Journal of Nonlinear Modeling and Analysis https://journal.global-sci.org/jnma
Volume 7, Number 2, April 2025, 622–640 DOI:10.12150/jnma.2025.622

Some Local Fractional Hermite-Hadamard-Type and Ostrowski-Type Inequalities for Exponentially s-Preconvex Functions with Generalized Mittag-Leffler Kernel*

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Abstract In this paper, we introduced the concepts of local fractional integral and generalized Mittag-Leffler kernel. Based on these, we establish Hermite-Hadamard integral inequalities and Ostrowski integral inequalities via exponentially s-preconvex functions and s-preconvex functions with generalized Mittag-Leffler kernel.

Keywords Local fractional integral, Mittag-Leffler kernel, exponentially s-preconvex functions, Hermite-Hadamard integral inequalities, Ostrowski integral inequalities

MSC(2010) 26A33, 26D10, 33E12.

1. Introduction

In the 19th century, Charles Hermite and Jacques Hadamard demonstrated in references [1, 2] the classical Hermite-Hadamard-type integral inequality of convex functions, which describes an estimate of the integral mean of convex functions:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \leq \frac{f(a)+f(b)}{2},$$

where$f : I \subseteq \mathbb{R} \to \mathbb{R}, a, b \in I, a < b$.

In 1938, Ostrowski gave another inequalities that are estimated for the difference between the value of a function and the mean of the integrals in [3], which is called Ostrowski-type integral inequality. Let$f : I \to \mathbb{R}$ be a differentiable function in I^1(the interior of I) and let $a, b \in I, a < b$. If $|f'(x)| \leq L$, for all $x \in [a, b]$, then:

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \right| \leq \frac{L}{b-a}\left[\frac{(x-a)^2-(b-x)^2}{2}\right],$$

where L is the Liphitz constant.

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*The authors were supported by National Natural Science Foundation of China (11671227).

Inequalities are known to be an important tool for solving many mathematical problems. These two types of inequalities above provide an estimate of the mean of the integrals of a function. They have a very wide range of applications in the field of mathematical and engineering calculations. For further research, most scholars have achieved some innovative results by considering the different convexity of functions, which can be referred to [4–8].

On the other hand, fractional calculus is an important branch of calculus theory. In 1993, Miller et al. established the theory of fractional differential equations on the basis of the fractional calculus operator of Riemann-Liouville. In 2006, Kilbas et al. gave the definitions of Riemann-Liouville left definite integral and right definite integral in [9]:

$$ {}^{\mathrm{RL}}\mathrm{I}_{a^+}^{\alpha}[f(x)] = \frac{1}{\Gamma(\alpha)} \int_a^x (x-\xi)^{\alpha-1} f(\xi)\mathrm{d}\xi, \quad x \geq a, $$

$$ {}^{\mathrm{RL}}\mathrm{I}_{b^-}^{\alpha}[f(x)] = \frac{1}{\Gamma(\alpha)} \int_x^b (\xi-x)^{\alpha-1} f(\xi)\mathrm{d}\xi, \quad x \leq b. $$

Subsequently, many scholars have improved and generalized the definition of fractional calculus operators. For example, Hadamard fractional integral operator, katugampola fractional integral operator, integrated fractional integral operator, ψ-Caputo fractional integral operator, etc.

The phenomenon of fractal is almost everywhere in nature, and the fractal problem is a non-differentiable function problem in mathematics, which is also known as the "mathematical pathological problem" in the world. Fractional calculus can deal with continuous classical power law phenomena, but it cannot solve discontinuities. Therefore, focusing on the mathematical-pathological problems that Newton-Leibniz calculus cannot handle, Yang proposed local fractional derivatives and local fractional integrals of non-differentiable functions in references [10, 11], which is also known as Yang's fractal theory. At present, Yang's fractal theory has been widely used in the field of engineering mechanics and differential equation calculation, which can be referred to in [12–17]. Based on the theory of local fractional calculus, the study of integral inequality has also achieved new results. For example, according to Yang's fractal theory, Sun proposed a local fractional integral operator with Mittag-Leffler kernel in [18], and obtained some inequalities about the h-preconvex function. Subsequently, Sun and Xu et al. used them to study the Hermite-Hadamard local fractional integral inequality [19, 20] . In 2024, Sun studied the local fractional integral inequalities of the Hermite-Hadamard type and Ostrowski type of the generalized h-preconvex function in [21].Wei gave a new fractal modeling for the nerve impulses based on local fractional derivative in [22].More recent results can be found in References [23–25].

Therefore, inspired by the existing results, this paper will construct some local fractional Hermite-Hadamard-type and Ostrowski-type inequalities for exponentially s-preconvex functions with generalized Mittag-Leffler kernel on the Yang's fractal sets. By taking some specific values for the parameters in the main results, it is possible to obtain some known results or new conclusions in the references.

2. Preliminarise

Firstly, using Yang's idea [10, 11], let's review Yang's fractal sets E^s, $s \in (0,1]$, where the set E is the base set of fractional set.

The s-type integers set is

$$\mathbb{Z}^s = \{0^s, \pm 1^s, \pm 2^s, \pm 3^s, \ldots\}.$$

The s-type rational numbers set is

$$\mathrm{Q}^s = \left\{q^s = \left(\frac{c}{d}\right)^s : c, d \in \mathbb{Z}, d \neq 0\right\}.$$

The s-type irrational numbers set is

$$\partial^s = \left\{\varrho^s \neq \left(\frac{c}{d}\right)^s : c, d \in \mathbb{Z}, d \neq 0\right\}.$$

The s-type real line numbers set is

$$\mathbb{R}^s = \mathrm{Q}^s \cup \partial^s.$$

Remark 2.1. If s=1, then the above sets respectively are integer sets, rational number sets, irrational number sets and real number sets.

The following are opration propertise on $\mathbb{R}^s$. Note that s represents the fractal dimension, not an exponential symbol.

If $\theta^s, \vartheta^s, \iota^s \in \mathbb{R}^s$, then:

(1) $\theta^s + \vartheta^s \in \mathbb{R}^s, \theta^s\vartheta^s \in \mathbb{R}^s$.

(2) $\theta^s + \vartheta^s = \vartheta^s + \theta^s = (\theta + \vartheta)^s = (\vartheta + \theta)^s$.

(3) $\theta^s + (\vartheta^s + \iota^s) = (\theta + \vartheta)^s + \iota^s$.

(4) $\theta^s\vartheta^s = \vartheta^s\theta^s = (\theta\vartheta)^s = (\vartheta\theta)^s$.

(5) $\theta^s(\vartheta^s\iota^s) = (\theta^s\vartheta^s)\iota^s$.

(6) $\theta^s(\vartheta^s + \iota^s) = \theta^s\vartheta^s + \theta^s\iota^s$.

(7) $\theta^s + 0^s = 0^s + \theta^s = \theta^s$, $\theta^s 1^s = 1^s\theta^s = \theta^s$.

(8) $(\theta - \vartheta)^s = \theta^s - \vartheta^s$.

(9)For each $\theta^s \in \mathbb{R}^s$, its inverse element $(-\theta)^s$ may be written as $-\theta^s$; for each $\vartheta^s \in \mathbb{R}^s \mid 0^s$, its inverse element $\left(\frac{1}{\vartheta}\right)^s$ may be written as $1^s/\vartheta^s$ but not as $1/\vartheta^s$.

Here are some definitions and lemmas that will be used in this article.

Definition 2.1. [10] Mittag-Leffler function on Yang's fractal sets is defined by:

$$E_s(\xi^s) = \sum_{\lambda=0}^{\infty} \frac{\xi^{\lambda s}}{\Gamma(1+\lambda s)}, \quad \xi \in \mathbb{R}.$$

Definition 2.2. [18] Let $G : [a_1, a_2] \to \mathbb{R}^s$ be a function on Yang's fractal sets and $G(\xi)$ be a local fractional integrable function. The left-side integral operator $I^s_{a_1^+}G$ and the right-side integral operator $I^s_{a_2^-}G$ of order $s \in (0,1)$ are, respectively, described as:

$$I^s_{a_1^+}G(\xi) = \frac{1}{s^s\Gamma(1+s)} \int_{a_1}^{\xi} E_s\left((-\frac{1-s}{s}(\xi - \tau))^s\right) G(\tau)(d\tau)^s, \xi > a_1,$$

and

$$I^s_{a_2^-}G(\xi) = \frac{1}{s^s\Gamma(1+s)} \int_{\xi}^{a_2} E_s\left((-\frac{1-s}{s}(\tau - \xi))^s\right) G(\tau)(d\tau)^s, \xi < a_2.$$

Remark 2.2. If s=1, then:

$$\lim_{s\to 1} I^{s}_{a_1^+} G(\xi) = \int_{a_1}^{\xi} G(\tau) d\tau,$$

$$\lim_{s\to 1} I^{s}_{a_2^-} G(\xi) = \int_{\xi}^{a_2} G(\tau) d\tau.$$

Definition 2.3. [10, 11] A non-differentiable function $G : \mathbb{R} \to \mathbb{R}^s, \xi \to G(\xi)$ is called local fractional continuous at ξ_0, if for any $\varepsilon > 0$, $|\xi - \xi_0| < \varepsilon$, there exists $\varepsilon > 0$, such that

$$|G(\xi) - G(\xi_0)| < \varepsilon^s,$$

If $G(\xi)$ is local fractional continuous on(a_1, a_2), then it is denoted by $G(\xi) \in C_s(a_1, a_2)$.

