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Vector Fixed Point Theorem with Application to Systems of Nonlinear Elastic Beams Equations

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Abstract In this work, we establish a new existence and uniqueness of vector fixed point for a class of sum-type vector operators with some mixed monotone property in partially ordered product Banach spaces. The technique used is Thompson's part metric, and our goal is to extend and improve existing works in the scalar case vector case. As an application, we study the existence and uniqueness of solutions for systems of nonlinear singular fourth-order elastic beam equations with nonlinear boundary conditions.

Keywords Mixed monotone vector operators, Meir-Keeler type, systems of nonlinear elastic beams equations, Thompson metric, ε-chainable metric space

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1. Introduction

The theory of fixed points represents a growing field of research and development, intelligently combining different disciplines of knowledge such as geometry, topology and analysis. It is among the most powerful and fruitful tools of modern mathematics and can be considered a central subject of nonlinear analysis. In particular, when it comes to the solvability of a functional equation (whether it is a differential equation, a fractional differential equation, or an integral equation...), the problem is formulated in terms of finding a fixed point of a certain mapping. This theory has numerous applications, notably in biology, chemistry, economics and physics. For example, in [20], the authors demonstrated some fixed point theorems and used them to prove the solvability of certain fractional differential equations. It is noteworthy that these types of differential equations are frequently used in engineering sciences. See also [21], where the authors confirmed the effectiveness of fixed points theory in physics by applying it to the equation of motion.

Recently, in the context of the development of fixed point theory, the mixed monotone operators, which were first introduced in 1987 by Guo and Lakshmikantham [9], have provided some existence theorems for coupled fixed points for both continuous and discontinuous operators with coupled upper-lower solutions. They then proposed some applications to initial value problems of ordinary differential equations with discontinuous right-hand sides. As an extension of [9], an existence and uniqueness theorem was established in 1988 by Dajun Guo [8] for an operator

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$A : \overset{\circ}{P} \times \overset{\circ}{P} \to \overset{\circ}{P}$ under the following condition: there exists $0 \leq \alpha < 1$ such that $A(tx, t^{-1}y) \succeq t^{\alpha} A(x, y)$ for each $x, y \in \overset{\circ}{P}$ and $0 < t < 1$. These results have been developed and generalized in [22], where the authors prove new fixed point theorems for mixed monotone operators $A : P \times P \to P$ under the following conditions:

$i)$ There exists $h \in P$ with $h \neq \theta$ such that $A(h, h) \in P_h$;

$ii)$ For any $x, y \in P$ and $t \in (0, 1)$, there exists $\varphi(t) \in (t, 1]$ such that
$A(tx, t^{-1}y) \succeq \varphi(t) A(x, y)$.

Using the main results obtained, they give the local existence and uniqueness of positive solutions for the following nonlinear boundary value problems:

$$\begin{cases} -u''(t) + m^2 u(t) = \lambda f\left(t, u(t), u(t)\right), & 0 < t < 1, \\ u'(0) = u'(1) = 0. \end{cases}$$

The powerful role of the theory of fixed points has driven its extension in various directions. For example, one new direction involves extending the Banach contraction principle to metric spaces endowed with a partial order. See [5], where the authors established a fixed point theorem for a new class of mixed monotone operators, which are nearly asymptotically nonexpansive.

In this context, H. Wang et al. [18] obtained the existence and uniqueness of fixed point of the nonlinear sum operators $Ax + Bx + C(x, x)$, where A is an increasing α-concave (or sub-homogeneous) operator, B is a decreasing operator and C is a mixed monotone operator and they applied their results to a fractional differential equation. Thereafter, the authors [19] studied another abstract related to sum-type operator equation $A(x, x) + B(x, x) + Cx = x$, where A and B are two mixed monotone operators and C is an increasing operator, then the authors applied the result to a nonlinear fractional differential equation with multi-point fractional boundary conditions. In [14], Y. Sang et al. established the existence and uniqueness of solution for the operator equation $A(x, x) + B(x, x) + Cx + e = x$. The authors generalized the results obtained in [24] on the cone mappings to non-cone case. However, as far as we know, the fixed point results concerning vector operators with mixed monotone properties are still very limited. In [11], the authors established the existence and uniqueness a fixed point for the abstract vector operator equation $\Phi(x, y, x, y) = (A_1(x, x, y), A_2(x, y, y)) = (x, y)$, where $\Phi : P_h \times P_k \times P_h \times P_k \to P_h \times P_k$ has some mixed monotone properties with respect to the operators $A_1 : P_h \times P_h \times P_k \to P_h$ and $A_2 : P_h \times P_k \times P_k \to P_k$. Then they applied their results to obtain the positive solution for a system of nonlinear Neumman boundary value problems.

Being motivated by [11,15,23] and other works, we intend to study the existence and uniqueness of fixed point for the following system of operators, which we are going to consider it as an adequate vector operator with certain mixed monotone properties

$$\begin{aligned} A_1(x, x, y) + B_1(x, x, y) + e_1 = x, \\ A_2(x, y, y) + B_2(x, y, y) + e_2 = y. \end{aligned} \tag{1.1}$$

Then, we apply our result to show the existence and uniqueness of solutions to the system (4.2). The main result can be considered, to some extent, as a generalization of most of the results obtained in the cited references, in addition to being a transition from the scalar case to the vector case.

This article is organized as follows. In the first section, we recall some basic facts to be used throughout this work. In the second section, we will prove our results concerning the existence and uniqueness of fixed point for sum of two mixed monotone vector operators of Meir-Keeler type. In the third section, we demonstrate the applicability of our abstract theorem by giving an application to systems of nonlinear elastic beams equations.

2. Preliminaries and some results

Let $(E, \|.\|)$ be a real Banach space and P be a cone in E. Recall that a non-empty closed and convex set $P \subset E$ is a cone if it satisfies (i) $x \in P, \lambda \geq 0 \Rightarrow \lambda x \in P$, (ii) $x \in P, -x \in P \Rightarrow x = \theta$, where θ is the zero element in E. A cone P induces a partial ordering $\preceq$ in E by $x \preceq y$ if and only if $y - x \in P$. A cone P is called normal if there exists a constant $N > 0$ such that $\theta \preceq x \preceq y$ implies $\|x\| \leq N\|y\|$; in this case N is called the normality constant of P. A cone P is said to be solid if it's interior $\overset{\circ}{P}$ is non-empty.

Letting $h \succ \theta$ (i.e. $h \succeq \theta$ and $h \neq \theta$), we denote P_h by

$$P_h = \{x \in E : \text{ there exist } \lambda > 0, \mu > 0 \text{ such that } \lambda h \preceq x \preceq \mu h\}.$$

It is easy to see that $P_h \subset P$ is convex and $\lambda P_h = P_h$ for all $\lambda > 0$. If $\overset{\circ}{P} \neq \varnothing$ and $h \in \overset{\circ}{P}$, then $P_h = \overset{\circ}{P}$.

For an element $h \in P$ with $h \neq \theta$ and $e \in P$ with $\theta \preceq e \preceq h$, we denote

$$P_{h,e} = \{x \in E \ : x + e \in P_h\}.$$

For every $x, y \in P_{h,e}$, let

$$M_e\big(\frac{x}{y}\big) = \inf\left\{\lambda > 0, \ x + e \preceq \lambda(y + e)\right\},$$

and define e-Thompson's metric d_e by

$$d_e(x, y) = \ln\big(\max\big\{M_e\big(\frac{x}{y}\big), M_e\big(\frac{y}{x}\big)\big\}\big).$$

Let us give some remarks.

Remark 2.1. (i) It is clear that $P_{h,\theta} = P_h$ for each $h \succ \theta$.

(ii) P_h and $P_{h,e}$ are of different nature. In fact, one can observe that $P_h \subset P \setminus \{\theta\}$ for any $h \succ \theta$, while $P_{h,e}$ need not be even a subset of the cone P for some $h \succ \theta$, $e \succeq \theta$ with $h \succeq e$.

iii) If $x \in P_{h,e}$, then $x + e \in P_h$.

iv) For any $x, y \in P_{h,e}$, we have $d_e(x, y) = d_\theta(x + e, y + e)$.

