coefficient between the body and the foundation, the condition (2.15) reduces to the equation

$$q(t) \cdot \nu = k_c(\theta(t) - \theta_F) \qquad \text{on} \quad \Gamma_3.$$

For more details, see [27]. Finally, (2.16) specifies the initial conditions of the problem, where u_0 and θ_0 are given functions representing the initial displacement and initial temperature, respectively.

3. Variational formulation

In this section, we derive a weak formulation of **Problem 2.1**. To achieve this, we first need to introduce some notations. Let H, $\mathbf{H}^1(\Omega)$ and $\mathcal{H}$ be the following spaces

$$H = [L^2(\Omega)]^d, \quad \mathbf{H}^1(\Omega) = [H^1(\Omega)]^d, \qquad \mathcal{H} = \{\sigma = (\sigma_{ij}); \sigma_{ij} = \sigma_{ji} \in L^2(\Omega)\}.$$

The spaces H, $\mathcal{H}$ and $\mathbf{H}^1(\Omega)$ are real Hilbert spaces endowed with the following inner products

$$(u, v)_H = \int_\Omega u_i v_i dx, \ \forall u, v \in H,$$

$$(\sigma,\tau)_{\mathcal{H}} = \int_\Omega \sigma_{ij}\tau_{ij}dx,\ \forall \sigma,\tau \in \mathcal{H},$$
$$(u,v)_{\mathbf{H}^1(\Omega)} = (u,v)_H + (\varepsilon(u),\varepsilon(v))_{\mathcal{H}},\ \forall u,v \in \mathbf{H}^1(\Omega),$$

and let $\|\cdot\|_H$, $\|\cdot\|_{\mathcal{H}}$ and $\|\cdot\|_{\mathbf{H}^1(\Omega)}$ be their associated norms, respectively. Keeping in mind (2.7), we introduce the closed subspace of $\mathbf{H}^1(\Omega)$

$$V = \{v \in \mathbf{H}^1(\Omega) \mid v = 0 \ \text{ on } \Gamma_1\},$$

and the set of admissible velocity fields K defined by

$$K = \{v \in V \mid v_\nu - g \leq 0 \text{ on } \Gamma_3\}.$$

We define over the space V the inner product

$$(u,v)_V = (\varepsilon(u),\varepsilon(v))_{\mathcal{H}}, \quad \forall u,v \in V,$$

and its associated norm

$$\|v\|_V = \|\varepsilon(v)\|_{\mathcal{H}}. \tag{3.1}$$

Since $|\Gamma_1| > 0$, the following Korn's inequality [12] holds: there exists $c_k > 0$ such that

$$\|\varepsilon(v)\|_{\mathcal{H}} \geq c_k \|v\|_{\mathbf{H}^1(\Omega)}, \quad \forall v \in V. \tag{3.2}$$

It follows from (3.1) and (3.2) that $\|\cdot\|_V$ is equivalent on V to the usual norm $\|\cdot\|_{\mathbf{H}^1(\Omega)}$, therefore $(V,\|\cdot\|_V)$ is a real Hilbert space. For simplicity, for an element $\omega \in \mathbf{H}^1(\Omega)$, we still denote by ω its trace $\gamma(\omega)$ on Γ. By trace theorem, there exists a constant $c_0 > 0$ such that

$$\|v\|_{[L^2(\Gamma_3)]^d} \leq c_0 \|v\|_V, \quad \forall v \in V.$$

For the electric potential, we introduce the closed function subspace of $H^1(\Omega)$

$$W = \{\xi \in H^1(\Omega) \mid \xi = 0 \quad \text{on} \quad \Gamma_a\}.$$

Over W, we consider the following inner product

$$(\varphi,\xi)_W = (\nabla\varphi,\nabla\xi)_H, \quad \forall \varphi,\xi \in W, \tag{3.3}$$

and the associated norm

$$\|\xi\|_W = \|\nabla\xi\|_H. \tag{3.4}$$

Since $|\Gamma_a| > 0$, Friedrichs-Poincaré inequality holds, i.e. there exists a constant $c_{p1} > 0$ such that

$$\|\nabla\xi\|_H \geq c_{p1}\|\xi\|_{H^1(\Omega)}, \quad \forall \xi \in W. \tag{3.5}$$

It follows from (3.4)-(3.5) that $\|\cdot\|_W$ is equivalent on W with $\|\cdot\|_{H^1(\Omega)}$ and then $(W,\|\cdot\|_W)$ is a real Hilbert space. Moreover, by trace theorem, there exists a constants $c_1 > 0$ such that

$$\|\xi\|_{L^2(\Gamma_3)} \leq c_1 \|\xi\|_W, \quad \forall \xi \in W. \tag{3.6}$$

For the temperature field, we introduce the closed function subspace of $H^1(\Omega)$

$$Q = \{\eta \in H^1(\Omega) \mid \eta = 0 \quad \text{on} \quad \Gamma_1 \cup \Gamma_2\}.$$

Over Q, we consider the following inner product

$$(\theta, \eta)_Q = (\nabla\theta, \nabla\eta)_H, \quad \forall \theta, \eta \in Q, \tag{3.7}$$

and the associated norm

$$\|\eta\|_Q = \|\nabla\eta\|_H . \tag{3.8}$$

Since $|\Gamma_1| > 0$, Friedrichs-Poincaré inequality holds, i.e. there exists a constant $c_{p2} > 0$ such that

$$\|\nabla\eta\|_H \geq c_{p2} \|\eta\|_{H^1(\Omega)}, \quad \forall \eta \in Q. \tag{3.9}$$

It follows from (3.8)-(3.9) that $\|\cdot\|_Q$ is equivalent on Q with $\|\cdot\|_{H^1(\Omega)}$ and then $(Q, \|\cdot\|_Q)$ is a real Hilbert space. Moreover, by trace theorem, there exists a constants $c_2 > 0$ such that

$$\|\eta\|_{L^2(\Gamma_3)} \leq c_2 \|\eta\|_Q, \quad \forall \eta \in Q. \tag{3.10}$$

To study **Problem 2.1** we need the following assumptions on its data.

(H_1) The viscosity tensor $\mathcal{A} : \Omega \times (0,T) \times \mathbb{S}^d \to \mathbb{S}^d$ satisfies

1. $\mathcal{A}(\cdot, t, \zeta)$ is continuous for all $t \in (0,T)$ and $\zeta \in \mathbb{S}^d$.
2. $\mathcal{A}(x, \cdot, \cdot)$ is continuous for a.e. $x \in \Omega$.
3. There exists $m_{\mathcal{A}} > 0$ such that $(\mathcal{A}(x,t,\zeta_1) - \mathcal{A}(x,t,\zeta_2))(\zeta_1 - \zeta_2) \geq m_{\mathcal{A}} \|\zeta_1 - \zeta_2\|^2$ for all $\zeta_1, \zeta_2 \in \mathbb{S}^d$ and a.e. $(x,t) \in \Omega \times (0,T)$.
4. There exists $L_{\mathcal{A}} > 0$ such that $\|\mathcal{A}(x,t,\zeta_1) - \mathcal{A}(x,t,\zeta_2)\| \leq L_{\mathcal{A}} \|\zeta_1 - \zeta_2\|$ for all $\zeta_1, \zeta_2 \in \mathbb{S}^d$ and a.e. $(x,t) \in \Omega \times (0,T)$.
5. There exist $\bar{a}_0 \in C(0,T; L^2(\Omega))_+$ and $\bar{a}_1 > 0$ such that $\|\mathcal{A}(x,t,\zeta)\| \leq \bar{a}_0(x,t) + \bar{a}_1 \|\zeta\|$ for all $\zeta \in \mathbb{S}^d$ and a.e. $(x,t) \in \Omega \times (0,T)$.
6. $\mathcal{A}(x,t,0) = 0$ for a.e. $(x,t) \in \Omega \times (0,T)$.

(H_2) The elasticity tensor $\mathcal{F} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d$ satisfies

1. $\mathcal{F}(\cdot, \zeta)$ is measurable on Ω for all $\zeta \in \mathbb{S}^d$.
2. There exists $m_{\mathcal{F}} > 0$ such that $(\mathcal{F}(x\zeta_1) - \mathcal{F}(x,\zeta_2))(\zeta_1 - \zeta_2) \geq m_{\mathcal{F}} \|\zeta_1 - \zeta_2\|^2$ for all $\zeta_1, \zeta_2 \in \mathbb{S}^d$ and a.e. $x \in \Omega$.
3. There exists $L_{\mathcal{F}} > 0$ such that $\|\mathcal{F}(x,\zeta_1) - \mathcal{F}(x,\zeta_2)\| \leq L_{\mathcal{F}} \|\zeta_1 - \zeta_2\|$ for all $\zeta_1, \zeta_2 \in \mathbb{S}^d$ and a.e. $x \in \Omega$.
4. $\mathcal{F}(x,t,0) = 0$ for a.e. $(x,t) \in \Omega \times (0,T)$.

(H_3) The thermal conductivity tensor $\mathcal{K} : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ satisfies

1. $\mathcal{K}(\cdot, \zeta)$ is measurable on Ω for all $\zeta \in \mathbb{R}^d$.
2. $\mathcal{K}(x, \cdot)$ is continuous on $\mathbb{R}^d$ for a.e. $x \in \Omega$.
3. There exists $m_{\mathcal{K}} > 0$ such that $(\mathcal{K}(x,\zeta_1) - \mathcal{K}(x,\zeta_2))(\zeta_1 - \zeta_2) \geq m_{\mathcal{K}} \|\zeta_1 - \zeta_2\|^2$ for all $\zeta_1, \zeta_2 \in \mathbb{R}^d$, a.e. $x \in \Omega$.
4. There exist $\bar{k}_0 \in L^2(\Omega)_+$ and $\bar{k}_1 > 0$ such that $\|\mathcal{K}(x,\zeta)\| \leq \bar{k}_0(x) + \bar{k}_1 \|\zeta\|$ for all $\zeta \in \mathbb{R}^d$, a.e. $x \in \Omega$.
5. $\mathcal{K}(x,0) = 0$ for a.e. $x \in \Omega$.

(H_4) The electric permittivity tensor $\mathcal{B} : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ satisfies

1. There exists $m_{\mathcal{B}} > 0$ such that $(\mathcal{B}(x, \zeta_1) - \mathcal{B}(x, \zeta_2))(\zeta_1 - \zeta_2) \geq m_{\mathcal{B}} \|\zeta_1 - \zeta_2\|^2$ for all $\zeta_1, \zeta_2 \in \mathbb{R}^d$ and a.e. $x \in \Omega$.
2. $\mathcal{B}(x, 0) = 0$ for a.e. $x \in \Omega$.

