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The Influence of Environmental Variability and Media Coverage on the Dynamics of an Epidemic Model*

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Abstract Various infectious diseases seriously affect human health and social economy. It is a never-ending battle that human beings fight against infectious diseases. Media coverage has been an important weapon in virus war and has contributed to the epidemic prevention. In this paper, we focus on the dynamics of a stochastic SIRS epidemic model with media coverage and two mean-reverting Ornstein-Uhlenbeck processes. Firstly, we present the existence and uniqueness of the solution. Then, the sufficient condition for disease extinction is provided. In order to get the condition for disease persistence, we verify the existence of stationary distribution by constructing appropriate Lyapunov functions. Moreover, it is theoretically proved that the solution follows a normal probability density function around the endemic equilibrium of corresponding deterministic model. Finally, some numerical simulations are carried out to confirm theoretical results.

Keywords Media coverage, Ornstein-Uhlenbeck process, stationary distribution, density function

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1. Introduction

Infectious diseases seem to have become an unavoidable problem for mankind. The history of the development of human society is also a history of constant struggle against infectious diseases. In recent years, the outbreak of Ebola virus, Middle East respiratory syndrome (MERS), Corona-virus diseases 2019 (COVID-19) and other infectious diseases led to serious damage to human health worldwide. Several epidemics have brought great panic to the whole world, and even caused catastrophic consequences in many regions, such as economic recession and social shutdown. Therefore, a full understanding of transmission rules and control strategies of the

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disease is urgently needed. Mathematical modeling provides a reliable theoretical analysis for studying the pathogenesis of infectious diseases and predicting the development trend. It has become an effective tool to solve various phenomena and problems caused by the infectious disease.

For the epidemic dynamics model, Kermack and Mckendrick [1] proposed a landmark SIR epidemic model, which makes the epidemic model enter the era of quantitative analysis. After that, many authors [2–8] assumed that the total population $N(t)$ is divided into three categories at time t, including susceptible individuals $S(t)$, infected individuals $I(t)$ and recovered individuals $R(t)$. Recovered individuals in the SIR epidemic model are permanent immunity. But in fact, for most infectious diseases, such as cholera, influenza and malaria, acquired immunity wears off over time, which is a phenomenon well described by the SIRS infectious disease model. Ma et al. [2] considered that acquired immunity may disappear after a period of time and proposed a standard SIRS model:

$$\begin{cases} \dot{S}(t) = \mu - \bar{\beta}S(t)I(t) - \mu S(t) + \gamma R(t), \\ \dot{I}(t) = \bar{\beta}S(t)I(t) - (\lambda + \mu)I(t), \\ \dot{R}(t) = \lambda I(t) - (\mu + \gamma)R(t), \end{cases} \tag{1.1}$$

where μ is the natural birth and death rate coefficient. $\bar{\beta}$ denotes the average incidence rate. γ represents the immunity loss rate of recovered individuals. λ is the recovery rate of infected individuals. All the parameters are considered to be positive. Up to now, various versions of SIRS models have been studied and many research results have been achieved [6–11].

It is now widely noted that the mass media (television, Internet, microblog, Tik Tok, billboards and wechat) plays a key role in the spread of the disease [12–14]. For the public, people can keep abreast of the transmission route and the epidemic data through the media at any time. From the government's point of view, they can make full use of the powerful force of the media to publish essential disease prevention measures as soon as possible and broadcast the latest public health policies. The information reported by the media can affect the change of people's social behavior. People may go out less, receive vaccinations, self-isolation and wear masks, which will indirectly reduce the number of infected individuals or incidence rate among the population.

Research has found that the epidemic models with nonlinear incidence have more accurate and complex dynamics than those with bilinear incidence ($\bar{\beta}SI$) or standard incidence ($\bar{\beta}SI/N$) [14, 15]. The contact rates commonly used, bilinear and standard, cannot describe the impact of media coverage on the dynamics of the infectious disease well. Cai et al. [16] and Tchuenche et al. [17] held that the incidence rate in a model considering media coverage should be a monotonically decreasing nonlinear function with respect to $I(t)$. Li et al. [18] and Tchuenche et al. [19] constructed a modified nonlinear incidence rate $\beta(I)$ with media coverage:

$$\beta(I(t)) = \bar{\beta} - \frac{\bar{\beta}_e I(t)}{a + I(t)},$$

where $\bar{\beta}_e$ is the maximum reduced contact rate due to the presence of infected individuals. a is the half-saturation constant which reflects the impact of media

coverage on the contact transmission. $\frac{\bar{\beta}_e I}{a+I}$ is used to measure the effect of reduction of the transmission rate when infected individuals are reported by the media. When $I \to +\infty$, $\beta(I(t))$ approaches $\bar{\beta} - \bar{\beta}_e$. If $I = 0$, $\beta(I(t)) = \bar{\beta}$. It is important to recognize that media coverage cannot totally prevent the spread of the disease and the incidence rate is non-negative. Assume $\bar{\beta} \geq \bar{\beta}_e$. A SIRS epidemic model with media coverage has the following form [12]:

$$\begin{cases} \dot{S}(t) = \mu - \left(\bar{\beta} - \dfrac{\bar{\beta}_e I(t)}{a+I(t)}\right) S(t)I(t) - \mu S(t) + \gamma R(t), \\ \dot{I}(t) = \left(\bar{\beta} - \dfrac{\bar{\beta}_e I(t)}{a+I(t)}\right) S(t)I(t) - (\lambda+\mu)I(t), \\ \dot{R}(t) = \lambda I(t) - (\mu+\gamma)R(t), \end{cases} \tag{1.2}$$

where parameters have the same definitions as model (1.1) expect for a and $\bar{\beta}_e$. The basic reproduction number of system (1.2) is $R_0 = \frac{\bar{\beta}}{\lambda+\mu}$. If $R_0 < 1$, there is a disease-free equilibrium $E_0(1,0,0)$, which is locally asymptotically stable. While, if $R_0 > 1$, E_0 becomes unstable and there exists an endemic equilibrium $E^* = \left(\frac{(a+I^*)(\lambda+\mu)}{a\bar{\beta}+(\bar{\beta}-\bar{\beta}_e)I^*}, I^*, \frac{\lambda I^*}{\mu+\gamma}\right)$, which is locally asymptotically stable. I^* is the positive root of the following equation:

$$\begin{aligned} &\left[\left(\frac{\gamma\lambda}{\mu+\gamma} - \lambda - \mu\right)(\bar{\beta}-\bar{\beta}_e)\right] I^2 \\ &+ \left[\mu(\bar{\beta}-\bar{\beta}_e) - (\lambda+\mu)(a\bar{\beta}+\mu) + \frac{a\bar{\beta}\gamma\lambda}{\mu+\gamma}\right] I + a\mu(\lambda+\mu)(R_0-1) = 0. \end{aligned}$$