Remark 2.3. If the function G is local fractional continuous at the endpoint value a_1 and a_2, then we denote $G(\xi) \in C_s[a_1, a_2]$.

Definition 2.4. [10,11] The local fractional derivative of $G(\xi)$ of order s at $\xi = \xi_0$ is defined by:

$$G^{(s)}(\xi_0) = \left.\frac{\mathrm{d}^s G(\xi)}{\mathrm{d}\xi^s}\right|_{\xi=\xi_0} = \lim_{\xi\to\xi_0} \frac{\Gamma(s+1)\left(G(\xi) - G(\xi_0)\right)}{(\xi - \xi_0)^s}.$$

$D_s(a_1, a_2)$ represents the s-local fractional derivative set.

Definition 2.5. [10, 11] Let $G(\xi) \in C_s[a_1, a_2]$. The local fractional integral of $G(\xi)$ of order s is defined by:

$${}_{a_1}I^{(s)}_{a_2} G(\xi) = \frac{1}{\Gamma(s+1)} \int_{a_1}^{a_2} G(\chi)(\mathrm{d}\chi)^s = \frac{1}{\Gamma(s+1)} \lim_{\Delta\chi\to 0} \sum_{j=0}^{N-1} G(\chi_j)(\Delta\chi_j)^s,$$

where $a_1 = \chi_0 < \chi_1 < \cdots < \chi_{N-1} < \chi_N = a_2, [\chi_j, \chi_{j+1}]$ is a partition of the $[a_1, a_2]$, $\Delta\chi_j = \chi_{j+1} - \chi_j, \Delta\chi = \max\{\Delta\chi_0, \Delta\chi_1, \ldots, \Delta\chi_{N-1}\}$.

We denote $G(\xi) \in I^{(s)}_{\xi}[a_1, a_2]$ if there exits $a_1 I^{(s)}_{\xi} G(\xi)$ for any $\xi \in [a_1, a_2]$.

Definition 2.6. [26] Let $\Omega \subseteq \mathbb{R}$. If the set Ω satisfies:

$$\xi_1, \xi_2 \in \Omega, 0 \leq \iota \leq 1 \Rightarrow \xi_2 + \iota\eta(\xi_1, \xi_2) \in \Omega,$$

then Ω is called the invex set regarding $\eta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$.

Definition 2.7. [27] Let $S \subseteq \mathbb{R}^n$ be an invex set with respect to $\eta : S \times S \to \mathbb{R}^n$. A function $\psi : S \to \mathbb{R}_0 = [0, \infty)$ is said to be s-preinvex with respect to η and $s \in (0, 1]$ if for every $m_1, m_2 \in S$ and $\tau \in [0, 1]$

$$\psi(m_1 + \tau\eta(m_2, m_1)) \leq \tau^s \psi(m_2) + (1 - \tau)^s \psi(m_1).$$

Remark 2.4. If $s = 1$, then the s-preconvex derives preconvex.

Definition 2.8. [28] Let $s \in (0,1]$ and a real-valued mapping ψ on the invex set Ω is said to be exponentially s-preinvex with respect to $\eta(.,.)$, if the inequality

$$\psi\left(m_1+\tau\eta\left(m_2,m_1\right)\right) \leq (1-\tau)^s \frac{\psi\left(m_1\right)}{e^{\alpha m_1}}+\tau^s \frac{\psi\left(m_2\right)}{e^{\alpha m_2}}$$

holds for all $m_1, m_1+\eta\left(m_2,m_1\right) \in \Omega, \tau \in [0,1]$, and $\alpha \in \mathbb{R}$.

Remark 2.5. If $\alpha=0$, then the exponentially s-preconvex derives the s-preconvex.

Condition C: Let $A \subseteq \mathbb{R}^n$ be an open invex subset with respect to $\eta: A \times A \to \mathbb{R}$. We say that the function η satisfies the condition C if for any $m_1, m_2 \in A$ and $t \in [0,1]$,

$$\eta\left(m_2, m_2+t\eta\left(m_1,m_2\right)\right)=-t\eta\left(m_1,m_2\right),$$

$$\eta\left(m_1, m_2+t\eta\left(m_1,m_2\right)\right)=(1-t)\eta\left(m_1,m_2\right),$$

and from condition C:

$$\eta\left(m_2+t_2\eta\left(m_1,m_2\right), m_2+t_1\eta\left(m_1,m_2\right)\right)=\left(t_2-t_1\right)\eta\left(m_1,m_2\right).$$

Lemma 2.1 (Hölder-Yang's inequality, [10])**.** *If $f, g \in C_s\left[a_1,a_2\right], p, q>1$, and $\frac{1}{p}+\frac{1}{q}=1$, then*

$$\begin{aligned}
&\frac{1}{\Gamma(s+1)} \int_{a_1}^{a_2}|f(\tau) g(\tau)|(d\tau)^s \\
&\leq\left(\frac{1}{\Gamma(s+1)} \int_{a_1}^{a_2}|f(\tau)|^p(d\tau)^s\right)^{\frac{1}{p}}\left(\frac{1}{\Gamma(s+1)} \int_{a_1}^{a_2}|g(\tau)|^q(d\tau)^s\right)^{\frac{1}{q}}.
\end{aligned}$$

Lemma 2.2 (lemma 3.1, [21])**.** *Let $\Omega \subseteq R$ be an open invex set regarding $\eta: \Omega \times \Omega \to R$ and $G: \Omega \to \mathbb{R}^s (0<s<1)$ be a function with $\eta\left(a_2,a_1\right)>0, a_1, a_2 \in \Omega$ and $a_1<a_2$. If $G^{(s)}(x) \in I_x^{(s)}\left[a_1, a_1+\eta\left(a_2,a_1\right)\right]$, then for all $x \in \left[a_1, a_1+\eta\left(a_2,a_1\right)\right]$, the following local fractional integral identity holds*

$$\begin{aligned}
&\frac{(1-s)^s}{2^s-E_s\left(\left(-\frac{1-s}{s}\left(x-a_1\right)\right)^s\right)-E_s\left(\left(-\frac{1-s}{s}\left(a_1+\eta\left(a_2,a_1\right)-x\right)\right)^s\right)} \\
&\times\left(I_{x^-}^s G\left(a_1\right)+I_{x^+}^s G\left(a_1+\eta\left(a_2,a_1\right)\right)\right)-G(x) \\
=&\frac{\eta^s\left(a_2,a_1\right)}{2^s-E_s\left(\left(-\frac{1-s}{s}\left(x-a_1\right)\right)^s\right)-E_s\left(\left(-\frac{1-s}{s}\left(a_1+\eta\left(a_2,a_1\right)-x\right)\right)^s\right)} \\
&\times\left[\frac{1}{\Gamma(1+s)} \int_0^{\frac{x-a_1}{\eta\left(a_2,a_1\right)}}\left(E_s\left((-\rho\tau)^s\right)-1^s\right) G^{(s)}\left(a_1+\tau\eta\left(a_2,a_1\right)\right)(\mathrm{d}\tau)^s\right. \\
&\left.+\frac{1}{\Gamma(1+s)} \int_{\frac{x-a_1}{\eta\left(a_2,a_1\right)}}^1\left(1^s-E_s(-\rho(1-\tau))^s\right) G^{(s)}\left(a_1+\tau\eta\left(a_2,a_1\right)\right)(\mathrm{d}\tau)^s\right].
\end{aligned}$$

Remark 2.6. For $s \to 1$,

$$\lim_{s \to 1} \frac{(1-s)^s}{2^s-E_s\left(\left(-\frac{1-s}{s}\left(x-a_1\right)\right)^s\right)-E_s\left(\left(-\frac{1-s}{s}\left(a_1+\eta\left(a_2,a_1\right)-x\right)\right)^s\right)}=\frac{1}{\eta(a_2,a_1)},$$

$$\lim_{s\to 1}\frac{E_s\left((-\rho\tau)^s\right)-1^s}{2^s-E_s\left((-\frac{1-s}{s}\left(x-a_1\right))^s\right)-E_s\left((-\frac{1-s}{s}\left(a_1+\eta\left(a_2,a_1\right)-x\right))^s\right)}=-\tau,$$

$$\lim_{s\to 1}\frac{1^s-E_s\left((-\rho(1-\tau))^s\right)}{2^s-E_s\left((-\frac{1-s}{s}\left(x-a_1\right))^s\right)-E_s\left((-\frac{1-s}{s}\left(a_1+\eta\left(a_2,a_1\right)-x\right))^s\right)}=1-\tau.$$

Thus, for $s\to 1$, identity in Lemma 2.2 becomes

$$\begin{aligned}&\frac{1}{\eta(a_2,a_1)}\int_{a_1}^{a_1+\eta(a_2,a_1)}G(u)du-G(x)\\ =&\eta(a_2,a_1)\left[\int_{\frac{x-a_1}{\eta(a_2,a_1)}}^{1}(1-\tau)G'(a_1+\eta(a_2,a_1))d\tau-\int_0^{\frac{x-a_1}{\eta(a_2,a_1)}}\tau G'(a_1+\eta(a_2,a_1))d\tau\right].\end{aligned}$$

3. Main results

Firstly, we will give the local fractional Hermite-Hadamard-type integral inequality with respect to the exponentially s-preconvex function G(x). In the following assumptions, $\rho=\frac{1-s}{s}\eta(a_2,a_1)$.