(H_5) The piezoelectric tensor $\mathcal{E} : \Omega \times \mathbb{S}^d \to \mathbb{R}^d$ satisfies

1. There exists $L_{\mathcal{E}} > 0$ such that $\|\mathcal{E}(x, \zeta_1) - \mathcal{E}(x, \zeta_2)\| \leq L_{\mathcal{E}} \|\zeta_1 - \zeta_2\|$ for all $\zeta_1, \zeta_2 \in \mathbb{S}^d$ and a.e. $x \in \Omega$.
2. $\mathcal{E}(x, 0) = 0$ for a.e. $x \in \Omega$.

(H_6) The thermal expansion tensor $\mathcal{M} : \Omega \times \mathbb{R} \to \mathbb{S}^d$ satisfies

1. $\mathcal{M}(\cdot, \zeta)$ is measurable on Ω for all $\zeta \in \mathbb{R}$.
2. There exists $L_{\mathcal{M}} > 0$ such that $\|\mathcal{M}(x, \zeta_1) - \mathcal{M}(x, \zeta_2)\| \leq L_{\mathcal{M}} \|\zeta_1 - \zeta_2\|$ for all $\zeta_1, \zeta_2 \in \mathbb{R}$ and a.e. $x \in \Omega$.
3. $\mathcal{M}(x, 0) = 0$ for a.e. $x \in \Omega$.

(H_7) The pyroelectric tensor $\mathcal{P} : \Omega \times \mathbb{R} \to \mathbb{R}^d$ satisfies

1. There exists $L_{\mathcal{P}} > 0$ such that $\|\mathcal{P}(x, \zeta_1) - \mathcal{P}(x, \zeta_2)\| \leq L_{\mathcal{P}} \|\zeta_1 - \zeta_2\|$ for all $\zeta_1, \zeta_2 \in \mathbb{R}$ and a.e. $x \in \Omega$.
2. $\mathcal{P}(x, 0) = 0$ for a.e. $x \in \Omega$.

(H_8) The function $\mathcal{N} : \mathbb{R}^d \to \mathbb{R}$ satisfies

1. $\mathcal{N}(\zeta) \in L^2(\Omega)$ for all $\zeta \in \mathbb{S}^d$.
2. There exists $L_{\mathcal{N}} > 0$ such that $\|\mathcal{N}(\zeta_1) - \mathcal{N}(\zeta_2)\|_{L^2(\Omega)} \leq L_{\mathcal{N}} \|\zeta_1 - \zeta_2\|_V$ for all $\zeta_1, \zeta_2 \in \mathbb{R}^d$.

(H_9) The friction bound function $F_b : \Gamma_3 \times (0, T) \times \mathbb{R} \to \mathbb{R}_+$ satisfies

1. $F_b(\cdot, t, r)$ is measurable for all $t \in (0, T)$ and $r \in \mathbb{R}$.
2. $F_b(x, \cdot, r)$ is continuous for all $r \in \mathbb{R}$ and a.e. $x \in \Gamma_3$.
3. There exists $L_F > 0$ such that $\|F_b(x, t, r_1) - F_b(x, t, r_2)\| \leq L_F \|r_1 - r_2\|$ for all $r_1, r_2 \in \mathbb{R}$, $t \in (0, T)$ and a.e. $x \in \Gamma_3$.
4. $x \mapsto F_b(x, t, 0)$ belongs to $L^2(\Gamma_3)$ for all $t \in (0, T)$.

(H_{10}) The functional $j_\nu : \mathbb{R} \to \mathbb{R}$ satisfies

1. j_ν is locally Lipschitz.
2. There exists $c_{0\nu} \geq 0$ such that $\|\partial j_\nu(r)\| \leq c_{0\nu}$ for all $r \in \mathbb{R}$.
3. There exists $\alpha_\nu > 0$ such that $j_\nu^0(r_1; r_2 - r_1) + j_\nu^0(r_2; r_1 - r_2) \leq \alpha_\nu \|r_1 - r_2\|^2$ for all $r_1, r_2 \in \mathbb{R}$.
4. Either j_ν or $-j_\nu$ is regular.

(H_{11}) For $\pi = e, c$, the functional $j_\pi : \Gamma_3 \times \mathbb{R} \to \mathbb{R}$ satisfies

1. $j_\pi(\cdot, r)$ is measurable on Γ_3 for all $r \in \mathbb{R}$.

2. $j_\pi(\cdot, 0) \in L^1(\Gamma_3)$.
3. $j_\pi(x, \cdot)$ is locally Lipschitz on $\mathbb{R}$ for a.e. $x \in \Gamma_3$.
4. There exists $c_{0\pi} \geq 0$ such that $\|\partial j_\pi(x, r)\| \leq c_{0\pi}$ for all $r \in \mathbb{R}$ and a.e. $x \in \Gamma_3$.
5. There exists $m_\pi \geq 0$ such that $(\zeta_1 - \zeta_2)(\xi_1 - \xi_2) \geq -m_\pi \|\xi_1 - \xi_2\|^2$ for all $\zeta_i, \xi_i \in \mathbb{R}$, $\zeta_i \in \partial j_\pi(x, \xi_i)$, i=1,2, for a.e $x \in \Gamma_3$.
6. There exists $\alpha_\pi > 0$ such that $j^0_\pi(x, r_1; r_2 - r_1) + j^0_\pi(x, r_2; r_1 - r_2) \leq \alpha_\pi \|r_1 - r_2\|^2$ for all $r_1, r_2 \in \mathbb{R}$ and a.e. $x \in \Gamma_3$.
7. Either $j_\pi(x, \cdot)$ or $-j_\pi(x, \cdot)$ is regular for a.e. $x \in \Gamma_3$.

(H_{12}) The function $h_\nu : \Gamma_3 \times (0, T) \times \mathbb{R} \to \mathbb{R}_+$ satisfies

1. $h_\nu(\cdot, t, r)$ is continuous for all $t \in (0, T)$ and $r \in \mathbb{R}$.
2. There exists $M_h > 0$ such that $0 < h_\nu(x, t, r) \leq M_h$ for all $r \in \mathbb{R}$, $t \in (0, T)$ and a.e. $x \in \Gamma_3$.
3. There exists $L_h > 0$ such that $\|h_\nu(x, t, r_1) - h_\nu(x, t, r_2)\| \leq L_h \|r_1 - r_2\|$ for all $r_1, r_2 \in \mathbb{R}$, $t \in (0, T)$ and a.e. $x \in \Gamma_3$.

(H_{13}) The forces, traction, bound g, volume and surface charge densities, heat source strength, electric potential of the foundation, temperature of the foundation, and initial conditions are assumed to exhibit the following regularity.

1. $f_0 \in L^2(0, T; H)$, $\quad f_2 \in L^2(0, T; [L^2(\Gamma_2)]^d)$, $\quad g \in L^2(\Gamma_3)$, $\quad u_0 \in V$.
2. $q_0 \in L^2(0, T; L^2(\Omega))$, $\quad q_b \in L^2(0, T; L^2(\Gamma_b))$, $\quad \varphi_F \in L^2(\Gamma_3)$.
3. $q_{th} \in L^2(0, T; L^2(\Omega))$, $\quad \theta_F \in L^2(\Gamma_3)$, $\quad \theta_0 \in L^2(\Omega)$.

(H_{14}) The following smallness conditions are satisfied

1. $m_\mathcal{A} > M_h \alpha_\nu c_0^2 + \dfrac{L_\mathcal{E} + L_\mathcal{M}}{2}$.
2. $m_\mathcal{B} > \alpha_e c_1^2 + \dfrac{L_\mathcal{E} + L_\mathcal{P}}{2}$.
3. $m_\mathcal{K} > \alpha_c c_2^2$.

Next, we define the elements $f(t) \in V^*$, $q_e(t) \in W^*$ and $q_c(t) \in Q^*$ by

$$(f(t), v)_{V^* \times V} = \int_\Omega f_0(t) \cdot v dx + \int_{\Gamma_2} f_2(t) \cdot v d\Gamma, \tag{3.11}$$

$$(q_e(t), \xi)_{W^* \times W} = \int_\Omega q_0(t) \xi dx - \int_{\Gamma_b} q_b(t) \xi d\Gamma, \tag{3.12}$$

$$(q_c(t), \eta)_{Q^* \times Q} = \int_\Omega q_{th}(t) \eta dx, \tag{3.13}$$

for all $w \in V$, $\xi \in W$ and $\eta \in Q$.

Finally, from [16, Theorem 15] we show that there exists a unique element $\varphi_0 \in W$ such that for all $\xi \in W$

$$\begin{aligned}(\mathcal{B}\nabla\varphi_0, \nabla\xi)_H - (\mathcal{E}\varepsilon(u_0), \nabla\xi)_H + (\theta_0 \mathcal{P}, \nabla\xi)_H \\ + \int_{\Gamma_3} j^0_e(\varphi_0 - \varphi_F; \xi) d\Gamma \geq (q_e(0), \xi)_{W^* \times W}.\end{aligned} \tag{3.14}$$

This equation ensures compatibility between the initial displacement and initial temperature fields.

Now, by applying Green's formula and the definition of the Clarke subdifferential, we derive the following variational formulation of **Problem 2.1**, which is expressed in terms of the displacement field, electric potential and temperature field.

Problem 3.1. *Find a displacement field $u : (0,T) \to V$, an electric potential $\varphi : (0,T) \to W$ and a temperature field $\theta : (0,T) \to Q$ a.e. $t \in (0,T)$ such that for all $w \in K$, $\xi \in W$ and $\eta \in Q$*

$$\begin{aligned}(\mathcal{A}\varepsilon(\dot{u}(t)), \varepsilon(w - \dot{u}(t)))_{\mathcal{H}} &+ (\mathcal{F}\varepsilon(u(t)), \varepsilon(w - \dot{u}(t)))_{\mathcal{H}} \\ &+ (\mathcal{E}^T \nabla\varphi(t), \varepsilon(w - \dot{u}(t)))_{\mathcal{H}} - (\theta(t)\mathcal{M}, \varepsilon(w - \dot{u}(t)))_{\mathcal{H}} \\ &+ \int_{\Gamma_3} F_b\left(t, \int_0^t \|u_\tau(s)\|\, ds\right)(\|w_\tau\| - \|\dot{u}_\tau(t)\|)d\Gamma \\ &+ \int_{\Gamma_3} h_\nu(t, u_\nu(t)) j_\nu^0(\dot{u}_\nu(t); w_\nu - \dot{u}_\nu(t))d\Gamma \\ \geq &(f(t), w - \dot{u}(t))_{V^*\times V},\end{aligned} \tag{3.15}$$

$$\begin{aligned}(\mathcal{B}\nabla\varphi(t), \nabla\xi)_H &- (\mathcal{E}\varepsilon(u(t)), \nabla\xi)_H + (\theta(t)\mathcal{P}, \nabla\xi)_H \\ &+ \int_{\Gamma_3} j_e^0(\varphi(t) - \varphi_F; \xi)d\Gamma \geq (q_e(t), \xi)_{W^*\times W},\end{aligned} \tag{3.16}$$

$$\begin{aligned}(\dot{\theta}(t), \eta)_{L^2(\Omega)} &+ (\mathcal{K}\nabla\theta(t), \nabla\eta)_H - (\mathcal{N}(\dot{u}(t)), \eta)_{L^2(\Omega)} \\ &+ \int_{\Gamma_3} j_c^0(\theta(t) - \theta_F; \eta)d\Gamma \geq (q_c(t), \eta)_{Q^*\times Q},\end{aligned} \tag{3.17}$$

$$u(0) = u_0, \qquad \theta(0) = \theta_0. \tag{3.18}$$

4. An existence and uniqueness result

Our main existence and uniqueness of the solutions of **Problem 3.1** is the following.