Definition 2.1. [4] Let (X, d) be a metric space and $\varepsilon > 0$ be fixed. We say that (X, d) is ε-chainable if for any $x, y \in X$, there exist $x_0 = x, x_1, ..., x_{n-1}, x_n = y$ (n may depend on both x and y) such that $d(x_i, x_{i+1}) < \varepsilon$ for all $i \in \{0, 1, ..., n-1\}$.

Lemma 2.1. *[16] Let P be a normal cone in E. Let $h \in P$, with $h \succ \theta$. Then, P_h is complete with θ-Thompson's metric.*

Lemma 2.2. *[3] Let P be a normal cone in E. Let $h \in P$, with $h \succ \theta$. Let $\varepsilon > 0$. Then (P_h, d_θ) is ε-chainable.*

Lemma 2.3. *Let P be a normal cone in E. Let $h, e \in P$, with $h \succ \theta$, $h \succeq e$. Let $\varepsilon > 0$. Then $(P_{h,e}, d_e)$ is ε-chainable.*

Proof. Let $x, y \in P_{h,e}$. Then $x+e, y+e \in P_h$. From Lemma 2.4, there exist $x_0 = x+e, x_1, x_2, ..., x_n = y+e \in P_h$ such that $d_\theta(x_i, x_{i+1}) \leq \varepsilon$ for any $i = 1, 2, ..., n-1$. Therefore, $x_0 - e = x, x_1 - e, x_2 - e, ..., x_n - e = y \in P_{h,e}$ and by Remark 2.1, we have

$$d_e(x_i - e, x_{i+1} - e) \leq \varepsilon,$$

which gives the result. □

Lemma 2.4. *Let P be a normal cone in E. Let $h, e \in P$, with $h \succ \theta$, $h \succeq e$. Let $\varepsilon > 0$. Then $(P_{h,e}, d_e)$ is a complete metric space.*

Proof. Let $\{x_n\}$ be a Cauchy sequence in $(P_{h,e}, d_e)$. Then $\{x_n + e\}$ is a Cauchy sequence in (P_h, d_θ). It follows from Lemma 2.3 that there exists $x \in P_h$ satisfying $\lim_{n\to\infty} d_\theta(x_n + e, x) = 0$ and from Remark 2.1, $\lim_{n\to\infty} d_e(x_n, x - e) = 0$. Thus, (x_n) converges to $(x - e)$ in $(P_{h,e}, d_e)$. □

Lemma 2.5. *[15] If $x \in P_{h,e}$, then $\lambda x + (\lambda - 1)e \in P_{h,e}$ for all $\lambda > 0$.*
If $x, y \in P_{h,e}$, then there exists $r \in (0,1)$, such that $ry + (r-1)e \preceq x \preceq \frac{1}{r}y + (\frac{1}{r} - 1)e$.

To prove our result, we apply the following fixed point theorem.

Theorem 2.1. *[23] Let (E, d) be a complete ε-chainable metric space and let $T : E \longrightarrow E$ be an operator. Suppose that for every $a \in (0, \varepsilon)$, there exists $b \in (a, \varepsilon)$ such that*

$$x, y \in E,\ a < d(x,y) < b \Rightarrow d(T(x), T(y)) < a.$$

Then T has a unique fixed point $x^* \in E$ with $\{T^n x\}$ converges to x^* for any $x \in E$.

3. Fixed point theorem

In this section, we present the main fixed point result of this paper. If $(E, \preceq)$ is an ordered Banach space, we define a partial order in $E^2 = E \times E$, denoted $\precsim$, as follows

$$(x,y), (u,v) \in E \times E,\ (x,y) \precsim (u,v) \ \Leftrightarrow\ x \preceq u \text{ and } y \preceq v.$$

Definition 3.1. [1] Let $(X, \preceq)$ be a partially ordered set and $T : X \times X \longrightarrow X$ be an operator. We say that T has the mixed monotone property if $T(x,y)$ is monotone non-decreasing in x and is monotone non-increasing in y, that is, for any $x, y \in X$,

$$\begin{aligned} x_1, x_2 \in X, \quad x_1 \preceq x_2 \ &\Rightarrow\ T(x_1, y) \preceq T(x_2, y), \\ y_1, y_2 \in X, \quad y_1 \preceq y_2 \ &\Rightarrow\ T(x, y_2) \preceq T(x, y_1). \end{aligned}$$

Theorem 3.1. *Let P be a normal cone in a Banach space E. Let $e_1, e_2 \in P$ and $h, k \in P$, with $h \succ 0, k \succ 0$, $h \succeq e_1, k \succeq e_2$. Assume that $A_1, B_1 : P_{h,e_1} \times P_{h,e_1} \times P_{k,e_2} \longrightarrow E$ and $A_2, B_2 : P_{h,e_1} \times P_{k,e_2} \times P_{k,e_2} \longrightarrow E$ are operators such that $A_1(h,h,k), B_1(h,h,k) \in P_{h,e_1}$ and $A_2(h,k,k), B_2(h,k,k) \in P_{k,e_2}$. Suppose that the following assumptions are verified.*

(H_1) *$A_1(., u, y), B_1(., u, y)$ are non-decreasing and $A_1(x, ., y), A_1(x, u, .), B_1(x, ., y)$, $B_1(x, u, .)$ are non-increasing;*
$A_2(., u, y), A_2(x, ., y), B_2(., u, y), B_2(x, ., y)$ are non-increasing and $A_2(x, u, .)$, $B_2(x, u, .)$ are non-decreasing.

(H_2) *There exist constants $n \geq 1, \eta > 0, \lambda > 1$ and functions $\varphi_i : (\eta, 1) \longrightarrow (0, 1)$ satisfying $\varphi_i(t) \geq t + \frac{t-t^n}{\lambda}$ $(i = 1, 2)$ such that*

$$\begin{aligned}
&A_1\big(tx + (t-1)e_1, \frac{1}{t}x + (\frac{1}{t} - 1)e_1, \frac{1}{t}y + (\frac{1}{t} - 1)e_2\big)\\
&\succeq \varphi_1(t)A_1(x, x, y) + (\varphi_1(t) - 1)e_1,\\
&A_2\big(\frac{1}{t}x + (\frac{1}{t} - 1)e_1, \frac{1}{t}y + (\frac{1}{t} - 1)e_2, ty + (t-1)e_2\big)\\
&\succeq \varphi_2(t)A_2(x, y, y) + (\varphi_2(t) - 1)e_2,
\end{aligned}$$

for every $(x, y) \in P_{h,e_1} \times P_{k,e_2}$.

(H_3) *For any $t \in (\eta, 1)$ and for every $(x, y) \in P_{h,e_1} \times P_{k,e_2}$*

$$\begin{aligned}
&B_1\big(tx + (t-1)e_1, \frac{1}{t}x + (\frac{1}{t} - 1)e_1, \frac{1}{t}y + (\frac{1}{t} - 1)e_2\big)\\
&\succeq t^n B_1(x, x, y) + (t^n - 1)e_1,\\
&B_2\big(\frac{1}{t}x + (\frac{1}{t} - 1)e_1, \frac{1}{t}y + (\frac{1}{t} - 1)e_2, ty + (t-1)e_2\big)\\
&\succeq t^n B_2(x, y, y) + (t^n - 1)e_2.
\end{aligned}$$

(H_4) *There exists a constant $\delta > \lambda$ such that*

$$\begin{aligned}
A_1(x, x, y) &\succeq \delta B_1(x, x, y) + (\delta - 1)e_1,\\
A_2(x, y, y) &\succeq \delta B_2(x, y, y) + (\delta - 1)e_2,
\end{aligned}$$

for all $(x, y) \in P_{h,e_1} \times P_{k,e_2}$.

Then, there exists a unique couple $(x^*, y^*) \in P_{h,e_1} \times P_{k,e_2}$ such that

$$\begin{aligned}
A_1(x^*, x^*, y^*) + B_1(x^*, x^*, y^*) + e_1 = x^*,\\
A_2(x^*, y^*, y^*) + B_2(x^*, y^*, y^*) + e_2 = y^*.
\end{aligned}$$

Proof. Define the operator $T : (P_{h,e_1} \times P_{k,e_2})^2 \longrightarrow P_{h,e_1} \times P_{k,e_2}$ by

$$T((x, y), (u, v)) = \big(T_1((x, y), (u, v)), T_2((x, y), (u, v))\big), \tag{3.1}$$

where

$$\begin{aligned}
T_1((x, y), (u, v)) = A_1(x, u, v) + B_1(x, u, v) + e_1,\\
T_2((x, y), (u, v)) = A_2(u, v, y) + B_2(u, v, y) + e_2,
\end{aligned}$$

for every $x, u \in P_{h,e_1}$ and $v, y \in P_{k,e_2}$.