However, it is now well known that the real world is stochastic. Environmental random variation has a significance impact on the spread of an epidemic [20, 21]. A deterministic model with fixed parameter values has certain limitations because it does not take random factor into account. May [22] argued that parameter values in the model are inevitably disturbed by environmental noise. Therefore, it is valuable and meaningful to study stochastic SIRS epidemic model [23–25]. At present, there are two common approaches to incorporate environmental variability by modifying the parameters in epidemic models: linear function of Gaussian white noise [7, 8, 13, 16, 23–28] and the mean-reverting Ornstein-Uhlenbeck process [29–32]. A few authors [31, 32] pointed out that the first way to involve environmental variability is problematic because the variance of the parameter tends to infinity as time shortens, which implies that the mean value of the parameter becomes more and more volatile as the time interval decreases. While, in the second way, the variance of the parameter will approach zero. This means that the second way is more consistent with the actual biological situation. Allen [33] compared these two approaches to modify parameters from different aspects and found that the mean-reverting Ornstein-Uhlenbeck process has the advantages of continuity, strong practicability, non-negativity, asymptotic property and ease of modifying the parameters for environmental data. Zhou et al. [32] devoted to a stochastic SIR epidemic model considering media coverage and Ornstein-Uhlenbeck process. In this paper, on the basis of the existing model (1.2), we will formulate a SIRS model with media coverage and two Ornstein-Uhlenbeck processes. For system (1.2), we assume

that $\beta(t)$ and $\beta_e(t)$ are affected by the following Ornstein-Uhlenbeck processes:

$$d\beta(t) = \theta_1(\bar{\beta} - \beta(t))dt + \sigma_1 dB_1(t), \quad d\beta_e(t) = \theta_2(\bar{\beta}_e - \beta_e(t))dt + \sigma_2 dB_2(t), \tag{1.3}$$

where $\theta_j > 0$ is the speed of reversion and $\sigma_j > 0$ is the intensity of volatility $(j = 1, 2)$. $B_1(t)$ and $B_2(t)$ are two independent Brownian motions defined on a complete probability space $\{\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P}\}$, where $\{\mathscr{F}_t\}_{t\geq 0}$ is a $\sigma-$ filtration satisfying the usual conditions (i.e. it is right continuous and $\mathscr{F}_0$ contains all $\mathbb{P}$-null sets). Dixit and Pindyck [34] were the first to propose this form of mean-reverting process in financial economics.

By adding both ends of the three equations of the model (1.2), we obtain

$$d(S + I + R) = [\mu - \mu(S + I + R)]dt.$$

It can easily be seen that the size of population is constant. In other words, there is an invariant set Γ for any $t \geq 0$ and the Γ satisfies

$$\Gamma = \{(S, I, R) \in \mathbb{R}^3_+ : S + I + R = 1\}.$$

Hence, we only need to study the dynamics of the following epidemic model for I and R:

$$\begin{cases} dI(t) = \left[\left(\bar{\beta} - \dfrac{\bar{\beta}_e I(t)}{a + I(t)}\right)(1 - I(t) - R(t))I(t) - (\lambda + \mu)I(t)\right]dt, \\ dR(t) = [\lambda I(t) - (\mu + \gamma)R(t)]dt, \end{cases} \tag{1.4}$$

where

$$(I, R) \in \Gamma_1 := \{(I, R) \in \mathbb{R}^2_+ : I + R < 1\}.$$

Considering the Ornstein-Uhlenbeck process, system (1.4) with random variable ($\bar{\beta}$, $\bar{\beta}_e$) can be expressed as

$$\begin{cases} dI(t) = \left[\left(\beta(t) - \dfrac{\beta_e(t) I(t)}{a + I(t)}\right)(1 - I(t) - R(t))I(t) - (\lambda + \mu)I(t)\right]dt, \\ dR(t) = [\lambda I(t) - (\mu + \gamma)R(t)]dt, \\ d\beta(t) = \theta_1(\bar{\beta} - \beta(t))dt + \sigma_1 dB_1(t), \\ d\beta_e(t) = \theta_2(\bar{\beta}_e - \beta_e(t))dt + \sigma_2 dB_2(t). \end{cases} \tag{1.5}$$

For the sake of simplicity, letting $r_1(t) = \beta(t) - \bar{\beta}, r_2(t) = \beta_e(t) - \bar{\beta}_e$, system (1.5) can be transformed into the following form:

$$\begin{cases} dI(t) = \left[\left(\left(\bar{\beta} - \dfrac{\bar{\beta}_e I(t)}{a + I(t)}\right) + \left(r_1 - \dfrac{r_2 I(t)}{a + I(t)}\right)\right)(1 - I(t) - R(t))I(t) \right. \\ \qquad \left. - (\lambda + \mu)I(t)\right]dt, \\ dR(t) = [\lambda I(t) - (\mu + \gamma)R(t)]dt, \\ dr_1(t) = -\theta_1 r_1 dt + \sigma_1 dB_1(t), \\ dr_2(t) = -\theta_2 r_2 dt + \sigma_2 dB_2(t), \end{cases} \tag{1.6}$$

where

$$\Gamma_* := \{(I, R, r_1, r_2) \in \mathbb{R}^2_+ \times \mathbb{R}^2 : I + R < 1\}.$$

For simplicity, there are some mathematical notations. $\mathbb{R}^n_+ = \{(x_1 \cdots x_n) \in \mathbb{R}^n | x_i > 0,\ 1 \le i \le n\}$. $\mathbf{I}_n$ denotes the n-dimensional unit matrix and $\mathbf{I}_{\mathbb{X}}$ is the indicator function of the set $\mathbb{X}$. $\|\cdot\|$ is the Euclidean norm. If A is a matrix, A^T and A^{-1} represent its transpose and inverse matrix, respectively. Denote $v_1 \vee v_2 = \max\{v_1, v_2\}$.

The aim of this paper is to probe into the long-time behavior of stochastic system (1.6). The structure of the paper is arranged as follows. In section 2, we prove that system (1.6) exists a unique solution. Sufficient condition for the extinction of the disease is obtained in section 3. Section 4 gives the condition for the existence of stationary distribution. The concrete expression of density function of the stationary distribution is derived in section 5. In section 6, we use some examples and numerical simulations to confirm our theoretical results. Lastly, the brief conclusion is given.

2. The existence and uniqueness of the solution

In this section, we prove the existence and uniqueness of the solution to (1.6).