Theorem 4.1. *Assume the hypotheses $(H_1) - (H_{14})$. Then **Problem 3.1** has a unique solution (u, φ, θ) which satisfies the following regularity conditions*

$$u \in C(0,T;V), \quad \dot{u} \in L^2(0,T;V) \text{ and } \dot{u}(t) \in K \text{ for a.e. } t \in (0,T), \tag{4.1}$$

$$\varphi \in L^2(0,T;W), \tag{4.2}$$

$$\theta \in L^2(0,T;Q), \quad \dot{\theta} \in L^2(0,T;Q^*). \tag{4.3}$$

The proof of **Theorem 4.1** will be conducted in several steps, utilizing arguments based on fixed point theory as well as hemivariational inequalities.

Step 1. Let $z \in L^2(0,T;V^*)$. We consider the following intermediate problem.

Problem 4.1. *Find a displacement field $u_z(t) \in V$ for a.e $t \in (0,T)$ such that*

$$\begin{aligned}
&(\mathcal{A}\varepsilon(\dot{u}_z(t)), \varepsilon(w - \dot{u}_z(t)))_{\mathcal{H}} + (\mathcal{F}\varepsilon(u_z(t)), \varepsilon(w - \dot{u}_z(t)))_{\mathcal{H}} \\
&+ \int_{\Gamma_3} F_b\left(t, \int_0^t \|u_{z_\tau}(s)\|\, ds\right)(\|w_\tau\| - \|\dot{u}_{z_\tau}(t)\|)d\Gamma \\
&+ \int_{\Gamma_3} h_\nu(t, u_{z_\nu}(t)) j_\nu^0(\dot{u}_{z_\nu}(t); w_\nu - \dot{u}_{z_\nu}(t))d\Gamma + (z(t), w - \dot{u}_z(t))_{V^*\times V} \\
&\geq (f(t), w - \dot{u}_z(t))_{V^*\times V}, \quad \forall w \in K,
\end{aligned} \tag{4.4}$$

$$u_z(0) = u_0. \tag{4.5}$$

Lemma 4.1. *Assume the hypotheses (H_1), (H_2), (H_9), (H_{12}), (H_{10}), $(H_{13})(1)$ and $(H_{14})(1)$. Then,* ***Problem 4.1*** *has a unique solution $u_z \in C(0,T;V)$ with $\dot{u}_z \in L^2(0,T;V)$ and $\dot{u}_z(t) \in K$ for a.e. $t \in (0,T)$. Moreover, for $z_1, z_2 \in L^2(0,T;V^*)$, let us denote by u_{z_1} and u_{z_2} the solutions of* ***Problem 4.1*** *corresponding to z_1 and z_2, respectively. Then there exists a constant $c > 0$ such that for all $t \in (0,T)$*

$$\|u_{z_1}(t) - u_{z_2}(t)\|_V^2 + \int_0^t \|\dot{u}_{z_1}(s) - \dot{u}_{z_2}(s)\|_V^2\, ds \leq c \int_0^t \|z_1(s) - z_2(s)\|_{V^*}^2\, ds. \tag{4.6}$$

Proof. We will apply Theorem 5. in [23] with the following functional framework: $E = \mathcal{H}$, $X = Y = Z = L^2(\Gamma_3)$.

We introduce the operators $A : (0,T) \times E \times V \to V^*$, $I : L^2(0,T;V) \to L^2(0,T;V)$, $R_1 : L^2(0,T;V) \to L^2(0,T;E)$, $R_2 = 0$, $R_3 : L^2(0,T;V) \to L^2(0,T;Y)$, $R_4 : L^2(0,T;V) \to L^2(0,T;Z)$ and $M : V \to X$ given by

$$\begin{aligned}
&(A(t,\lambda,v), y) = (\mathcal{A}(t, \varepsilon(v)) + \lambda, \varepsilon(y))_{\mathcal{H}}, \text{ for all } t \in (0,T), \lambda \in E \text{ and } v, y \in V, \\
&(Iv)(t) = u_0 + \int_0^t v(s)ds, \text{ for all } v \in L^2(0,T;V) \text{ and } t \in (0,T), \\
&(R_1 v)(t) = \mathcal{F}\varepsilon((Iv)(t)), \text{ for all } v \in L^2(0,T;V) \text{ and } t \in (0,T), \\
&(R_3 v)(t) = \int_0^t \left\| \int_0^s v_\tau(r)dr + u_{0_\tau} \right\| ds, \text{ for all } v \in L^2(0,T;V) \text{ and } t \in (0,T), \\
&(R_4 v)(t) = [(Iv)(t)]_\nu, \text{ for all } v \in L^2(0,T;V) \text{ and } t \in (0,T), \\
&Mv = v_\nu \text{ for all } v \in V.
\end{aligned}$$

Let the functions $\psi : (0,T) \times Y \times V \times V \to \mathbb{R}$ and $j : (0,T) \times Z \times X \to \mathbb{R}$ be defines by

$$\begin{aligned}
&\psi(t,u,w,v) = \int_{\Gamma_3} F_b(t,u)\, \|v_\tau\|\, d\Gamma, \text{ for all } t \in (0,T), u \in Y \text{ and } w, v \in V, \\
&j(t,\zeta,v) = \int_{\Gamma_3} h_\nu(t,\zeta) j_\nu(v) d\Gamma, \text{ for all } t \in (0,T), \zeta \in Z \text{ and } v \in X.
\end{aligned}$$

We define the element $f_z(t) \in V^*$ by

$$(f_z(t), w)_{V^*\times V} = (f(t) - z(t), w)_{V^*\times V}, \quad \text{for all } w \in V \text{ and } t \in (0,T).$$

Now, we put $v_z(t) = \dot{u}_z(t)$ for all $t \in (0,T)$. Then, with the above notations, the inequality (4.4) can be written as follows

$$\begin{aligned}&(A(t,(R_1v_z)(t),v_z(t)) - f_z(t), w - v_z(t))_{V^*\times V} + \psi(t,(R_3v_z)(t),v_z(t),w)\\&+\psi(t,(R_3v_z)(t),v_z(t),v_z(t)) + j^0(t,(R_4v_z)(t),Mv_z(t);Mw - Mv_z(t)) \geq 0,\end{aligned} \tag{4.7}$$

for all $w \in K$. It follows from the hypotheses (H_1), (H_2), (H_9), (H_{12}), (H_{10}), $(H_{13})(1)$ and $(H_{14})(1)$ that we are able to apply Theorem 5 in [23], from which we infer that problem (4.7) combined with the initial condition (4.5) has a unique solution $u_z \in C(0,T;V)$ with $\dot{u}_z \in L^2(0,T;V)$ and $\dot{u}_z(t) \in K$ for a.e. $t \in (0,T)$. We turn now to show the estimate (4.6). Let $z_1, z_2 \in L^2(0,T;V^*)$. We write the variational-hemivariational inequality (4.4) successively for z_1 and z_2 taking $w = \dot{u}_{z_2}(t)$ in the first inequality and $w = \dot{u}_{z_1}(t)$ in the second one, and add the resulting inequalities to obtain that for all $t \in (0,T)$

$$\begin{aligned}&(\mathcal{A}\varepsilon(\dot{u}_{z_1}(t) - \dot{u}_{z_2}(t)), \varepsilon(\dot{u}_{z_1}(t) - \dot{u}_{z_2}(t)))_{\mathcal{H}}\\&+(\mathcal{F}\varepsilon(u_{z_1}(t) - u_{z_2}(t)), \varepsilon(\dot{u}_{z_1}(t) - \dot{u}_{z_2}(t)))_{\mathcal{H}}\\\leq&\int_{\Gamma_3}\left[F_b\Big(t,\int_0^t \|u_{z_{1_\tau}}(s)\|\,ds\Big) - F_b\Big(t,\int_0^t \|u_{z_{2_\tau}}(s)\|\,ds\Big)\right]\left(\|\dot{u}_{z_{2_\tau}}(t))\| - \|\dot{u}_{z_{1_\tau}}(t)\|\right)d\Gamma\\&+\int_{\Gamma_3}\big[h_\nu(t,u_{z_{1_\nu}}(t))j_\nu^0(\dot{u}_{z_{1_\nu}}(t);\dot{u}_{z_{2_\nu}}(t) - \dot{u}_{z_{1_\nu}}(t))\\&+h_\nu(t,u_{z_{2_\nu}}(t))j_\nu^0(\dot{u}_{z_{2_\nu}}(t);\dot{u}_{z_{1_\nu}}(t) - \dot{u}_{z_{2_\nu}}(t))\big]\,d\Gamma\\&+(z_1(t) - z_2(t), \dot{u}_{z_2}(t) - \dot{u}_{z_1}(t))_{V^*\times V}.\end{aligned} \tag{4.8}$$