Theorem 3.1. *Let $\Omega\subseteq R$ be an open invex set regarding $\eta:\Omega\times\Omega\to R$, and $G:\Omega\to\mathbb{R}^s(0<s<1)$ be an exponentially s-preinvex function on Ω with $\eta\left(a_2,a_1\right)>0, a_1,a_2\in\Omega$ for $G(x)\in I_x^{(s)}\left[a_1,a_1+\eta\left(a_2,a_1\right)\right]$. If $\eta(\cdot,\cdot)$ satisfies Condition C, then the following local fractional integral inequalities hold*

$$\begin{aligned}&G\left(a_1+\frac{1}{2}\eta\left(a_2,a_1\right)\right)\\ \leq&\frac{(1-s)^s(\frac{1}{2})^s}{1^s-E_s\left((-\frac{\rho}{2})^s\right)}\left[I^s_{\left(a_1+\frac{\eta(a_2,a_1)}{2}\right)^-}\frac{G\left(a_1\right)}{e^{\alpha a_1}}+I^s_{\left(a_1+\frac{\eta(a_2,a_1)}{2}\right)^+}\frac{G\left(a_1+\eta\left(a_2,a_1\right)\right)}{e^{\alpha(a_1+\eta(a_2,a_1))}}\right]\\ \leq&\frac{\rho^s(\frac{1}{2})^s}{1^s-E_s\left((-\frac{\rho}{2})^s\right)}\left[\frac{G\left(a_2\right)}{e^{\alpha a_2}}+\frac{G\left(a_1\right)}{e^{\alpha a_1}}\right]\frac{1}{\Gamma(1+s)}\int_0^{\frac{1}{2}}\left[\tau^s+(1-\tau)^s\right]E_s\left((-\rho\tau)^s\right)(\mathrm{d}\tau)^s.\end{aligned}\tag{3.1}$$

Proof. Since $\Omega\subseteq R$ is an open set regarding η and $a_1,a_2\in\Omega$, then $a_1+\eta\left(a_2,a_1\right)\in\Omega$. Owing to the exponentially s-preinvexity of G on Ω, then for $v,\omega\in\Omega$, setting $\tau=\frac{1}{2}$ in Definition 2.8, we obtain

$$G\left(v+\frac{1}{2}\eta(\omega,v)\right)\leq\left(\frac{1}{2}\right)^s\left[\frac{G(v)}{e^{\alpha v}}+\frac{G(\omega)}{e^{\alpha\omega}}\right].$$

Using the variable substitutions with $\omega=a_1+\tau\eta\left(a_2,a_1\right)$ and $v=a_1+(1-\tau)\eta\left(a_2,a_1\right)$, we have

$$\begin{aligned}&G\left(a_1+(1-\tau)\eta\left(a_2,a_1\right)+\frac{1}{2}\eta\left(a_1+\tau\eta\left(a_2,a_1\right),a_1+(1-\tau)\eta\left(a_2,a_1\right)\right)\right)\\ \leq&\left(\frac{1}{2}\right)^s\left[\frac{1}{e^{\alpha(a_1+\tau\eta(a_2,a_1))}}G\left(a_1+\tau\eta\left(a_2,a_1\right)\right)\right.\\ &\left.+\frac{1}{e^{\alpha(a_1+(1-\tau)\eta(a_2,a_1))}}G\left(a_1+(1-\tau)\eta\left(a_2,a_1\right)\right)\right].\end{aligned}\tag{3.2}$$

Form Condition C, we have

$$\eta\left(a_1+\tau\eta\left(a_2,a_1\right),a_1+(1-\tau)\eta\left(a_2,a_1\right)\right)=(2\tau-1)\eta\left(a_2,a_1\right).$$

Bring the above equation into equation (3.2):

$$\begin{aligned}&G\left(a_1+\frac{1}{2}\eta\left(a_2,a_1\right)\right)\\&\leq\left(\frac{1}{2}\right)^s\left[\frac{1}{e^{\alpha(a_1+\tau\eta(a_2,a_1))}}G\left(a_1+\tau\eta\left(a_2,a_1\right)\right)\right.\\&\left.+\frac{1}{e^{\alpha(a_1+(1-\tau)\eta(a_2,a_1))}}G\left(a_1+(1-\tau)\eta\left(a_2,a_1\right)\right)\right].\end{aligned}\tag{3.3}$$

Multiplying both sides of the equation (3.3) by $E_s((-\rho\tau)^s)$, and local fractional integrating the resulting inequality regarding τ over $\left[0,\frac{1}{2}\right]$, we obtain

$$\begin{aligned}&\frac{G\left(a_1+\frac{1}{2}\eta\left(a_2,a_1\right)\right)}{\left(\frac{1}{2}\right)^s}\frac{1}{\Gamma(1+s)}\int_0^{\frac{1}{2}}E_s((-\rho\tau)^s)(\mathrm{d}\tau)^s\\\leq&\frac{1}{\Gamma(1+s)}\int_0^{\frac{1}{2}}E_s((-\rho\tau)^s)\frac{1}{e^{\alpha(a_1+\tau\eta(a_2,a_1))}}G\left(a_1+\tau\eta\left(a_2,a_1\right)\right)(\mathrm{d}\tau)^s\\&+\frac{1}{\Gamma(1+s)}\int_0^{\frac{1}{2}}E_s((-\rho\tau)^s)\frac{1}{e^{\alpha(a_1+(1-\tau)\eta(a_2,a_1))}}G\left(a_1+(1-\tau)\eta\left(a_2,a_1\right)\right)(\mathrm{d}\tau)^s.\end{aligned}$$

In the above equation, we set $\omega=a_1+\tau\eta\left(a_2,a_1\right)$, then $\tau=\frac{\omega-a_1}{\eta(a_2,a_1)}$. Similarly, in the second integral, seting $v=a_1+(1-\tau)\eta\left(a_2,a_1\right)$, then $\tau=\frac{\eta(a_2,a_1)+a_1-v}{\eta(a_2,a_1)}$, we obtain

$$\begin{aligned}&\frac{G\left(a_1+\frac{1}{2}\eta\left(a_2,a_1\right)\right)}{\left(\frac{1}{2}\right)^s}\frac{1^s-E_s\left((-\frac{\rho}{2})^s\right)}{\rho^s}\\\leq&\frac{s^s}{\eta^s\left(a_2,a_1\right)}\left[\frac{1}{s^s\Gamma(1+s)}\int_{a_1}^{a_1+\frac{\eta(a_2,a_1)}{2}}E_s\left((-\frac{1-s}{s}\left(\omega-a_1\right))^s\right)\frac{G(\omega)}{e^{\alpha\omega}}(\mathrm{d}\omega)^s\right.\\&\left.+\frac{1}{s^s\Gamma(1+s)}\int_{a_1+\frac{\eta(a_2,a_1)}{2}}^{a_1+\eta(a_2,a_1)}E_s\left((-\frac{1-s}{s}\left(a_1+\eta\left(a_2,a_1\right)-v\right))^s\right)\frac{G(v)}{e^{\alpha v}}(dv)^s\right]\\=&\frac{s^s}{\eta^s\left(a_2,a_1\right)}\left[I^s_{\left(a_1+\frac{\eta(a_2,a_1)}{2}\right)^-}\frac{G\left(a_1\right)}{e^{\alpha a_1}}+I^s_{\left(a_1+\frac{\eta(a_2,a_1)}{2}\right)^+}\frac{G\left(a_1+\eta\left(a_2,a_1\right)\right)}{e^{\alpha(a_1+\eta(a_2,a_1))}}\right].\end{aligned}\tag{3.4}$$

Thus, the left-side inequality of (3.1) holds.

For the right-side inequality, using the exponentially s-preinvexity of G, we have

$$G\left(a_1+\tau\eta\left(a_2,a_1\right)\right)\leq\tau^s\frac{G\left(a_2\right)}{e^{\alpha a_2}}+(1-\tau)^s\frac{G\left(a_1\right)}{e^{\alpha a_1}}$$

and

$$G\left(a_1+(1-\tau)\eta\left(a_2,a_1\right)\right)\leq(1-\tau)^s\frac{G\left(a_2\right)}{e^{\alpha a_2}}+\tau^s\frac{G\left(a_1\right)}{e^{\alpha a_1}}.$$

Adding the aforementioned two inequalities, we obtain

$$G\left(a_1+\tau\eta\left(a_2,a_1\right)\right)+G\left(a_1+(1-\tau)\eta\left(a_2,a_1\right)\right)\leq\left[\tau^s+(1-\tau)^s\right]\left[\frac{G\left(a_2\right)}{e^{\alpha a_2}}+\frac{G\left(a_1\right)}{e^{\alpha a_1}}\right].$$