Theorem 2.1. *For any initial value $(I(0), R(0), r_1(0), r_2(0)) \in \Gamma_*$, system (1.6) has a unique global solution $(I(t), R(t), r_1(t), r_2(t))$, and this solution will also exist in Γ_* almost surely.*

Proof. Clearly, the coefficients of system (1.6) all have locally Lipschitz property. Consequently, for any initial value $(I(0), R(0), r_1(0), r_2(0)) \in \Gamma_*$, system (1.6) exists a unique local solution $(I(t), R(t), r_1(t), r_2(t))$ on $t \in [0, \psi_w)$, where ψ_w denotes the explosion time. To show that the solution is global, we should prove $\psi_w = \infty$ a.s. Select a sufficiently large integer n_0 such that $I(0)$, $R(0)$, $e^{r_1(0)}$ and $e^{r_2(0)} \in [\frac{1}{n_0}, n_0]$. For each integer $n \ge n_0$, a stopping time can be defined as follows

$$\begin{aligned}\psi_n = \inf\Big\{ & t \in [0, \psi_w) : \min\{I(t), R(t), e^{r_1(t)}, e^{r_2(t)}\} \le \frac{1}{n} \\ & \text{or } \max\{I(t), R(t), e^{r_1(t)}, e^{r_2(t)}\} \ge n\Big\},\end{aligned}$$

where ψ_n is increasing as n approaches infinity. We denote $\inf\{\varnothing\} = \infty$ and $\psi_\infty = \lim\limits_{n\to\infty} \psi_n$, thus $\psi_\infty \le \psi_w$ a.s. If $\psi_\infty = \infty$ a.s., then $\psi_w = \infty$ a.s. and $(I(t), R(t), r_1(t), r_2(t)) \in \Gamma_*$ a.s. for all $t \ge 0$. But, if this assertion is incorrect, then there exist two constants $T > 0$ and $\epsilon \in (0, 1)$ such that

$$\mathbb{P}\{\psi_\infty \le T\} > \epsilon.$$

Therefore, there is an integer $n_1 \ge n_0$ such that

$$\mathbb{P}\{\psi_n \le T\} \ge \varepsilon, \ \ \forall n \ge n_1.$$

Using the formula $y - 1 - \log y \ge 0$ for any $y > 0$, we construct an non-negative C^2-function $\widehat{V}(I, R, r_1, r_2)$ as follows:

$$\widehat{V} = I - 1 - \log I + R - 1 - \log R + (1 - I - R) - 1 - \log(1 - I - R) + \frac{r_1^2}{2} + \frac{r_2^2}{2}.$$

Applying $It\widehat{o}'s$ formula derives

$$\mathcal{L}(-\log I) = -\left(\bar{\beta} - \frac{\bar{\beta}_e I}{a+I}\right)(1-I-R) + (\lambda+\mu) - \left(r_1 - \frac{r_2 I}{a+I}\right)(1-I-R). \tag{2.1}$$

$$\mathcal{L}(-\log R) = -\frac{\lambda I}{R} + \mu + \gamma. \tag{2.2}$$

$$\begin{aligned}\mathcal{L}(-\log(1-I-R)) = &-\frac{\mu}{1-I-R} + \mu - \frac{\gamma R}{1-I-R} \\ &+ \left(\bar{\beta} - \frac{\bar{\beta}_e I}{a+I}\right) I + \left(r_1 - \frac{r_2 I}{a+I}\right) I.\end{aligned} \tag{2.3}$$

$$\mathcal{L}(\frac{r_1^2}{2}) = -\theta_1 r_1^2 + \frac{1}{2}\sigma_1^2. \tag{2.4}$$

$$\mathcal{L}(\frac{r_2^2}{2}) = -\theta_2 r_2^2 + \frac{1}{2}\sigma_2^2. \tag{2.5}$$

Combining (2.1)-(2.5) gets

$$\begin{aligned}\mathcal{L}\widehat{V} =&\mu - \left(\bar{\beta} - \frac{\bar{\beta}_e I}{a+I}\right)(1-I-R) + (\lambda+\mu) - \left(r_1 - \frac{r_2 I}{a+I}\right)(1-I-R) \\ &- \frac{\lambda I}{R} + (\mu+\gamma) - \frac{\mu}{1-I-R} + \mu - \frac{\gamma R}{1-I-R} + \left(\bar{\beta} - \frac{\bar{\beta}_e I}{a+I}\right) I \\ &+ \left(r_1 - \frac{r_2 I}{a+I}\right) I - \theta_1 r_1^2 + \frac{\sigma_1^2}{2} - \theta_2 r_2^2 + \frac{\sigma_2^2}{2} \\ \leq&4\mu + \lambda + \gamma + \frac{\sigma_1^2+\sigma_2^2}{2} + \bar{\beta} I - \left(r_1 - \frac{r_2 I}{a+I}\right)(1-I-R) + \left(r_1 - \frac{r_2 I}{a+I}\right) I \\ &- \theta_1 r_1^2 - \theta_2 r_2^2 \\ \leq&4\mu + \lambda + \gamma + \frac{\sigma_1^2+\sigma_2^2}{2} + \bar{\beta} + \left(|r_1| + \frac{|r_2| I}{a+I}\right)(1-I-R) + |r_1| I + \frac{|r_2| I^2}{a+I} \\ &- \theta_1 r_1^2 - \theta_2 r_2^2 \\ \leq&4\mu + \lambda + \gamma + \frac{\sigma_1^2+\sigma_2^2}{2} + \bar{\beta} + 2|r_1| + 2|r_2| - \theta_1 r_1^2 - \theta_2 r_2^2 \\ \leq&4\mu + \lambda + \gamma + \frac{\sigma_1^2+\sigma_2^2}{2} + \bar{\beta} + \frac{1}{\theta_1} + \frac{1}{\theta_2} := G^*,\end{aligned}$$

where G^* is a positive constant. We omit the rest of the proof process, which can be proved in a similar way in [26]. This completes the proof. □

3. Extinction

In this part, we deduce the condition for infectious disease extinction.

Theorem 3.1. *Assume that* $(I(t), R(t), r_1(t), r_2(t))$ *is the solution of system* (1.6) *with any initial value* $(I(0), R(0), r_1(0), r_2(0))$ *in* Γ_*. *If*

$$R_0^* := R_0 + \frac{1}{2\sqrt{\pi}(\lambda+\mu)}\left(\frac{\sigma_1}{\sqrt{\theta_1}} + \frac{\sigma_2}{(a+1)\sqrt{\theta_2}}\right) < 1,$$

then

$$\limsup_{t\to\infty} \frac{\log I(t)}{t} \le (\lambda+\mu)(R_0^* - 1) < 0, \ a.s.$$

Namely, the disease of system (1.6) *will eventually die out almost surely.*

Proof. Applying *Itô's* formula to the first equation of system (1.6) gets

$$\begin{aligned}\frac{d\log I}{dt} &= \left(\bar{\beta} - \frac{\bar{\beta}_e I}{a+I}\right)(1-I-R) - (\lambda+\mu) + \left(r_1 - \frac{r_2 I}{a+I}\right)(1-I-R)\\ &\le \bar{\beta} - (\lambda+\mu) + r_1(1-I-R) + \frac{r_2 I}{a+I}(1-I-R)\\ &\le (\lambda+\mu)(R_0-1) + (r_1 \vee 0) + \frac{1}{a+1}(r_2 \vee 0). \end{aligned} \tag{3.1}$$