Let $(x_0, y_0), (x_1, y_1), (u, v) \in P_{h,e_1} \times P_{k,e_2}$ such that $(x_0, y_0) \precsim (x_1, y_1)$ i.e., $x_0 \preceq x_1$ and $y_0 \preceq y_1$. Then by (H_1), we have

$$\begin{aligned}
A_1(x_0, u, v) \preceq A_1(x_1, u, v)\ ,\ B_1(x_0, u, v) \preceq B_1(x_1, u, v),\\
A_2(u, v, y_0) \preceq A_2(u, v, y_1)\ ,\ B_2(u, v, y_0) \preceq B_2(u, v, y_1),
\end{aligned}$$

which implies that $T((x_0, y_0), (u, v)) \precsim T((x_1, y_1), (u, v))$.

Similarly, we get $T((u, v), (x_1, y_1)) \precsim T((u, v), (x_0, y_0))$. Hence, T is a mixed monotone vector operator.

Now, we prove that $T\big((x, y), (x, y)\big) \in P_{h,e_1} \times P_{k,e_2}$, for all $(x, y) \in P_{h,e_1} \times P_{k,e_2}$. Using Lemma 2.5 and (H_2), we have

$$\begin{aligned}
&\varphi_1(t)A_1\left(\frac{1}{t}x + (\frac{1}{t} - 1)e_1, tx + (t-1)e_1, ty + (t-1)e_2\right) + (\varphi_1(t) - 1)e_1 \\
&\preceq A_1\bigg(t(\frac{1}{t}x + (\frac{1}{t} - 1)e_1) + (t-1)e_1, \frac{1}{t}(tx + (t-1)e_1) + (\frac{1}{t} - 1)e_1, \\
&\qquad\qquad \frac{1}{t}(ty + (t-1)e_2) + (\frac{1}{t} - 1)e_2\bigg).
\end{aligned}$$

Thus,

$$\begin{aligned}
&A_1\left(\frac{1}{t}x + (\frac{1}{t} - 1)e_1, tx + (t-1)e_1, ty + (t-1)e_2\right) \\
&\preceq \frac{1}{\varphi_1(t)}A_1\big(x, x, y\big) - \frac{1}{\varphi_1(t)}(\varphi_1(t) - 1)e_1 \\
&\preceq \frac{1}{\varphi_1(t)}A_1\big(x, x, y\big) + (\frac{1}{\varphi_1(t)} - 1)e_1.
\end{aligned}$$

By the same reasoning we get,

$$B_1\big(\frac{1}{t}x + (\frac{1}{t} - 1)e_1, tx + (t-1)e_1, ty + (t-1)e_2\big) \preceq \frac{1}{t^n}B_1(x, x, y) + (\frac{1}{t^n} - 1)e_1.$$

On the other hand, since $A_1(h, h, k), B_1(h, h, k) \in P_{h,e_1}$, there exist constants a, b in $(0, 1)$ such that

$$\begin{aligned}
ah + (a-1)e_1 \preceq A_1(h, h, k) \preceq \frac{1}{a}h + (\frac{1}{a} - 1)e_1, \\
bh + (b-1)e_1 \preceq B_1(h, h, k) \preceq \frac{1}{b}h + (\frac{1}{b} - 1)e_1.
\end{aligned}$$

Let $x, u \in P_{h,e_1}$ and $v \in P_{k,e_2}$. Then there exist constants c, d, f in $(0, 1)$, such that

$$\begin{aligned}
ch + (c-1)e_1 \preceq x \preceq \frac{1}{c}h + (\frac{1}{c} - 1)e_1, \\
dh + (d-1)e_1 \preceq u \preceq \frac{1}{d}h + (\frac{1}{d} - 1)e_1, \\
fk + (f-1)e_2 \preceq v \preceq \frac{1}{f}k + (\frac{1}{f} - 1)e_2.
\end{aligned}$$

We choose $\ell = \min\{c, d, f\}$, then

$$\begin{aligned}
A_1(x, u, v) &\succeq A_1\big(\ell h + (\ell - 1)e_1, \frac{1}{\ell}h + (\frac{1}{\ell} - 1)e_1, \frac{1}{\ell}k + (\frac{1}{\ell} - 1)e_2\big) \\
&\succeq \varphi_1(l)A_1(h, h, k) + (\varphi_1(l) - 1)e_1 \\
&\succeq \varphi_1(l)\big(ah + (a-1)e_1\big) + (\varphi_1(l) - 1)e_1 \\
&\succeq \varphi_1(l)ah + (\varphi_1(l)a - 1)e_1
\end{aligned}$$

and

$$\begin{aligned}A_1(x,u,v) &\preceq A_1\big(\frac{1}{\ell}h+(\frac{1}{\ell}-1)e_1, lh+(l-1)e_1, lk+(l-1)e_2\big)\\ &\preceq \frac{1}{\varphi_1(\ell)}A_1(h,k,k)+(\frac{1}{\varphi_1(\ell)}-1)e_1\\ &\preceq \frac{1}{\varphi_1(\ell)}\big(\frac{1}{a}h+(\frac{1}{a}-1)e_1\big)+(\frac{1}{\varphi_1(\ell)}-1)e_1\\ &\preceq \frac{1}{\varphi_1(\ell)}\frac{1}{a}h+(\frac{1}{\varphi_1(\ell)}\frac{1}{a}-1)e_1,\end{aligned}$$

which implies that $A_1(x,u,v)\in P_{h,e_1}$ and similarly we obtain $B_1(x,u,v)\in P_{h,e_1}$. Analogously, we prove that $A_2(u,v,y)\in P_{k,e_2}$ and $B_2(u,v,y)\in P_{k,e_2}$ for every $x\in P_{h,e_1}$ and $v,y\in P_{k,e_2}$. Hence, $T\big((x,y),(u,v)\big)\in P_{h,e_1}\times P_{k,e_2}$, $\forall (x,y),(u,v)\in P_{h,e_1}\times P_{k,e_2}$.

Next, we set $\psi(t)=\min\Big\{\big(\frac{\delta\varphi_1(t)+t^n}{\delta+1}\big),\big(\frac{\delta\varphi_2(t)+t^n}{\delta+1}\big)\Big\}$ and we prove that for any $t\in(\eta,1)$ and for all $(x,y)\in P_{h,e_1}\times P_{k,e_2}$

$$\begin{aligned}&T\Big(t(x,y)+(t-1)(e_1,e_2),\frac{1}{t}(x,y)+(\frac{1}{t}-1)(e_1,e_2)\Big)\\ &\succsim \psi(t)T\Big((x,y),(x,y)\Big)+\big(\psi(t)-1\big)(e_1,e_2).\end{aligned}$$

Let $(x,y)\in P_{h,e_1}\times P_{k,e_2}$. By (H_4), we have

$$\begin{aligned}A_1(x,x,y)+\delta A_1(x,x,y) &\succeq \delta B_1(x,x,y)+(\delta-1)e_1+\delta A_1(x,x,y)\\ &\succeq \delta T_1\big((x,y),(x,y)\big)-e_1,\end{aligned}$$

which implies that

$$A_1(x,x,y)\succeq \frac{\delta}{\delta+1}T_1\big((x,y),(x,y)\big)-\frac{e_1}{\delta+1}. \tag{3.2}$$