Multiplying both sides of the above equation by $E_s((-\rho\tau)^s)$, and local fractional integrating the resulting inequality regarding τ over $\left[0, \frac{1}{2}\right]$, we deduce that

$$\begin{aligned}
&\frac{1}{\Gamma(1+s)} \int_0^{\frac{1}{2}} E_s\left((-\rho\tau)^s\right) G\left(a_1+\tau\eta\left(a_2, a_1\right)\right) (\mathrm{d}\tau)^s \\
&+\frac{1}{\Gamma(1+s)} \int_0^{\frac{1}{2}} E_s\left((-\rho\tau)^s\right) G\left(a_1+(1-\tau)\eta\left(a_2, a_1\right)\right) (\mathrm{d}\tau)^s \\
\leq &\left[\frac{G\left(a_2\right)}{e^{\alpha a_2}}+\frac{G\left(a_1\right)}{e^{\alpha a_1}}\right] \frac{1}{\Gamma(1+s)} \int_0^{\frac{1}{2}}\left[\tau^s+(1-\tau)^s\right] E_s\left((-\rho\tau)^s\right) (\mathrm{d}\tau)^s .
\end{aligned} \tag{3.5}$$

According to equations (3.4) and (3.5), we have

$$\begin{aligned}
&\frac{s^s}{\eta^s\left(a_2, a_1\right)}\left[I^s_{\left(a_1+\frac{\eta\left(a_2, a_1\right)}{2}\right)^-} \frac{G\left(a_1\right)}{e^{\alpha a_1}}+I^s_{\left(a_1+\frac{\eta\left(a_2, a_1\right)}{2}\right)^+} \frac{G\left(a_1+\eta\left(a_2, a_1\right)\right)}{e^{\alpha\left(a_1+\eta\left(a_2, a_1\right)\right)}}\right] \\
\leq &\left[\frac{G\left(a_2\right)}{e^{\alpha a_2}}+\frac{G\left(a_1\right)}{e^{\alpha a_1}}\right] \frac{1}{\Gamma(1+s)} \int_0^{\frac{1}{2}}\left[\tau^s+(1-\tau)^s\right] E_s\left((-\rho\tau)^s\right) (\mathrm{d}\tau)^s .
\end{aligned} \tag{3.6}$$

By equations (3.4) and (3.6), we obtain

$$\begin{aligned}
&\frac{G\left(a_1+\frac{1}{2}\eta\left(a_2, a_1\right)\right)}{\left(\frac{1}{2}\right)^s} \frac{1^s-E_s\left((-\frac{\rho}{2})^s\right)}{\rho^s} \\
\leq &\frac{s^s}{\eta^s\left(a_2, a_1\right)}\left[I^s_{\left(a_1+\frac{\eta\left(a_2, a_1\right)}{2}\right)^-} \frac{G\left(a_1\right)}{e^{\alpha a_1}}+I^s_{\left(a_1+\frac{\eta\left(a_2, a_1\right)}{2}\right)^+} \frac{G\left(a_1+\eta\left(a_2, a_1\right)\right)}{e^{\alpha\left(a_1+\eta\left(a_2, a_1\right)\right)}}\right] \\
\leq &\left[\frac{G\left(a_2\right)}{e^{\alpha a_2}}+\frac{G\left(a_1\right)}{e^{\alpha a_1}}\right] \frac{1}{\Gamma(1+s)} \int_0^{\frac{1}{2}}\left[\tau^s+(1-\tau)^s\right] E_s\left((-\rho\tau)^s\right) (\mathrm{d}\tau)^s .
\end{aligned}$$

This completes the proof.

Remark 3.1. For $s \to 1$, we have

$$\lim_{s\to 1} \frac{(1-s)^s}{1^s-E_s\left((-\frac{\rho}{2})^s\right)}=\frac{2}{\eta\left(a_2, a_1\right)}, \quad \lim_{s\to 1} \frac{\rho^s}{1^s-E_s\left((-\frac{\rho}{2})^s\right)}=2.$$

Thus, let $s \to 1$ under the condition of Theorem 3.1, equation (3.1) becomes:

$$G\left(a_1+\frac{1}{2}\eta\left(a_2, a_1\right)\right) \leq \frac{1}{\eta\left(a_2, a_1\right)} \int_{a_1}^{a_1+\eta\left(a_2, a_1\right)} \frac{G(x)}{e^{\alpha x}} dx \leq \frac{1}{2}\left[\frac{G\left(a_2\right)}{e^{\alpha a_2}}+\frac{G\left(a_1\right)}{e^{\alpha a_1}}\right].$$

Corollary 3.1. *If we set $\alpha = 0$ in Theorem 3.1, then we obtain the following Hermite-Hadamard-type local fractional inequality for s-preconvex functions:*

$$\begin{aligned}
&G\left(a_1+\frac{1}{2}\eta\left(a_2, a_1\right)\right) \\
\leq &\frac{(1-s)^s(\frac{1}{2})^s}{1^s-E_s\left((-\frac{\rho}{2})^s\right)}\left[I^s_{\left(a_1+\frac{\eta\left(a_2, a_1\right)}{2}\right)^-} G\left(a_1\right)+I^s_{\left(a_1+\frac{\eta\left(a_2, a_1\right)}{2}\right)^+} G\left(a_1+\eta\left(a_2, a_1\right)\right)\right] \\
\leq &\frac{\rho^s(\frac{1}{2})^s}{1^s-E_s\left((-\frac{\rho}{2})^s\right)}\left[G\left(a_2\right)+G\left(a_1\right)\right] \frac{1}{\Gamma(1+s)} \int_0^{\frac{1}{2}}\left[\tau^s+(1-\tau)^s\right] E_s\left((-\rho\tau)^s\right) (\mathrm{d}\tau)^s .
\end{aligned}$$

Corollary 3.2. *If we set $\eta(a_2,a_1)=a_2-a_1$ in Theorem 3.1 with $a_2>a_1$, then we obtain the following Hermite-Hadamard-type local fractional inequality:*

$$\begin{aligned}
&G\left(\frac{a_2+a_1}{2}\right)\\
\le&\frac{(1-s)^s(\frac{1}{2})^s}{1^s-E_s\left((-\frac{\rho}{2})^s\right)}\left[I^s_{(\frac{a_2+a_1}{2})^-}\frac{G(a_1)}{e^{\alpha a_1}}+I^s_{(\frac{a_2+a_1}{2})^+}\frac{G(a_2)}{e^{\alpha a_2}}\right]\\
\le&\frac{\rho^s(\frac{1}{2})^s}{1^s-E_s\left((-\frac{\rho}{2})^s\right)}\left[\frac{G(a_2)}{e^{\alpha a_2}}+\frac{G(a_1)}{e^{\alpha a_1}}\right]\frac{1}{\Gamma(1+s)}\int_0^{\frac{1}{2}}\left[\tau^s+(1-\tau)^s\right]E_s\left((-\rho\tau)^s\right)(\mathrm{d}\tau)^s.
\end{aligned}$$

The Ostrowski-type integral inequalities for the local fractional integral of the exponentially s-preconvex functions are given below.

Theorem 3.2. *Let $\Omega\subseteq R$ be an open invex set regarding $\eta:\Omega\times\Omega\to R$ and $G:\Omega\to\mathbb{R}^s(0<s<1)$ be a funtion with $\eta(a_2,a_1)>0, a_1,a_2\in\Omega$, and $G^{(s)}(x)\in I_x^{(s)}[a_1,a_1+\eta(a_2,a_1)]$. If $\left|G^{(s)}\right|$ is an exponentially s-preinvex function on Ω, then the following inequality holds*

$$\begin{aligned}
&\left|\frac{(1-s)^s}{2^s-E_s\left((-\frac{1-s}{s}(x-a_1))^s\right)-E_s\left((-\frac{1-s}{s}(a_1+\eta(a_2,a_1)-x))^s\right)}\right.\\
&\left.\times\left(I^s_{x^-}G(a_1)+I^s_{x^+}G(a_1+\eta(a_2,a_1))\right)-G(x)\right|\\
\le&\frac{\eta^s(a_2,a_1)}{2^s-E_s\left((-\frac{1-s}{s}(x-a_1))^s\right)-E_s\left((-\frac{1-s}{s}(a_1+\eta(a_2,a_1)-x))^s\right)}\\
&\times\left[\left(\frac{1}{\Gamma(1+s)}\int_0^{\frac{x-a_1}{\eta(a_2,a_1)}}\left(1^s-E_s\left((-\rho\tau)^s\right)\right)\tau^s(\mathrm{d}\tau)^s\right.\right.\\
&\left.+\frac{1}{\Gamma(1+s)}\int_{\frac{x-a_1}{\eta(a_2,a_1)}}^1\left(1^s-E_s\left((-\rho(1-\tau))^s\right)\right)\tau^s(\mathrm{d}\tau)^s\right)\frac{|G^{(s)}(a_2)|}{e^{\alpha a_2}}\\
&+\left(\frac{1}{\Gamma(1+s)}\int_0^{\frac{x-a_1}{\eta(a_2,a_1)}}\left(1^s-E_s\left((-\rho\tau)^s\right)\right)(1-\tau)^s(\mathrm{d}\tau)^s\right.\\
&\left.\left.+\frac{1}{\Gamma(1+s)}\int_{\frac{x-a_1}{\eta(a_2,a_1)}}^1\left(1^s-E_s\left((-\rho(1-\tau))^s\right)\right)(1-\tau)^s(\mathrm{d}\tau)^s\right)\frac{|G^{(s)}(a_1)|}{e^{\alpha a_1}}\right].
\end{aligned}$$

