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Error Bounds for Corrected Euler-Maclaurin Formula in Tempered Fractional Integrals

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Abstract In this paper, an equality is established for tempered fractional integrals. With the help of this equality, we prove several corrected Euler-Maclaurin-type inequalities for the case of differentiable convex functions involving tempered fractional integrals. Moreover, we provide our results by using special cases of obtained theorems.

Keywords Quadrature formulae, corrected Maclaurin's formula, tempered fractional integrals, convex functions

MSC(2010) 26D07, 26D10, 26D15, 65D32.

1. Introduction & preliminaries

Simpson-type inequalities are inequalities that are created from Simpson's following rules:

i. Simpson's quadrature formula (Simpson's 1/3 rule) is formulated as follows:

$$\int_a^b f(x)\,dx \approx \frac{b-a}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]. \tag{1.1}$$

ii. Simpson's second formula or Newton-Cotes quadrature formula (Simpson's 3/8 rule (cf. [8])) is formulated as follows:

$$\int_a^b f(x)\,dx \approx \frac{b-a}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]. \tag{1.2}$$

iii. There is also the corresponding dual Simpson's 3/8 formula - the Maclaurin rule based on the Maclaurin formula (cf. [8]):

$$\int_a^b f(x)\,dx \approx \frac{b-a}{8}\left[3f\left(\frac{5a+b}{6}\right)+2f\left(\frac{a+b}{2}\right)+3f\left(\frac{a+5b}{6}\right)\right]. \tag{1.3}$$

Formulae (1.1), (1.2) and (1.3) are satisfied for any function f with continuous 4^{th} derivative on $[a,b]$.

The most popular Newton-Cotes quadrature involving three-point is Simpson-type inequality as follows:

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Theorem 1.1. *Let $f:[a,b]\to\mathbb{R}$ be a four times continuously differentiable function on (a,b), and let $\left\|f^{(4)}\right\|_{\infty}=\sup_{x\in(a,b)}\left|f^{(4)}(x)\right|<\infty$. Then, one has the following inequality*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq\frac{1}{2880}\left\|f^{(4)}\right\|_{\infty}(b-a)^4.$$

One of the classical closed type quadrature rules is the Simpson 3/8 rule based on the Simpson 3/8 inequality as follows:

Theorem 1.2 (See [8])**.** *If $f:[a,b]\to\mathbb{R}$ is a four times continuously differentiable function on (a,b) and $\left\|f^{(4)}\right\|_{\infty}=\sup_{x\in(a,b)}\left|f^{(4)}(x)\right|<\infty$, then one has the inequality*

$$\begin{aligned}&\left|\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right|\\ \leq&\frac{1}{6480}\left\|f^{(4)}\right\|_{\infty}(b-a)^4.\end{aligned}$$

The corresponding dual Simpson's 3/8 formula - the Maclaurin rule based on the Maclaurin inequality is as follows:

Theorem 1.3 (See [8])**.** *Assume that $f:[a,b]\to\mathbb{R}$ is a four times continuously differentiable function on (a,b) and $\left\|f^{(4)}\right\|_{\infty}=\sup_{x\in(a,b)}\left|f^{(4)}(x)\right|<\infty$. Then, the following inequality holds:*

$$\begin{aligned}&\left|\frac{1}{8}\left[3f\left(\frac{5a+b}{6}\right)+2f\left(\frac{a+b}{2}\right)+3f\left(\frac{a+5b}{6}\right)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right|\\ \leq&\frac{7}{51840}\left\|f^{(4)}\right\|_{\infty}(b-a)^4.\end{aligned}$$

Finally, three-point open formula known as corrected Euler-Maclaurin's inequalities (see [15]) is as follows:

Theorem 1.4 (See [15])**.** *Suppose that $f:[a,b]\to\mathbb{R}$ is a four times continuously differentiable function on (a,b) and $\left\|f^{(4)}\right\|_{\infty}=\sup_{x\in(a,b)}\left|f^{(4)}(x)\right|<\infty$. Then, it yields*

$$\begin{aligned}&\left|\frac{1}{80}\left[27f\left(\frac{5a+b}{6}\right)+26f\left(\frac{a+b}{2}\right)+27f\left(\frac{a+5b}{6}\right)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right|\\ \leq&\frac{2401}{28800}\left\|f^{(4)}\right\|_{\infty}(b-a)^4.\end{aligned}$$

Fractional calculus has been grown interest because of its applications in a wide range of disparate domains of science. Because of the importance of fractional calculus, mathematicians have investigated different fractional integral inequalities. Riemann-Liouville fractional integrals, tempered fractional, conformable fractional integrals, and many types of fractional integrals have been considered with several important types of inequalities. The bounds of new formulas can be established by using not only Hermite-Hadamard and Simpson type inequalities but also Newton and Euler-Maclaurin-type inequalities.

Dragomir et al. [11] proved new inequalities of Simpson-type and their application to quadrature inequalities in numerical analysis. Dragomir [12] presented an estimation of remainder for Simpson's quadrature formula for functions of bounded variation and applications in the theory of special means. Several fractional Simpson-type inequalities were established in the case of a function whose second derivatives in absolute value are convex as given in [20]. Budak et al. [1] proved some variants of Simpson-type for differentiable convex functions using generalized fractional integrals. See Refs. [2,5,42] and the references therein for further information.

Simpson's second rule has the rule of three-point Newton-Cotes quadrature, hence evaluations for the case of three steps quadratic kernel are usually called Newton type results. These results are known as Newton-type inequalities in the literature. Many mathematicians have investigated Newton type inequalities extensively. For instance, Erden et al. presented several Newton-type inequalities for the case of functions whose first derivative in absolute value at certain power are arithmetically-harmonically convex in paper [14]. Moreover, several Newton-type inequalities for the case of differentiable convex functions through the well-known Riemann-Liouville fractional integrals were established and several inequalities of Riemann-Liouville fractional Newton's type for functions of bounded variation were given in [22]. Furthermore, new Newton-type inequalities based on convexity were presented and several applications for special cases of real functions were also proved in paper [18]. It can be referred to [4,6,7,23–25,35,39] and the references therein to the case of more information associated with Newton-type inequalities including convex differentiable functions.

Sets of inequalities are established by applying the Euler-Maclaurin formulae and the results are applied in order to obtain several error estimates for the case of the Maclaurin quadrature rules in [9]. Upon this, Franjić and Pečarić [15] investigated the corrected Euler-Maclaurin's formulae, i.e. open type quadrature formulae where the integral is approximated not only with the values of the function at points $(5a+b)/6$, $(a+b)/2$, and $(a+5b)/6$, but also with values of the first derivative at end points of the interval. These formulae will have a higher degree of exactness than the ones obtained in [9]. Furthermore, a set of inequalities is proved by using the Euler-Simpson 3/8 formulae. The results are applied to get several error estimates for the Simpson 3/8 quadrature rules in [10]. Thereupon, Franjić and Pečarić [16] established the corrected Euler-Simpson's 3/8 formulae, i.e. closed type quadrature formulae where the integral is approximated not only with the values of the function at points a , $(2a+b)/3$, $(a+2b)/3$, and b, but also with values of the first derivative at boundary points of the interval. These formulae will have a higher degree of exactness than those obtained in [10]. With the help of the derived inequalities, several inequalities for the case of different classes of functions are presented. We refer to [21,32,33,36,37] and the references therein for further information about these kinds of inequalities.

Recall that the gamma function, incomplete gamma function, λ-incomplete gamma function are described by

$$\Gamma(\alpha) := \int_0^{\infty} t^{\alpha-1} e^{-t} dt,$$

$$\Upsilon(\alpha, x) := \int\limits_0^x t^{\alpha-1} e^{-t} dt,$$

and

$$\Upsilon_\lambda(\alpha, x) := \int\limits_0^x t^{\alpha-1} e^{-\lambda t} dt,$$

respectively. Here, $0 < \alpha < \infty$ and $\lambda \geq 0$.