Integrating (3.1) from 0 to t and dividing by t on both sides, one yields

$$\begin{aligned}\frac{\log I(t)}{t} - \frac{\log I(0)}{t} \le& (\lambda+\mu)(R_0-1) + \frac{1}{t}\int_0^t (r_1(\tau)\vee 0)d\tau\\ &+ \frac{1}{t}\int_0^t \frac{1}{a+1}(r_2(\tau)\vee 0)d\tau. \end{aligned} \tag{3.2}$$

According to the results of [27], $r_1(t)$ and $r_2(t)$ will converge weakly to the invariant densities

$$k_1(y) = \frac{\sqrt{\theta_1}}{\sqrt{\pi}\sigma_1} e^{-\frac{\theta_1 y^2}{\sigma_1^2}}, (y \in \mathbb{R}).$$

$$k_2(z) = \frac{\sqrt{\theta_2}}{\sqrt{\pi}\sigma_2} e^{-\frac{\theta_2 z^2}{\sigma_2^2}}, (z \in \mathbb{R}).$$

That is $\mathbb{N}(0, \frac{\sigma_1^2}{2\theta_1})$ and $\mathbb{N}(0, \frac{\sigma_2^2}{2\theta_2})$. Then, we get

$$\begin{aligned}\int_{-\infty}^{+\infty} (y\vee 0)k_1(y)dy &= \int_0^{+\infty} \frac{\sqrt{\theta_1}y}{\sqrt{\pi}\sigma_1} e^{-\frac{\theta_1 y^2}{\sigma_1^2}} dy\\ =\frac{\sigma_1}{2\sqrt{\pi\theta_1}} \int_0^{+\infty} e^{-\left(\frac{\sqrt{\theta_1}y}{\sigma_1}\right)^2} d\left(\frac{\sqrt{\theta_1}y}{\sigma_1}\right)^2 &= \frac{\sigma_1}{2\sqrt{\pi\theta_1}}, a.s. \end{aligned} \tag{3.3}$$

Similarly,

$$\int_{-\infty}^{+\infty} (z\vee 0)k_2(z)dz = \frac{\sigma_2}{2\sqrt{\pi\theta_2}}, a.s. \tag{3.4}$$

Taking the superior limit on both sides of the inequation (3.2) and combining (3.3)-(3.4), one obtains

$$\begin{aligned}
&\limsup_{t\to\infty}\frac{\log I(t)}{t}\\
\leq&(\lambda+\mu)(R_0-1)+\lim_{t\to\infty}\frac{1}{t}\int_0^t(r_1(\tau)\vee 0)d\tau+\lim_{t\to\infty}\frac{1}{t}\int_0^t\frac{1}{a+1}(r_2(\tau)\vee 0)d\tau\\
=&(\lambda+\mu)(R_0-1)+\int_{-\infty}^{+\infty}(y\vee 0)k_1(y)dy+\frac{1}{a+1}\int_{-\infty}^{+\infty}(z\vee 0)k_2(z)dz\\
=&(\lambda+\mu)(R_0-1)+\frac{\sigma_1}{2\sqrt{\pi\theta_1}}+\frac{\sigma_2}{2(a+1)\sqrt{\pi\theta_2}}\\
=&(\lambda+\mu)(R_0^*-1)<0, a.s. \qquad (3.5)
\end{aligned}$$

It implies that $\lim_{t\to\infty} I(t)=0$. That is to say, the disease will go to extinction with probability 1. This completes the proof. □

4. Stationary distribution

In this section, the conditions for infectious disease persistence will be studied. For deterministic model (1.2), we investigate the persistence of the disease by endemic equilibrium point. But the endemic equilibrium does not exist for stochastic model. We can prove that stochastic system (1.6) admits stationary distribution, which also shows the persistence of the disease. We define

$$R_0^s=R_0-\frac{1}{2\sqrt{\pi}(\lambda+\mu)}\left(\frac{\sigma_1}{\sqrt{\theta_1}}+\frac{\sigma_2}{(a+1)\sqrt{\theta_2}}\right).$$

Condider an n-dimensional stochastic differential equation with initial value $\Psi(0)$:

$$d\Psi(t)=h_1(\Psi(t))dt+h_2(\Psi(t))dB(t), \qquad (4.1)$$

where $h_1:\mathbb{R}^n\to\mathbb{R}^n$ and $h_2:\mathbb{R}^n\to\mathbb{R}^{n\times m}$ are Borel measurable. $B(t)$ is an m-dimensional Brownian motion on the space $\{\Omega,\mathscr{F},\{\mathscr{F}_t\}_{t\geq 0},\mathbb{P}\}$. Based on the searches of Dieu [28] and Zhou et al. [32], we can obtain the sufficient condition for the existence of stationary distribution as Lemma 4.1.

Lemma 4.1. *If there is a bounded closed domain $\Xi\subset\mathbb{R}^n$ with a regular boundary Λ, for any initial value $\Psi(0)\in\mathbb{R}^n$,*

$$\liminf_{t\to+\infty}\frac{1}{t}\int_0^t\mathbb{P}(\tau,\Psi(0),\Xi)d\tau>0 \ \ a.s.,$$

where $\mathbb{P}(\tau,\Psi(0),\cdot)$ is the transition probability of $\Psi(t)$. Then system (4.1) has a solution and admits at least one invariant probability measure on $\mathbb{R}^n$, which implies system (4.1) has at least one stationary distribution on $\mathbb{R}^n$.

Theorem 4.1. *If $R_0^s>1$, there exists at least one stationary distribution of system (1.6) on Γ_*.*

Proof. According to Lemma 4.1, the proof of Theorem 4.1 is divided into three steps.

Step 1. (Construction of a C^2-function):
Construct a C^2-function $\bar{V}(I,R,r_1,r_2):\Gamma_*\to\mathbb{R}$ as follows:

$$\begin{aligned}\bar{V}&=M_0\left(-\log I+\frac{\bar{\beta}}{\mu+\gamma}R\right)-\log R-\log(1-I-R)+\frac{r_1^2}{2}+\frac{r_2^2}{2}\\&:=-M_0V_1+V_2+V_3,\end{aligned}$$

where $V_1=-\log I+\dfrac{\bar{\beta}}{\mu+\gamma}R$, $V_2=-\log R-\log(1-I-R)$, $V_3=\dfrac{r_1^2}{2}+\dfrac{r_2^2}{2}$, and M_0 is a sufficiently large positive number satisfying the following inequality:

$$\begin{aligned}&-M_0(\lambda+\mu)(R_0^s-1)+\bar{\beta}+2\mu+\gamma+\frac{\sigma_1^2+\sigma_2^2}{2}\\&+\sup_{(r_1,r_2)\in\mathbb{R}^2}\left\{|r_1|+\frac{|r_2|}{a+1}-\frac{\theta_1}{2}r_1^2-\frac{\theta_2}{2}r_2^2\right\}\le-2.\end{aligned}\tag{4.2}$$