From assumptions (H_1) and (H_2), it is easy to see that

$$\begin{aligned}&\int_0^t (\mathcal{A}\varepsilon(\dot{u}_{z_1}(s) - \dot{u}_{z_2}(s)),\varepsilon(\dot{u}_{z_1}(s) - \dot{u}_{z_2}(s)))_{\mathcal{H}}ds\\&\qquad\qquad\geq m_{\mathcal{A}}\int_0^t \|\dot{u}_{z_1}(s) - \dot{u}_{z_2}(s)\|_V^2\,ds,\end{aligned} \tag{4.9}$$

$$\int_0^t (\mathcal{F}\varepsilon(u_{z_1}(s) - u_{z_2}(s)),\varepsilon(\dot{u}_{z_1}(s) - \dot{u}_{z_2}(s)))_{\mathcal{H}} \geq \frac{m_{\mathcal{F}}}{2}\|u_{z_1}(t) - u_{z_2}(t)\|_V^2, \tag{4.10}$$

for all $t \in (0,T)$. Moreover, by (H_9) we find that

$$\begin{aligned}&\int_{\Gamma_3}\left[F_b\left(t,\int_0^t \|u_{z_{1_\tau}}(s)\|\,ds\right) - F_b\left(t,\int_0^t \|u_{z_{2_\tau}}(s)\|\,ds\right)\right](\|\dot{u}_{z_{2_\tau}}(t)\| - \|\dot{u}_{z_{1_\tau}}(t)\|)d\Gamma\\\leq& L_F c_0^2\|\dot{u}_{z_1}(t) - \dot{u}_{z_2}(t)\|_V\int_0^t \|u_{z_1}(s) - u_{z_2}(s)\|_V\,ds.\end{aligned} \tag{4.11}$$

We integrate (4.8) over $(0,t)$, keeping in mind (4.9)-(4.11) and the assumptions

(H_{10}) to find that for all $t \in (0,T)$

$$\begin{aligned} &\frac{m_{\mathcal{F}}}{2} \|u_{z_1}(t) - u_{z_2}(t)\|_V^2 + (m_{\mathcal{A}} - M_h \alpha_\nu c_0^2) \int_0^t \|\dot{u}_{z_1}(s) - \dot{u}_{z_2}(s)\|_V^2 \, ds \\ &\leq \int_0^t \|z_1(s) - z_2(s)\|_{V^*} \|\dot{u}_{z_1}(s) - \dot{u}_{z_2}(s)\|_V \, ds \\ &\quad + L_F c_0^2 \int_0^t \left[\|\dot{u}_{z_1}(s) - \dot{u}_{z_2}(s)\|_V \int_0^s \|u_{z_1}(r) - u_{z_2}(r)\|_V \, dr \right] ds. \end{aligned} \tag{4.12}$$

Remember the following modified Cauchy–Schwarz inequality

$$xy \leq \epsilon x^2 + \frac{1}{4\epsilon} y^2, \quad \forall x, y \in \mathbb{R}, \epsilon > 0. \tag{4.13}$$

Then, we deduce from (4.12) that there exists $c > 0$ such that for all $t \in (0,T)$

$$\begin{aligned} &\frac{m_{\mathcal{F}}}{2} \|u_{z_1}(t) - u_{z_2}(t)\|_V^2 + (m_{\mathcal{A}} - M_h \alpha_\nu c_0^2 - 2\epsilon) \int_0^t \|\dot{u}_{z_1}(s) - \dot{u}_{z_2}(s)\|_V^2 \, ds \\ &\leq c \left(\int_0^t \|z_1(s) - z_2(s)\|_{V^*}^2 \, ds + \int_0^t \|u_{z_1}(s) - u_{z_2}(s)\|_V \, ds \right). \end{aligned} \tag{4.14}$$

Finally, we choose ϵ to be small enough, keeping in mind the smallness condition $(H_{14})(1)$, we then apply Gronwall's lemma to get the estimate (4.6). □

Step 2. we use the displacement field u_z obtained in **Lemma 4.1** to construct the following auxiliary problem in terms of temperature field.

Problem 4.2. *Find a temperature field $\theta_z(t) \in Q$ for a.e. $t \in (0,T)$ such that*

$$\begin{aligned} &(\dot{\theta}_z(t), \eta)_{L^2(\Omega)} + (\mathcal{K}\nabla\theta_z(t), \nabla\eta)_H - (\mathcal{N}(\dot{u}_z(t)), \eta)_{L^2(\Omega)} \\ &\qquad + \int_{\Gamma_3} j_c^0(\theta_z(t) - \theta_F; \eta) d\Gamma \geq (q_c(t), \eta)_{Q^* \times Q}, \quad \forall \eta \in Q, \end{aligned} \tag{4.15}$$

$$\theta_z(0) = \theta_0. \tag{4.16}$$

Lemma 4.2. *Assume the hypotheses (H_3), (H_8), $(H_{13})(3)$, (H_{11}) and $(H_{14})(3)$. Then, **Problem 4.2** has a unique solution $\theta_z \in L^2(0,T;Q)$ with $\dot{\theta}_z \in L^2(0,T;Q^*)$. Moreover, for $z_1, z_2 \in L^2(0,T;V^*)$, let us denote by θ_{z_1} and θ_{z_2} the solutions of **Problem 4.2** corresponding to z_1 and z_2, respectively. Then there exists a constant $c > 0$ such that for all $t \in (0,T)$*

$$\begin{aligned} \|\theta_{z_1}(t) - \theta_{z_2}(t)\|_{L^2(\Omega)}^2 &+ \int_0^t \|\theta_{z_1}(s) - \theta_{z_2}(s)\|_Q^2 \, ds \\ &\leq c \int_0^t \|z_1(s) - z_2(s)\|_{V^*}^2 \, ds. \end{aligned} \tag{4.17}$$

Proof. We use the Riesz's representation theorem to define the element $\widehat{q}_c(t) \in Q^*$ by

$$(\widehat{q}_c(t), \eta)_{Q^* \times Q} = (q_c(t), \eta)_{Q^* \times Q} + (\mathcal{N}(\varepsilon(\dot{u}(t))), \eta)_{L^2(\Omega)}.$$

Then, we rewrite the inequality (4.15) as

$$(\dot{\theta}_z(t),\eta)_{L^2(\Omega)}+(\mathcal{K}\nabla\theta_z(t),\nabla\eta)_H + \int_{\Gamma_3} j_c^0(\theta_z(t)-\theta_F;\eta)d\Gamma \geq (\widehat{q}_c(t),\eta)_{Q^*\times Q}, \tag{4.18}$$

for all $\eta \in Q$. Under hypotheses (H_3), (H_8), $(H_{13})(3)$ and (H_{11}) and $(H_{14})(3)$ it follows by [27, Lemma 11] that the problem (4.18) combined with the initial condition (4.16) has a unique solution $\theta_z \in L^2(0,T;Q)$ with $\dot{\theta}_z \in L^2(0,T;Q^*)$.On the other hand, let $z_1, z_2 \in L^2(0,T;V^*)$. We write the inequality (4.15) successively for z_1 and z_2, taking $\eta = \theta_{z_2}(t)-\theta_{z_1}(t)$ in the first inequality and $\eta = \theta_{z_1}(t)-\theta_{z_2}(t)$ in the second one, and add the resulting inequalities to find that for all $t \in (0,T)$

$$\begin{aligned}
&(\dot{\theta}_{z_1}(t)-\dot{\theta}_{z_2}(t),\theta_{z_1}(t)-\theta_{z_2}(t))_{L^2(\Omega)} \\
&\quad+(\mathcal{K}\nabla(\theta_{z_1}(t)-\theta_{z_2}(t)),\nabla(\theta_{z_1}(t)-\theta_{z_2}(t)))_H \\
&\leq \int_{\Gamma_3}\left[j_c^0(\theta_{z_1}(t)-\theta_F;\theta_{z_2}(t)-\theta_{z_1}(t))+j_c^0(\theta_{z_2}(t)-\theta_F;\theta_{z_1}(t)-\theta_{z_2}(t))\right]d\Gamma \\
&\quad+(\mathcal{N}(\dot{u}_{z_1}(t))-\mathcal{N}(\dot{u}_{z_2}(t)),\theta_{z_2}(t)-\theta_{z_1}(t))_{L^2(\Omega)}.
\end{aligned} \tag{4.19}$$

We integrate this inequality over $(0,t)$ and keep (H_3), (H_{11}) and the inequality (4.13) to obtain that there exists $c>0$ such that for all $t \in (0,T)$

$$\frac{1}{2}\|\theta_{z_1}(t)-\theta_{z_2}(t)\|^2_{L^2(\Omega)}+(m_{\mathcal{K}}-\alpha_c c_2^2-\epsilon)\int_0^t\|\theta_{z_1}(s)-\theta_{z_2}(s)\|_Q^2\,ds \leq c\int_0^t\|\dot{u}_{z_1}(s)-\dot{u}_{z_2}(s)\|_V^2\,ds, \tag{4.20}$$

where ϵ is a positive real parameter. Finally, we combine (4.20) with the estimate (4.6) and keep in mind the smallness condition $(H_{14})(3)$ and we chose ϵ to be small enough to get the estimation (4.17). □

<u>**Step 3.**</u> We use the displacement field u_z and the temperature field θ_z obtained in **Lemma 4.1** and **Lemma 4.2** respectively, to construct the following auxiliary problem in terms of electric potential.

Problem 4.3. *Find an electric potential $\varphi_z(t) \in W$ for a.e. $t \in (0,T)$ such that*

$$(\mathcal{B}\nabla\varphi_z(t),\nabla\xi)_H-(\mathcal{E}\varepsilon(u_z(t)),\nabla\xi)_H+(\theta_z(t)\mathcal{P},\nabla\xi)_H + \int_{\Gamma_3} j_e^0(\varphi_z(t)-\varphi_F;\xi)d\Gamma \geq (q_e(t),\xi)_{W^*\times W}, \quad \forall \xi \in W. \tag{4.21}$$

Lemma 4.3. *Assume the hypotheses (H_4), (H_5), (H_7), $(H_{13})(2)$, (H_{11}) and $(H_{14})(2)$. Then **Problem 4.3** has a unique solution $\varphi_z \in L^2(0,T;W)$. Moreover, for $z_1, z_2 \in L^2(0,T;V^*)$, let us denote by φ_{z_1} and φ_{z_2} the solutions of **Problem 4.3** corresponding to z_1 and z_2, respectively. Then there exists a constant $c>0$ such that for all $t \in (0,T)$*

$$\|\varphi_{z_1}(t)-\varphi_{z_2}(t)\|_W^2 \leq c\int_0^t\|z_1(s)-z_2(s)\|_{V^*}^2\,ds. \tag{4.22}$$

Proof. By using the Riesz's representation theorem we define the element $\widehat{q}_e(t) \in W^*$ by

$$(\widehat{q}_e(t), \xi)_{W^* \times W} = (q_e(t), \xi)_{W^* \times W} + (\mathcal{E}\varepsilon(u_z(t)), \nabla\xi)_H - (\theta_z(t)\mathcal{P}, \nabla\xi)_H,$$

for all $\xi \in W$ and $t \in (0, T)$. Then, the inequality (4.21) can be written as

$$(\mathcal{B}\nabla\varphi_z(t), \nabla\xi)_H + \int_{\Gamma_3} j_e^0(\varphi_z(t) - \varphi_F; \xi)d\Gamma \geq (\widehat{q}_e(t), \xi)_{W^* \times W}, \quad \forall\xi \in W. \quad (4.23)$$