There are some properties λ-incomplete gamma function as follows:

Remark 1.1. [34] For the real numbers $\alpha > 0$; $x, \lambda \geq 0$ and $a < b$, we readily have

i. $\Upsilon_{\lambda(b-a)}(\alpha, 1) = \int\limits_0^1 t^{\alpha-1} e^{-\lambda(b-a)t} dt = \frac{1}{(b-a)^\alpha} \Upsilon_\lambda(\alpha, b-a)$,

ii. $\int\limits_0^1 \Upsilon_{\lambda(b-a)}(\alpha, x)\, dx = \frac{\Upsilon_\lambda(\alpha, b-a)}{(b-a)^\alpha} - \frac{\Upsilon_\lambda(\alpha+1, b-a)}{(b-a)^{\alpha+1}}$.

Recall that the Riemann-Liouville integrals of order $\alpha > 0$ are given by

$$J_{a+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt, \quad x > a \tag{1.4}$$

and

$$J_{b-}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t) dt, \quad x < b \tag{1.5}$$

for $f \in L_1[a,b]$. See [19, 28] for details and unexplained subjects. Note that the Riemann-Liouville integrals become classical integrals for the condition $\alpha = 1$.

We shall now present the fundamental definitions and new notations of tempered fractional operators.

Definition 1.1. [29, 31] The fractional tempered integral operators $\mathcal{J}_{a+}^{(\alpha,\lambda)} f$ and $\mathcal{J}_{b-}^{(a,\lambda)} f$ of order $\alpha > 0$ and $\lambda \geq 0$ are given by

$$\mathcal{J}_{a+}^{(\alpha,\lambda)} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} e^{-\lambda(x-t)} f(t) dt, \quad x \in [a,b] \tag{1.6}$$

and

$$\mathcal{J}_{b-}^{(\alpha,\lambda)} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} e^{-\lambda(t-x)} f(t) dt, \quad x \in [a,b], \tag{1.7}$$

respectively for $f \in L_1[a,b]$.

If we assign $\lambda = 0$, then the fractional integrals in (1.6) and (1.7) coincide with the Riemann-Liouville fractional integrals in (1.4) and (1.5), respectively.

Tempered fractional calculus can be specified as the generalization of fractional calculus. The definitions of fractional integration with weak singular and exponential kernels were firstly reported in Buschman's earlier work [3]. See the Refs. [17, 26, 27, 30, 38, 40, 41] and references therein for more information connected with the different definitions of the tempered fractional integration. In paper [34], Mohammed et al. established several Hermite–Hadamard-type inequalities associated with the tempered fractional integrals for the case of convex functions which cover the previously published results such as Riemann integrals, Riemann-Liouville fractional integrals.

The aim of this paper is to derive corrected Euler-Maclaurin-type inequalities for the case of differentiable convex functions by tempered fractional integrals. The fundamental definition of fractional calculus and other relevant research in this discipline are presented in above. We will prove an integral equality in Section 2 that is critical in establishing the primary results of the presented paper. Furthermore, we will prove some corrected Euler-Maclaurin-type inequalities for the case of differentiable convex functions involving tempered fractional integrals. By using the special cases of the obtained results, several important inequalities will be presented. In Section 3, we will give several ideas about corrected Euler-Maclaurin-type inequalities for further directions of research.

2. Principal outcomes

Lemma 2.1. *Let us consider that $f:[a,b]\to\mathbb{R}$ is an absolutely continuous function (a,b) so that $f'\in L_1[a,b]$. Then, the following equality holds:*

$$\begin{aligned}&\frac{1}{80}\left[27f\left(\frac{5a+b}{6}\right)+26f\left(\frac{a+b}{2}\right)+27f\left(\frac{a+5b}{6}\right)\right]\\&-\frac{\Gamma(\alpha)}{2\,\Upsilon_{\lambda}(\alpha,b-a)}\left[\mathcal{J}_{b-}^{(\alpha,\lambda)}f(a)+\mathcal{J}_{a+}^{(\alpha,\lambda)}f(b)\right]\\=&\frac{(b-a)^{\alpha+1}}{2\,\Upsilon_{\lambda}(\alpha,b-a)}\sum_{i=1}^{4}I_i.\end{aligned}\tag{2.1}$$

Here,

$$\begin{cases}I_1=\int_0^{\frac{1}{6}}\Upsilon_{\lambda(b-a)}(\alpha,t)\left[f'(tb+(1-t)a)-f'(ta+(1-t)b)\right]dt,\\I_2=\int_{\frac{1}{6}}^{\frac{1}{2}}\left\{\Upsilon_{\lambda(b-a)}(\alpha,t)-\frac{27}{80}\Upsilon_{\lambda(b-a)}(\alpha,1)\right\}\left[f'(tb+(1-t)a)-f'(ta+(1-t)b)\right]dt,\\I_3=\int_{\frac{1}{2}}^{\frac{5}{6}}\left\{\Upsilon_{\lambda(b-a)}(\alpha,t)-\frac{53}{80}\Upsilon_{\lambda(b-a)}(\alpha,1)\right\}\left[f'(tb+(1-t)a)-f'(ta+(1-t)b)\right]dt,\\I_4=\int_{\frac{5}{6}}^{1}\left\{\Upsilon_{\lambda(b-a)}(\alpha,t)-\Upsilon_{\lambda(b-a)}(\alpha,1)\right\}\left[f'(tb+(1-t)a)-f'(ta+(1-t)b)\right]dt.\end{cases}$$

Proof. From the facts of integration by parts, we get

$$\begin{aligned}I_1&=\int_0^{\frac{1}{6}}\Upsilon_{\lambda(b-a)}(\alpha,t)\left[f'(tb+(1-t)a)-f'(ta+(1-t)b)\right]dt\\&=\frac{1}{b-a}\Upsilon_{\lambda(b-a)}(\alpha,t)\left[f(tb+(1-t)a)+f(ta+(1-t)b)\right]\Big|_0^{\frac{1}{6}}\\&\quad-\frac{1}{b-a}\int_0^{\frac{1}{6}}t^{\alpha-1}e^{-\lambda(b-a)t}\left[f(tb+(1-t)a)+f(ta+(1-t)b)\right]dt\end{aligned}\tag{2.2}$$

$$= \frac{1}{b-a}\,\Upsilon_{\lambda(b-a)}\left(\alpha,\frac{1}{6}\right)\left[f\left(\frac{5a+b}{6}\right)+f\left(\frac{a+5b}{6}\right)\right]$$
$$-\frac{1}{b-a}\int_{0}^{\frac{1}{6}} t^{\alpha-1}e^{-\lambda(b-a)t}\left[f\left(tb+(1-t)a\right)+f\left(ta+(1-t)b\right)\right]dt.$$