We make the differential operator $\mathcal{L}$ act on $-\log I$ to get

$$\begin{aligned}\mathcal{L}(-\log I)&=-\left(\bar{\beta}-\frac{\bar{\beta}_eI}{a+I}\right)(1-I-R)-\left(r_1-\frac{r_2I}{a+I}\right)(1-I-R)+(\lambda+\mu)\\&\le-\bar{\beta}+(\lambda+\mu)+\bar{\beta}I+\bar{\beta}R+\frac{\bar{\beta}_eI}{a+I}-r_1(1-I-R)+\frac{r_2I}{a+I}(1-I-R)\\&\le-(\lambda+\mu)(R_0-1)+\bar{\beta}I+\bar{\beta}R+\frac{\bar{\beta}_eI}{a+I}-r_1(1-I-R)+\frac{r_2I}{a+I}\\&\le-(\lambda+\mu)(R_0-1)+\bar{\beta}I+\bar{\beta}R+\frac{\bar{\beta}_eI}{a}+(r_1\vee0)+\frac{1}{a+1}(r_2\vee0).\end{aligned}\tag{4.3}$$

Combining (3.3)-(3.4) and (4.3) leads to

$$\begin{aligned}\mathcal{L}V_1\le&-(\lambda+\mu)(R_0-1)+\bar{\beta}(I+R)+\frac{\bar{\beta}_eI}{a}+(r_1\vee0)\\&+\frac{1}{a+1}(r_2\vee0)+\frac{\bar{\beta}}{\mu+\gamma}[\lambda I-(\mu+\gamma)R]\\\le&-(\lambda+\mu)(R_0-1)+\left(\bar{\beta}+\frac{\bar{\beta}_e}{a}+\frac{\lambda\bar{\beta}}{\mu+\gamma}\right)I+(r_1\vee0)+\frac{1}{a+1}(r_2\vee0)\\=&-(\lambda+\mu)(R_0-1)+\left(\bar{\beta}+\frac{\bar{\beta}_e}{a}+\frac{\lambda\bar{\beta}}{\mu+\gamma}\right)I\\&+\frac{\sigma_1}{2\sqrt{\pi\theta_1}}+(r_1\vee0)-\int_{-\infty}^{+\infty}(y\vee0)k_1(y)dy\\&+\frac{\sigma_2}{2(a+1)\sqrt{\pi\theta_2}}+\frac{1}{a+1}\left[(r_2\vee0)-\int_{-\infty}^{+\infty}(z\vee0)k_2(z)dz\right]\\=&-(\lambda+\mu)\left(R_0-1-\frac{\sigma_1}{2\sqrt{\pi\theta_1}(\lambda+\mu)}-\frac{\sigma_2}{2(a+1)\sqrt{\pi\theta_2}(\lambda+\mu)}\right)\\&+\left(\bar{\beta}+\frac{\bar{\beta}_e}{a}+\frac{\lambda\bar{\beta}}{\mu+\gamma}\right)I+(r_1\vee0)-\int_{-\infty}^{+\infty}(y\vee0)k_1(y)dy\\&+\frac{1}{a+1}\left[(r_2\vee0)-\int_{-\infty}^{+\infty}(z\vee0)k_2(z)dz\right]\end{aligned}$$

$$= -(\lambda+\mu)(R_0^s-1)+\left(\bar{\beta}+\frac{\bar{\beta}_e}{a}+\frac{\lambda\bar{\beta}}{\mu+\gamma}\right)I+(r_1\vee 0)$$
$$-\int_{-\infty}^{+\infty}(y\vee 0)k_1(y)dy+\frac{1}{a+1}\left[(r_2\vee 0)-\int_{-\infty}^{+\infty}(z\vee 0)k_2(z)dz\right]. \quad (4.4)$$

Making use of $Itô's$ formula to V_2 and V_3, one obtains

$$\begin{aligned}\mathcal{L}V_2 =& -\frac{\lambda I}{R}+(\mu+\gamma)-\frac{\mu}{1-I-R}+\left(\bar{\beta}-\frac{\bar{\beta}_e I}{a+I}\right)I\\ &+\left(r_1-\frac{r_2 I}{a+I}\right)I+\mu-\frac{\gamma R}{1-I-R}\\ \le& -\frac{\lambda I}{R}+2\mu+\gamma-\frac{\mu}{1-I-R}+\bar{\beta}I+|r_1|I+\frac{|r_2|I^2}{a+I}\\ \le& -\frac{\lambda I}{R}+2\mu+\gamma-\frac{\mu}{1-I-R}+\bar{\beta}+|r_1|+\frac{|r_2|}{a+1}, \end{aligned} \quad (4.5)$$

and

$$\mathcal{L}V_3=\frac{\sigma_1^2+\sigma_2^2}{2}-\theta_1 r_1^2-\theta_2 r_2^2. \quad (4.6)$$

Combining (4.4)-(4.6), one derives

$$\begin{aligned}\mathcal{L}\bar{V} =& -M_0(\lambda+\mu)(R_0^s-1)+M_0\left(\bar{\beta}+\frac{\bar{\beta}_e}{a}+\frac{\lambda\bar{\beta}}{\mu+\gamma}\right)I-\frac{\lambda I}{R}-\frac{\mu}{1-I-R}\\ &+\bar{\beta}+2\mu+\gamma+\frac{\sigma_1^2+\sigma_2^2}{2}+|r_1|+\frac{|r_2|}{a+1}-\theta_1 r_1^2-\theta_2 r_2^2\\ &+M_0\left[(r_1\vee 0)-\int_{-\infty}^{+\infty}(y\vee 0)k_1(y)dy\right]\\ &+\frac{M_0}{a+1}\left[(r_2\vee 0)-\int_{-\infty}^{+\infty}(z\vee 0)k_2(z)dz\right]\\ \le& -M_0(\lambda+\mu)(R_0^s-1)+M_0\left(\bar{\beta}+\frac{\bar{\beta}_e}{a}+\frac{\lambda\bar{\beta}}{\mu+\gamma}\right)I+\bar{\beta}+2\mu+\gamma\\ &+\sup_{(r_1,r_2)\in\mathbb{R}^2}\left\{|r_1|+\frac{|r_2|}{a+1}-\frac{\theta_1}{2}r_1^2-\frac{\theta_2}{2}r_2^2\right\}-\frac{\mu}{1-I-R}-\frac{\lambda I}{R}+\frac{\sigma_1^2+\sigma_2^2}{2}\\ &-\frac{\theta_1}{2}r_1^2-\frac{\theta_2}{2}r_2^2+M_0\left[(r_1\vee 0)-\int_{-\infty}^{+\infty}(y\vee 0)k_1(y)dy\right]\\ &+\frac{M_0}{a+1}\left[(r_2\vee 0)-\int_{-\infty}^{+\infty}(z\vee 0)k_2(z)dz\right]\\ :=&F(I,R,r_1,r_2)+M_0\left[(r_1\vee 0)-\int_{-\infty}^{+\infty}(y\vee 0)k_1(y)dy\right]\\ &+\frac{M_0}{a+1}\left[(r_2\vee 0)-\int_{-\infty}^{+\infty}(z\vee 0)k_2(z)dz\right], \end{aligned} \quad (4.7)$$