Proof. Since $\Omega\subseteq R$ is an open set regarding η and $a_1,a_2\in\Omega$, then $a_1+\eta(a_2,a_1)\in\Omega$. Owing to the exponentially s-preinvexity of $\left|G^{(s)}\right|$ on Ω, then

$$|G^{(s)}(a_1+\tau\eta(a_2,a_1))|\le\tau^s\frac{|G^{(s)}(a_2)|}{e^{\alpha a_2}}+(1-\tau)^s\frac{|G^{(s)}(a_1)|}{e^{\alpha a_1}}.$$

By Lemma 2.2, we have

$$
\begin{aligned}
&\Bigg|\frac{(1-s)^s}{2^s - E_s\left((-\frac{1-s}{s}(x-a_1))^s\right) - E_s\left((-\frac{1-s}{s}(a_1+\eta(a_2,a_1)-x))^s\right)}\\
&\times\left(I_{x^-}^s G(a_1) + I_{x^+}^s G(a_1+\eta(a_2,a_1))\right) - G(x)\Bigg|\\
\leq&\frac{\eta^s(a_2,a_1)}{2^s - E_s\left((-\frac{1-s}{s}(x-a_1))^s\right) - E_s\left((-\frac{1-s}{s}(a_1+\eta(a_2,a_1)-x))^s\right)}\\
&\times\left[\frac{1}{\Gamma(1+s)}\int_0^{\frac{x-a_1}{\eta(a_2,a_1)}}\left(1^s - E_s\left((-\rho\tau)^s\right)\right)\left|G^{(s)}(a_1+\tau\eta(a_2,a_1))\right|(\mathrm{d}\tau)^s\right.\\
&\left.+\frac{1}{\Gamma(1+s)}\int_{\frac{x-a_1}{\eta(a_2,a_1)}}^1\left(1^s - E_s\left((-\rho(1-\tau))^s\right)\right)\left|G^{(s)}(a_1+\tau\eta(a_2,a_1))\right|(\mathrm{d}\tau)^s\right]\\
\leq&\frac{\eta^s(a_2,a_1)}{2^s - E_s\left((-\frac{1-s}{s}(x-a_1))^s\right) - E_s\left((-\frac{1-s}{s}(a_1+\eta(a_2,a_1)-x))^s\right)}\\
&\times\left[\frac{1}{\Gamma(1+s)}\int_0^{\frac{x-a_1}{\eta(a_2,a_1)}}\left(1^s - E_s\left((-\rho\tau)^s\right)\right)\right.\\
&\times\left(\tau^s\frac{|G^{(s)}(a_2)|}{e^{\alpha a_2}} + (1-\tau)^s\frac{|G^{(s)}(a_1)|}{e^{\alpha a_1}}\right)(\mathrm{d}\tau)^s\\
&+\frac{1}{\Gamma(1+s)}\int_{\frac{x}{\eta(a_2,a_1)}}^1\left(1^s - E_s\left((-\rho(1-\tau))^s\right)\right)\\
&\left.\times\left(\tau^s\frac{|G^{(s)}(a_2)|}{e^{\alpha a_2}} + (1-\tau)^s\frac{|G^{(s)}(a_1)|}{e^{\alpha a_1}}\right)(\mathrm{d}\tau)^\delta\right]\\
=&\frac{\eta^s(a_2,a_1)}{2^s - E_s\left((-\frac{1-s}{s}(x-a_1))^s\right) - E_s\left((-\frac{1-s}{s}(a_1+\eta(a_2,a_1)-x))^s\right)}\\
&\times\left[\left(\frac{1}{\Gamma(1+s)}\int_0^{\frac{x-a_1}{\eta(a_2,a_1)}}\left(1^s - E_s\left((-\rho\tau)^s\right)\right)\tau^s(\mathrm{d}\tau)^s\right.\right.\\
&\left.+\frac{1}{\Gamma(1+s)}\int_{\frac{x-a_1}{\eta(a_2,a_1)}}^1\left(1^s - E_s\left((-\rho(1-\tau))^s\right)\right)\tau^s(\mathrm{d}\tau)^s\right)\frac{|G^{(s)}(a_2)|}{e^{\alpha a_2}}\ ,\\
&+\left(\frac{1}{\Gamma(1+s)}\int_0^{\frac{x-a_1}{\eta(a_2,a_1)}}\left(1^s - E_s\left((-\rho\tau)^s\right)\right)(1-\tau)^s(\mathrm{d}\tau)^s\right.\\
&\left.\left.+\frac{1}{\Gamma(1+s)}\int_{\frac{x-a_1}{\eta(a_2,a_1)}}^1\left(1^s - E_s\left((-\rho(1-\tau))^s\right)\right)(1-\tau)^s(\mathrm{d}\tau)^s\right)\frac{|G^{(s)}(a_1)|}{e^{\alpha a_1}}\right].
\end{aligned}
$$

This completes the proof.

Remark 3.2. For $s \to 1$, we have

$$\begin{aligned}&\left|\frac{1}{\eta(a_2,a_1)}\int_{a_1}^{a_1+\eta(a_2,a_1)} G(u)du - G(x)\right|\\ \leq&\eta(a_2,a_1)\left[\left(\int_0^{\frac{x-a_1}{\eta(a_2,a_1)}} \tau^2 d\tau + \int_{\frac{x-a_1}{\eta(a_2,a_1)}}^1 (1-\tau)\tau d\tau\right)\frac{|G'(a_2)|}{e^{\alpha a_2}}\right.\\ &\left.+\left(\int_0^{\frac{x-a_1}{\eta(a_2,a_1)}} \tau(1-\tau) d\tau + \int_{\frac{x-a_1}{\eta(a_2,a_1)}}^1 (1-\tau)^2 d\tau\right)\frac{|G'(a_1)|}{e^{\alpha a_1}}\right].\end{aligned}$$

Corollary 3.3. *If we set $\alpha = 0$ in Theorem 3.2, then we obtain the following Ostrowski-type local fractional inequality for s-preconvex functions:*

$$\begin{aligned}&\left|\frac{(1-s)^s}{2^s - E_s\left((-\frac{1-s}{s}(x-a_1))^s\right) - E_s\left((-\frac{1-s}{s}(a_1+\eta(a_2,a_1)-x))^s\right)}\right.\\ &\times\left(I_{x^-}^s G(a_1) + I_{x^+}^s G(a_1+\eta(a_2,a_1))\right) - G(x)\Big|\\ \leq&\frac{\eta^s(a_2,a_1)}{2^s - E_s\left((-\frac{1-s}{s}(x-a_1))^s\right) - E_s\left((-\frac{1-s}{s}(a_1+\eta(a_2,a_1)-x))^s\right)}\\ &\times\left[\left(\frac{1}{\Gamma(1+s)}\int_0^{\frac{x-a_1}{\eta(a_2,a_1)}}\left(1^s - E_s((-\rho\tau)^s)\right)\tau^s(\mathrm{d}\tau)^s\right.\right.\\ &\left.+\frac{1}{\Gamma(1+s)}\int_{\frac{x-a_1}{\eta(a_2,a_1)}}^1\left(1^s - E_s((-\rho(1-\tau))^s)\right)\tau^s(\mathrm{d}\tau)^s\right)|G^{(s)}(a_2)|\,,\\ &+\left(\frac{1}{\Gamma(1+s)}\int_0^{\frac{x-a_1}{\eta(a_2,a_1)}}\left(1^s - E_s((-\rho\tau)^s)\right)(1-\tau)^s(\mathrm{d}\tau)^s\right.\\ &\left.\left.+\frac{1}{\Gamma(1+s)}\int_{\frac{x-a_1}{\eta(a_2,a_1)}}^1\left(1^s - E_s((-\rho(1-\tau))^s)\right)(1-\tau)^s(\mathrm{d}\tau)^s\right)|G^{(s)}(a_1)|\right].\end{aligned}$$

Corollary 3.4. *If we set $\eta(a_2,a_1) = a_2 - a_1$ in Theorem 3.2 with $a_2 > a_1$, then we obtain the following Ostrowski-type local fractional inequality:*

$$\begin{aligned}&\left|\frac{(1-s)^s}{2^s - E_s\left((-\frac{1-s}{s}(x-a_1))^s\right) - E_s\left((-\frac{1-s}{s}(a_2-x))^s\right)}\right.\\ &\times\left(I_{x^-}^s G(a_1) + I_{x^+}^s G(a_2)\right) - G(x)\Big|\\ \leq&\frac{(a_2-a_1)^s}{2^s - E_s\left((-\frac{1-s}{s}(x-a_1))^s\right) - E_s\left((-\frac{1-s}{s}(a_2-x))^s\right)}\\ &\times\left[\left(\frac{1}{\Gamma(1+s)}\int_0^{\frac{x-a_1}{a_2-a_1}}\left(1^s - E_s((-\rho\tau)^s)\right)\tau^s(\mathrm{d}\tau)^s\right.\right.\\ &\left.+\frac{1}{\Gamma(1+s)}\int_{\frac{x-a_1}{a_2-a_1}}^1\left(1^s - E_s((-\rho(1-\tau))^s)\right)\tau^s(\mathrm{d}\tau)^s\right)\frac{|G^{(s)}(a_2)|}{e^{\alpha a_2}}\,,\end{aligned}$$

$$+\left(\frac{1}{\Gamma(1+s)}\int_0^{\frac{x-a_1}{a_2-a_1}}\left(1^s-E_s\left((-\rho\tau)^s\right)\right)(1-\tau)^s(\mathrm{d}\tau)^s\right.$$
$$\left.+\frac{1}{\Gamma(1+s)}\int_{\frac{x-a_1}{a_2-a_1}}^1\left(1^s-E_s\left((-\rho(1-\tau))^s\right)\right)(1-\tau)^s(\mathrm{d}\tau)^s\right)\left.\frac{\left|G^{(s)}\left(a_1\right)\right|}{e^{\alpha a_1}}\right].$$