Then, similar to foregoing process, we readily obtain

$$I_2 = \frac{2}{b-a}\left\{\Upsilon_{\lambda(b-a)}\left(\alpha,\frac{1}{2}\right)-\frac{27}{80}\,\Upsilon_{\lambda(b-a)}\left(\alpha,1\right)\right\}f\left(\frac{a+b}{2}\right)$$
$$-\frac{1}{b-a}\left\{\Upsilon_{\lambda(b-a)}\left(\alpha,\frac{1}{6}\right)-\frac{27}{80}\,\Upsilon_{\lambda(b-a)}\left(\alpha,1\right)\right\}\left[f\left(\frac{5a+b}{6}\right)+f\left(\frac{a+5b}{6}\right)\right]$$
$$-\frac{1}{b-a}\int_{\frac{1}{6}}^{\frac{1}{2}} t^{\alpha-1}e^{-\lambda(b-a)t}\left[f\left(tb+(1-t)a\right)+f\left(ta+(1-t)b\right)\right]dt, \tag{2.3}$$

$$I_3 = \frac{1}{b-a}\left\{\Upsilon_{\lambda(b-a)}\left(\alpha,\frac{5}{6}\right)-\frac{53}{80}\,\Upsilon_{\lambda(b-a)}\left(\alpha,1\right)\right\}\left[f\left(\frac{5a+b}{6}\right)+f\left(\frac{a+5b}{6}\right)\right]$$
$$-\frac{2}{b-a}\left\{\Upsilon_{\lambda(b-a)}\left(\alpha,\frac{1}{2}\right)-\frac{53}{80}\,\Upsilon_{\lambda(b-a)}\left(\alpha,1\right)\right\}f\left(\frac{a+b}{2}\right)$$
$$-\frac{1}{b-a}\int_{\frac{1}{2}}^{\frac{5}{6}} t^{\alpha-1}e^{-\lambda(b-a)t}\left[f\left(tb+(1-t)a\right)+f\left(ta+(1-t)b\right)\right]dt, \tag{2.4}$$

and

$$I_4 = -\frac{1}{b-a}\left\{\Upsilon_{\lambda(b-a)}\left(\alpha,\frac{5}{6}\right)-\Upsilon_{\lambda(b-a)}\left(\alpha,1\right)\right\}\left[f\left(\frac{5a+b}{6}\right)+f\left(\frac{a+5b}{6}\right)\right]$$
$$-\frac{1}{b-a}\int_{\frac{5}{6}}^{1} t^{\alpha-1}e^{-\lambda(b-a)t}\left[f\left(tb+(1-t)a\right)+f\left(ta+(1-t)b\right)\right]dt. \tag{2.5}$$

If we add (2.2) to (2.5), then we have

$$\sum_{i=1}^{4} I_i = \frac{\Upsilon_{\lambda}\left(\alpha,b-a\right)}{40\left(b-a\right)^{\alpha+1}}\left[27f\left(\frac{5a+b}{6}\right)+26f\left(\frac{a+b}{2}\right)+27f\left(\frac{5a+b}{6}\right)\right]$$
$$-\frac{1}{b-a}\int_{0}^{1} t^{\alpha-1}e^{-\lambda(b-a)t}\left[f\left(tb+(1-t)a\right)+f\left(ta+(1-t)b\right)\right]dt. \tag{2.6}$$

With the help of the change of the variable $x = tb + (1-t)a$ and $x = ta + (1-t)b$ for $t \in [0,1]$ respectively, equality (2.6) can be rewritten as follows

$$\sum_{i=1}^{4} I_i = \frac{\Upsilon_{\lambda}\left(\alpha,b-a\right)}{40\left(b-a\right)^{\alpha+1}}\left[27f\left(\frac{5a+b}{6}\right)+26f\left(\frac{a+b}{2}\right)+27f\left(\frac{5a+b}{6}\right)\right]$$

$$-\frac{\Gamma(\alpha)}{(b-a)^{\alpha+1}}\left[\mathcal{J}_{b-}^{(\alpha,\lambda)}f(a)+\mathcal{J}_{a+}^{(\alpha,\lambda)}f(b)\right]. \tag{2.7}$$

Multiplying both sides of (2.7) by $\frac{(b-a)^{\alpha+1}}{2\Upsilon_{\lambda}(\alpha,b-a)}$, the equality (2.1) is obtained. □

Theorem 2.1. *If the assumptions of Lemma 2.1 hold and the function $|f'|$ is convex on $[a,b]$, then we have the following corrected Euler-Maclaurin-type inequality*

$$\begin{aligned}&\left|\frac{1}{80}\left[27f\left(\frac{5a+b}{6}\right)+26f\left(\frac{a+b}{2}\right)+27f\left(\frac{a+5b}{6}\right)\right]\right.\\&\left.-\frac{\Gamma(\alpha)}{2\Upsilon_{\lambda}(\alpha,b-a)}\left[\mathcal{J}_{b-}^{(\alpha,\lambda)}f(a)+\mathcal{J}_{a+}^{(\alpha,\lambda)}f(b)\right]\right|\\\leq&\frac{(b-a)^{\alpha+1}}{2\Upsilon_{\lambda}(\alpha,b-a)}\left(\Omega_1(\alpha,\lambda)+\Omega_2(\alpha,\lambda)+\Omega_3(\alpha,\lambda)+\Omega_4(\alpha,\lambda)\right)\left[|f'(a)|+|f'(b)|\right].\end{aligned}$$

Here,

$$\begin{cases}\Omega_1(\alpha,\lambda)=\int_0^{\frac{1}{6}}\left|\Upsilon_{\lambda(b-a)}(\alpha,t)\right|dt,\\ \Omega_2(\alpha,\lambda)=\int_{\frac{1}{6}}^{\frac{1}{2}}\left|\Upsilon_{\lambda(b-a)}(\alpha,t)-\frac{27}{80}\Upsilon_{\lambda(b-a)}(\alpha,1)\right|dt,\\ \Omega_3(\alpha,\lambda)=\int_{\frac{1}{2}}^{\frac{5}{6}}\left|\Upsilon_{\lambda(b-a)}(\alpha,t)-\frac{53}{80}\Upsilon_{\lambda(b-a)}(\alpha,1)\right|dt,\\ \Omega_4(\alpha,\lambda)=\int_{\frac{5}{6}}^{1}\left|\Upsilon_{\lambda(b-a)}(\alpha,t)-\Upsilon_{\lambda(b-a)}(\alpha,1)\right|dt.\end{cases}$$

Proof. Let us take modulus in Lemma 2.1. Then, we have

$$\begin{aligned}&\left|\frac{1}{80}\left[27f\left(\frac{5a+b}{6}\right)+26f\left(\frac{a+b}{2}\right)+27f\left(\frac{a+5b}{6}\right)\right]\right.\\&\left.-\frac{\Gamma(\alpha)}{2\Upsilon_{\lambda}(\alpha,b-a)}\left[\mathcal{J}_{b-}^{(\alpha,\lambda)}f(a)+\mathcal{J}_{a+}^{(\alpha,\lambda)}f(b)\right]\right|\\\leq&\frac{(b-a)^{\alpha+1}}{2\Upsilon_{\lambda}(\alpha,b-a)}\left[\int_0^{\frac{1}{6}}\left|\Upsilon_{\lambda(b-a)}(\alpha,t)\right|\left|f'(tb+(1-t)a)-f'(ta+(1-t)b)\right|dt\right.\\&+\int_{\frac{1}{6}}^{\frac{1}{2}}\left|\Upsilon_{\lambda(b-a)}(\alpha,t)-\frac{27}{80}\Upsilon_{\lambda(b-a)}(\alpha,1)\right|\left|f'(tb+(1-t)a)-f'(ta+(1-t)b)\right|dt\\&+\int_{\frac{1}{2}}^{\frac{5}{6}}\left|\Upsilon_{\lambda(b-a)}(\alpha,t)-\frac{53}{80}\Upsilon_{\lambda(b-a)}(\alpha,1)\right|\left|f'(tb+(1-t)a)-f'(ta+(1-t)b)\right|dt\\&\left.+\int_{\frac{5}{6}}^{1}\left|\Upsilon_{\lambda(b-a)}(\alpha,t)-\Upsilon_{\lambda(b-a)}(\alpha,1)\right|\left|f'(tb+(1-t)a)-f'(ta+(1-t)b)\right|dt\right].\end{aligned} \tag{2.8}$$