Under hypotheses (H_4), (H_5), (H_7), $(H_{13})(2)$, (H_{11}) and $(H_{14})(2)$ it follows from [24, Lemma 9] that there exists a unique solution to problem (4.23). Let $z_1, z_2 \in L^2(0, T; V^*)$. We write the hemivariational inequality (4.21) successively for z_1 and z_2, taking $\xi = \varphi_{z_2}(t) - \varphi_{z_1}(t)$ in the first inequality and $\xi = \varphi_{z_1}(t) - \varphi_{z_2}(t)$ in the second one, and add the resulting inequalities to find that for all $t \in (0, T)$

$$\begin{aligned}
&(\mathcal{B}\nabla(\varphi_{z_1}(t) - \varphi_{z_2}(t)), \nabla(\varphi_{z_1}(t) - \varphi_{z_2}(t)))_H \\
\leq &(\mathcal{E}\varepsilon(u_{z_1}(t) - u_{z_2}(t)), \nabla(\varphi_{z_1}(t) - \varphi_{z_2}(t)))_H \\
&+ ((\theta_{z_1}(t) - \theta_{z_2}(t))\mathcal{P}, \nabla(\varphi_{z_2}(t) - \varphi_{z_1}(t)))_H \\
&+ \int_{\Gamma_3} \big[j_e^0(\varphi_{z_1}(t) - \varphi_F; \varphi_{z_2}(t) - \varphi_{z_1}(t)) \\
&+ j_e^0(\varphi_{z_2}(t) - \varphi_F; \varphi_{z_1}(t) - \varphi_{z_2}(t))\big]\, d\Gamma.
\end{aligned} \quad (4.24)$$

We use the assumptions (H_4), (H_5) and (H_7), (H_{11}) and the inequality (4.13) several times to obtain that there exists $c > 0$ such that for all $t \in (0, T)$

$$\begin{aligned}
&(m_{\mathcal{B}} - \alpha_e c_1^2 - 2\epsilon) \|\varphi_{z_1}(t) - \varphi_{z_2}(t)\|_W^2 \\
&\qquad \leq c\left(\|u_{z_1}(t) - u_{z_2}(t)\|_V^2 + \|\theta_{z_1}(t) - \theta_{z_2}(t)\|_{L^2(\Omega)}^2\right).
\end{aligned} \quad (4.25)$$

Finally, for ϵ sufficiently small, by combining (4.25) with the estimates (4.6), (4.17) and the smallness condition $(H_{14})(2)$, (4.22) holds. □

Step 4. For $z \in L^2(0, T; V^*)$ we denote by θ_z and φ_z the solutions of **Problem 4.2** and **Problem 4.3**, respectively, and we consider the operator $\Lambda : L^2(0, T; V^*) \to L^2(0, T; V^*)$ defined by

$$(\Lambda z(t), w)_{V^* \times V} = (\mathcal{E}^T\nabla\varphi_z(t) - \theta_z(t)\mathcal{M}, \varepsilon(w))_{\mathcal{H}}, \quad (4.26)$$

for all $w \in V$ and a.e. $t \in (0, T)$.

Lemma 4.4. *There exists a unique $\bar{z} \in L^2(0, T; V^*)$ such that $\Lambda\bar{z} = \bar{z}$.*

Proof. Let $z_1, z_2 \in L^2(0, T; V^*)$. From (4.26) and after some algebra we find that there exists $c > 0$ such that for all $t \in (0, T)$

$$\|\Lambda z_1(t) - \Lambda z_2(t)\|_{V^*}^2 \leq c\left(\|\theta_{z_1}(t) - \theta_{z_2}(t)\|_{L^2(\Omega)}^2 + \|\varphi_{z_1}(t) - \varphi_{z_2}(t)\|_W^2\right). \quad (4.27)$$

We combine this inequality with (4.17) and (4.22) to conclude that there exists $c > 0$ such that

$$\|\Lambda z_1(t) - \Lambda z_2(t)\|_{V^*}^2 \leq c\int_0^t \|z_1(s) - z_2(s)\|_{V^*}^2\, ds, \quad (4.28)$$

for all $t \in (0,T)$. It follows from [21, Lemma 7] that Λ has a unique fixed point $\bar{z} \in L^2(0,T;V^*)$. □

Finally, we turn to prove **Theorem 4.1**. We denote by $u_{\bar{z}}$, $\theta_{\bar{z}}$ and $\varphi_{\bar{z}}$ the solutions of **Problem 4.1**, **Problem 4.2** and **Problem 4.3** corresponding to $\bar{z}$, respectively, where $\bar{z} \in L^2(0,T;V^*)$ is the unique fixed point of the operator Λ. Since $(\bar{z}, w)_{V^*\times V} = (\mathcal{E}^T \nabla\varphi_{\bar{z}}(t) - \theta_{\bar{z}}(t)\mathcal{M}, \varepsilon(w))_{\mathcal{H}}$ for all $w \in V$, we conclude that the triplet $(u_{\bar{z}}, \varphi_{\bar{z}}, \theta_{\bar{z}})$ is a solution to **Problem 3.1**. The uniqueness part of the solution follows from uniqueness of the fixed point of operator Λ.

5. Fully discrete approximation: error estimates

In this section, we present a fully discrete approximation for **Problem 3.1** and establish an error estimate for the approximate solution.

Let Ω be a polygonal domain. We consider a finite element triangulation $\mathcal{T}^h = \{\mathcal{T}_r\}_r$ of $\overline{\Omega}$ that is compatible with the boundary partitions, where h denotes the spatial discretization parameter. Let $\mathbb{P}^1(\mathcal{T}_r)$ represent the space of polynomials of degree at most 1 on $\mathcal{T}_r$. We then define the following finite-dimensional spaces to approximate V, W and Q, respectively:

$$V^h = \{w^h \in [C(\overline{\Omega})]^d \mid w^h_{|\mathcal{T}_r} \in [\mathbb{P}^1(\mathcal{T}_r)]^d,\ w^h = 0 \text{ on } \Gamma_1\} \subset V, \tag{5.1}$$

$$W^h = \{\xi^h \in C(\overline{\Omega}) \mid \xi^h_{|\mathcal{T}_r} \in \mathbb{P}^1(\mathcal{T}_r),\ \xi^h = 0 \text{ on } \Gamma_a\} \subset W, \tag{5.2}$$

$$Q^h = \{\eta^h \in C(\overline{\Omega}) \mid \eta^h_{|\mathcal{T}_r} \in \mathbb{P}^1(\mathcal{T}_r),\ \eta^h = 0 \text{ on } \Gamma_1 \cup \Gamma_2\} \subset Q. \tag{5.3}$$

We also define a related finite element subset K^h of the space V^h to approximate K

$$K^h = \{w^h \in V^h \mid w^h_\nu - g \le 0 \text{ on } \Gamma_3\} \subset K. \tag{5.4}$$

We consider a uniform partition $t_0 = 0 < t_1 < ... < t_N = T$ of $[0,T]$. We denote by k the time step size given by $k = \frac{T}{N}$. Moreover, for a continuous function f we denote $f(t_n) = f_n$, and for a sequence $\{z_n\}_{n=0}^N$ we denote $\delta z_n = \frac{z_n - z_{n-1}}{k}$. Let u_0^h and θ_0^h be the appropriate approximations of the initial conditions u_0 and θ_0, respectively.

Using the backward Euler scheme, the fully discrete approximation of **Problem 3.1** is formulated as follows.

Problem 5.1. *Find a discrete velocity field* $v^{hk} = \{v_n^{hk}\}_{n=0}^N \subset V^h$*, a discrete electric potential* $\varphi^{hk} = \{\varphi_n^{hk}\}_{n=0}^N \subset W^h$ *and a discrete temperature field* $\theta^{hk} = \{\theta_n^{hk}\}_{n=0}^N \subset Q^h$ *such that for* $n = 1, 2, ..., N$*,*

$$\begin{aligned}
&(\mathcal{A}\varepsilon(v_n^{hk}), \varepsilon(w^h - v_n^{hk}))_{\mathcal{H}} + (\mathcal{F}\varepsilon(u_n^{hk}), \varepsilon(w^h - v_n^{hk}))_{\mathcal{H}} \\
&\quad + (\mathcal{E}^T \nabla \varphi_n^{hk}, \varepsilon(w^h - v_n^{hk}))_{\mathcal{H}} - (\theta_n^{hk}\mathcal{M}, \varepsilon(w^h - v_n^{hk}))_{\mathcal{H}} \\
&\quad + \int_{\Gamma_3} F_b\left(t_n, k\sum_{i=0}^{n-1} \|u_i^{hk}\|\right)\left(\|w_\tau^h\| - \|v_{n_\tau}^{hk}\|\right) d\Gamma \\
&\quad + \int_{\Gamma_3} h_\nu(t_n, u_{n_\nu}^{hk}) j_\nu^0(v_{n_\nu}^{hk}; w_\nu^h - v_{n_\nu}^{hk}) d\Gamma \ge (f_n, w^h - v_n^{hk})_{V^*\times V}, \quad \forall w^h \in K^h,
\end{aligned} \tag{5.5}$$

$$(\mathcal{B}\nabla\varphi_n^{hk},\nabla\xi^h)_H-(\mathcal{E}\varepsilon(u_n^{hk}),\nabla\xi^h)_H+(\theta_n^{hk}\mathcal{P},\nabla\xi^h)_H$$
$$+\int_{\Gamma_3}j_e^0(\varphi_n^{hk}-\varphi_F;\xi^h)d\Gamma\geq(q_{e_n},\xi^h)_{W^*\times W},\quad\forall\xi^h\in W^h, \tag{5.6}$$

$$(\delta\theta_n^{hk},\eta^h)_{L^2(\Omega)}+(\mathcal{K}\nabla\theta_n^{hk},\nabla\eta^h)_H-(\mathcal{N}(v_n^{hk}),\eta^h)_{L^2(\Omega)}$$
$$+\int_{\Gamma_3}j_c^0(\theta_n^{hk}-\theta_F;\eta^h)d\Gamma\geq(q_{c_n},\eta^h)_{Q^*\times Q},\quad\forall\eta^h\in Q^h, \tag{5.7}$$

$$u_0^{hk}=u_0^h,\qquad\theta_0^{hk}=\theta_0^h. \tag{5.8}$$

Here the discrete velocity field $\{v_n^{hk}\}_{n=0}^N$ is related with the discrete displacement $\{u_n^{hk}\}_{n=0}^N$ field by the following equalities

$$v_n^{hk}=\delta u_n^{hk}\quad\text{and}\quad u_n^{hk}=u_0^h+\sum_{i=1}^n kv_i^{hk},\qquad n=1,...,N. \tag{5.9}$$

Using the same arguments presented in the previous section, it can be shown that **Problem 5.1** has a unique solution $(u^{hk},\varphi^{hk},\theta^{hk})$. Our objective here is to estimate the following numerical errors $\|u_n-u_n^{hk}\|_V$, $\|v_n-v_n^{hk}\|_V$, $\|\varphi_n-\varphi_n^{hk}\|_W$, $\|\theta_n-\theta_n^{hk}\|_{L^2(\Omega)}$ and $\|\theta_n-\theta_n^{hk}\|_Q$. Throughout this section, we will denote by c the various positive constants that may depend on the solution and the problem's data, but are independent of the discretization parameters h and k. The value of c may change from line to line.