Theorem 3.3. *Let $\Omega\subseteq R$ be an open invex set regarding $\eta:\Omega\times\Omega\to R$ and $G:\Omega\to\mathbb{R}^s(0<s<1)$ be a function with $\eta\left(a_2,a_1\right)>0, a_1,a_2\in\Omega$, and $G^{(s)}(x)\in I_x^{(s)}\left[a_1,a_1+\eta\left(a_2,a_1\right)\right]$. If $\left|G^{(s)}\right|^q$ is an exponentially s-preinvex function on Ω with $\frac{1}{p}+\frac{1}{q}=1$, then the following inequality holds*

$$\left|\frac{(1-s)^s}{2^s-E_s\left((-\frac{1-s}{s}\left(x-a_1\right))^s\right)-E_s\left((-\frac{1-s}{s}\left(a_1+\eta\left(a_2,a_1\right)-x\right))^s\right)}\right.$$
$$\left.\times\left(I_{x^-}^sG\left(a_1\right)+I_{x^+}^sG\left(a_1+\eta\left(a_2,a_1\right)\right)\right)-G(x)\right|$$
$$\leq\frac{\eta^s\left(a_2,a_1\right)}{2^s-E_s\left((-\frac{1-s}{s}\left(x-a_1\right))^s\right)-E_s\left((-\frac{1-s}{s}\left(a_1+\eta\left(a_2,a_1\right)-x\right))^s\right)}$$
$$\times\left(\frac{\left(\frac{x-a_1}{\eta(a_2,a_1)}\right)^s}{\Gamma(1+s)}+\frac{E_s\left(\left(-p\rho\frac{x-a_1}{\eta(a_2,a_1)}\right)^s\right)-1^s}{(p\rho)^s}\right)^{\frac{1}{p}}$$
$$\times\left(\frac{1}{\Gamma(1+s)}\int_0^{\frac{x-a_1}{\eta(a_2,a_1)}}\left(\tau^s\frac{\left|G^{(s)}\left(a_2\right)\right|^q}{e^{\alpha a_2}}+(1-\tau)^s\frac{\left|G^{(s)}\left(a_1\right)\right|^q}{e^{\alpha a_1}}\right)(\mathrm{d}\tau)^s\right)^{\frac{1}{q}}$$
$$\times\left(\frac{\left(1-\frac{x-a_1}{\eta(a_2,a_1)}\right)^s}{\Gamma(1+s)}+\frac{E_s\left(\left(-p\rho\left(1-\frac{x-a_1}{\eta(a_2,a_1)}\right)\right)^s\right)-1^s}{(p\rho)^s}\right)^{\frac{1}{p}}$$
$$\times\left(\frac{1}{\Gamma(1+s)}\int_{\frac{x-a_1}{\eta(a_2,a_1)}}^1\left(\tau^s\frac{\left|G^{(s)}\left(a_2\right)\right|^q}{e^{\alpha a_2}}+(1-\tau)^s\frac{\left|G^{(s)}\left(a_1\right)\right|^q}{e^{\alpha a_1}}\right)(\mathrm{d}\tau)^s\right)^{\frac{1}{q}}.$$

Proof. Since $\Omega\subseteq R$ is an open invex set regarding η and $a_2,a_1\in\Omega$, then $a_1+\eta(a_2,a_1)\in\Omega$. Using Lemma 2.2, by the properties of modules, it follows that

$$\left|\frac{(1-s)^s}{2^s-E_s\left((-\frac{1-s}{s}\left(x-a_1\right))^s\right)-E_s\left((-\frac{1-s}{s}\left(a_1+\eta\left(a_2,a_1\right)-x\right))^s\right)}\right.$$
$$\left.\times\left(I_{x^-}^sG\left(a_1\right)+I_{x^+}^sG\left(a_1+\eta\left(a_2,a_1\right)\right)\right)-G(x)\right|$$
$$\leq\frac{\eta^s\left(a_2,a_1\right)}{2^s-E_s\left((-\frac{1-s}{s}\left(x-a_1\right))^s\right)-E_s\left((-\frac{1-s}{s}\left(a_1+\eta\left(a_2,a_1\right)-x\right))^s\right)}$$
$$\times\left[\frac{1}{\Gamma(1+s)}\int_0^{\frac{x-a_1}{\eta(a_2,a_1)}}\left(1^s-E_s\left((-\rho\tau)^s\right)\right)\left|G^{(s)}\left(a_1+\tau\eta\left(a_2,a_1\right)\right)\right|(\mathrm{d}\tau)^s\right.$$
$$\left.+\frac{1}{\Gamma(1+s)}\int_{\frac{x-a_1}{\eta(a_2,a_1)}}^1\left(1^s-E_s\left((-\rho(1-\tau))^s\right)\right)\left|G^{(s)}\left(a_1+\tau\eta\left(a_2,a_1\right)\right)\right|(\mathrm{d}\tau)^s\right]. \tag{3.7}$$

Using the fact that $(m-n)^p \le m^p - n^p$ for $m > n \ge 0$, $p \ge 1$, we have

$$\begin{aligned}
&\frac{1}{\Gamma(1+s)}\int_0^{\frac{x-a_1}{\eta(a_2,a_1)}} (1^s - E_s((-\rho\tau)^s))^p\,(\,\mathrm{d}\tau)^s \\
\le&\frac{1}{\Gamma(1+s)}\int_0^{\frac{x-a_1}{\eta(a_2,a_1)}} (1^{ps} - E_s((-p\rho\tau)^s))\,(\mathrm{d}\tau)^s \\
=&\frac{1}{\Gamma(1+s)}\int_0^{\frac{x-a_1}{\eta(a_2,a_1)}} 1^{ps}(\mathrm{d}\tau)^s - \frac{1}{\Gamma(1+s)}\int_0^{\frac{x-a_1}{\eta(a_2,a_1)}} E_s((-p\rho\tau)^s)\,(\mathrm{d}\tau)^s \\
=&\frac{\left(\frac{x-a_1}{\eta(a_2,a_1)}\right)^s}{\Gamma(1+s)} + \frac{E_s\left((-p\rho\frac{x-a_1}{\eta(a_2,a_1)})^s\right) - 1^s}{(p\rho)^s},
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&\frac{1}{\Gamma(1+s)}\int_{\frac{x-a_1}{\eta(a_2,a_1)}}^1 (1^s - E_s((-\rho(1-\tau))^s))^p\,(\,\mathrm{d}\tau)^s \\
\le&\frac{1}{\Gamma(1+s)}\int_{\frac{x-a_1}{\eta(a_2,a_1)}}^1 (1^{ps} - E_s((-p\rho(1-\tau))^s))\,(\mathrm{d}\tau)^s \\
=&\frac{1}{\Gamma(1+s)}\int_{\frac{x-a_1}{\eta(a_2,a_1)}}^1 1^{ps}(\mathrm{d}\tau)^s - \frac{1}{\Gamma(1+s)}\int_{\frac{x-a_1}{\eta(a_2,a_1)}}^1 E_s((-p\rho(1-\tau))^s)\,(\mathrm{d}\tau)^s \\
=&\frac{\left(1-\frac{x-a_1}{\eta(a_2,a_1)}\right)^s}{\Gamma(1+s)} + \frac{E_s\left((-p\rho\left(1-\frac{x-a_1}{\eta(a_2,a_1)}\right))^s\right) - 1^s}{(p\rho)^s}.
\end{aligned}$$

By Lemma 2.1 and the exponentially s-preinvexity of $\left|G^{(s)}\right|^q$, we obtain

$$\begin{aligned}
&\frac{1}{\Gamma(1+s)}\int_0^{\frac{x-a_1}{\eta(a_2,a_1)}} (1^s - E_s((-\rho\tau)^s))\left|G^{(s)}(a_1+\tau\eta(a_2,a_1))\right|(\mathrm{d}\tau)^s \\
\le&\left(\frac{1}{\Gamma(1+s)}\int_0^{\frac{x-a_1}{\eta(a_2,a_1)}} (1^s - E_s((-\rho\tau)^s))^p\,(\,\mathrm{d}\tau)^s\right)^{\frac{1}{p}} \\
&\times\left(\frac{1}{\Gamma(1+s)}\int_0^{\frac{x-a_1}{\eta(a_2,a_1)}} \left|G^{(s)}(a_1+\tau\eta(a_2,a_1))\right|^q(\,\mathrm{d}\tau)^s\right)^{\frac{1}{q}} \\
\le&\left(\frac{\left(\frac{x-a_1}{\eta(a_2,a_1)}\right)^s}{\Gamma(1+s)} + \frac{E_s\left((-p\rho\frac{x-a_1}{\eta(a_2,a_1)})^s\right) - 1^s}{(p\rho)^s}\right)^{\frac{1}{p}} \\
&\times\left(\frac{1}{\Gamma(1+s)}\int_0^{\frac{x-a_1}{\eta(a_2,a_1)}} \left(\tau^s\frac{\left|G^{(s)}(a_2)\right|^q}{e^{\alpha a_2}} + (1-\tau)^s\frac{\left|G^{(s)}(a_1)\right|^q}{e^{\alpha a_1}}\right)(\mathrm{d}\tau)^s\right)^{\frac{1}{q}},
\end{aligned} \tag{3.8}$$