It is known that $|f'|$ is convex. Then, it yields

$$\begin{aligned}
&\left|\frac{1}{80}\left[27f\left(\frac{5a+b}{6}\right)+26f\left(\frac{a+b}{2}\right)+27f\left(\frac{a+5b}{6}\right)\right]\right. \\
&\left.-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)\right]\right| \\
\leq&\frac{(b-a)^{\alpha+1}}{2\,\Upsilon_{\lambda}(\alpha,b-a)} \\
&\times\left[\int_{0}^{\frac{1}{6}}\left|\Upsilon_{\lambda(b-a)}(\alpha,t)\right|\left[t\left|f'(b)\right|+(1-t)\left|f'(a)\right|+t\left|f'(a)\right|+(1-t)\left|f'(b)\right|\right]dt\right. \\
&+\int_{\frac{1}{6}}^{\frac{1}{2}}\left|\Upsilon_{\lambda(b-a)}(\alpha,t)-\frac{27}{80}\Upsilon_{\lambda(b-a)}(\alpha,1)\right| \\
&\times\left[t\left|f'(b)\right|+(1-t)\left|f'(a)\right|+t\left|f'(a)\right|+(1-t)\left|f'(b)\right|\right]dt \\
&+\int_{\frac{1}{2}}^{\frac{5}{6}}\left|\Upsilon_{\lambda(b-a)}(\alpha,t)-\frac{53}{80}\Upsilon_{\lambda(b-a)}(\alpha,1)\right| \\
&\times\left[t\left|f'(b)\right|+(1-t)\left|f'(a)\right|+t\left|f'(a)\right|+(1-t)\left|f'(b)\right|\right]dt \\
&+\int_{\frac{5}{6}}^{1}\left|\Upsilon_{\lambda(b-a)}(\alpha,t)-\Upsilon_{\lambda(b-a)}(\alpha,1)\right| \\
&\left.\times\left[t\left|f'(b)\right|+(1-t)\left|f'(a)\right|+t\left|f'(a)\right|+(1-t)\left|f'(b)\right|\right]dt\right] \\
=&\frac{(b-a)^{\alpha+1}}{2\,\Upsilon_{\lambda}(\alpha,b-a)}\left(\Omega_1(\alpha)+\Omega_2(\alpha)+\Omega_3(\alpha)+\Omega_4(\alpha)\right)\left[\left|f'(a)\right|+\left|f'(b)\right|\right].
\end{aligned}$$

This finishes the proof of Theorem 2.1. □

Remark 2.1. Let us consider $\lambda=0$ in Theorem 2.1. Then, the following corrected Euler-Maclaurin-type inequality holds:

$$\begin{aligned}
&\left|\frac{1}{80}\left[27f\left(\frac{5a+b}{6}\right)+26f\left(\frac{a+b}{2}\right)+27f\left(\frac{a+5b}{6}\right)\right]\right. \\
&\left.-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{b-}^{\alpha}f(a)+J_{a+}^{\alpha}f(b)\right]\right| \\
\leq&\frac{\alpha(b-a)}{2}\left(\Omega_1(\alpha,0)+\Omega_2(\alpha,0)+\Omega_3(\alpha,0)+\Omega_4(\alpha,0)\right)\left[\left|f'(a)\right|+\left|f'(b)\right|\right],
\end{aligned}$$

which is given in paper [21].

Remark 2.2. If we assign $\lambda=0$ and $\alpha=1$ in Theorem 2.1, then we get the following corrected Euler-Maclaurin-type inequality

$$\left|\frac{1}{80}\left[27f\left(\frac{5a+b}{6}\right)+26f\left(\frac{a+b}{2}\right)+27f\left(\frac{a+5b}{6}\right)\right]-\frac{1}{b-a}\int_{a}^{b}f(t)\,dt\right|$$

$$\leq \frac{2401(b-a)}{5760}\left[\left|f'(a)\right|+\left|f'(b)\right|\right],$$

which is given in paper [21].

Theorem 2.2. *Suppose that the assumptions of Lemma 2.1 hold and the function $|f'|^q$, $q>1$ is convex on $[a,b]$. Then, the following corrected Euler-Maclaurin-type inequality holds:*

$$\begin{aligned}
&\left|\frac{1}{80}\left[27f\left(\frac{5a+b}{6}\right)+26f\left(\frac{a+b}{2}\right)+27f\left(\frac{a+5b}{6}\right)\right]\right.\\
&\left.-\frac{\Gamma(\alpha)}{2\Upsilon_{\lambda}(\alpha,b-a)}\left[\mathcal{J}_{b-}^{(\alpha,\lambda)}f(a)+\mathcal{J}_{a+}^{(\alpha,\lambda)}f(b)\right]\right|\\
\leq&\frac{(b-a)^{\alpha+1}}{2\Upsilon_{\lambda}(\alpha,b-a)}\left\{\left(\varphi_1^p(\alpha,\lambda)+\varphi_4^p(\alpha,\lambda)\right)\right.\\
&\times\left[\left(\frac{11\left|f'(a)\right|^q+\left|f'(b)\right|^q}{72}\right)^{\frac{1}{q}}+\left(\frac{\left|f'(a)\right|^q+11\left|f'(b)\right|^q}{72}\right)^{\frac{1}{q}}\right]\\
&+\left(\varphi_2^p(\alpha,\lambda)+\varphi_3^p(\alpha,\lambda)\right)\\
&\left.\times\left[\left(\frac{2\left|f'(a)\right|^q+\left|f'(b)\right|^q}{9}\right)^{\frac{1}{q}}+\left(\frac{\left|f'(a)\right|^q+2\left|f'(b)\right|^q}{9}\right)^{\frac{1}{q}}\right]\right\},
\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$ *and*

$$\begin{cases}
\varphi_1^p(\alpha,\lambda)=\left(\int\limits_0^{\frac{1}{6}}\left|\Upsilon_{\lambda(b-a)}(\alpha,t)\right|^p dt\right)^{\frac{1}{p}},\\
\varphi_2^p(\alpha,\lambda)=\left(\int\limits_{\frac{1}{6}}^{\frac{1}{2}}\left|\Upsilon_{\lambda(b-a)}(\alpha,t)-\frac{27}{80}\Upsilon_{\lambda(b-a)}(\alpha,1)\right|^p dt\right)^{\frac{1}{p}},\\
\varphi_3^p(\alpha,\lambda)=\left(\int\limits_{\frac{1}{2}}^{\frac{5}{6}}\left|\Upsilon_{\lambda(b-a)}(\alpha,t)-\frac{53}{80}\Upsilon_{\lambda(b-a)}(\alpha,1)\right|^p dt\right)^{\frac{1}{p}},\\
\varphi_4^p(\alpha,\lambda)=\left(\int\limits_{\frac{5}{6}}^{1}\left|\Upsilon_{\lambda(b-a)}(\alpha,t)-\Upsilon_{\lambda(b-a)}(\alpha,1)\right|^p dt\right)^{\frac{1}{p}}.
\end{cases}$$