It follows from (H_2), (H_3) and (3.2) that

$$\begin{aligned}&T_1\big(t(x,y)+(t-1)(e_1,e_2),\frac{1}{t}(x,y)+(\frac{1}{t}-1)(e_1,e_2)\big)-t^nT_1\big((x,y),(x,y)\big)\\ &=A_1\big(tx+(t-1)e_1,\frac{1}{t}x+(\frac{1}{t}-1)e_1,\frac{1}{t}y+(\frac{1}{t}-1)e_2\big)\\ &\quad+B_1\big(tx+(t-1)e_1,\frac{1}{t}x+(\frac{1}{t}-1)e_1,\frac{1}{t}y+(\frac{1}{t}-1)e_2\big)\\ &\quad+e_1-t^n\big(A_1(x,x,y)+B_1(x,x,y)+e_1\big)\\ &\succeq \varphi_1(t)A_1(x,x,y)+(\varphi_1(t)-1)e_1+t^nB_1(x,x,y)+(t^n-1)e_1\\ &\quad+e_1-t^nA_1(x,x,y)-t^nB_1(x,x,y)-t^ne_1\\ &\succeq \big(\varphi_1(t)-t^n\big)A_1(x,x,y)+(\varphi_1(t)-1)e_1\\ &\succeq \big(\varphi_1(t)-t^n\big)\left(\frac{\delta}{\delta+1}T_1\big((x,y),(x,y)\big)-\frac{e_1}{\delta+1}\right)+(\varphi_1(t)-1)e_1\\ &\succeq \big(\varphi_1(t)-t^n\big)\frac{\delta}{\delta+1}T_1\big((x,y),(x,y)\big)+\left(\frac{\delta\varphi_1(t)+t^n}{\delta+1}-1\right)e_1.\end{aligned}$$

Then,

$$\begin{aligned}
&T_1\big(t(x,y)+(t-1)(e_1,e_2),\frac{1}{t}(x,y)+(\frac{1}{t}-1)(e_1,e_2)\big)\\
&\succeq \big(\varphi_1(t)-t^n\big)\frac{\delta}{\delta+1}T_1\big((x,y),(x,y)\big)+\left(\frac{\delta\varphi_1(t)+t^n}{\delta+1}-1\right)e_1+t^nT_1\big((x,y),(x,y)\big)\\
&\succeq \left(\frac{\delta\varphi_1(t)+t^n}{\delta+1}\right)T_1\big((x,y),(x,y)\big)+\left(\frac{\delta\varphi_1(t)+t^n}{\delta+1}-1\right)e_1.
\end{aligned}$$

Analogously, we show that

$$\begin{aligned}
&T_2\big(t(x,y)+(t-1)(e_1,e_2),\frac{1}{t}(x,y)+(\frac{1}{t}-1)(e_1,e_2)\big)\\
&\succeq \left(\frac{\delta\varphi_2(t)+t^n}{\delta+1}\right)T_2\big((x,y),(x,y)\big)+\left(\frac{\delta\varphi_2(t)+t^n}{\delta+1}-1\right)e_2.
\end{aligned}$$

Hence the result.

Let $\varepsilon=-\ln\eta$. Then, by Lemma 2.3, $\big(P_{h,e_1}\times P_{k,e_2},d_{(e_1,e_2)}\big)$ is ε-chainable (with $d_{(e_1,e_2)}$ is the (e_1,e_2)-Thompson's metric). Choose $a\in(0,\varepsilon)$ and $\alpha=\exp(-a)$. Then $\alpha\in(\eta,1)$ and for every $\gamma\in(0,\alpha)$, we have

$$\psi(\alpha-\gamma)\geq\frac{\delta\Big((\alpha-\gamma)+\frac{(\alpha-\gamma)-(\alpha-\gamma)^n}{\lambda}\Big)+(\alpha-\gamma)^n}{1+\delta}.$$

Now, for all $\alpha\in(\eta,1)$, define the real valued function f_α on $(0,\alpha)$ by

$$f_\alpha(\gamma)=\frac{\delta\Big((\alpha-\gamma)+\frac{(\alpha-\gamma)-(\alpha-\gamma)^n}{\lambda}\Big)+(\alpha-\gamma)^n}{1+\delta}-\alpha,\quad\forall\gamma\in(0,\alpha).\tag{3.3}$$

Then,

$$\begin{aligned}
f_\alpha(0)&=\frac{\delta\Big((\alpha-0)+\frac{(\alpha-0)-(\alpha-0)^n}{\lambda}\Big)+(\alpha-0)^n}{1+\delta}-\alpha\\
&=\frac{\delta\Big(\frac{\lambda\alpha+\alpha-\alpha^n}{\lambda}\Big)+\alpha^n-(1+\delta)\alpha}{1+\delta}\\
&=\frac{\delta(\lambda\alpha+\alpha-\alpha^n)+\lambda\alpha^n-\lambda(1+\delta)\alpha}{\lambda(1+\delta)}\\
&=\frac{\delta\lambda\alpha+\delta\alpha-\delta\alpha^n+\lambda\alpha^n-\lambda\alpha-\lambda\delta\alpha}{\lambda(1+\delta)}\\
&=\frac{(\delta-\lambda)(\alpha-\alpha^n)}{\lambda(1+\delta)}>0.
\end{aligned}$$

It's clear that f_α is a continuous function. Thus, there exists a constant $\xi>0$ such that $f_\alpha(\gamma)>0$, for all $\gamma\in(0,\xi)$. Since $\psi(\alpha-\gamma)-\alpha\geq f_\alpha(\gamma)$, then $\psi(\alpha-\gamma)-\alpha>0$, for every $\gamma\in(0,\xi)$, that is, $\psi(\alpha-\gamma)>\alpha$.

For $0<\gamma<\alpha-\eta$ and $b=-\ln(\alpha-\gamma)$, we have $a<b<\varepsilon$. Let $x,y\in P_{h,e_1}\times P_{k,e_2}$ with

$$a=-\ln\alpha<d_{(e_1,e_2)}\big((x,y),(u,v)\big)<b=-\ln(\alpha-\gamma).\tag{3.4}$$

If $M_{(e_1,e_2)}\big(\frac{(x,y)}{(u,v)}\big) \geq M_{(e_1,e_2)}\big(\frac{(u,v)}{(x,y)}\big)$, then $d_{(e_1,e_2)}\big((x,y),(u,v)\big) = \ln\big[M_{(e_1,e_2)}\big(\frac{(x,y)}{(u,v)}\big)\big]$. Therefore, by (3.4) we get

$$\frac{1}{\alpha} < M_{(e_1,e_2)}\left(\frac{(x,y)}{(u,v)}\right) < \frac{1}{\alpha-\gamma}.$$

Let $M_1 = M_{(e_1,e_2)}\big(\frac{(x,y)}{(u,v)}\big)$ and $M_2 = M_{(e_1,e_2)}\big(\frac{(u,v)}{(x,y)}\big)$. Then

$$\begin{aligned}
&T\big((x,y),(x,y)\big) \succsim T\Big(M_2^{-1}(u,v) + \big(M_2^{-1}-1\big)(e_1,e_2), M_1\ (u,v) + \big(M_1-1\big)(e_1,e_2)\Big)\\
&\succsim T\Big(M_1^{-1}(u,v) + \big(M_1^{-1}-1\big)(e_1,e_2), M_1\ (u,v) + \big(M_1-1\big)(e_1,e_2)\Big)\\
&\succsim T\Big((\alpha-\gamma)(u,v) + \big((\alpha-\gamma)-1\big)(e_1,e_2), \frac{1}{(\alpha-\gamma)}(u,v) + \Big(\frac{1}{(\alpha-\gamma)}-1\Big)(e_1,e_2)\Big)\\
&\succsim \psi(\alpha-\gamma)T\big((u,v),(u,v)\big) + \big(\psi(\alpha-\gamma)-1\big)(e_1,e_2),
\end{aligned}$$

which implies that

$$T\big((x,y),(x,y)\big) + (e_1,e_2) \succsim \psi(\alpha-\gamma)\Big(T\big((u,v),(u,v)\big) + (e_1,e_2)\Big).$$

Consequently,

$$M_{(e_1,e_2)}\left(\frac{T\big((u,v),(u,v)\big)}{T\big((x,y),(x,y)\big)}\right) \leq \psi(\alpha-\gamma)^{-1} < \frac{1}{\alpha}. \tag{3.5}$$

Similarly, we have

$$\begin{aligned}
&T\big((u,v),(u,v)\big)\\
&\succsim T\Big(M_1^{-1}(x,y) + \big(M_1^{-1}-1\big)(e_1,e_2), M_2\ (x,y) + \big(M_2-1\big)(e_1,e_2)\Big)\\
&\succsim T\Big(M_1^{-1}(x,y) + \big(M_1^{-1}-1\big)(e_1,e_2), M_1\ (x,y) + \big(M_1-1\big)(e_1,e_2)\Big)\\
&\succsim T\Big((\alpha-\gamma)(x,y) + \big((\alpha-\gamma)-1\big)(e_1,e_2), \frac{1}{(\alpha-\gamma)}\ (x,y) + \Big(\frac{1}{(\alpha-\gamma)}-1\Big)(e_1,e_2)\Big)\\
&\succsim \psi(\alpha-\gamma)T\big((x,y),(x,y)\big) + \big(\psi(\alpha-\gamma)-1\big)(e_1,e_2).
\end{aligned}$$