and

$$
\begin{aligned}
&\frac{1}{\Gamma(1+s)} \int_{\frac{x-a_1}{\eta(a_2,a_1)}}^{1} \left(1^s - E_s\left((-\rho(1-\tau))^s\right)\right) \left|G^{(s)}\left(a_1+\tau\eta\left(a_2,a_1\right)\right)\right| (\mathrm{d}\tau)^s \\
&\leq \left(\frac{1}{\Gamma(1+s)} \int_{\frac{x-a_1}{\eta(a_2,a_1)}}^{1} \left(1^s - E_s\left((-\rho(1-\tau))^s\right)\right)^p (\,\mathrm{d}\tau)^s\right)^{\frac{1}{p}} \\
&\quad \times \left(\frac{1}{\Gamma(1+s)} \int_{\frac{x-a_1}{\eta(a_2,a_1)}}^{1} \left|G^{(s)}\left(a_1+\tau\eta\left(a_2,a_1\right)\right)\right|^q (\,\mathrm{d}\tau)^s\right)^{\frac{1}{q}} \qquad (3.9)\\
&\leq \left(\frac{\left(1-\frac{x-a_1}{\eta(a_2,a_1)}\right)^s}{\Gamma(1+s)} + \frac{E_s\left((-p\rho\left(1-\frac{x-a_1}{\eta(a_2,a_1)}\right))^s\right) - 1^s}{(p\rho)^s}\right)^{\frac{1}{p}} \\
&\quad \times \left(\frac{1}{\Gamma(1+s)} \int_{\frac{x-a_1}{\eta(a_2,a_1)}}^{1} \left(\tau^s \frac{\left|G^{(s)}\left(a_2\right)\right|^q}{e^{\alpha a_2}} + (1-\tau)^s \frac{\left|G^{(s)}\left(a_1\right)\right|^q}{e^{\alpha a_1}}\right) (\mathrm{d}\tau)^s\right)^{\frac{1}{q}}.
\end{aligned}
$$

Substituting equations (3.8) and (3.9) into equation (3.7), we achieve the desired result. This completes the proof.

Corollary 3.5. *If we set $\alpha = 0$ in Theorem 3.3, then we obtain the following Ostrowski-type local fractional inequality for s-preconvex functions:*

$$
\begin{aligned}
&\left|\frac{(1-s)^s}{2^s - E_s\left((-\frac{1-s}{s}\left(x-a_1\right))^s\right) - E_s\left((-\frac{1-s}{s}\left(a_1+\eta\left(a_2,a_1\right)-x\right))^s\right)}\right. \\
&\quad \left. \times \left(I_{x^-}^s G\left(a_1\right) + I_{x^+}^s G\left(a_1+\eta\left(a_2,a_1\right)\right)\right) - G(x)\right| \\
&\leq \frac{\eta^s\left(a_2,a_1\right)}{2^s - E_s\left((-\frac{1-s}{s}\left(x-a_1\right))^s\right) - E_s\left((-\frac{1-s}{s}\left(a_1+\eta\left(a_2,a_1\right)-x\right))^s\right)} \\
&\quad \times \left(\frac{\left(\frac{x-a_1}{\eta(a_2,a_1)}\right)^s}{\Gamma(1+s)} + \frac{E_s\left((-p\rho\frac{x-a_1}{\eta(a_2,a_1)})^s\right) - 1^s}{(p\rho)^s}\right)^{\frac{1}{p}} \\
&\quad \times \left(\frac{1}{\Gamma(1+s)} \int_0^{\frac{x-a_1}{\eta(a_2,a_1)}} \left(\tau^s \left|G^{(s)}\left(a_2\right)\right|^q + (1-\tau)^s \left|G^{(s)}\left(a_1\right)\right|^q\right) (\mathrm{d}\tau)^s\right)^{\frac{1}{q}} \\
&\quad \times \left(\frac{\left(1-\frac{x-a_1}{\eta(a_2,a_1)}\right)^s}{\Gamma(1+s)} + \frac{E_s\left(\left(-p\rho\left(1-\frac{x-a_1}{\eta(a_2,a_1)}\right)\right)^s\right) - 1^s}{(p\rho)^s}\right)^{\frac{1}{p}} \\
&\quad \times \left(\frac{1}{\Gamma(1+s)} \int_{\frac{x-a_1}{\eta(a_2,a_1)}}^{1} \left(\tau^s \left|G^{(s)}\left(a_2\right)\right|^q + (1-\tau)^s \left|G^{(s)}\left(a_1\right)\right|^q\right) (\mathrm{d}\tau)^s\right)^{\frac{1}{q}}.
\end{aligned}
$$

Corollary 3.6. *If we set $\eta(a_2, a_1) = a_2 - a_1$ in Theorem 3.3 with $a_2 > a_1$, then we* *obtain the following Ostrowski-type local fractional inequality:*

$$\begin{aligned}
&\left| \frac{(1-s)^s}{2^s - E_s\left((-\frac{1-s}{s}(x-a_1))^s\right) - E_s\left((-\frac{1-s}{s}(a_2-x))^s\right)} \right. \\
&\times \left(I_{x^-}^s G(a_1) + I_{x^+}^s G(a_2)\right) - G(x)\Big| \\
\leq &\frac{(a_2-a_1)^s}{2^s - E_s\left((-\frac{1-s}{s}(x-a_1))^s\right) - E_s\left((-\frac{1-s}{s}(a_2-x))^s\right)} \\
&\times \left(\frac{\left(\frac{x-a_1}{a_2-a_1}\right)^s}{\Gamma(1+s)} + \frac{\left(E_s\left(\left(-p\rho\frac{x-a_1}{a_2-a_1}\right)\right)^s\right) - 1^s}{(p\rho)^s} \right)^{\frac{1}{p}} \\
&\times \left(\frac{1}{\Gamma(1+s)} \int_0^{\frac{x-a_1}{a_2-a_1}} \left(\tau^s \frac{\left|G^{(s)}(a_2)\right|^q}{e^{\alpha a_2}} + (1-\tau)^s \frac{\left|G^{(s)}(a_1)\right|^q}{e^{\alpha a_1}} \right) (\mathrm{d}\tau)^s \right)^{\frac{1}{q}} \\
&\times \left(\frac{\left(\frac{a_2-x}{a_2-a_1}\right)^s}{\Gamma(1+s)} + \frac{E_s\left(\left(-p\rho\left(\frac{a_2-x}{a_2-a_1}\right)\right)^s\right) - 1^s}{(p\rho)^s} \right)^{\frac{1}{p}} \\
&\times \left(\frac{1}{\Gamma(1+s)} \int_{\frac{x-a_1}{a_2-a_1}}^1 \left(\tau^s \frac{\left|G^{(s)}(a_2)\right|^q}{e^{\alpha a_2}} + (1-\tau)^s \frac{\left|G^{(s)}(a_1)\right|^q}{e^{\alpha a_1}} \right) (\mathrm{d}\tau)^s \right)^{\frac{1}{q}}.
\end{aligned}$$

4. Applications

In numerical analysis and algorithm optimization,the Hermite-Hadamard and Ostrowski inequality inequalities are of great importance.The Hermite-Hadamard inequality can be used to estimate the upper and lower bounds of the integral value.And Ostrowski inequality can be used to estimate the error between the value of a function and its mean.The results we have obtained in this article are important for this.Next,several examples are given to illustrate the main results.

Example 4.1. By Remark 3.1, we obtain

$$G\left(a_1 + \frac{1}{2}\eta(a_2, a_1)\right) \leq \frac{1}{\eta(a_2, a_1)} \int_{a_1}^{a_1+\eta(a_2,a_1)} \frac{G(x)}{e^{\alpha x}} dx \leq \frac{1}{2}\left[\frac{G(a_2)}{e^{\alpha a_2}} + \frac{G(a_1)}{e^{\alpha a_1}}\right].$$

Let $G : (0, \infty) \to \mathbb{R}, G(u) = ln u$ be an exponentially s-preinvex function for all $\alpha \leq -1$ and $u \in [a_1, a_1 + \eta(a_2, a_1)]$ with $a_2 > a_1$. Then we choose $\eta(a_2, a_1) = a_2 - a_1$,and take $a_1 = 1$ and $a_2 = 3$. We have

$$\begin{aligned}
&G\left(a_1 + \frac{1}{2}\eta(a_2, a_1)\right) = G(2) = ln2 = 0.6931, \\
&\frac{1}{\eta(a_2, a_1)} \int_{a_1}^{a_1+\eta(a_2,a_1)} \frac{G(x)}{e^{\alpha x}} dx = \frac{1}{2}\int_1^3 G(x)e^x dx = 2.1142, \\
&\frac{1}{2}\left[\frac{G(a_2)}{e^{\alpha a_2}} + \frac{G(a_1)}{e^{\alpha a_1}}\right] = \frac{1}{2} \times ln3 \times e^3 = 11.0330.
\end{aligned}$$

Obviously, $0.6931 < 2.1142 < 11.0330$, i.e., then Remark 3.1 holds in this case.