Proof. If we apply Hölder inequality in (2.8), then it follows

$$\begin{aligned}
&\left|\frac{1}{80}\left[27f\left(\frac{5a+b}{6}\right)+26f\left(\frac{a+b}{2}\right)+27f\left(\frac{a+5b}{6}\right)\right]\right.\\
&\left.-\frac{\Gamma(\alpha)}{2\Upsilon_{\lambda}(\alpha,b-a)}\left[\mathcal{J}_{b-}^{(\alpha,\lambda)}f(a)+\mathcal{J}_{a+}^{(\alpha,\lambda)}f(b)\right]\right|\\
\leq&\frac{(b-a)^{\alpha+1}}{2\Upsilon_{\lambda}(\alpha,b-a)}\left\{\left(\int\limits_0^{\frac{1}{6}}\left|\Upsilon_{\lambda(b-a)}(\alpha,t)\right|^p dt\right)^{\frac{1}{p}}\left[\left(\int\limits_0^{\frac{1}{6}}\left|f'(tb+(1-t)a)\right|^q dt\right)^{\frac{1}{q}}\right.\right.
\end{aligned}$$

$$+\left(\int_{0}^{\frac{1}{6}}\left|f'\left(ta+(1-t)b\right)\right|^{q}dt\right)^{\frac{1}{q}}\Bigg]+\left(\int_{\frac{1}{6}}^{\frac{1}{2}}\left|\Upsilon_{\lambda(b-a)}\left(\alpha,t\right)-\frac{27}{80}\Upsilon_{\lambda(b-a)}\left(\alpha,1\right)\right|^{p}dt\right)^{\frac{1}{p}}$$

$$\times\left[\left(\int_{\frac{1}{6}}^{\frac{1}{2}}\left|f'\left(tb+(1-t)a\right)\right|^{q}dt\right)^{\frac{1}{q}}+\left(\int_{\frac{1}{6}}^{\frac{1}{2}}\left|f'\left(ta+(1-t)b\right)\right|^{q}dt\right)^{\frac{1}{q}}\right]$$

$$+\left(\int_{\frac{1}{2}}^{\frac{5}{6}}\left|\Upsilon_{\lambda(b-a)}\left(\alpha,t\right)-\frac{53}{80}\Upsilon_{\lambda(b-a)}\left(\alpha,1\right)\right|^{p}dt\right)^{\frac{1}{p}}\left[\left(\int_{\frac{1}{2}}^{\frac{5}{6}}\left|f'\left(tb+(1-t)a\right)\right|^{q}dt\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_{\frac{1}{2}}^{\frac{5}{6}}\left|f'\left(ta+(1-t)b\right)\right|^{q}dt\right)^{\frac{1}{q}}\right]+\left(\int_{\frac{5}{6}}^{1}\left|\Upsilon_{\lambda(b-a)}\left(\alpha,t\right)-\Upsilon_{\lambda(b-a)}\left(\alpha,1\right)\right|^{p}dt\right)^{\frac{1}{p}}$$

$$\left.\times\left[\left(\int_{\frac{5}{6}}^{1}\left|f'\left(tb+(1-t)a\right)\right|^{q}dt\right)^{\frac{1}{q}}+\left(\int_{\frac{5}{6}}^{1}\left|f'\left(ta+(1-t)b\right)\right|^{q}dt\right)^{\frac{1}{q}}\right]\right\}.$$

By using convexity of $|f'|^{q}$, we readily get

$$\left|\frac{1}{80}\left[27f\left(\frac{5a+b}{6}\right)+26f\left(\frac{a+b}{2}\right)+27f\left(\frac{a+5b}{6}\right)\right]\right.$$
$$\left.-\frac{\Gamma(\alpha)}{2\,\Upsilon_{\lambda}\left(\alpha,b-a\right)}\left[\mathcal{J}_{b-}^{(\alpha,\lambda)}f(a)+\mathcal{J}_{a+}^{(\alpha,\lambda)}f(b)\right]\right|$$
$$\leq\frac{(b-a)^{\alpha+1}}{2\,\Upsilon_{\lambda}\left(\alpha,b-a\right)}\left\{\left(\int_{0}^{\frac{1}{6}}\left|\Upsilon_{\lambda(b-a)}\left(\alpha,t\right)\right|^{p}dt\right)^{\frac{1}{p}}\right.$$
$$\times\left[\left(\int_{0}^{\frac{1}{6}}t\left|f'(b)\right|^{q}+(1-t)\left|f'(a)\right|^{q}dt\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_{0}^{\frac{1}{6}}t\left|f'(a)\right|^{q}+(1-t)\left|f'(b)\right|^{q}dt\right)^{\frac{1}{q}}\right]$$
$$+\left(\int_{\frac{1}{6}}^{\frac{1}{2}}\left|\Upsilon_{\lambda(b-a)}\left(\alpha,t\right)-\frac{27}{80}\Upsilon_{\lambda(b-a)}\left(\alpha,1\right)\right|^{p}dt\right)^{\frac{1}{p}}$$

$$\times \left[\left(\int_{\frac{1}{6}}^{\frac{1}{2}} t\left|f'(b)\right|^q + (1-t)\left|f'(a)\right|^q dt \right)^{\frac{1}{q}} \right.$$

$$\left. + \left(\int_{\frac{1}{6}}^{\frac{1}{2}} t\left|f'(a)\right|^q + (1-t)\left|f'(b)\right|^q dt \right)^{\frac{1}{q}} \right]$$

$$+ \left(\int_{\frac{1}{2}}^{\frac{5}{6}} \left| \Upsilon_{\lambda(b-a)}(\alpha,t) - \frac{53}{80}\Upsilon_{\lambda(b-a)}(\alpha,1) \right|^p dt \right)^{\frac{1}{p}}$$

$$\times \left[\left(\int_{\frac{1}{2}}^{\frac{5}{6}} t\left|f'(b)\right|^q + (1-t)\left|f'(a)\right|^q dt \right)^{\frac{1}{q}} \right.$$

$$\left. + \left(\int_{\frac{1}{2}}^{\frac{5}{6}} t\left|f'(a)\right|^q + (1-t)\left|f'(b)\right|^q dt \right)^{\frac{1}{q}} \right]$$

$$+ \left(\int_{\frac{5}{6}}^{1} \left| \Upsilon_{\lambda(b-a)}(\alpha,t) - \Upsilon_{\lambda(b-a)}(\alpha,1) \right|^p dt \right)^{\frac{1}{p}}$$

$$\times \left[\left(\int_{\frac{5}{6}}^{1} t\left|f'(b)\right|^q + (1-t)\left|f'(a)\right|^q dt \right)^{\frac{1}{q}} \right.$$

$$\left. \left. + \left(\int_{\frac{5}{6}}^{1} t\left|f'(a)\right|^q + (1-t)\left|f'(b)\right|^q dt \right)^{\frac{1}{q}} \right] \right\}$$

$$= \frac{(b-a)^{\alpha+1}}{2\Upsilon_\lambda(\alpha,b-a)}$$

$$\times \left\{ \left(\left(\int_0^{\frac{1}{6}} \left| \Upsilon_{\lambda(b-a)}(\alpha,t) \right|^p dt \right)^{\frac{1}{p}} + \left(\int_{\frac{5}{6}}^{1} \left| \Upsilon_{\lambda(b-a)}(\alpha,t) - \Upsilon_{\lambda(b-a)}(\alpha,1) \right|^p dt \right)^{\frac{1}{p}} \right) \right.$$

$$\times \left[\left(\frac{11\left|f'(a)\right|^q + \left|f'(b)\right|^q}{72} \right)^{\frac{1}{q}} + \left(\frac{\left|f'(a)\right|^q + 11\left|f'(b)\right|^q}{72} \right)^{\frac{1}{q}} \right]$$

$$+ \left(\left(\int_{\frac{1}{6}}^{\frac{1}{2}} \left| \Upsilon_{\lambda(b-a)}(\alpha,t) - \frac{27}{80}\Upsilon_{\lambda(b-a)}(\alpha,1) \right|^p dt \right)^{\frac{1}{p}} \right.$$

$$+\left(\int_{\frac{1}{2}}^{\frac{5}{6}}\left|\Upsilon_{\lambda(b-a)}(\alpha,t)-\frac{53}{80}\Upsilon_{\lambda(b-a)}(\alpha,1)\right|^{p}dt\right)^{\frac{1}{p}}\Bigg)$$
$$\times\left[\left(\frac{2|f'(a)|^{q}+|f'(b)|^{q}}{9}\right)^{\frac{1}{q}}+\left(\frac{|f'(a)|^{q}+2|f'(b)|^{q}}{9}\right)^{\frac{1}{q}}\right]\Bigg\},$$