where

$$F(I,R,r_1,r_2)$$

$$
\begin{aligned}
&= - M_0(\lambda+\mu)(R_0^s-1) + \bar{\beta} + 2\mu + \gamma + \frac{\sigma_1^2+\sigma_2^2}{2} + M_0\left(\bar{\beta} + \frac{\bar{\beta}_e}{a} + \frac{\lambda\bar{\beta}}{\mu+\gamma}\right) I \\
&\quad + \sup_{(r_1,r_2)\in\mathbb{R}^2}\left\{|r_1| + \frac{|r_2|}{a+1} - \frac{\theta_1}{2}r_1^2 - \frac{\theta_2}{2}r_2^2\right\} - \frac{\mu}{1-I-R} - \frac{\lambda I}{R} - \frac{\theta_1}{2}r_1^2 - \frac{\theta_2}{2}r_2^2 \\
&:= - M_0(\lambda+\mu)(R_0^s-1) + K + M_0\left(\bar{\beta} + \frac{\bar{\beta}_e}{a} + \frac{\bar{\beta}\lambda}{\mu+\gamma}\right) I \\
&\quad - \frac{\mu}{1-I-R} - \frac{\lambda I}{R} - \frac{\theta_1}{2}r_1^2 - \frac{\theta_2}{2}r_2^2, \qquad (4.8)
\end{aligned}
$$

in which $K = \bar{\beta} + 2\mu + \gamma + \dfrac{\sigma_1^2+\sigma_2^2}{2} + \sup\limits_{(r_1,r_2)\in\mathbb{R}^2}\left\{|r_1| + \dfrac{|r_2|}{a+1} - \dfrac{\theta_1}{2}r_1^2 - \dfrac{\theta_2}{2}r_2^2\right\}$.

Step 2. (Construction of a compact set):
Construct a bounded set $\mathbb{D}_\varepsilon$

$$
\mathbb{D}_\varepsilon = \left\{(I,R,r_1,r_2) \in \Gamma_* | I \geq \varepsilon,\, R \geq \varepsilon^2,\, I+R \leq 1-\varepsilon,\, |r_1| \leq \frac{1}{\varepsilon},\, |r_2| \leq \frac{1}{\varepsilon}\right\},
$$

where ε is a sufficiently small positive number satisfying

$$
M_0\left(\bar{\beta} + \frac{\bar{\beta}_e}{a} + \frac{\lambda\bar{\beta}}{\mu+\gamma}\right)\varepsilon \leq 1. \qquad (4.9)
$$

$$
K + M_0\left(\bar{\beta} + \frac{\bar{\beta}_e}{a} + \frac{\lambda\bar{\beta}}{\mu+\gamma}\right) - \frac{\lambda}{\varepsilon} \leq -1. \qquad (4.10)
$$

$$
K + M_0\left(\bar{\beta} + \frac{\bar{\beta}_e}{a} + \frac{\lambda\bar{\beta}}{\mu+\gamma}\right) - \frac{\mu}{\varepsilon} \leq -1. \qquad (4.11)
$$

$$
K + M_0\left(\bar{\beta} + \frac{\bar{\beta}_e}{a} + \frac{\lambda\bar{\beta}}{\mu+\gamma}\right) - \frac{\theta_1}{2\varepsilon^2} \leq -1. \qquad (4.12)
$$

$$
K + M_0\left(\bar{\beta} + \frac{\bar{\beta}_e}{a} + \frac{\lambda\bar{\beta}}{\mu+\gamma}\right) - \frac{\theta_2}{2\varepsilon^2} \leq -1. \qquad (4.13)
$$

Next, we divide the set $\Gamma_* \setminus \mathbb{D}_\varepsilon$ into five subsets $\mathbb{D}_{\varepsilon,j}^c$, $j = 1,2,\ldots,5$, where

$$
\begin{aligned}
&\mathbb{D}_{\varepsilon,1}^c = \{(I,R,r_1,r_2) \in \Gamma_* | I < \varepsilon\}, \\
&\mathbb{D}_{\varepsilon,2}^c = \{(I,R,r_1,r_2) \in \Gamma_* | R < \varepsilon^2,\, I \geq \varepsilon\}, \\
&\mathbb{D}_{\varepsilon,3}^c = \{(I,R,r_1,r_2) \in \Gamma_* | I+R > 1-\varepsilon\}, \quad \mathbb{D}_{\varepsilon,4}^c = \left\{(I,R,r_1,r_2) \in \Gamma_* | |r_1| > \frac{1}{\varepsilon}\right\}, \\
&\mathbb{D}_{\varepsilon,5}^c = \left\{(I,R,r_1,r_2) \in \Gamma_* | |r_2| > \frac{1}{\varepsilon}\right\}.
\end{aligned}
$$

We discuss the following five cases:

(I) If $(I,R,r_1,r_2) \in \mathbb{D}_{\varepsilon,1}^c$, combining (4.8) and (4.9) gets

$$
F(I,R,r_1,r_2) \leq -M_0(\lambda+\mu)(R_0^s-1) + K + M_0\left(\bar{\beta} + \frac{\bar{\beta}_e}{a} + \frac{\lambda\bar{\beta}}{\mu+\gamma}\right) I
$$

$$\leq -2 + M_0 \left(\bar{\beta} + \frac{\bar{\beta}_e}{a} + \frac{\lambda \bar{\beta}}{\mu + \gamma} \right) \varepsilon$$
$$\leq -1.$$

(II) If $(I, R, r_1, r_2) \in \mathbb{D}^c_{\varepsilon,2}$, from (4.8) and (4.10), we have

$$F(I, R, r_1, r_2) \leq K + M_0 \left(\bar{\beta} + \frac{\bar{\beta}_e}{a} + \frac{\lambda \bar{\beta}}{\mu + \gamma} \right) I - \frac{\lambda I}{R}$$
$$\leq K + M_0 \left(\bar{\beta} + \frac{\bar{\beta}_e}{a} + \frac{\lambda \bar{\beta}}{\mu + \gamma} \right) - \frac{\lambda}{\varepsilon}$$
$$\leq -1.$$

(III) If $(I, R, r_1, r_2) \in \mathbb{D}^c_{\varepsilon,3}$, combining (4.8) and (4.11), we get

$$F(I, R, r_1, r_2) \leq K + M_0 \left(\bar{\beta} + \frac{\bar{\beta}_e}{a} + \frac{\lambda \bar{\beta}}{\mu + \gamma} \right) I - \frac{\mu}{1 - I - R}$$
$$\leq K + M_0 \left(\bar{\beta} + \frac{\bar{\beta}_e}{a} + \frac{\lambda \bar{\beta}}{\mu + \gamma} \right) - \frac{\mu}{\varepsilon}$$
$$\leq -1.$$