Consequently,

$$M_{(e_1,e_2)}\left(\frac{T\big((x,y),(x,y)\big)}{T\big((u,v),(u,v)\big)}\right) \leq \psi(\alpha-\gamma)^{-1} < \frac{1}{\alpha}. \tag{3.6}$$

Next, from (3.5) and (3.6), we have

$$\begin{aligned}
&d_{(e_1,e_2)}\Big(T\big((x,y),(x,y)\big), T\big((u,v),(u,v)\big)\Big)\\
&= \ln\left(\max\left\{M_{(e_1,e_2)}\left(\frac{T\big((x,y),(x,y)\big)}{T\big((u,v),(u,v)\big)}\right), M_{(e_1,e_2)}\left(\frac{T\big((u,v),(u,v)\big)}{T\big((x,y),(x,y)\big)}\right)\right\}\right)
\end{aligned}$$

$< -\ln(\alpha) = a.$

In the same manner, if $M_{(e_1,e_2)}\big(\frac{(x,y)}{(u,v)}\big) \leq M_{(e_1,e_2)}\big(\frac{(u,v)}{(x,y)}\big)$, we show that

$$d_{(e_1,e_2)}\Big(T\big((x,y),(x,y)\big), T\big((u,v),(u,v)\big)\Big) < a.$$

Finally, by applying Theorem 2.1, T has a unique fixed point (x^*, y^*) in $P_{h,e_1} \times P_{k,e_2}$, that is, $T\big((x^*,y^*),(x^*,y^*)\big) = (x^*,y^*)$. □

Example 3.1. Take the ordered Banach space $\mathbb{R}$ with the normal cone $P = \mathbb{R}_+$. Define the operators A_1, A_2, B_1 and B_2 for every $x, y, z \in \mathbb{R}_+^* = (0, \infty)$ by

$$A_1(x,y,z) = 6,\ A_2(x,y,z) = 4,\ B_1(x,y,z) = 2, \text{and } B_2(x,y,z) = 1.$$

It's clear that all the hypotheses of Theorem 3.1 are verified with, $h = k = 1,\ e_1 = e_2 = 0,\ n = 1,\ \eta = \frac{1}{2},\ \lambda = 2,\ \delta = 3$ and $\varphi(t) = t$. Moreover,

$$\begin{aligned} A_1(8,8,5) + B_1(8,8,5) &= 8, \\ A_2(8,5,5) + B_2(8,5,5) &= 5. \end{aligned}$$

4. Applications

In the past few years, there has been significant focus on elastic beam equations. A variety of tools and methods have been utilized to investigate the existence, uniqueness and multiplicity of solutions for this type of equation. In [15] by applying the main result regarding fixed point theorem, the authors studied the existence and uniqueness of solutions to the following boundary value problem:

$$\begin{cases} x^{(4)}(t) = f(t, x(t), (Hx)(t)) - b,\ t \in (0,1), \\ x(0) = x'(0) = x''(1) = 0, \\ x'''(1) = g(x(1)). \end{cases} \tag{4.1}$$

In this section, we extend and generalize the results obtained in [15], demonstrating the existence and uniqueness of solutions for the following system of fourth-order nonlinear boundary value problems:

$$\begin{cases} x^{(4)}(t) = f_1(t, x(t), x(t), y(t)) + \alpha_1(t, x(t)) - b_1,\ 0 < t < 1, \\ y^{(4)}(t) = f_2(t, x(t), y(t), y(t)) + \alpha_2(t, y(t)) - b_2,\ 0 < t < 1, \\ x(0) = y(0) = x'(0) = y'(0) = x''(1) = y''(1) = 0, \\ x'''(1) = g_1\big(x(1), y(1)\big)\ ,\ y'''(1) = g_2\big(x(1), y(1)\big), \end{cases} \tag{4.2}$$

where b_1, b_2 are positive constants and $f_1, f_2, g_1, g_2, \alpha_1, \alpha_2$ are appropriate functions specified later.

Consider the Banach space $C\big([0,1]\big)$ of all continuous functions from $[0,1]$ to $\mathbb{R}$ equipped with the sup norm and set

$$P = \big\{x \in C\big([0,1]\big),\ x(t) \geq 0,\ t \in [0,1]\big\}.$$

Recall that P is a normal cone in $C\big([0,1]\big)$ of which the normality constant is 1.

Lemma 4.1. *[2] Suppose that $h \in C([0,1])$. Then the boundary value problem*

$$\begin{cases} x^{(4)}(t) = h(t), \ 0 < t < 1, \\ x(0) = x'(0) = x''(1) = x'''(1) = 0, \end{cases}$$

has a unique solution $x(t) = \int_0^1 G(t,s)h(s)ds$, where G is the Green function given by

$$G(t,s) = \frac{1}{6}\begin{cases} s^2(3t-s), \ 0 \le s \le t \le 1, \\ t^2(3s-t), \ 0 \le t \le s \le 1. \end{cases}$$

Remark 4.1. It's easy to show that

$$\frac{1}{3}t^2s^2 \le G(t,s) \le \frac{1}{2}t^2 s, \ \forall t,s \in [0,1].$$

Now, we are ready to present and prove the main result of this section.

Theorem 4.1. *Suppose that*

(C_1) *(1)* $f_1 : [0,1] \times \left[-\frac{b_1}{8}, +\infty\right) \times \left[-\frac{b_1}{8}, +\infty\right) \times \left[-\frac{b_2}{8}, +\infty\right) \longrightarrow \mathbb{R}$,
$f_2 : [0,1] \times \left[-\frac{b_1}{8}, +\infty\right) \times \left[-\frac{b_2}{8}, +\infty\right) \times \left[-\frac{b_2}{8}, +\infty\right) \longrightarrow \mathbb{R}$,
(2) $g_1, g_2 : \left[-\frac{b_1}{8}, +\infty\right) \times \left[-\frac{b_2}{8}, +\infty\right) \longrightarrow (-\infty, 0]$,
(3) $\alpha_1 : [0,1] \times \left[-\frac{b_1}{8}, +\infty\right) \longrightarrow [0, +\infty)$,
$\alpha_2 : [0,1] \times \left[-\frac{b_2}{8}, +\infty\right) \longrightarrow [0, +\infty)$
are continuous functions.

(C_2) *(1)* *$f_1(s,.,u,y)$ is non-decreasing and $f_1(s,x,.,y), f_1(s,x,u,.)$ are non-increasing for all $s \in [0,1]$, $x,u \in \left[-\frac{b_1}{8}, +\infty\right)$ and $y \in \left[-\frac{b_2}{8}, +\infty\right)$;*
(2) *$f_2(s,.,v,y), f_2(s,x,.,y)$ are non-increasing and $f_2(s,x,v,.)$ is non-decreasing for all $s \in [0,1]$, $x \in \left[-\frac{b_1}{8}, +\infty\right)$ and $v,y \in \left[-\frac{b_2}{8}, +\infty\right)$;*
(3) *$g_1(.,y)$, $g_1(x,.)$ are non-decreasing, $g_2(.,y)$ is non-decreasing and $g_2(x,.)$ is non-increasing for every $x \in \left[-\frac{b_1}{8}, +\infty\right)$ and $y \in \left[-\frac{b_2}{8}, +\infty\right)$;*
(4) *$\alpha_1(s,.)$ is non-decreasing and $\alpha_2(s,.)$ is non-increasing for all $s \in [0,1]$.*