which completes the proof of Theorem 2.2. □

Remark 2.3. Let us consider $\lambda=0$ in Theorem 2.2. Then, the following corrected Euler-Maclaurin-type inequality holds:

$$\begin{aligned}&\Bigg|\frac{1}{80}\left[27f\left(\frac{5a+b}{6}\right)+26f\left(\frac{a+b}{2}\right)+27f\left(\frac{a+5b}{6}\right)\right]\\&-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{b-}^{\alpha}f(a)+J_{a+}^{\alpha}f(b)\right]\Bigg|\\&\leq\frac{\alpha(b-a)}{2}\Big\{\left(\varphi_1^{p}(\alpha,0)+\varphi_4^{p}(\alpha,0)\right)\\&\times\left[\left(\frac{11|f'(a)|^{q}+|f'(b)|^{q}}{72}\right)^{\frac{1}{q}}+\left(\frac{|f'(a)|^{q}+11|f'(b)|^{q}}{72}\right)^{\frac{1}{q}}\right]\\&+\left(\varphi_2^{p}(\alpha,0)+\varphi_3^{p}(\alpha,0)\right)\\&\left[\left(\frac{2|f'(a)|^{q}+|f'(b)|^{q}}{9}\right)^{\frac{1}{q}}+\left(\frac{|f'(a)|^{q}+2|f'(b)|^{q}}{9}\right)^{\frac{1}{q}}\right]\Big\},\end{aligned}$$

which is given in paper [21].

Remark 2.4. If we choose $\lambda=0$ and $\alpha=1$ in Theorem 2.2, then we obtain the following corrected Euler-Maclaurin-type inequality

$$\begin{aligned}&\left|\frac{1}{80}\left[27f\left(\frac{5a+b}{6}\right)+26f\left(\frac{a+b}{2}\right)+27f\left(\frac{a+5b}{6}\right)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\\&\leq(b-a)\Bigg[\left(\frac{1}{(p+1)\,6^{p+1}}\right)^{\frac{1}{p}}\\&\times\left[\left(\frac{11|f'(a)|^{q}+|f'(b)|^{q}}{72}\right)^{\frac{1}{q}}+\left(\frac{|f'(a)|^{q}+11|f'(b)|^{q}}{72}\right)^{\frac{1}{q}}\right]\\&+\left(\frac{1}{(p+1)}\left(\left(\frac{41}{240}\right)^{p+1}+\left(\frac{13}{80}\right)^{p+1}\right)\right)^{\frac{1}{p}}\\&\times\left[\left(\frac{2|f'(a)|^{q}+|f'(b)|^{q}}{9}\right)^{\frac{1}{q}}+\left(\frac{|f'(a)|^{q}+2|f'(b)|^{q}}{9}\right)^{\frac{1}{q}}\right]\Bigg],\end{aligned}$$

which is presented in paper [21].

Theorem 2.3. *Assume that the assumptions of Lemma 2.1 are valid. Assume also that the function $|f'|^{q}$, $q\geq 1$ is convex on $[a,b]$. Then, the following corrected*

Euler-Maclaurin-type inequality holds:

$$\begin{aligned}
&\left|\frac{1}{80}\left[27f\left(\frac{5a+b}{6}\right)+26f\left(\frac{a+b}{2}\right)+27f\left(\frac{a+5b}{6}\right)\right]\right. \\
&\left.-\frac{\Gamma(\alpha)}{2\,\Upsilon_{\lambda}(\alpha,b-a)}\left[\mathcal{J}_{b-}^{(\alpha,\lambda)}f(a)+\mathcal{J}_{a+}^{(\alpha,\lambda)}f(b)\right]\right| \\
\leq&\frac{(b-a)^{\alpha+1}}{2\,\Upsilon_{\lambda}(\alpha,b-a)}\Big\{(\Omega_1(\alpha,\lambda))^{1-\frac{1}{q}} \\
&\times\Big[\left(\Omega_5(\alpha,\lambda)\left|f'(b)\right|^q+(\Omega_1(\alpha,\lambda)-\Omega_5(\alpha,\lambda))\left|f'(a)\right|^q\right)^{\frac{1}{q}} \\
&+\left(\Omega_5(\alpha,\lambda)\left|f'(a)\right|^q+(\Omega_1(\alpha,\lambda)-\Omega_5(\alpha,\lambda))\left|f'(b)\right|^q\right)^{\frac{1}{q}}\Big] \\
&+(\Omega_2(\alpha,\lambda))^{1-\frac{1}{q}}\Big[\left(\Omega_6(\alpha,\lambda)\left|f'(b)\right|^q+(\Omega_2(\alpha,\lambda)-\Omega_6(\alpha,\lambda))\left|f'(a)\right|^q\right)^{\frac{1}{q}} \\
&+\left(\Omega_6(\alpha,\lambda)\left|f'(a)\right|^q+(\Omega_2(\alpha,\lambda)-\Omega_6(\alpha,\lambda))\left|f'(b)\right|^q\right)^{\frac{1}{q}}\Big] \\
&+(\Omega_3(\alpha,\lambda))^{1-\frac{1}{q}}\Big[\left(\Omega_7(\alpha,\lambda)\left|f'(b)\right|^q+(\Omega_3(\alpha,\lambda)-\Omega_7(\alpha,\lambda))\left|f'(a)\right|^q\right)^{\frac{1}{q}} \\
&+\left(\Omega_7(\alpha,\lambda)\left|f'(a)\right|^q+(\Omega_3(\alpha,\lambda)-\Omega_7(\alpha,\lambda))\left|f'(b)\right|^q\right)^{\frac{1}{q}}\Big] \\
&+(\Omega_4(\alpha,\lambda))^{1-\frac{1}{q}}\Big[\left(\Omega_8(\alpha,\lambda)\left|f'(b)\right|^q+(\Omega_4(\alpha,\lambda)-\Omega_8(\alpha,\lambda))\left|f'(a)\right|^q\right)^{\frac{1}{q}} \\
&+\left(\Omega_8(\alpha,\lambda)\left|f'(a)\right|^q+(\Omega_4(\alpha,\lambda)-\Omega_8(\alpha,\lambda))\left|f'(b)\right|^q\right)^{\frac{1}{q}}\Big]\Big\}.
\end{aligned}$$

Here, $\Omega_1(\alpha,\lambda)$, $\Omega_2(\alpha,\lambda)$, $\Omega_3(\alpha,\lambda)$ *and* $\Omega_4(\alpha,\lambda)$ *are defined in Theorem 2.1 and*

$$\begin{cases}
\Omega_5(\alpha,\lambda)=\int\limits_0^{\frac{1}{6}}\left|\Upsilon_{\lambda(b-a)}(\alpha,t)\right|tdt, \\
\Omega_6(\alpha,\lambda)=\int\limits_{\frac{1}{6}}^{\frac{1}{2}}\left|\Upsilon_{\lambda(b-a)}(\alpha,t)-\frac{27}{80}\,\Upsilon_{\lambda(b-a)}(\alpha,1)\right|tdt, \\
\Omega_7(\alpha,\lambda)=\int\limits_{\frac{1}{2}}^{\frac{5}{6}}\left|\Upsilon_{\lambda(b-a)}(\alpha,t)-\frac{53}{80}\,\Upsilon_{\lambda(b-a)}(\alpha,1)\right|tdt, \\
\Omega_8(\alpha,\lambda)=\int\limits_{\frac{5}{6}}^{1}\left|\Upsilon_{\lambda(b-a)}(\alpha,t)-\Upsilon_{\lambda(b-a)}(\alpha,1)\right|tdt.
\end{cases}$$

