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Common Fixed Point of the Commutative F-Contraction Self-mappings with Uniquely Bounded Sequence*

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Abstract We establish the existence of a common fixed point for mappings that satisfy and extend the F-contraction condition. To support our findings, we present pertinent definitions and properties associated with F-contraction mappings. Additionally, we establish an analogue to the Banach contraction theorem. Our results contribute to the broader understanding of this field by extending and generalizing existing findings in the literature.

Keywords Contraction mapping, fixed point, common fixed point

MSC(2010) 54H25, 47H10.

1. Introduction

In 1976, Jungck [3] pioneered the proof that if two continuous functions, f and g are defined on a complete metric space, with the additional property of being commuting functions and g being an F-contraction such that the range of g is included in that of f, then f and g must possess a unique common fixed point. It is essential to note that the inclusion condition in Jungck's theorem is sufficient but not necessary for the existence of common fixed points. In this study, we maintain all the aforementioned conditions in Jungck's theorem and replace the inclusion, which is an algebraic condition of Jungck's hypotheses, with another topological condition formulated in the form of a bounded Picard sequence. Thus, we obtain the same result as Jungck's theorem but with different hypotheses. In this case, we assure the existence and uniqueness of the common fixed point under the necessary and sufficient conditions.These results are also a generalization of the results discussed in the article [1].

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2. Preliminaries

Let us revisit certain definitions and established results pertaining to common fixed points [6].

Definition 2.1. [1] Consider two metric spaces, (X, d) equipped with distance d. Let f and g be mappings defined from X into itself. A mapping g is deemed an F-contraction if there exists a real positive constant $0 < k < 1$ satisfying the condition:

$$d(g(x), g(y)) \leq kd(f(x), f(y)), \forall x, y \in X. \tag{2.1}$$

We denote this relationship as $g - k - f$. If f is continuous, we refer to $g - k - f$ as a continuous contraction.

The objective of this work is to provide a generalization of the following theorem by eliminating the convergent condition from the hypothesis and streamlining the assumptions.

Theorem 2.1. *[1] Consider a continuous contraction mapping $g - k - f$ in the complete metric space X in itself. Additionally, assume that the mappings f and g commute with each other and there exists an element $x_0 \in X$ such that the Picard sequence $\{f^n(x_0)\}_{n\geq 0}$ converges to $t_0 \in X$. In this case, the Picard sequence $\{g^n(x_0)\}_{n\geq 0}$ converges to a point $r \in X$. Furthermore, the Picard sequence $\{f^n(r)\}_{n\geq 0}$ is bounded, then the sequence $\{g^n(t)\}_{n\geq 0}$ converges to r, which stands as the unique common fixed point of the mappings f and g.*

Remark 2.1. If we substitute the mapping f with an identity mapping in condition (2.1), we retrieve the classical contraction mapping scenario, leading us to the well-known Banach fixed-point theorem, as discussed in [2].

3. Main results and theorems

The proof of our theorems relies on the establishment of certain definitions, properties, propositions, and lemmas.

In what follows, let x_0 denote an element of a non-empty complete metric space X. For the sake of notation, we introduce $g^0(x_0) = x_0, f^0(x_0)) = x_0$ and inductively $g^{n+1}(x_0) = g(g^n(x_0)), f^{n+1}(x_0) = f(f^n(x_0))$, where $n \in \{1, 2, ...\}$.

The results presented in this subsection are commonly referred to as a variant of Banach's contraction principle.

Theorem 3.1. *Let $g - k - f$ be a continuous contraction mapping on the complete metric space X in itself, such that f and g commute with each other. If there exists an element $x_0 \in X$ such that the sequence $\{f^n(x_0)\}_{n\geq 0}$ is bounded, then the maps f and g possess a unique common fixed point in X.*

To establish the proof of the main theorem, we require the following lemmas in succession.