Theorem 5.1. *Let assumptions of **Theorem 4.1** still hold, and let the following condition*

$$L_{\mathcal{M}}+L_{\mathcal{P}}<2 \tag{5.10}$$

*be satisfied. Let (u,φ,θ) and $(u^{hk},\varphi^{hk},\theta^{hk})$ denote the solutions to **Problem 3.1** and **Problem 5.1**, respectively. Then, the following error estimates hold for all $\{w_i^h\}_{i=1}^N\subset K^h$, $\{\xi_i^h\}_{i=1}^N\subset W^h$ and $\{\eta_i^h\}_{i=1}^N\subset Q^h$*

$$\begin{aligned}&\max_{1\leq n\leq N}\Big\{\|u_n-u_n^{hk}\|_V^2+\|v_n-v_n^{hk}\|_V^2+\|\varphi_n-\varphi_n^{hk}\|_W^2\\&+\|\theta_n-\theta_n^{hk}\|_{L^2(\Omega)}^2\Big\}+k\sum_{i=1}^N\|\theta_i-\theta_i^{hk}\|_Q^2\leq c\Big[\|u_0-u_0^h\|_V^2\\&+\|\theta_0-\theta_0^h\|_{L^2(\Omega)}^2+\|\theta_1-\eta_1^h\|_{L^2(\Omega)}^2+k\sum_{i=1}^N\Big(\|\dot\theta_i-\delta\theta_i\|_{L^2(\Omega)}^2\\&+\|\theta_i-\eta_i^h\|_{L^2(\Omega)}^2+\|\theta_i-\eta_i^h\|_Q^2+\|\eta_i^h-\theta_i\|_{L^2(\Gamma_3)}\Big)\\&+\max_{1\leq n\leq N}\Big\{\|v_n-w_n^h\|_V^2+\|\varphi_n-\xi_n^h\|_W^2+\|\theta_n-\eta_n^h\|_{L^2(\Omega)}^2\\&+\|v_n-w_n^h\|_{[L^2(\Gamma_3)]^d}+\|\varphi_n-\xi_n^h\|_{L^2(\Gamma_3)}+I_n^2\Big\}\\&+\frac{1}{k}\sum_{i=1}^{N-1}\|\theta_i-\eta_i^h-(\theta_{i+1}-\eta_{i+1}^h)\|_{L^2(\Omega)}\Big],\end{aligned} \tag{5.11}$$

where the integration error I_n is given by

$$I_n = \left\| k\sum_{j=1}^{n} v_j - \int_0^{t_n} v(s)ds \right\|_V . \tag{5.12}$$

Proof. By rearranging the terms, we rewrite (5.5) in the following form

$$\begin{aligned}
&(\mathcal{A}\varepsilon(v_n^{hk}),\varepsilon(v_n - v_n^{hk}))_{\mathcal{H}} + (\mathcal{F}\varepsilon(u_n^{hk}),\varepsilon(v_n - v_n^{hk}))_{\mathcal{H}} \\
&\quad + (\mathcal{E}^T\nabla\varphi_n^{hk},\varepsilon(v_n - v_n^{hk}))_{\mathcal{H}} - (\theta_n^{hk}\mathcal{M},\varepsilon(v_n - v_n^{hk}))_{\mathcal{H}} \\
&\quad + \int_{\Gamma_3} F_b\left(t_n, k\sum_{i=0}^{n-1}\|u_i^{hk}\|\right)\left(\|w_\tau^h\| - \|v_{n_\tau}^{hk}\|\right)d\Gamma \\
&\quad + \int_{\Gamma_3} h_\nu(t_n,u_{n_\nu}^{hk})j_\nu^0(v_{n_\nu}^{hk};w_\nu^h - v_{n_\nu}^{hk})d\Gamma \geq (f_n, w^h - v_n^{hk})_{V^*\times V} \\
&\quad + (\mathcal{A}\varepsilon(v_n^{hk}),\varepsilon(v_n - w^h))_{\mathcal{H}} + (\mathcal{F}\varepsilon(u_n^{hk}),\varepsilon(v_n - w^h))_{\mathcal{H}} \\
&\quad + (\mathcal{E}^T\nabla\varphi_n^{hk},\varepsilon(v_n - w^h))_{\mathcal{H}} - (\theta_n^{hk}\mathcal{M},\varepsilon(v_n - w^h))_{\mathcal{H}},
\end{aligned} \tag{5.13}$$

for all $w^h \in K^h$. Adding it to the inequality (3.15) at time $t = t_n$ with taking $w = v_n^{hk}$, and using the boundedness of the functions F_b and h_ν, we find easily that for all $w^h \in K^h$

$$\begin{aligned}
&(\mathcal{A}\varepsilon(v_n - v_n^{hk}),\varepsilon(v_n - v_n^{hk}))_{\mathcal{H}} \leq (\mathcal{A}\varepsilon(v_n - v_n^{hk}),\varepsilon(w^h - v_n))_{\mathcal{H}} \\
&+ (\mathcal{F}\varepsilon(u_n - u_n^{hk}),\varepsilon(v_n^{hk} - v_n))_{\mathcal{H}} + (\mathcal{E}^T\nabla(\varphi_n - \varphi_n^{hk}),\varepsilon(v_n^{hk} - v_n))_{\mathcal{H}} \\
&+ ((\theta_n - \theta_n^{hk})\mathcal{M},\varepsilon(v_n^{hk} - v_n))_{\mathcal{H}} + (\mathcal{F}\varepsilon(u_n - u_n^{hk}),\varepsilon(w^h - v_n))_{\mathcal{H}} \\
&+ (\mathcal{E}^T\nabla(\varphi_n - \varphi_n^{hk}),\varepsilon(w^h - v_n))_{\mathcal{H}} + ((\theta_n - \theta_n^{hk})\mathcal{M},\varepsilon(w^h - v_n))_{\mathcal{H}} \\
&+ M_h\int_{\Gamma_3}\left[j_\nu^0(v_{n_\nu};v_{n_\nu}^{hk} - v_{n_\nu}) + j_\nu^0(v_{n_\nu}^{hk};w_\nu^h - v_{n_\nu}^h k)\right. \\
&\left.- j_\nu^0(v_{n_\nu};w_\nu^h - v_{n_\nu})\right]d\Gamma + R(w^h,v_n),
\end{aligned} \tag{5.14}$$

where the residual $R(w,v)$ is defined by

$$\begin{aligned}
R(w,v) =& (\mathcal{A}\varepsilon(\dot{u}_n),\varepsilon(w - v))_{\mathcal{H}} + (\mathcal{F}\varepsilon(u_n),\varepsilon(w - v))_{\mathcal{H}} \\
&+ (\mathcal{E}^T\nabla\varphi_n,\varepsilon(w - v))_{\mathcal{H}}) - (\theta_n\mathcal{M},\varepsilon(w - v))_{\mathcal{H}} \\
&\int_{\Gamma_3} F_b\left(t_n,\int_0^{t_n}\|u_\tau(s)\|\,ds\right)\left(\|w_\tau\| - \|v_\tau\|\right)d\Gamma \\
&+ \int_{\Gamma_3} h_\nu(u_{n_\nu})j_\nu^0(v_{n_\nu};w_\nu - v_\nu)d\Gamma - (f_n, w - v)_{V^*\times V}.
\end{aligned} \tag{5.15}$$

By using the subadditivity of the function $j_\nu(x,r;\cdot)$ for all $r \in \mathbb{R}$ a.e. $x \in \Gamma_3$, and assumption (H_{10}), we can easily find that

$$\begin{aligned}
&\int_{\Gamma_3}\left[j_\nu^0(v_{n_\nu};v_{n_\nu}^{hk} - v_{n_\nu}) + j_\nu^0(v_{n_\nu}^{hk};w_\nu^h - v_{n_\nu}^{hk}) - j_\nu^0(v_{n_\nu};w_\nu^h - v_{n_\nu})\right]d\Gamma \\
\leq& \alpha_\nu\int_{\Gamma_3}\left\|v_n - v_n^{hk}\right\|^2 d\Gamma + \int_{\Gamma_3}\left[j_\nu^0(v_{n_\nu}^{hk};w_\nu^h - v_{n_\nu}) - j_\nu^0(v_{n_\nu};w_\nu^h - v_{n_\nu})\right]d\Gamma.
\end{aligned} \tag{5.16}$$

From the definition of the generalized gradient we get

$$\begin{aligned} j_\nu^0(v_{n_\nu}^{hk}; w_\nu^h - v_{n_\nu}) &= \max\left\{\zeta \cdot (w_\nu^h - v_{n_\nu}) \mid \zeta \in \partial j_\nu(v_{n_\nu}^{hk})\right\} \\ &\leq \left\|w_\nu^h - v_{n_\nu}\right\| \left\|\partial j_\nu(v_{n_\nu}^{hk})\right\|. \end{aligned} \tag{5.17}$$

Keeping in mind hypothesis (H_{10}), we obtain that

$$j_\nu^0(v_{n_\nu}^{hk}; w_\nu^h - v_{n_\nu}) \leq c_{0\nu} \left\|w_\nu^h - v_{n_\nu}\right\|. \tag{5.18}$$

Thus, we deduce from (5.16) that

$$\begin{aligned} &\int_{\Gamma_3} \left[j_\nu^0(v_{n_\nu}; v_{n_\nu}^{hk} - v_{n_\nu}) + j_\nu^0(v_{n_\nu}^{hk}; w_\nu^h - v_{n_\nu}^{hk}) - j_\nu^0(v_{n_\nu}; w_\nu^h - v_{n_\nu})\right] d\Gamma \\ &\quad \leq \alpha_\nu c_0^2 \left\|v_n - v_n^{hk}\right\|_V^2 + 2c_{0\nu}\sqrt{|\Gamma_3|}\left\|w^h - v_n\right\|_{[L^2(\Gamma_3)]^d}. \end{aligned} \tag{5.19}$$