Proof. Let us first apply power-mean inequality in (2.8). Then, we get

$$\begin{aligned}
&\left|\frac{1}{80}\left[27f\left(\frac{5a+b}{6}\right)+26f\left(\frac{a+b}{2}\right)+27f\left(\frac{a+5b}{6}\right)\right]\right. \\
&\left.-\frac{\Gamma(\alpha)}{2\,\Upsilon_{\lambda}(\alpha,b-a)}\left[\mathcal{J}_{b-}^{(\alpha,\lambda)}f(a)+\mathcal{J}_{a+}^{(\alpha,\lambda)}f(b)\right]\right| \\
\leq&\frac{(b-a)^{\alpha+1}}{2\,\Upsilon_{\lambda}(\alpha,b-a)}\left\{\left(\int\limits_0^{\frac{1}{6}}\left|\Upsilon_{\lambda(b-a)}(\alpha,t)\right|dt\right)^{1-\frac{1}{q}}\right.
\end{aligned}$$

$$\times\left(\int_{0}^{\frac{1}{6}}\left|\Upsilon_{\lambda(b-a)}(\alpha,t)\right|\left|f'(tb+(1-t)a)\right|^{q}dt\right)^{\frac{1}{q}}$$

$$+\left(\int_{0}^{\frac{1}{6}}\left|\Upsilon_{\lambda(b-a)}(\alpha,t)\right|dt\right)^{1-\frac{1}{q}}\left(\int_{0}^{\frac{1}{6}}\left|\Upsilon_{\lambda(b-a)}(\alpha,t)\right|\left|f'(ta+(1-t)b)\right|^{q}dt\right)^{\frac{1}{q}}$$

$$+\left(\int_{\frac{1}{6}}^{\frac{1}{2}}\left|\Upsilon_{\lambda(b-a)}(\alpha,t)-\frac{27}{80}\Upsilon_{\lambda(b-a)}(\alpha,1)\right|dt\right)^{1-\frac{1}{q}}$$

$$\times\left(\int_{\frac{1}{6}}^{\frac{1}{2}}\left|\Upsilon_{\lambda(b-a)}(\alpha,t)-\frac{27}{80}\Upsilon_{\lambda(b-a)}(\alpha,1)\right|\left|f'(tb+(1-t)a)\right|^{q}dt\right)^{\frac{1}{q}}$$

$$+\left(\int_{\frac{1}{6}}^{\frac{1}{2}}\left|\Upsilon_{\lambda(b-a)}(\alpha,t)-\frac{27}{80}\Upsilon_{\lambda(b-a)}(\alpha,1)\right|dt\right)^{1-\frac{1}{q}}$$

$$\times\left(\int_{\frac{1}{6}}^{\frac{1}{2}}\left|\Upsilon_{\lambda(b-a)}(\alpha,t)-\frac{27}{80}\Upsilon_{\lambda(b-a)}(\alpha,1)\right|\left|f'(ta+(1-t)b)\right|^{q}dt\right)^{\frac{1}{q}}$$

$$+\left(\int_{\frac{1}{2}}^{\frac{5}{6}}\left|\Upsilon_{\lambda(b-a)}(\alpha,t)-\frac{53}{80}\Upsilon_{\lambda(b-a)}(\alpha,1)\right|dt\right)^{1-\frac{1}{q}}$$

$$\times\left(\int_{\frac{1}{2}}^{\frac{5}{6}}\left|\Upsilon_{\lambda(b-a)}(\alpha,t)-\frac{53}{80}\Upsilon_{\lambda(b-a)}(\alpha,1)\right|\left|f'(tb+(1-t)a)\right|^{q}dt\right)^{\frac{1}{q}}$$

$$+\left(\int_{\frac{1}{2}}^{\frac{5}{6}}\left|\Upsilon_{\lambda(b-a)}(\alpha,t)-\frac{53}{80}\Upsilon_{\lambda(b-a)}(\alpha,1)\right|dt\right)^{1-\frac{1}{q}}$$

$$\times\left(\int_{\frac{1}{2}}^{\frac{5}{6}}\left|\Upsilon_{\lambda(b-a)}(\alpha,t)-\frac{53}{80}\Upsilon_{\lambda(b-a)}(\alpha,1)\right|\left|f'(ta+(1-t)b)\right|^{q}dt\right)^{\frac{1}{q}}$$

$$+\left(\int_{\frac{5}{6}}^{1}\left|\Upsilon_{\lambda(b-a)}(\alpha,t)-\Upsilon_{\lambda(b-a)}(\alpha,1)\right|dt\right)^{1-\frac{1}{q}}$$

$$\times\left(\int_{\frac{5}{6}}^{1}\left|\Upsilon_{\lambda(b-a)}(\alpha,t)-\Upsilon_{\lambda(b-a)}(\alpha,1)\right|\left|f'(tb+(1-t)a)\right|^{q}dt\right)^{\frac{1}{q}}$$

$$+\left(\int_{\frac{5}{6}}^{1}\left|\Upsilon_{\lambda(b-a)}(\alpha,t)-\Upsilon_{\lambda(b-a)}(\alpha,1)\right|dt\right)^{1-\frac{1}{q}}$$

$$\left.\times\left(\int_{\frac{5}{6}}^{1}\left|\Upsilon_{\lambda(b-a)}(\alpha,t)-\Upsilon_{\lambda(b-a)}(\alpha,1)\right|\left|f'(ta+(1-t)b)\right|^{q}dt\right)^{\frac{1}{q}}\right\}.$$

Using the fact that $|f'|^q$ is convex, it follows

$$\begin{aligned}&\left|\frac{1}{80}\left[27f\left(\frac{5a+b}{6}\right)+26f\left(\frac{a+b}{2}\right)+27f\left(\frac{a+5b}{6}\right)\right]\right.\\&\left.-\frac{\Gamma(\alpha)}{2\,\Upsilon_{\lambda}(\alpha,b-a)}\left[\mathcal{J}_{b-}^{(\alpha,\lambda)}f(a)+\mathcal{J}_{a+}^{(\alpha,\lambda)}f(b)\right]\right|\\&\leq\frac{(b-a)^{\alpha+1}}{2\,\Upsilon_{\lambda}(\alpha,b-a)}\left\{\left(\int_{0}^{\frac{1}{6}}\left|\Upsilon_{\lambda(b-a)}(\alpha,t)\right|dt\right)^{1-\frac{1}{q}}\right.\\&\times\left[\left(\int_{0}^{\frac{1}{6}}\left|\Upsilon_{\lambda(b-a)}(\alpha,t)\right|\left[t\left|f'(b)\right|^{q}+(1-t)\left|f'(a)\right|^{q}\right]dt\right)^{\frac{1}{q}}\right.\\&\left.+\left(\int_{0}^{\frac{1}{6}}\left|\Upsilon_{\lambda(b-a)}(\alpha,t)\right|\left[t\left|f'(a)\right|^{q}+(1-t)\left|f'(b)\right|^{q}\right]dt\right)^{\frac{1}{q}}\right]\\&+\left(\int_{\frac{1}{6}}^{\frac{1}{2}}\left|\Upsilon_{\lambda(b-a)}(\alpha,t)-\frac{27}{80}\Upsilon_{\lambda(b-a)}(\alpha,1)\right|dt\right)^{1-\frac{1}{q}}\\&\times\left[\left(\int_{\frac{1}{6}}^{\frac{1}{2}}\left|\Upsilon_{\lambda(b-a)}(\alpha,t)-\frac{27}{80}\Upsilon_{\lambda(b-a)}(\alpha,1)\right|\left[t\left|f'(b)\right|^{q}+(1-t)\left|f'(a)\right|^{q}\right]dt\right)^{\frac{1}{q}}\right.\\&\left.+\left(\int_{\frac{1}{6}}^{\frac{1}{2}}\left|\Upsilon_{\lambda(b-a)}(\alpha,t)-\frac{27}{80}\Upsilon_{\lambda(b-a)}(\alpha,1)\right|\left[t\left|f'(a)\right|^{q}+(1-t)\left|f'(b)\right|^{q}\right]dt\right)^{\frac{1}{q}}\right]\\&+\left(\int_{\frac{1}{2}}^{\frac{5}{6}}\left|\Upsilon_{\lambda(b-a)}(\alpha,t)-\frac{53}{80}\Upsilon_{\lambda(b-a)}(\alpha,1)\right|dt\right)^{1-\frac{1}{q}}\end{aligned}$$