Lemma 3.1. *Let $g-k-f$ be a contraction mapping on the complete metric space X in itself, such that f and g commute with each other. Then the following inequality holds.*

$$d(g^n(x), g^n(y)) \leq k^n d(f^n(x), f^n(y)), \quad \forall x, y \in X, \forall n \in \mathbb{N}. \tag{3.1}$$

Proof. Consider two elements x, y in the space X. From condition (2.1), we have:

$$d(g \circ g^{n-1}(x), g \circ g^{n-1}(y)) \leq kd(f \circ g^{n-1}(x), f \circ g^{n-1}(y)).$$

Utilizing the commutativity, we observe that

$$f \circ g^n = g^n \circ f, \forall n \in \mathbb{N}. \tag{3.2}$$

The same line of reasoning yields:

$$d(f \circ g^{n-1}(x), f \circ g^{n-1}(y)) = d(g^{n-1} \circ f(x), g^{n-1} \circ f(y)).$$

Applying conditions (3.2) and (2.1), once again, we obtain:

$$d(g^{n-1} \circ f(x), g^{n-1} \circ f(y)) \leq kd(g^{n-2} \circ f^2(x), g^{n-2} \circ f^2(y)).$$

Through induction, we derive:

$$d(g^n(x), g^n(y)) \leq k^n d(f^n(x), f^n(y)).$$

This concludes our proof. □

Lemma 3.2. *Let $g - k - f$ be a contraction mapping on the complete metric space X in itself, such that f and g commute with each other. Then the ensuing inequality is established:*

$$d((f \circ g)^n(x), (f \circ g)^{n-1}(y)) \leq k^{n-1} d(g \circ f^{2n-1}(x), f^{2n-2}(y)) \leq sk^{n-1}, \forall x, y \in X. \tag{3.3}$$

Proof. Let x_0 be an element in X. Applying Lemma 3.1, we deduce:

$$\begin{aligned} d((f \circ g)^n(x_0), (f \circ g)^{n-1}(x_0)) &= d(g^{n-1}(g \circ f^n)(x_0)), g^{n-1}(f^{n-1}(x_0))) \\ &\leq k^{n-1} d(g \circ f^{2n-1}(x_0), f^{2n-2}(x_0)) \\ &\leq sk^{n-1}. \end{aligned}$$

By virtue of Lemma 3.1 and the boundedness of $\{f^n(x_0)\}_{n \geq 0}$, we deduce that the distance $d(g \circ f^{2n-1}(x), f^{2n-2}(y)$ is also bounded. Let

$$s = \sup_{n \geq 1} d(g \circ f^{2n-1}(x_0), f^{2n-2}(x_0))$$

This completes the proof. □

Lemma 3.3. *Under the hypothesis of theorem 3.1, the sequence $\{(f \circ g)^n(x_0)\}_{n \geq 0}$ converges in the complete metric space X.*

Proof. To establish the convergence, we first demonstrate that the sequence is a Cauchy sequence. Consider integers n, m, where $n > m$. By employing the triangle inequality and Lemma 3.2, we find

$$d((f \circ g)^n(x_0), (f \circ g)^m(x_0)) \leq \sum_{j=m}^{j=n-1} d((f \circ g)^j(x_0), (f \circ g)^{j+1}(x_0))$$

$$\leq \sum_{j=m}^{j=n-1} sk^j$$
$$\leq s\frac{k^m - k^n}{1-k}. \tag{3.4}$$

By allowing $n \to \infty$ and $m \to \infty$, the sequence becomes a Cauchy sequence. Therefore, by completeness, it converges to a limit l. Leveraging the continuity of the maps f and g, we observe that $f \circ g(l) = l$. This completes the proof of the lemma. □

Corollary 3.1. *Under the hypothesis of Theorem 3.1, if the sequence* $\{f^n(x_0)\}_{n\geq 0}$ *is bounded, it follows that the sequence* $\{(f \circ g)^n(x_0)\}_{n\geq 0}$ *is also bounded in the metric space* X.