Example 4.2. By Remark 3.2, we obtain

$$\begin{aligned}&\left|\frac{1}{\eta(a_2,a_1)}\int_{a_1}^{a_1+\eta(a_2,a_1)}G(u)du-G(x)\right|\\ \leq&\eta(a_2,a_1)\left[\left(\int_0^{\frac{x-a_1}{\eta(a_2,a_1)}}\tau^2d\tau+\int_{\frac{x-a_1}{\eta(a_2,a_1)}}^1(1-\tau)\tau d\tau\right)\frac{|G'(a_2)|}{e^{\alpha a_2}}\right.\\ &\left.+\left(\int_0^{\frac{x-a_1}{\eta(a_2,a_1)}}\tau(1-\tau)d\tau+\int_{\frac{x-a_1}{\eta(a_2,a_1)}}^1(1-\tau)^2d\tau\right)\frac{|G'(a_1)|}{e^{\alpha a_1}}\right].\end{aligned}$$

Let $G:(0,\infty)\to\mathbb{R},G(u)=lnu$ be an exponentially s-preinvex function for all $\alpha\leq-1$ and $u\in[a_1,a_1+\eta(a_2,a_1)]$ with $a_2>a_1$. Then we choose $\eta(a_2,a_1)=a_2-a_1$,and take $a_1=1$ and $a_2=3$. We have

$$\left|\frac{1}{\eta(a_2,a_1)}\int_{a_1}^{a_1+\eta(a_2,a_1)}G(u)du-G(x)\right|=|3ln3-2-lnx|,$$

and

$$\begin{aligned}&\eta(a_2,a_1)\left[\left(\int_0^{\frac{x-a_1}{\eta(a_2,a_1)}}\tau^2d\tau+\int_{\frac{x-a_1}{\eta(a_2,a_1)}}^1(1-\tau)\tau d\tau\right)\frac{|G'(a_2)|}{e^{\alpha a_2}}\right.\\ &\left.+\left(\int_0^{\frac{x-a_1}{\eta(a_2,a_1)}}\tau(1-\tau)d\tau+\int_{\frac{x-a_1}{\eta(a_2,a_1)}}^1(1-\tau)^2d\tau\right)\frac{|G'(a_1)|}{e^{\alpha a_1}}\right]\\ =&(\frac{1}{18}(x-1)^3-\frac{1}{12}(x-1)^2+\frac{1}{9})e^3+(-\frac{1}{6}(x-1)^3+\frac{3}{4}(x-1)^2-x+\frac{5}{3})e.\end{aligned}$$

We use MATLAB software to prove that the above inequality holds for $x\in(a,b)$ with $0<a<1<b$. This also shows that Remark 3.2 is true in this case.

Example 4.3. Let X be a continuous random variable with the generalized probability density function $f:\Omega\to R^s$, where Ω is an open invex set regarding $\eta:\Omega\times\Omega\to R$ and $\eta(a_2,a_1)>0,a_1,a_2\in\Omega$. We define the generalized n-th central moment about any $u\in R$ of $X,n\geq0$ as:

$$M_s^n(u)=\frac{1}{\Gamma(1+s)}\int_{a_1}^{a_1+\eta(a_2,a_1)}(x-u)^{ns}f(x)(\mathrm{d}x)^s,\quad n=1,2,3,\ldots$$

Moreover, we have

$$\begin{aligned}(M_s^n(u))^{(s)}&=-\frac{\Gamma(1+ns)\Gamma(1+s)}{\Gamma(1+(n-1)s)}\frac{1}{\Gamma(1+s)}\int_{a_1}^{a_1+\eta(a_2a_1)}(x-u)^{(n-1)s}f(x)(\mathrm{d}x)^s\\ &=-\frac{\Gamma(1+ns)\Gamma(1+s)}{\Gamma(1+(n-1)s)}M_s^{n-1}(u).\end{aligned}$$

Based on the above example 4.3, we obtain the following two properties:

Proposition 4.1. *Let $G(u)=M_s^n(u)$. If the conditions of Corollary 3.3 are satisfied, then we obtain the midpointtype inequalities involving generalized moment as* *follows:*

$$\left|\frac{(1-s)^s}{2^s - E_s\left((-\frac{1-s}{s}(x-a_1))^s\right) - E_s\left((-\frac{1-s}{s}(a_1+\eta(a_2,a_1)-x))^s\right)}\right.$$

$$\times \left(I_{x^-}^s M_s^n(a_1) + I_{x^+}^s M_s^n(a_1+\eta(a_2,a_1))\right) - M_s^n(x)\Big|$$

$$\leq \frac{\eta^s(a_2,a_1)}{2^s - E_s\left((-\frac{1-s}{s}(x-a_1))^s\right) - E_s\left((-\frac{1-s}{s}(a_1+\eta(a_2,a_1)-x))^s\right)}$$

$$\times \frac{\Gamma(1+ns)\Gamma(1+s)}{\Gamma(1+(n-1)s)}\left[\left(\frac{1}{\Gamma(1+s)}\int_0^{\frac{x-a_1}{\eta(a_2,a_1)}}\left(1^s - E_s\left((-\rho\tau)^s\right)\right)\tau^s(\mathrm{d}\tau)^s\right.\right.$$

$$\left.+\frac{1}{\Gamma(1+s)}\int_{\frac{x-a_1}{\eta(a_2,a_1)}}^1\left(1^s - E_s\left((-\rho(1-\tau))^s\right)\right)\tau^s(\mathrm{d}\tau)^s\right)\left|M_s^{n-1}(a_2)\right|,$$

$$+\left(\frac{1}{\Gamma(1+s)}\int_0^{\frac{x-a_1}{\eta(a_2,a_1)}}\left(1^s - E_s\left((-\rho\tau)^s\right)\right)(1-\tau)^s(\mathrm{d}\tau)^s\right.$$

$$\left.\left.+\frac{1}{\Gamma(1+s)}\int_{\frac{x-a_1}{\eta(a_2,a_1)}}^1\left(1^s - E_s\left((-\rho(1-\tau))^s\right)\right)(1-\tau)^s(\mathrm{d}\tau)^s\right)\left|M_s^{n-1}(a_1)\right|\right].$$

Proposition 4.2. *Let $G(u) = M_s^n(u)$. If the conditions of Corollary 3.5 are satisfied, then we obtain the midpointtype inequalities involving generalized moment as follows:*

$$\left|\frac{(1-s)^s}{2^s - E_s\left((-\frac{1-s}{s}(x-a_1))^s\right) - E_s\left((-\frac{1-s}{s}(a_1+\eta(a_2,a_1)-x))^s\right)}\right.$$

$$\times \left(I_{x^-}^s M_s^n(a_1) + I_{x^+}^s M_s^n(a_1+\eta(a_2,a_1))\right) - M_s^n(x)\Big|$$

$$\leq \frac{\eta^s(a_2,a_1)}{2^s - E_s\left((-\frac{1-s}{s}(x-a_1))^s\right) - E_s\left((-\frac{1-s}{s}(a_1+\eta(a_2,a_1)-x))^s\right)}$$

$$\times \left(\frac{\Gamma(1+ns)\Gamma(1+s)}{\Gamma(1+(n-1)s)}\right)^2$$

$$\times \left(\frac{\left(\frac{x-a_1}{\eta(a_2,a_1)}\right)^s}{\Gamma(1+s)} + \frac{E_s\left((-p\rho\frac{x-a_1}{\eta(a_2,a_1)})^s\right) - 1^s}{(p\rho)^s}\right)^{\frac{1}{p}}$$

$$\times \left(\frac{1}{\Gamma(1+s)}\int_0^{\frac{x-a_1}{\eta(a_2,a_1)}}\left(\tau^s\left|M_s^{n-1}(a_2)\right|^q + (1-\tau)^s\left|M_s^{n-1}(a_1)\right|^q\right)(\mathrm{d}\tau)^s\right)^{\frac{1}{q}}$$

$$\times \left(\frac{\left(1-\frac{x-a_1}{\eta(a_2,a_1)}\right)^s}{\Gamma(1+s)} + \frac{E_s\left(\left(-p\rho\left(1-\frac{x-a_1}{\eta(a_2,a_1)}\right)\right)^s\right) - 1^s}{(p\rho)^s}\right)^{\frac{1}{p}}$$

$$\times \left(\frac{1}{\Gamma(1+s)}\int_{\frac{x-a_1}{\eta(a_2,a_1)}}^1\left(\tau^s\left|M_s^{n-1}(a_2)\right|^q + (1-\tau)^s\left|M_s^{n-1}(a_1)\right|^q\right)(\mathrm{d}\tau)^s\right)^{\frac{1}{q}}.$$

5. Conclusions

This paper is mainly based on the fractal theory of Yang. First of all, we generalize the previous results and study the Hermite-Hadamard integral inequality and the Ostrowski integral inequality of the local fractional integral of the exponentially s-preconvex function. Then the corresponding inequality of the s-preconvex function can be obtained by degeneration, and some new results can also be obtained by taking some special values. Finally, the combination of some examples and applications illustrates the correctness and significance of the results we obtain.The results obtained in this paper can provide some references for the study of inequalities and the bounded estimation of mathematical problems involving the engineering field, and can be further generalized by using the different convexity of functions to obtain results with a wider range of applications.

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