$$\times\left[\left(\int_{\frac{1}{2}}^{\frac{5}{6}}\left|\Upsilon_{\lambda(b-a)}(\alpha,t)-\frac{53}{80}\Upsilon_{\lambda(b-a)}(\alpha,1)\right|\left[t\left|f'(b)\right|^{q}+(1-t)\left|f'(a)\right|^{q}\right]dt\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_{\frac{1}{2}}^{\frac{5}{6}}\left|\Upsilon_{\lambda(b-a)}(\alpha,t)-\frac{53}{80}\Upsilon_{\lambda(b-a)}(\alpha,1)\right|\left[t\left|f'(a)\right|^{q}+(1-t)\left|f'(b)\right|^{q}\right]dt\right)^{\frac{1}{q}}\right]$$

$$+\left(\int_{\frac{5}{6}}^{1}\left|\Upsilon_{\lambda(b-a)}(\alpha,t)-\Upsilon_{\lambda(b-a)}(\alpha,1)\right|dt\right)^{1-\frac{1}{q}}$$

$$\times\left[\left(\int_{\frac{5}{6}}^{1}\left|\Upsilon_{\lambda(b-a)}(\alpha,t)-\Upsilon_{\lambda(b-a)}(\alpha,1)\right|\left[t\left|f'(b)\right|^{q}+(1-t)\left|f'(a)\right|^{q}\right]dt\right)^{\frac{1}{q}}\right.$$

$$\left.\left.+\left(\int_{\frac{5}{6}}^{1}\left|\Upsilon_{\lambda(b-a)}(\alpha,t)-\Upsilon_{\lambda(b-a)}(\alpha,1)\right|\left[t\left|f'(a)\right|^{q}+(1-t)\left|f'(b)\right|^{q}\right]dt\right)^{\frac{1}{q}}\right]\right\}.$$

Finally, we obtain the desired result of Theorem 2.3. □

Remark 2.5. Consider $\lambda = 0$ in Theorem 2.3. Then, the following corrected Euler-Maclaurin-type inequality holds:

$$\left|\frac{1}{80}\left[27f\left(\frac{5a+b}{6}\right)+26f\left(\frac{a+b}{2}\right)+27f\left(\frac{a+5b}{6}\right)\right]\right.$$
$$\left.-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{b-}^{\alpha}f(a)+J_{a+}^{\alpha}f(b)\right]\right|$$
$$\leq\frac{\alpha(b-a)}{2}\left\{\left(\Omega_{1}(\alpha,0)\right)^{1-\frac{1}{q}}\right.$$
$$\times\left[\left(\Omega_{5}(\alpha,0)\left|f'(b)\right|^{q}+\left(\Omega_{1}(\alpha,0)-\Omega_{5}(\alpha,0)\right)\left|f'(a)\right|^{q}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\Omega_{5}(\alpha,0)\left|f'(a)\right|^{q}+\left(\Omega_{1}(\alpha,0)-\Omega_{5}(\alpha,0)\right)\left|f'(b)\right|^{q}\right)^{\frac{1}{q}}\right]$$
$$+\left(\Omega_{2}(\alpha,0)\right)^{1-\frac{1}{q}}\left[\left(\Omega_{6}(\alpha,0)\left|f'(b)\right|^{q}+\left(\Omega_{2}(\alpha,0)-\Omega_{6}(\alpha,0)\right)\left|f'(a)\right|^{q}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\Omega_{6}(\alpha,0)\left|f'(a)\right|^{q}+\left(\Omega_{2}(\alpha,0)-\Omega_{6}(\alpha,0)\right)\left|f'(b)\right|^{q}\right)^{\frac{1}{q}}\right]$$
$$+\left(\Omega_{3}(\alpha,0)\right)^{1-\frac{1}{q}}\left[\left(\Omega_{7}(\alpha,0)\left|f'(b)\right|^{q}+\left(\Omega_{3}(\alpha,0)-\Omega_{7}(\alpha,0)\right)\left|f'(a)\right|^{q}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\Omega_{7}(\alpha,0)\left|f'(a)\right|^{q}+\left(\Omega_{3}(\alpha,0)-\Omega_{7}(\alpha,0)\right)\left|f'(b)\right|^{q}\right)^{\frac{1}{q}}\right]$$
$$+\left(\Omega_{4}(\alpha,0)\right)^{1-\frac{1}{q}}\left[\left(\Omega_{8}(\alpha,0)\left|f'(b)\right|^{q}+\left(\Omega_{4}(\alpha,0)-\Omega_{8}(\alpha,0)\right)\left|f'(a)\right|^{q}\right)^{\frac{1}{q}}\right.$$
$$\left.\left.+\left(\Omega_{8}(\alpha,0)\left|f'(a)\right|^{q}+\left(\Omega_{4}(\alpha,0)-\Omega_{8}(\alpha,0)\right)\left|f'(b)\right|^{q}\right)^{\frac{1}{q}}\right]\right\},$$

which is presented in paper [21].

Remark 2.6. Let us consider $\lambda = 0$ and $\alpha = 1$ in Theorem 2.3. Then, the following corrected Euler-Maclaurin-type inequality holds:

$$\begin{aligned}
&\left|\frac{1}{80}\left[27f\left(\frac{5a+b}{6}\right)+26f\left(\frac{a+b}{2}\right)+27f\left(\frac{a+5b}{6}\right)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \\
\leq &\frac{(b-a)}{72}\left[\left(\frac{|f'(b)|^q+8|f'(a)|^q}{9}\right)^{\frac{1}{q}}+\left(\frac{|f'(a)|^q+8|f'(b)|^q}{9}\right)^{\frac{1}{q}}\right. \\
&+\left(\frac{1601}{800}\right)^{1-\frac{1}{q}}\left[\left(\frac{379441|f'(b)|^q+773279|f'(a)|^q}{576000}\right)^{\frac{1}{q}}\right. \\
&\left.\left.+\left(\frac{379441|f'(a)|^q+773279|f'(b)|^q}{576000}\right)^{\frac{1}{q}}\right]\right],
\end{aligned}$$

which is established in paper [21].

3. Summary & concluding remarks

Several new versions of corrected Euler-Maclaurin-type inequalities are presented for the case of differentiable convex functions by using tempered fractional integrals. More precisely, corrected Euler-Maclaurin-type inequalities are obtained by taking advantage of the convexity, the Hölder inequality, and the power mean inequality. Furthermore, previous and new results are presented by using special cases of the obtained theorems.

In future works, the ideas and strategies for our results related to corrected Euler-Maclaurin-type inequalities using tempered fractional integrals may open new ways for mathematicians in this field. In addition to this, one can try to generalize our results by utilizing a different version of convex function classes or other types fractional integral operators. Moreover, one can obtain likewise corrected Euler-Maclaurin-type inequalities using tempered fractional integrals for convex functions by using quantum calculus.

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