(IV) If $(I, R, r_1, r_2) \in \mathbb{D}^c_{\varepsilon,4}$, using (4.8) and (4.12) obtains

$$F(I, R, r_1, r_2) \leq K + M_0 \left(\bar{\beta} + \frac{\bar{\beta}_e}{a} + \frac{\lambda \bar{\beta}}{\mu + \gamma} \right) I - \frac{\theta_1}{2} r_1^2$$
$$\leq K + M_0 \left(\bar{\beta} + \frac{\bar{\beta}_e}{a} + \frac{\lambda \bar{\beta}}{\mu + \gamma} \right) - \frac{\theta_1}{2\varepsilon^2}$$
$$\leq -1.$$

(V) If $(I, R, r_1, r_2) \in \mathbb{D}^c_{\varepsilon,5}$, applying (4.8) and (4.13), we have

$$F(I, R, r_1, r_2) \leq K + M_0 \left(\bar{\beta} + \frac{\bar{\beta}_e}{a} + \frac{\lambda \bar{\beta}}{\mu + \gamma} \right) I - \frac{\theta_2}{2} r_2^2$$
$$\leq K + M_0 \left(\bar{\beta} + \frac{\bar{\beta}_e}{a} + \frac{\lambda \bar{\beta}}{\mu + \gamma} \right) - \frac{\theta_2}{2\varepsilon^2}$$
$$\leq -1.$$

Taking the above five cases into consideration, we can conclude

$$F(I, R, r_1, r_2) \leq -1, \ \ \forall (I, R, r_1, r_2) \in \Gamma_* \setminus \mathbb{D}_\varepsilon. \tag{4.14}$$

Let

$$\hat{U} := \sup_{(I,R,r_1,r_2) \in \Gamma_*} \left\{ -2 + M_0 \left(\bar{\beta} + \frac{\bar{\beta}_e}{a} + \frac{\lambda \bar{\beta}}{\mu + \gamma} \right) I - \frac{\mu}{1 - I - R} - \frac{\lambda I}{R} - \frac{\theta_1}{2} r_1^2 - \frac{\theta_2}{2} r_2^2 \right\}.$$

Then, there exists $U > 0$, such that $U > \hat{U}$ and

$$F(I, R, r_1, r_2) \leq U < \infty, \qquad \forall (I, R, r_1, r_2) \in \Gamma_*. \tag{4.15}$$

Since the function $\bar{V}(I,R,r_1,r_2)$ tends to $+\infty$ as (I,R) approaches the boundary of $\mathbb{R}^2_+$ or as $\|(I,R,r_1,r_2)\| \to +\infty$, there is a point (I^*,R^*,r_1^*,r_2^*) in the interior of Γ_*, which makes $\bar{V}(I,R,r_1,r_2)$ be minimized. Therefore, a non-negative C^2-function $V(I,R,r_1,r_2)$ is defined by

$$V(I,R,r_1,r_2) = \bar{V}(I,R,r_1,r_2) - \bar{V}(I^*,R^*,r_1^*,r_2^*).$$

Using (4.7), we draw

$$\begin{aligned}\mathcal{L}V \leq & F(I,R,r_1,r_2) + M_0\left[(r_1 \vee 0) - \int_{-\infty}^{+\infty}(y \vee 0)k_1(y)dy\right] \\ & + \frac{M_0}{a+1}\left[(r_2 \vee 0) - \int_{-\infty}^{+\infty}(z \vee 0)k_2(z)dz\right]. \end{aligned} \tag{4.16}$$

Step 3. (Existence):
For any initial value $(I(0),R(0),r_1(0),r_2(0)) \in \Gamma_*$ and the interval $[0,t]$, using $It\widehat{o}'s$ integral and mathematical expectation to $V(I,R,r_1,r_2)$, we can get

$$\begin{aligned} 0 \leq & \frac{\mathbb{E}V(I(t),R(t),r_1(t),r_2(t))}{t} \\ \leq & \frac{\mathbb{E}V(I(0),R(0),r_1(0),r_2(0)}{t} + \frac{1}{t}\int_0^t \mathbb{E}(F(I(\tau),R(\tau),r_1(\tau),r_2(\tau)))d\tau \\ & + M_0\mathbb{E}\left[\frac{1}{t}\int_0^t (r_1(\tau)\vee 0)d\tau - \int_{-\infty}^{+\infty}(y\vee 0)k_1(y)dy\right] \\ & + \frac{M_0}{a+1}\mathbb{E}\left[\frac{1}{t}\int_0^t (r_2(\tau)\vee 0)d\tau - \int_{-\infty}^{+\infty}(z\vee 0)k_2(z)dz\right]. \end{aligned} \tag{4.17}$$

Making use of the strong law of large numbers [32,35], we subsequently infer

$$\begin{aligned} & \lim_{t\to\infty}\mathbb{E}\left[\frac{1}{t}\int_0^t (r_1(\tau)\vee 0)d\tau - \int_{-\infty}^{+\infty}(y\vee 0)k_1(y)dy\right] \\ = & \mathbb{E}\left[\int_0^\infty yk_1(y)dy\right] - \int_0^\infty yk_1(y)dy = 0, \ \ a.s., \end{aligned} \tag{4.18}$$

and

$$\begin{aligned} & \lim_{t\to\infty}\mathbb{E}\left[\frac{1}{t}\int_0^t (r_2(\tau)\vee 0)d\tau - \int_{-\infty}^{+\infty}(z\vee 0)k_2(z)dz\right] \\ = & \mathbb{E}\left[\int_0^\infty zk_2(z)dz\right] - \int_0^\infty zk_2(z)dz = 0, \ \ a.s. \end{aligned} \tag{4.19}$$

On the one hand, taking the limit on both sides of (4.17) and using (4.18)-(4.19), we launch

$$\begin{aligned} 0 \leq & \lim_{t\to+\infty}\frac{\mathbb{E}V(I(0),R(0),r_1(0),r_2(0))}{t} + \lim_{t\to+\infty}\frac{1}{t}\int_0^t \mathbb{E}(F(I(\tau),R(\tau),r_1(\tau),r_2(\tau)))d\tau \\ & + \lim_{t\to+\infty}\left\{M_0\mathbb{E}\left[\frac{1}{t}\int_0^t (r_1(\tau)\vee 0)d\tau - \int_{-\infty}^{+\infty}(y\vee 0)k_1(y)dy\right]\right. \\ & \left. + \frac{M_0}{a+1}\mathbb{E}\left[\frac{1}{t}\int_0^t (r_2(\tau)\vee 0)d\tau - \int_{-\infty}^{+\infty}(z\vee 0)k_2(z)dz\right]\right\} \end{aligned}$$

$$= \lim_{t\to+\infty} \frac{1}{t}\int_0^t \mathbb{E}(F(I(\tau),R(\tau),r_1(\tau),r_2(\tau)))d\tau \ \ a.s. \tag{4.20}$$