(C_3) *There exist constants $n \ge 1$, $\lambda > 1, \eta \in (0,1)$ and functions $\varphi_1, \varphi_2 : (\eta,1) \to (0,1)$ satisfying*

$$\varphi_i(t) \ge t + \frac{t-t^n}{\lambda}, \ \forall t \in (\eta,1) \text{ and } (i=1,2),$$

such that for all $(x,y) \in \left[-\frac{b_1}{8}, +\infty\right) \times \left[-\frac{b_2}{8}, +\infty\right)$,

$$f_1\big(s, tx + (t-1)\frac{b_1}{8}, \frac{1}{t}x + (\frac{1}{t}-1)\frac{b_1}{8}, \frac{1}{t}y + (\frac{1}{t}-1)\frac{b_2}{8}\big) \ge \varphi_1(t) f_1(s,x,x,y),$$
$$f_2\big(s, \frac{1}{t}x + (\frac{1}{t}-1)\frac{b_1}{8}, \frac{1}{t}y + (\frac{1}{t}-1)\frac{b_2}{8}, ty + (t-1)\frac{b_2}{8}\big) \ge \varphi_2(t) f_2(s,x,y,y),$$
$$g_1\left(\frac{1}{t}x + (\frac{1}{t}-1)\frac{b_1}{8}, \frac{1}{t}y + (\frac{1}{t}-1)\frac{b_2}{8}\right) \le t^n g_1(x,y),$$
$$g_2\left(\frac{1}{t}x + (\frac{1}{t}-1)\frac{b_1}{8}, ty + (t-1)\frac{b_2}{8}\right) \le t^n g_2(x,y),$$
$$\alpha_1(s, tx + (t-1)\frac{b_1}{8}) \ge t^n \alpha_1(s,x) \text{ and } \alpha_2(s, \frac{1}{t}y + (\frac{1}{t}-1)\frac{b_2}{8}) \ge t^n \alpha_2(s,y).$$

(C_4) *(1)* $\int_0^1 s^2 f_1\big(s, 0, \frac{b_1}{2}, \frac{b_2}{2}\big)ds > 0$ *and* $\int_0^1 s^2 f_2\big(s, \frac{b_1}{2}, \frac{b_2}{2}, 0\big)ds > 0$;

(2) $g_1\big(\frac{b_1}{2}, \frac{b_2}{2}\big) < 0$ *and* $g_2\big(\frac{b_1}{2}, \frac{b_2}{2}\big) < 0$;

(3) There exists a constant $\delta > \lambda$, *such that, for all* $x, x_1 \in [-\frac{b_1}{8}, +\infty)$ *and for each* $y, y_1 \in [-\frac{b_2}{8}, +\infty)$,

$\frac{2}{3}\int_0^1 s^2 f_1(s, -\frac{b_1}{8}, x, y)ds \geq \delta \int_0^1 s\alpha_1(s, x)ds - \delta g_1\big(x_1, y_1\big)$,

$\frac{2}{3}\int_0^1 s^2 f_2(s, x, y, -\frac{b_2}{8})ds \geq \delta \int_0^1 s\alpha_2(s, y)ds - \delta g_2\big(x_1, y_1\big)$.

Then system (4.2) has a unique solution in $P_{h,e_1} \times P_{k,e_2}$.

h, k, e_1 *and* e_2 *will be determined later.*

Proof. Firstly, from Lemma 4.1, the integral formulation of the system (4.2) is given by

$$\begin{cases} x(t) = \int_0^1 G(t,s)\Bigg(f_1(s, x(s), x(s), y(s)) + \alpha_1(s, x(s)) - b_1\Bigg)ds \\ \qquad -g_1\big(x(1), y(1)\big)\Bigg(\frac{t^2}{2} - \frac{t^3}{6}\Bigg), \\ y(t) = \int_0^1 G(t,s)\Bigg(f_2(s, x(s), y(s), y(s)) + \alpha_2(s, y(s)) - b_2\Bigg)ds \\ \qquad -g_2\big(x(1), y(1)\big)\Bigg(\frac{t^2}{2} - \frac{t^3}{6}\Bigg). \end{cases} \tag{4.3}$$

Then, by a simple calculation we show that system (4.2) is equivalent to the following

$$\begin{cases} x(t) = \int_0^1 G(t,s)f_1\big(s, x(s), x(s), y(s)\big)ds + \int_0^1 G(t,s)\alpha_1(s, x(s))ds - e_1(t) \\ \qquad -g_1\big(x(1), y(1)\big)\Big(\frac{t^2}{2} - \frac{t^3}{6}\Big), \\ y(t) = \int_0^1 G(t,s)f_2\big(s, x(s), y(s), y(s)\big)ds + \int_0^1 G(t,s)\alpha_2(s, y(s))ds - e_2(t) \\ \qquad -g_2\big(x(1), y(1)\big)\Big(\frac{t^2}{2} - \frac{t^3}{6}\Big), \end{cases}$$

with $e_i(t) = \frac{b_i}{24}t^4 - \frac{b_i}{6}t^3 + \frac{b_i}{4}t^2$, $i \in \{1, 2\}$ and we have for each $t \in [0, 1]$,

$$e_i(t) = \frac{b_i}{24}t^4 - \frac{b_i}{6}t^3 + \frac{b_i}{4}t^2 \geq b_i t^2\Big(\frac{t^2}{36} - \frac{t}{6} + \frac{1}{4}\Big) = b_i t^2\Big(\frac{t}{6} - \frac{1}{2}\Big)^2 \geq 0,$$

which implies that $e_1, e_2 \in P$. Moreover, for each $t \in [0, 1]$ and $i = 1, 2$,

$$e_i(t) = \frac{b_i}{24}t^4 - \frac{b_i}{6}t^3 + \frac{b_i}{4}t^2 \leq \frac{b_i}{24}t^4 + \frac{b_i}{4}t^2 \leq \frac{b_i}{24}t^2 + \frac{b_i}{4}t^2 \leq \frac{b_i}{2}t^2, \tag{4.4}$$

that is, $e_1 \preceq h$ and $e_2 \preceq k$, with $h(t) = \frac{b_1}{2}t^2$ and $k(t) = \frac{b_2}{2}t^2$, $t \in [0, 1]$.

Define the operators $A_1, B_1 : P_{h,e_1} \times P_{h,e_1} \times P_{k,e_2} \longrightarrow E$ and $A_2, B_2 : P_{h,e_1} \times P_{k,e_2} \times P_{k,e_2} \longrightarrow E$ by

$$A_1(x, u, y)(t) = \int_0^1 G(t,s)f_1\big(s, x(s), u(s), y(s)\big)ds - e_1(t),$$

$$B_1(x, u, y)(t) = \int_0^1 G(t,s)\alpha_1(s, x(s))ds - g_1\big(u(1), y(1)\big)\Big(\frac{t^2}{2} - \frac{t^3}{6}\Big) - e_1(t),$$

$$A_2(x,y,v)(t) = \int_0^1 G(t,s) f_2\big(s, x(s), y(s), v(s)\big) ds - e_2(t),$$
$$B_2(x,y,v)(t) = \int_0^1 G(t,s)\alpha_2(s, y(s)) ds - g_2\big(x(1), v(1)\big)\Big(\frac{t^2}{2} - \frac{t^3}{6}\Big) - e_2(t).$$

From (C_1) and the definition of G one can show that the operators A_1, A_2, B_1 and B_2 are well defined.

It is a standard result that (x,y) is a solution of system (4.2) if and only if

$$A_1(x,x,y) + B_1(x,x,y) + e_1 = x,$$
$$A_2(x,y,y) + B_2(x,y,y) + e_2 = y.$$

To prove the existence and uniqueness of solution of system (4.2), we will apply Theorem 3.1 to the operators A_1, A_2, B_1 and B_2.