Lemma 3.4. *Consider the hypothesis of Lemma 3.2, which asserts the existence of a constant* c *such that:*

$$\forall n, m \in \mathbb{N}, n > m, \quad d(g^n(l), g^m(l)) \leq c\frac{k^{\frac{m}{2}} - k^{\frac{n}{2}}}{1-\sqrt{k}}. \tag{3.5}$$

Additionally, it is established that the sequence $\{g^n(l)\}_{n\geq 0}$ *is convergent.*

Proof. Let n be a positive integer. It is straightforward to verify the following inequality:

$$\begin{aligned} d(g^n(l), g^{n-1}(l)) &= d(g \circ g^{n-1}(l), g \circ g^{n-2}(l)); \\ \textit{By the condition (2.1)} &\leq kd(f \circ g^{n-1}(l), f \circ g^{n-2}(l)); \\ \textit{By exploiting the commutativity} &\leq kd(g^{n-2}(g \circ f(l)), g^{n-3}(g \circ f(l))), \\ \textit{and by lemma 3.3} &\leq kd(g^{n-2}(l), g^{n-3}(l)). \end{aligned}$$

Employing induction, we distinguish two cases:
1. If n is odd, then:

$$d(g^n(l), g^{n-1}(l)) \leq k^{\frac{n-1}{2}} d(g(l), l).$$

2. If n is even, then

$$d(g^n(l), g^{n-1}(l)) \leq k^{\frac{n}{2}} d(f(l), l).$$

Set

$$c = \max\{d(g(l), l), \sqrt{k}\ d(f(l), l)\}.$$

It follows:

$$d(g^n(l), g^{n-1}(l)) \leq ck^{\frac{n-1}{2}}.$$

For any positive integers n and m with $n > m$, we have:

$$\forall n > m, \quad d(g^n(l), g^m(l)) \leq \sum_{j=m}^{n-1} d(g^j(l), g^{j+1}(l)) \leq c\sum_{j=m}^{n-1} k^{\frac{j}{2}}$$
$$\leq c\frac{k^{\frac{m}{2}} - k^{\frac{n}{2}}}{1-\sqrt{k}}.$$

(3.6)

By letting $n \to \infty$ and $m \to \infty$, we obtain $d(g^n(l), g^m(l)) \to 0$. Therefore, the sequence $\{g^n(l)\}_{n\geq 0}$ is a Cauchy sequence and, by completeness, converges to a limit l_1. The continuity argument of g yields $g(l_1) = l_1$, concluding the proof of the lemma. □

Lemma 3.5. *For every integer $n \geq 1$, the following equality is satisfied:*

$$(g \circ f)(g^n(l)) = g^n(l).$$

Proof. Straightforward. □

Lemma 3.6. *Under the conditions of Theorem 3.1, the composed mapping $g \circ f$ possesses a fixed point in the space X.*

Proof. Utilising limit Lemmas 3.4 and 3.5, we establish that the sequence $(g \circ f)(g^n(l))$ is a Cauchy sequence in the complete space X. Consequently, it converges to the limit $l_1 \in X$. Taking the limit and exploiting the continuity of $g \circ f$, we deduce that:

$$\lim_{n\to+\infty} (g \circ f)(g^n)(l) = l_1 = g \circ f(l_1).$$

This concludes the proof. □

Now, we proceed to prove the main result of Theorem 3.1.

Proof. Proof of theorem 3.1.

The necessity of the condition is evident. If f and g have a common fixed point t, then the sequence $(f^n(t))_{n\geq 0}$ is bounded. Regarding sufficiency, suppose that there exists an element $x_0 \in X$, such that the sequence $(f^n(x_0))_{n\geq 0}$ is bounded. It is easy to see that l_1 is a common fixed point of the maps f and g. In fact, by lemmas 3.4, 3.6 and the commutativity, we have:

$$g \circ f(l_1) = f(g(l_1)) = f(l_1) = g(l_1) = l_1.$$

Next, we prove the uniqueness of the common fixed point of f and g.