Proceeding as in [19], we find that there exists $c_R > 0$ such that

$$\left\|R(w^h, v_n)\right\| \leq c_R \left\|w^h - v_n\right\|_{[L^2(\Gamma_3)]^d}. \tag{5.20}$$

Then, by using the Cauchy-Shwarz inequality and the inequality (4.13) several times, we deduce from (5.14) that there exists $c > 0$ such that for all $w^h \in K^h$

$$\begin{aligned} &\left\|u_n - u_n^{hk}\right\|_V^2 + \left(m_\mathcal{A} - M_h \alpha_\nu c_0^2 - \frac{L_\mathcal{E} + L_\mathcal{M}}{2} - 2\epsilon\right)\left\|v_n - v_n^{hk}\right\|_V^2 \\ &\qquad \leq c\left[\left\|u_n - u_n^{hk}\right\|_V^2 + \left\|v_n - w^h\right\|_V^2 + \left\|v_n - w^h\right\|_{[L^2(\Gamma_3)]^d}\right] \\ &\qquad\quad + \left(\frac{L_\mathcal{M}}{2} + \epsilon\right)\left\|\theta_n - \theta_n^{hk}\right\|_{L^2(\Omega)}^2 + \left(\frac{L_\mathcal{E}}{2} + \epsilon\right)\left\|\varphi_n - \varphi_n^{hk}\right\|_W^2, \end{aligned} \tag{5.21}$$

where ϵ is a positive real parameter.

Now, we write successively the hemivariational inequalities (3.16) at time $t = t_n$ and (5.6) with substitute ξ by $\varphi_n^{hk} - \xi^h$, and ξ^h by $\xi^h - \varphi_n^{hk}$ respectively, then we add the two obtained inequalities to find that for all $\xi^h \in W^h$

$$\begin{aligned} &(\mathcal{B}\nabla(\varphi_n - \varphi_n^{hk}), \nabla(\varphi_n - \varphi_n^{hk}))_H \leq (\mathcal{B}\nabla(\varphi_n - \varphi_n^{hk}), \nabla(\varphi_n - \xi^h))_H \\ &+ ((\theta_n - \theta_n^{hk})\mathcal{P}, \nabla(\varphi_n^{hk} - \xi^h))_H + (\mathcal{E}\varepsilon(u_n - u_n^{hk}), \nabla(\xi^h - \varphi_n^{hk}))_H \\ &+ \int_{\Gamma_3} \left[j_e^0(\varphi_n - \varphi_F; \varphi_n^{hk} - \xi^h) + j_e^0(\varphi_n^{hk} - \varphi_F; \xi^h - \varphi_n^{hk})\right] d\Gamma. \end{aligned} \tag{5.22}$$

We use the same idea used previously in the proof of (5.19), we also find that

$$\begin{aligned} j_e^0(\varphi_n - \varphi_F; \varphi_n^{hk} - \xi^h) + j_e^0(\varphi_n^{hk} - \varphi_F; \xi^h - \varphi_n^{hk}) &\leq \alpha_e c_1^2 \left\|\varphi_n - \varphi_n^{hk}\right\|_W^2 \\ &\quad + 2c_{0e}\sqrt{|\Gamma_3|}\left\|\varphi_n - \xi^h\right\|_{L^2(\Gamma_3)}. \end{aligned} \tag{5.23}$$

Then, we deduce from (5.22) that there exists $c > 0$ such that for all $\xi^h \in W^h$

$$\begin{aligned} &\left(m_\mathcal{B} - \alpha_e c_1^2 - \frac{L_\mathcal{P}}{2} - 3\epsilon\right)\left\|\varphi_n - \varphi_n^{hk}\right\|_W^2 \leq \left(\frac{L_\mathcal{P}}{2} + \epsilon\right)\left\|\theta_n - \theta_n^{hk}\right\|_{L^2(\Omega)}^2 \\ &+ c\left[\left\|u_n - u_n^{hk}\right\|_V^2 + \left\|\varphi_n - \xi^h\right\|_W^2 + \left\|\varphi_n - \xi^h\right\|_{L^2(\Gamma_3)}\right]. \end{aligned} \tag{5.24}$$

Proceeding as in [5], we find that there exists $c > 0$ such that

$$\left\|u_i - u_i^{hk}\right\|_V^2 \le c\left\{\left\|u_0 - u_0^h\right\|_V^2 + k\sum_{j=1}^{i}\left\|v_j - v_j^{hk}\right\|_V^2 + I_i^2\right\}. \tag{5.25}$$

Now, combining (5.21) and (5.24) and keeping in mind the previous inequality, we find that there exists $c > 0$ such that for all $w^h \in K^h$ and $\xi^h \in W^h$

$$\begin{aligned}
&\left\|u_n - u_n^{hk}\right\|_V^2 + \left(m_{\mathcal{A}} - M_h\alpha_\nu c_0^2 - \frac{L_{\mathcal{E}} + L_{\mathcal{M}}}{2} - 2\epsilon\right)\left\|v_n - v_n^{hk}\right\|_V^2\\
&+ \left(m_{\mathcal{B}} - \alpha_e c_1^2 - \frac{L_{\mathcal{P}} + L_{\mathcal{E}}}{2} - 4\epsilon\right)\left\|\varphi_n - \varphi_n^{hk}\right\|_W^2 \le c\left\{I_n^2 + \left\|u_0 - u_0^h\right\|_V^2\right.\\
&+ k\sum_{i=1}^{n}\left\|v_i - v_i^{hk}\right\|_V^2 + \left\|v_n - w^h\right\|_V^2 + \left\|\varphi_n - \xi^h\right\|_W^2 + \left\|v_n - w^h\right\|_{[L^2(\Gamma_3)]^d}\\
&\left. + \left\|\varphi_n - \xi^h\right\|_{L^2(\Gamma_3)}\right\} + \left(\frac{L_{\mathcal{M}} + L_{\mathcal{P}}}{2} - 2\epsilon\right)\left\|\theta_n - \theta_n^{hk}\right\|_{L^2(\Omega)}^2.
\end{aligned} \tag{5.26}$$

Next, we write successively the inequalities (3.17) at time $t = t_n$ and (5.7) with taking $\eta = \theta_n^{hk} - \eta^h$ and $\eta^h = \eta^h - \theta_n^{hk}$ respectively, and add the resulting inequalities to find that for all $\eta^h \in Q^h$

$$\begin{aligned}
&(\delta(\theta_n - \theta_n^{hk}), \theta_n - \theta_n^{hk})_{L^2(\Omega)} + (\mathcal{K}\nabla(\theta_n - \theta_n^{hk}), \nabla(\theta_n - \theta_n^{hk}))_H\\
\le&(\delta\theta_n - \dot{\theta}_n, \theta_n - \theta_n^{hk})_{L^2(\Omega)} + (\dot{\theta}_n - \delta\theta_n, \theta_n - \eta^h)_{L^2(\Omega)}\\
&+ (\mathcal{K}\nabla(\theta_n - \theta_n^{hk}), \nabla(\theta_n - \eta^h))_H + (\mathcal{N}(v_n) - \mathcal{N}(v_n^{hk}), \eta^h - \theta_n^{hk})_{L^2(\Omega)}\\
&+ \int_{\Gamma_3}\left[j_c^0(\theta_n - \theta_F; \theta_n^{hk} - \eta^h) + j_c^0(\theta_n^{hk} - \theta_F; \eta^h - \theta_n^{hk})\right] d\Gamma\\
&+ (\delta(\theta_n - \theta_n^{hk}), \theta_n - \eta^h)_{L^2(\Omega)}.
\end{aligned} \tag{5.27}$$

Using a similar manner to (5.19), we can easily find that

$$\begin{aligned}
&\int_{\Gamma_3}\left[j_c^0(\theta_n - \theta_F; \theta_n^{hk} - \eta^h) + j_c^0(\theta_n^{hk} - \theta_F; \eta^h - \theta_n^{hk})\right] d\Gamma\\
\le&\alpha_c c_2^2\left\|\theta_n - \theta_n^{hk}\right\|_Q^2 + 2c_{0c}\sqrt{|\Gamma_3|}\left\|\theta_n - \eta^h\right\|_{L^2(\Gamma_3)}.
\end{aligned} \tag{5.28}$$

On the other hand, we have (see [19, 31])

$$\begin{aligned}
&(\delta(\theta_n - \theta_n^{hk}), \theta_n - \theta_n^{hk})_{L^2(\Omega)}\\
&\qquad \ge \frac{1}{2k}\left(\left\|\theta_n - \theta_n^{hk}\right\|_{L^2(\Omega)}^2 - \left\|\theta_{n-1} - \theta_{n-1}^{hk}\right\|_{L^2(\Omega)}^2\right).
\end{aligned} \tag{5.29}$$

We substitute the two previous inequalities into (5.27) to deduce that there exists

$c > 0$ such that for all $\eta^h \in Q^h$

$$\begin{aligned}
&\frac{1}{2k}\left(\left\|\theta_n - \theta_n^{hk}\right\|_{L^2(\Omega)}^2 - \left\|\theta_{n-1} - \theta_{n-1}^{hk}\right\|_{L^2(\Omega)}^2\right) \\
&+ \left(m_{\mathcal{K}} - \alpha_c c_2^2 - \epsilon\right)\left\|\theta_n - \theta_n^{hk}\right\|_Q^2 \le c\left[\left\|v_n - v_n^{hk}\right\|_V^2 + \left\|\theta_n - \theta_n^{hk}\right\|_{L^2(\Omega)}^2\right. \\
&\left.+ \left\|\dot{\theta}_n - \delta\theta_n\right\|_{L^2(\Omega)}^2 + \left\|\theta_n - \eta^h\right\|_{L^2(\Omega)}^2 + \left\|\theta_n - \eta^h\right\|_Q^2 + \left\|\theta_n - \eta^h\right\|_{L^2(\Gamma_3)}\right] \\
&+ (\delta(\theta_n - \theta_n^{hk}), \theta_n - \eta^h)_{L^2(\Omega)}.
\end{aligned} \tag{5.30}$$