1) By Remark 4.1 and from (C_3), we have for every $t \in [0,1]$,

$$\begin{aligned} A_1(h,h,k)(t) + e_1(t) &= \int_0^1 G(t,s) f_1\big(s, h(s), h(s), k(s)\big) ds \\ &\le \int_0^1 G(t,s) f_1\big(s, \frac{b_1}{2}, 0, 0\big) ds \\ &\le \int_0^1 \frac{1}{2} t^2 s f_1\big(s, \frac{b_1}{2}, 0, 0\big) ds \\ &= \Big(\frac{1}{b_1}\int_0^1 s f_1\big(s, \frac{b_1}{2}, 0, 0\big) ds\Big) h(t) \end{aligned}$$

and

$$\begin{aligned} A_1(h,h,k)(t) + e_1(t) &= \int_0^1 G(t,s) f_1\big(s, h(s), h(s), k(s)\big) ds \\ &\ge \int_0^1 G(t,s) f_1\big(s, 0, \frac{b_1}{2}, \frac{b_2}{2}\big) ds \\ &\ge \int_0^1 \frac{1}{3} t^2 s^2 f_1\big(s, 0, \frac{b_1}{2}, \frac{b_2}{2}\big) ds \\ &= \Big(\frac{2}{3b_1}\int_0^1 s^2 f_1\big(s, 0, \frac{b_1}{2}, \frac{b_2}{2}\big) ds\Big) h(t). \end{aligned}$$

Using (C_4), we get

$$\frac{1}{b_1}\int_0^1 s f_1\big(s, \frac{b_1}{2}, 0, 0\big) ds \ge \frac{2}{3b_1}\int_0^1 s^2 f_1\big(s, 0, \frac{b_1}{2}, \frac{b_2}{2}\big) ds > 0.$$

Therefore, $A_1(h,h,k) \in P_{h,e_1}$. Moreover, since g_1 is a function with non-positive values, we have

$$\begin{aligned} B_1(h,h,k)(t) + e_1(t) &= \int_0^1 G(t,s)\alpha_1(s, h(s)) ds - g_1\big(h(1), k(1)\big)\Big(\frac{t^2}{2} - \frac{t^3}{6}\Big) \\ &\le \int_0^1 \frac{1}{2} t^2 s \alpha_1(s, \frac{b_1}{2}) ds - g_1\Big(\frac{b_1}{2}, \frac{b_2}{2}\Big)\Big(\frac{t^2}{2} - \frac{t^3}{6}\Big) \end{aligned}$$

$$
\begin{aligned}
&\le \frac{t^2}{2}\bigg(\int_0^1 s\alpha_1(s,\frac{b_1}{2})ds - g_1\Big(\frac{b_1}{2},\frac{b_2}{2}\Big)\bigg)\\
&\le \frac{1}{b_1}\bigg(\int_0^1 s\alpha_1(s,\frac{b_1}{2})ds - g_1\Big(\frac{b_1}{2},\frac{b_2}{2}\Big)\bigg)h(t)
\end{aligned}
$$

and

$$
\begin{aligned}
B_1(h,h,k)(t)+e_1(t) &= \int_0^1 G(t,s)\alpha_1(s,h(s))ds - g_1\big(h(1),k(1)\big)\Big(\frac{t^2}{2}-\frac{t^3}{6}\Big)\\
&\ge -g_1\Big(\frac{b_1}{2},\frac{b_2}{2}\Big)\Big(\frac{t^2}{2}-\frac{t^3}{6}\Big)\\
&\ge -g_1\Big(\frac{b_1}{2},\frac{b_2}{2}\Big)\frac{t^2}{3}\\
&\ge \Big(-g_1\Big(\frac{b_1}{2},\frac{b_2}{2}\Big)\frac{2}{3b_1}\Big)h(t).
\end{aligned}
$$

Again, from (C_4), we have

$$
\frac{1}{b_1}\bigg(\int_0^1 s\alpha_1(s,\frac{b_1}{2})ds - g_1\Big(\frac{b_1}{2},\frac{b_2}{2}\Big)\bigg) \ge -\frac{2}{3b_1}g_1\Big(\frac{b_1}{2},\frac{b_2}{2}\Big) > 0.
$$

Thus, $B_1(h,h,k)\in P_{h,e_1}$.

Analogously, we demonstrate that, $A_2(h,k,k), B_2(h,k,k)\in P_{k,e_2}$.

2) Using (C_2), it's easy to check that the hypothesis (H_1) of Theorem 3.1 is satisfied.

3) We show that hypotheses (H_2) and (H_3) of Theorem 3.1 are satisfied. Let $x\in P_{h,e_1}$, $y\in P_{k,e_2}$, $t\in[0,1]$ and $t'\in(\eta,1)$. Then by (C_4), we have

$$
\begin{aligned}
&A_1\big(t'x+(t'-1)e_1, \frac{1}{t'}x+(\frac{1}{t'}-1)e_1, \frac{1}{t'}y+(\frac{1}{t'}-1)e_2\big)(t)\\
&= \int_0^1 G(t,s)f_1\Big(s, t'x(s)+(t'-1)e_1(s), \frac{1}{t'}x(s)+(\frac{1}{t'}-1)e_1(s),\\
&\qquad\qquad \frac{1}{t'}y(s)+(\frac{1}{t'}-1)e_2(s)\Big)ds - e_1(t)\\
&\ge \int_0^1 G(t,s)f_1\Big(s, t'x(s)+(t'-1)\frac{b_1}{8}, \frac{1}{t'}x(s)+(\frac{1}{t'}-1)\frac{b_1}{8}, \frac{1}{t'}y(s)+(\frac{1}{t'}-1)\frac{b_2}{8}\Big)ds\\
&\quad - e_1(t)\\
&\ge \int_0^1 G(t,s)\varphi_1(t')f_1\big(s,x(s),x(s),y(s)\big)ds - e_1(t)\\
&\ge \varphi_1(t')\Big(\int_0^1 G(t,s)f_1\big(s,x(s),x(s),y(s)\big)ds - e_1(t)\Big) + \big(\varphi_1(t')-1\big)e_1(t)\\
&= \varphi_1(t')A_1(x,x,y)(t) + \big(\varphi_1(t')-1\big)e_1(t)
\end{aligned}
$$

and

$$
B_1\big(t'x+(t'-1)e_1, \frac{1}{t'}x+(\frac{1}{t'}-1)e_1, \frac{1}{t'}y+(\frac{1}{t'}-1)e_2\big)(t)
$$

$$= \int_0^1 G(t,s)\alpha_1(s, t'x(s) + (t'-1)e_1(s))ds$$
$$- g_1\left(\frac{1}{t'}x(1) + (\frac{1}{t'}-1)e_1(1), \frac{1}{t'}y(1) + (\frac{1}{t'}-1)e_2(1)\right)\left(\frac{t^2}{2} - \frac{t^3}{6}\right) - e_1(t).$$

From (C_3), we have

$$B_1\big(t'x + (t'-1)e_1, \frac{1}{t'}x + (\frac{1}{t'}-1)e_1, \frac{1}{t'}y + (\frac{1}{t'}-1)e_2\big)(t)$$
$$\geq \int_0^1 G(t,s)(t')^n\alpha_1(s, x(s))ds - (t')^n g_1\big(x(1), y(1)\big)\left(\frac{t^2}{2} - \frac{t^3}{6}\right) - e_1(t)$$
$$= (t')^n\left[\int_0^1 G(t,s)\alpha_1(s, x(s))ds - g_1\big(x(1), y(1)\big)\left(\frac{t^2}{2} - \frac{t^3}{6}\right) - e_1(t)\right]$$
$$+ \big((t')^n - 1\big)e_1(t)$$
$$= (t')^n B_1\big(x, x, y\big)(t) + \big((t')^n - 1\big)e_1(t).$$

We follow the same reasoning for A_2 and B_2.

4) Finally, we prove that (H_4) of Theorem 3.1 is satisfied. We have for all $x \in P_{h,e_1}$, $y \in P_{k,e_2}$ and $t \in [0,1]$,

$$A_1(x,x,y)(t) = \int_0^1 G(t,s)f_1\big(s, x(s), x(s), y(s)\big)ds - e_1(t)$$
$$\geq \frac{1}{3}t^2\int_0^1 s^2 f_1\big(s, -\frac{b_1}{8}, \|x\|, \|y\|)\big)ds - e_1(t)$$
$$\geq \delta\int_0^1 \frac{1}{2}t^2 s\alpha_1(s, \|x\|)ds - \delta g_1\big(x(1), y(1)\big)\frac{1}{2}t^2 - e_1(t)$$
$$\geq \delta\int_0^1 G(t,s)\alpha_1(s, x(s))ds - \delta g_1\big(x(1), y(1)\big)\left(\frac{t^2}{2} - \frac{t^3}{6}\right)$$
$$- \delta e_1(t) + (\delta - 1)e_1(t)$$
$$= \delta B_1(x,x,y)(t) + (\delta - 1)e_1(t).$$

Analogously, we prove that $A_2(x,y,y)(t) \geq \delta B_2(x,y,y)(t) + (\delta-1)e_2(t)$. The proof is complete. □