Let l_1 and l_2 be two common fixed points of f and g, i.e., $f(l_1) = g(l_1) = l_1$ and $f(l_2) = g(l_2) = l_2$. By computing the distance between l_1 and l_2, we obtain

$$d(l_1, l_2) = d(g(l_1), g(l_2)) \leq kd(f(l_1), f(l_2)) = kd(l_1, l_2).$$

From the last inequality, we deduce that $(1 - k)d(l_1, l_2) \leq 0$. Since $k < 1$, then $l_1 = l_2$. This shows the uniqueness of the common fixed point and concludes the proof of our theorem. □

Example 3.1. Let $X = [1, +\infty[$ with the usual metric. For integer numbers p, q such that $p < q$, define $f, g : X \to X, f(x) = x^p$ *and* $g(x) = x^q$.

All conditions of the theorem are fulfilled. In fact, f and g commute, the contraction is given by $|f(x) - f(y)| \leq \frac{p}{q}|g(x) - g(y)|, \forall x, y \in X$ and finally, the iterations of the Picard sequence $g^n(1) = 1$ are bounded at the point $x_0 = 1$. In this situation, the unique common fixed point of f and g is the element 1.

Remark 3.1. The following corollary is an example illustrating that the inclusion condition in Theorem 1.1 in [3] and the boundedness condition in our theorem are independent of each other.

Corollary 3.2. *For positive integers $n, m \geq 1$ and a positive real number $k > 1$, let f be a continuous map defined on the complete metric space X into itself. Assume that the inequality*

$$d(f^n(x), f^n(y)) \geq kd(x, y) \quad \text{holds.} \tag{3.7}$$

The map f has a unique fixed point in X if and only if there exists an element $x_0 \in X$ such that the sequence $\{f^m(x_0)\}_{m\geq 1}$ is bounded.

Remark 3.2. Here, the map f need not be onto. However, in corollary 2 of reference [3], f must be onto.

Corollary 3.3. *Let f and g be commuting mappings of a complete metric space (X, d) into itself where f is continuous. Consider positive integers $m, n,$and p. Suppose that the inequality*

$$d(g^m(x), g^m(y)) \leq kd(f^n(x), f^n(y)), \quad 0 < k < 1 \quad \text{holds.} \tag{3.8}$$

Then, the maps f and g have a unique common fixed point if and only if there exists an element $x_0 \in X$ such that the sequence $\{f^p(x_0)\}_{p\geq 0}$ is bounded.

The subsequent corollaries deal with the existence of a common fixed point for three commutative self-mappings.

Corollary 3.4. *Let f, g and h be three continuous and commuting mappings defined on the complete metric space into itself. Assume the condition*

$$d(g(x), g(y)) \leq kd(h(x), h(y)), \ \forall x, y \in X \quad \text{holds.} \tag{3.9}$$

The maps f, g and h have a unique common fixed point in the subspace X, if and only if there exists an element $x_0 \in X$ such that the sequence $\{h^n(x_0)\}_{n\geq 0}$ is bounded.

Proof. By hypothesis, g is an F-contraction relative to h. Since h is continuous and commutes with g, Theorem 3.1 guarantees that g and h share a unique common fixed point, say $z \in X$, i.e.,

$$g(z) = h(z) = z.$$

Next, we extend this result to include f. By the pairwise commutativity of f, g, and h:

$$g(f(z)) = f(g(z)) = f(z)$$

and

$$h(f(z)) = f(h(z)) = f(z).$$

This implies $f(z)$ is also a common fixed point of g and h. However, Theorem 3.1 guarantees the uniqueness of such a fixed point z. Therefore, we must have:

$$f(z) = z.$$

Combining these results, we conclude:

$$f(z) = g(z) = h(z) = z.$$

□

Corollary 3.5. *Under the assumption of corollary 3.4, if the map h is a bounded mapping, then the maps f, g, and h have a unique common fixed point in X.*

In what follows, we provide a more general version than theorem 3.23 given in reference [1].