We take $n = i$ in the previous inequality, and we sum the result inequality over i from 1 to n and keep in mind the following inequality

$$\begin{aligned}
&2k\sum_{i=1}^{n}(\delta(\theta_i - \theta_i^{hk}), \theta_i - \eta_i^h)_{L^2(\Omega)} \le \epsilon\left\|\theta_n - \theta_n^{hk}\right\|_{L^2(\Omega)}^2 \\
&+ c\left\{\left\|\theta_n - \eta_n^h\right\|_{L^2(\Omega)}^2 + \left\|\theta_0 - \theta_0^h\right\|_{L^2(\Omega)}^2 + \left\|\theta_1 - \eta_1^h\right\|_{L^2(\Omega)}^2\right\} \\
&+ k\sum_{i=1}^{n-1}\left\|\theta_i - \theta_i^{hk}\right\|_{L^2(\Omega)}^2 + \frac{1}{k}\sum_{i=1}^{n-1}\left\|\theta_i - \eta_i^h - (\theta_{i+1} - \eta_{i+1}^h)\right\|_{L^2(\Omega)}^2,
\end{aligned} \tag{5.31}$$

to obtain that there exists $c > 0$ such that for all $\{\eta_i\}_{i=1}^n \subset Q^h$

$$\begin{aligned}
&(1-\epsilon)\left\|\theta_n - \theta_n^{hk}\right\|_{L^2(\Omega)}^2 + k\left(m_{\mathcal{K}} - \alpha_c c_2^2 - \epsilon\right)\sum_{i=1}^{n}\left\|\theta_i - \theta_i^{hk}\right\|_Q^2 \\
\le& c\left[\left\|\theta_0 - \theta_0^h\right\|_{L^2(\Omega)}^2 + \left\|\theta_1 - \eta_1^h\right\|_{L^2(\Omega)}^2 + \left\|\theta_n - \eta_n^h\right\|_{L^2(\Omega)}^2\right. \\
&+ k\sum_{i=1}^{n}\left(\left\|\theta_i - \theta_i^{hk}\right\|_{L^2(\Omega)}^2 + \left\|v_i - v_i^{hk}\right\|_V^2\right) \\
&+ \frac{1}{k}\sum_{i=1}^{n-1}\left\|\theta_i - \eta_i^h - (\theta_{i+1} - \eta_{i+1}^h)\right\|_{L^2(\Omega)} + k\sum_{i=1}^{n}\left(\left\|\dot{\theta}_i - \delta\theta_i\right\|_{L^2(\Omega)}^2\right. \\
&\left.\left.+ \left\|\theta_i - \eta_i^h\right\|_{L^2(\Omega)}^2 + \left\|\theta_i - \eta_i^h\right\|_Q^2 + \left\|\theta_i - \eta_i^h\right\|_{L^2(\Gamma_3)}\right)\right].
\end{aligned} \tag{5.32}$$

Finally, we combine (5.26) with (5.32),then we choose ϵ to be small enough and keep in mind the smallness conditions (H_{14}) and (5.10) to obtain that there exists

$c > 0$ such that

$$\begin{aligned}
&\left\|u_n - u_n^{hk}\right\|_V^2 + \left\|v_n - v_n^{hk}\right\|_V^2 + \left\|\varphi_n - \varphi_n^{hk}\right\|_W^2 + \left\|\theta_n - \theta_n^{hk}\right\|_{L^2(\Omega)}^2 \\
&+ k\sum_{i=1}^{n} \left\|\theta_n - \theta_n^{hk}\right\|_Q^2 \le c\Big[\left\|\theta_0 - \theta_0^h\right\|_{L^2(\Omega)}^2 + \left\|u_0 - u_0^h\right\|_V^2 \\
&+ \left\|\theta_1 - \eta_1^h\right\|_{L^2(\Omega)}^2 + \left\|v_n - w_n^h\right\|_V^2 + \left\|\varphi_n - \xi_n^h\right\|_W^2 \\
&+ \left\|\theta_n - \eta_n^h\right\|_{L^2(\Omega)}^2 + \left\|v_n - w_n^h\right\|_{[L^2(\Gamma_3)]^d} + \left\|\varphi_n - \xi_n^h\right\|_{L^2(\Gamma_3)} \\
&+ I_n^2 + \frac{1}{k}\sum_{i=1}^{n-1} \left\|\theta_i - \eta_i^h - (\theta_{i+1} - \eta_{i+1}^h)\right\|_{L^2(\Omega)} + k\sum_{i=1}^{n}\Big(\left\|\dot{\theta}_i - \delta\theta_i\right\|_{L^2(\Omega)}^2 \\
&+ \left\|\theta_i - \eta_i^h\right\|_{L^2(\Omega)}^2 + \left\|\theta_i - \eta_i^h\right\|_Q^2 + \left\|\theta_i - \eta_i^h\right\|_{L^2(\Gamma_3)}\Big) \\
&+ k\sum_{i=1}^{n}\Big(\left\|\theta_i - \theta_i^{hk}\right\|_{L^2(\Omega)}^2 + \left\|v_i - v_i^{hk}\right\|_V^2\Big)\Big],
\end{aligned} \tag{5.33}$$

for all $\{w_i^h\}_{i=1}^n \subset K^h$, $\{\xi_i^h\}_{i=1}^n \subset W^h$ and $\{\eta_i^h\}_{i=1}^n \subset Q^h$. We apply the discrete Gronwall's inequality [30] to derive the desired estimate (5.11). □

Corollary 5.1. *Let the assumptions of **Theorem 5.1** hold. Under the following regularity conditions*

$$v \in C(0,T;[H^2(\Omega)]^d) \cap H^1(0,T;V), \qquad v_{|\Gamma_3} \in C(0,T;[H^2(\Gamma_3)]^d), \tag{5.34}$$

$$\varphi \in C(0,T;H^2(\Omega)) \cap C(0,T;W), \qquad \varphi_{|\Gamma_3} \in C(0,T;H^2(\Gamma_3)), \tag{5.35}$$

$$\begin{aligned}&\theta \in C(0,T;H^2(\Omega)) \cap C(0,T;Q) \cap H^1(0,T;Q) \cap H^2(0,T;L^2(\Omega)),\\ &\qquad \theta_{|\Gamma_3} \in C(0,T;H^2(\Gamma_3)),\end{aligned} \tag{5.36}$$

there exists $c > 0$ independent of h and k such that

$$\begin{aligned}
&\max_{1\le n\le N}\Big\{\left\|u_n - u_n^{hk}\right\|_V + \left\|v_n - v_n^{hk}\right\|_H + \left\|\varphi_n - \varphi_n^{hk}\right\|_W + \left\|\theta_n - \theta_n^{hk}\right\|_{L^2(\Omega)}\Big\} \\
&\qquad + k\sum_{i=1}^{N}\left\|\theta_i - \theta_i^{hk}\right\|_Q \le c(h+k).
\end{aligned} \tag{5.37}$$

Proof. For $n = 1, ..., N$, we choose $w_n^h = \Pi_V^h v_n$, $\xi_n^h = \Pi_W^h \varphi_n$ and $\eta_n^h = \Pi_Q^h \theta_n$ the finite element interpolant of v_n, φ_n and θ_n, respectively, where Π_B^h is the standard finite element interpolation operator over B [11]. By using standard finite element interpolation error estimates [6, 11, 17], we have the following approximation properties for $n = 1, ..., N$

$$\left\|v_n - w_n^h\right\|_V^2 \le ch^2 \left\|v\right\|_{C(0,T;[H^2(\Omega)]^d)}^2, \tag{5.38}$$

$$\left\|\varphi_n - \xi_n^h\right\|_W^2 \le ch^2 \left\|\varphi\right\|_{C(0,T;H^2(\Omega))}^2, \tag{5.39}$$

$$\left\|\theta_n - \eta_n^h\right\|_{L^2(\Omega)}^2 \le ch^2 \left\|\theta\right\|_{C(0,T;H^2(\Omega))}^2, \tag{5.40}$$

$$\left\|\theta_n - \eta_n^h\right\|_Q^2 \le ch^2 \left\|\theta\right\|_{C(0,T;H^2(\Omega))}^2, \tag{5.41}$$

$$\left\|v_n - w_n^h\right\|_{[L^2(\Gamma_3)]^d} \le ch^2 \left\|v\right\|_{C(0,T;[H^2(\Gamma_3)]^d)}, \tag{5.42}$$

$$\left\|\varphi_n - \xi_n^h\right\|_{L^2(\Gamma_3)} \le ch^2 \left\|\varphi\right\|_{C(0,T;H^2(\Gamma_3))}, \tag{5.43}$$

$$\left\|\theta_n - \eta_n^h\right\|_{L^2(\Gamma_3)} \le ch^2 \left\|\theta\right\|_{C(0,T;H^2(\Gamma_3))}. \tag{5.44}$$

We assume that the discrete initial conditions u_0^h and θ_0^h are chosen to be the finite element interpolants of u_0 and θ_0 respectively, i.e. $u_0^h = \Pi_V^h u_0$ and $\theta_0^h = \Pi_Q^h \theta_0$, then (see [11, 15])

$$\left\|u_0 - u_0^h\right\|_V^2 \le ch^2 \left\|u_0\right\|_{[H^2(\Omega)]^d}^2, \tag{5.45}$$

$$\left\|\theta_0 - \theta_0^h\right\|_{L^2(\Omega)}^2 \le ch^2 \left\|\theta_0\right\|_{H^2(\Omega)}^2. \tag{5.46}$$

We also have that [4, 6]

$$I_n \le ck \left\|v\right\|_{H^1(0,T;V)}, \tag{5.47}$$

$$\frac{1}{k}\sum_{i=1}^{N-1} \left\|\theta_i - \eta_i^h - (\theta_{i+1} - \eta_{i+1}^h)\right\|_Q^2 \le ch^2 \left\|\theta\right\|_{H^1(0,T;Q)}^2, \tag{5.48}$$

$$\sum_{i=1}^{n} \left\|\dot{\theta}_i - \delta\theta_i\right\|_{L^2(\Omega)}^2 \le ck \left\|\theta\right\|_{H^2(0,T;L^2(\Omega))}^2. \tag{5.49}$$

By combining the estimates (5.38)-(5.49) with the error estimate (5.11), we find that there exists a constant $c > 0$ such that

$$\max_{1\le n\le N}\left\{\left\|u_n - u_n^{hk}\right\|_V^2 + \left\|v_n - v_n^{hk}\right\|_V^2 + \left\|\varphi_n - \varphi_n^{hk}\right\|_W^2 + \left\|\theta_n - \theta_n^{hk}\right\|_{L^2(\Omega)}^2\right\} + k\sum_{i=1}^{N}\left\|\theta_i - \theta_i^{hk}\right\|_Q^2 \le c(h^2 + k^2). \tag{5.50}$$

Finally, keeping in mind the additional regularity (5.34)-(5.36) and the fact that $0 < k < 1$, we conclude the convergence rate (5.37). □

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