Example 4.1. Consider the system of nonlinear elastic beams equations (4.2) with $b_1 = 4, b_2 = \frac{8}{3}$ and, for $s \in [0,1]$, $x, u \in \big[-\frac{1}{2}, \infty\big)$, $y, v \in \big[-\frac{1}{3}, \infty\big)$, we set

$$f_1(s,x,u,y) = 25\sqrt{2}\Big(s + sx\Big)^{1/2} + \Big(\frac{5}{6}s + 1 + su + sy\Big)^{-1/2},$$
$$f_2(s,x,v,y) = 40\Big(\frac{s}{2} + sy\Big)^{1/3} + \Big(\frac{5}{6}s + 1 + sx + sv\Big)^{-1/3},$$
$$\alpha_1(s,x) = \begin{cases} \frac{1}{25}s\big(x + \frac{1}{2}\big)^2, & \frac{-1}{2} \leq x \leq \frac{9}{2}, \\ s, & x \geq \frac{9}{2}, \end{cases} \quad , \quad \alpha_2(s,y) = \frac{s^2}{2}\Big(y + \frac{3}{2}\Big)^{-2},$$
$$g_1(x,y) = \begin{cases} x + y \text{ if } \frac{-5}{6} \leq x + y \leq \frac{-4}{9}, \\ \frac{-4}{9} \quad \text{if } x + y \geq \frac{-4}{9}, \end{cases} \quad ,$$

$$g_2(x,y)=\begin{cases} x-y-\frac{1}{3} & \text{if } \frac{-1}{2}\le x\le \frac{-1}{6}, \frac{-1}{3}\le y\le 0,\\ -y-\frac{1}{2} & \text{if } \frac{-1}{6}\le x, \frac{-1}{3}\le y\le 0,\\ x-\frac{1}{3} & \text{if } \frac{-1}{2}\le x\le \frac{-1}{6}, 0\le y,\\ -\frac{1}{2} & \text{if } \frac{-1}{6}\le x, 0\le y. \end{cases}$$

Then, it is easy to check that the functions $f_1, f_2, g_1, g_2, \alpha_1$ and α_2 satisfy assumptions (C_1) and (C_2) of Theorem 4.1.

Next, for all $s\in[0,1]$, $t\in(0,1)$, $x,u\in\big[-\frac{1}{2},\infty\big)$, $y,v\in\big[-\frac{1}{3},\infty\big)$, one can show that

$$f_1\big(s, tx+(t-1)\frac{1}{2}, \frac{1}{t}x+(\frac{1}{t}-1)\frac{1}{2}, \frac{1}{t}y+(\frac{1}{t}-1)\frac{1}{3}\big)\ge \sqrt{t}f_1\big(s,x,x,y\big),$$
$$f_2\big(s, \frac{1}{t}x+(\frac{1}{t}-1)\frac{1}{2}, \frac{1}{t}y+(\frac{1}{t}-1)\frac{1}{3}, ty+(t-1)\frac{1}{3}\big)\ge \sqrt{t}f_2(s,x,y,y).$$

Notice that there exists $\eta_1\in(0,1)$ such that $\sqrt{t}\ge t+\frac{t-t^3}{5}$.

Also, if we put

$$h_1(x)=\begin{cases} x, & \frac{-1}{2}\le x\le\frac{-1}{6},\\ \frac{-1}{6}, & x\ge\frac{-1}{6}, \end{cases} \quad\text{and } h_2(y)=\begin{cases} -y-\frac{1}{3}, & \frac{-1}{3}\le y\le 0,\\ -\frac{1}{3}, & y\ge 0, \end{cases}$$

then we have $g_2(x,y)=h_1(x)+h_2(y)$, for all $x\in\big[-\frac{1}{2},\infty\big)$ and $y\in\big[-\frac{1}{3},\infty\big)$.

Let $t\in(0,1)$ and $x\in\big[-\frac{1}{2},\infty\big)$. Then we distinguish two cases.

First case. $x\ge-\frac{1}{6}$.

Then $\frac{1}{t}x+(\frac{1}{t}-1)\frac{1}{2}\ge-\frac{1}{6}$, therefore $h_1(\frac{1}{t}x+(\frac{1}{t}-1)\frac{1}{2})\le t^3h_1(x)$.

Second case. $x\le-\frac{1}{6}$.

i) If $\frac{1}{t}x+(\frac{1}{t}-1)\frac{1}{2}\le-\frac{1}{6}$, then

$$\frac{1}{t}x+(\frac{1}{t}-1)\frac{1}{2}\le t^3x \Leftrightarrow x(1-t^4)\le-(1-t)\frac{1}{2}$$
$$\Leftrightarrow x(1+t)(1+t^2)\le-\frac{1}{2}.$$

It's clear that there exists $\varepsilon_1\in(0,1)$, for all $t\in(\varepsilon_1,1)$, $(1+t)(1+t^2)>3$. Since $0\le x\le-\frac{1}{6}$ then, for all $t\in(\varepsilon_1,1)$, we have $x(1+t)(1+t^2)\le-\frac{1}{2}$. Furthermore

$$h_1(\frac{1}{t}x+(\frac{1}{t}-1)\frac{1}{2})\le t^3h_1(x).$$

ii) If $\frac{1}{t}x+(\frac{1}{t}-1)\frac{1}{2}\ge-\frac{1}{6}$, then

$$\frac{1}{t}x+(\frac{1}{t}-1)\frac{1}{2}\ge-\frac{1}{6}\Leftrightarrow t^3x+(t^3-t^4)\frac{1}{2}\ge-\frac{1}{6}t^4$$
$$\Leftrightarrow t^3x\ge-\frac{1}{6}t^4-(t^3-t^4)\frac{1}{2}=\frac{t^4}{3}-\frac{t^3}{2}.$$

It's easy to show that for all $t\in(0,1)$, we have $\frac{t^4}{3}-\frac{t^3}{2}\ge-\frac{1}{6}$. Hence, $t^3x\ge-\frac{1}{6}$, which implies that

$$h_1(\frac{1}{t}x+(\frac{1}{t}-1)\frac{1}{2})\le t^3h_1(x).$$

Again, by separating cases, we establish the existence of $\varepsilon_2 \in (0,1)$ such that, for all $t \in (\varepsilon_2, 1)$ and $y \in \left[-\frac{1}{3}, \infty\right)$, the following inequality holds

$$h_2(ty + (t-1)\frac{1}{3}) \leq t^3 h_2(y).$$

Let $\eta_2 = \max\{\varepsilon_1, \varepsilon_2\}$. Then we obtain

$$g_2\left(\frac{1}{t}x + (\frac{1}{t} - 1)\frac{1}{2}, ty + (t-1)\frac{1}{3}\right) \leq t^3 g_2(x, y),$$

for all $t \in (\eta_2, 1)$, $x \in \left[-\frac{1}{2}, \infty\right)$ and $y \in \left[-\frac{1}{3}, \infty\right)$. Analogously we prove the existence of $\eta_3 \in (0,1)$ satisfying

$$\begin{aligned}
&g_1\left(\frac{1}{t}x + (\frac{1}{t} - 1)\frac{1}{2}, \frac{1}{t}y + (\frac{1}{t} - 1)\frac{1}{3}\right) \leq t^3 g_1(x, y),\\
&\alpha_1(s, tx + (t-1)\frac{1}{2}) \geq t^3 \alpha_1(s, x),\\
&\alpha_2(s, \frac{1}{t}y + (\frac{1}{t} - 1)\frac{1}{3}) \geq t^3 \alpha_2(s, y),
\end{aligned}$$

for all $t \in (\eta_3, 1)$, $x \in \left[-\frac{1}{2}, \infty\right)$ and $y \in \left[-\frac{1}{3}, \infty\right)$. Thus, the hypothesis (C_3) of theorem 4.1 is satisfied for $\lambda = 5, n = 3$, $\eta = \max\{\eta_1, \eta_2, \eta_3\}$ and $\varphi_1(t) = \varphi_2(t) = \sqrt{t}$. Moreover, by a simple calculation we show that the assumption (C_4) is satisfied with $\delta = 6$.

Finally, all hypotheses of theorem 4.1 are verified. Consequently, system (4.2) with the functions given above has a unique solution in $P_{h,e_1} \times P_{k,e_2}$, where h, k and e_i $(i = 1, 2)$ are given by (4.4).

Open question

Given that theorem 3.1 depends primarily on the fixed point theorem for operators of Meir–Keeler type, how can we oversee (H_1) hypothesis which is related to the monotony?

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