Theorem 3.2. *Let $h-k-g$ and $g-kk'-f$ be two contraction mappings in the complete metric X into itself. Assume that they are continuous and commute with each other two by two, and there exists an element $x_0 \in X$ such that the Picard sequence $\{f^n(x_0)\}_{n\geq 0}$ is bounded in X. Then, the maps f, g, and h have a unique common fixed point in X.*

Proposition 3.1. *Let g_1-k-f_1 and $g_2-k'-f_2$ be two continuous contraction mappings in the complete metric space M into itself, where f_1 commutes with g_2. Assume that $g_1 \circ g_2$ and $f_2 \circ f_1$ commute with each other, and $\{(f_2 \circ f_1)^n(x_0)\}_{n\geq 0}$ is bounded in X. Then, $g_1 \circ g_2$ and $f_2 \circ f_1$ have a unique common fixed point in X.*

Lastly, we observe that the requirement 3.1 of our theorem can be weakened by demanding that (X,d) be compact.

Corollary 3.6. *For a positive integer $n \geq 1$ and a positive real number $k > 1$,let f be a continuous map defined on the compact metric space (X,d) into itself. If the inequality holds,*

$$d(f^n(x), f^n(y)) \geq kd(x,y), \tag{3.10}$$

then the map f has a unique fixed point in X.

Corollary 3.7. *Let f, g and h be three commuting mappings on the compact space (M,d) into itself. Assume that h and g are continuous and satisfy condition (3.9). Then, the maps f, g and h have a unique common fixed point in X.*

Corollary 3.8. *All mappings f that commutate with a contraction mapping g, defined on the compact metric space (M,d) into itself, have a unique common fixed point.*

Corollary 3.9. *Consider a sequence of continuous functions f_n defined on the compact set K converging uniformly to a function f. If there exists a contraction mapping g that commutes with each f_n for all positive integers n, then f and g possess a unique common fixed point.*

Example 3.2. Let the functions f and g be defined on the interval [2, 4] into itself, where

$$f(x) = \frac{7x+2}{3x-1} \quad , \quad g(x) = \frac{8x+2}{3x}.$$

Through straightforward computation, we observe that f and g commute on the interval [2,4]. The quotient of their derivatives is expressed as:

$$\left|\frac{g'(x)}{f'(x)}\right| = \frac{2}{39}\left(\frac{3x-1}{x}\right)^2.$$

The F-contraction constant k is established by the inequality below:

$$\left|\frac{g'(x)}{f'(x)}\right| \leq \frac{121}{312} < 1.$$

confirming that g satisfies the F-contraction condition.

Therefore, g is an F-contraction. For any initial point $x_0 \in [2,4]$, the function f satisfies $\mid f(x_0) \mid \leq \frac{16}{5}$, establishing that f is bounded by $\frac{16}{5}$. Consequently, the Picard sequence $\{f^n(x_0)\}_{n\geq 0}$ remains bounded within the interval $[2,4]$. All conditions of Theorem 3.1 are satisfied, which confirms the existence of a unique common fixed point at $x = 2.8968$ in $[2,4]$.

4. Conclusions

This study delves into the identification of common fixed points among commuting mappings, presenting a comprehensive examination of both necessary and sufficient conditions. This innovative approach introduces a paradigm shift, guaranteeing the existence of a shared fixed point. Its versatility spans across various disciplines, encompassing economic sciences as well as other realms of mathematics and physics [4, 5], where it can be effectively deployed through numerical programs.

5. Conflict of interest

No potential conflict of interest was reported by the authors.

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