On the other hand, combining (4.14) and (4.15) gets

$$\begin{aligned}
&\frac{1}{t}\int_0^t \mathbb{E}(F(I(\tau),R(\tau),r_1(\tau),r_2(\tau)))d\tau\\
=&\frac{1}{t}\int_0^t \mathbb{E}(F(I(\tau),R(\tau),r_1(\tau),r_2(\tau)))\mathbf{1}_{\{(I(\tau),R(\tau),r_1(\tau),r_2(\tau))\in\mathbb{D}_\varepsilon\}}d\tau\\
&+\frac{1}{t}\int_0^t \mathbb{E}(F(I(\tau),R(\tau),r_1(\tau),r_2(\tau)))\mathbf{1}_{\{(I(\tau),R(\tau),r_1(\tau),r_2(\tau))\in(\Gamma_*\setminus\mathbb{D}_\varepsilon)\}}d\tau\\
\le&\frac{U}{t}\int_0^t \mathbf{1}_{\{(I(\tau),R(\tau),r_1(\tau),r_2(\tau))\in\mathbb{D}_\varepsilon\}}d\tau-\frac{1}{t}\int_0^t \mathbf{1}_{\{(I(\tau),R(\tau),r_1(\tau),r_2(\tau))\in(\Gamma_*\setminus\mathbb{D}_\varepsilon)\}}d\tau\\
\le&-1+\frac{U+1}{t}\int_0^t \mathbf{1}_{\{(I(\tau),R(\tau),r_1(\tau),r_2(\tau))\in\mathbb{D}_\varepsilon\}}d\tau.
\end{aligned} \tag{4.21}$$

In view of (4.20) and (4.21), taking the inferior limit obtains

$$\liminf_{t\to+\infty}\frac{1}{t}\int_0^t \mathbf{1}_{\{(I(\tau),R(\tau),r_1(\tau),r_2(\tau))\in\mathbb{D}_\varepsilon\}}d\tau \ge \frac{1}{U+1} > 0 \ \ a.s. \tag{4.22}$$

By the definition of event probability and Fatou's lemma [28], we acquire the equivalent form of (4.22)

$$\liminf_{t\to+\infty}\frac{1}{t}\int_0^t \mathbb{P}(\tau,(I(\tau),R(\tau),r_1(\tau),r_2(\tau)),\mathbb{D}_\varepsilon)d\tau \ge \frac{1}{U+1} > 0 \ \ a.s., \tag{4.23}$$

where $\mathbb{P}(t,(I,R,r_1,r_2),\mathbb{D}_\varepsilon)$ is the transition probability of $(I(t),R(t),r_1(t),r_2(t)) \in \mathbb{D}_\varepsilon$. Thus, according to Lemma 4.1, system (1.6) has at least one stationary distribution on Γ_*. This completes the proof. □

Theorem 4.1 indicates that if $R_0^s > 1$, then the disease will prevail in a long term.

5. Probability density function

In this part, we will give the exact expression for the probability density function.

Lemma 5.1. *For the real algebraic equation $\mathcal{K}^2 + \hat{\mathcal{A}}\bar{\Sigma} + \bar{\Sigma}\hat{\mathcal{A}}^T = 0$, where $\mathcal{K} = diag(\tilde{\sigma},0,0)$, $\bar{\Sigma}$ is a real symmetric matrix, and*

$$\hat{\mathcal{A}} = \begin{pmatrix} -a_1 & -a_2 & -a_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

If $a_1 > 0$, $a_3 > 0$ and $a_1a_2 - a_3 > 0$, then $\bar{\Sigma}$ is positive definite. Here, a_1, a_2 and a_3 are the coefficients of characteristic polynomial $\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3$ of $\hat{\mathcal{A}}$. Appendix A gives the detail proof of Lemma 5.1.

The dynamical property of system (1.6) is the same as system (1.5). As a matter of convenience, we study the density function of system (1.5). Epidemic system (1.5) without stochastic noises is shown below:

$$\begin{cases} dI(t) &= \left[\left(\beta(t) - \dfrac{\beta_e(t)I(t)}{a+I(t)}\right)(1-I(t)-R(t))I(t) - (\lambda+\mu)I(t)\right]dt, \\ dR(t) &= [\lambda I(t) - (\mu+\gamma)R(t)]dt, \\ d\beta(t) &= \theta_1(\bar{\beta}-\beta(t))dt, \\ d\beta_e(t) &= \theta_2(\bar{\beta}_e - \beta_e(t))dt. \end{cases} \tag{5.1}$$

There exists a positive equilibrium $\widehat{E}^*$=($\hat{I}^*$, $\hat{R}^*$, $\bar{\beta}$, $\bar{\beta}_e$) in system (5.1) when $R_0 > 1$, where $\hat{I}^* = I^*$ and $\hat{R}^* = R^*$ are the same as Section 2. $\widehat{E}^*$ is the same as E^* without considering two mean-reverting Ornstein-Uhlenbeck processes. Letting $Z = (z_1, z_2, z_3, z_4) = (I - I^*, R - R^*, \beta - \bar{\beta}, \beta_e - \bar{\beta}_e)$, we can get the corresponding linearized system of (1.5) at point $\widehat{E}^*$, that is

$$\begin{cases} dz_1 = \left[\left((1-2I^*-R^*)\left(\bar{\beta} - \dfrac{\bar{\beta}_e I^*}{a+I^*}\right) - (1-I^*-R^*)\dfrac{a\bar{\beta}_e I^*}{(a+I^*)^2} - (\lambda+\mu)\right)z_1 \right. \\ \qquad \left. - \left(\bar{\beta} - \dfrac{\bar{\beta}_e I^*}{a+I^*}\right)I^* z_2 + (1-I^*-R^*)I^* z_3 - (1-I^*-R^*)\dfrac{(I^*)^2}{a+I^*}z_4\right]dt, \\ dz_2 = [\lambda z_1 - (\mu+\gamma)z_2]dt, \\ dz_3 = -\theta_1 z_3 dt + \sigma_1 dB_1(t), \\ dz_4 = -\theta_2 z_4 dt + \sigma_2 dB_2(t). \end{cases} \tag{5.2}$$

Let

$$\begin{aligned} a_{11} &= (2I^* + R^* - 1)\left(\bar{\beta} - \frac{\bar{\beta}_e I^*}{a+I^*}\right) + (1-I^*-R^*)\frac{a\bar{\beta}_e I^*}{(a+I^*)^2} + \lambda + \mu, \\ a_{12} &= \left(\bar{\beta} - \frac{\bar{\beta}_e I^*}{a+I^*}\right)I^*, \quad a_{13} = (1-I^*-R^*)I^*, \quad a_{14} = (1-I^*-R^*)\frac{(I^*)^2}{a+I^*}, \\ a_{21} &= \lambda, \qquad a_{22} = \mu+\gamma, \qquad a_{33} = \theta_1, \qquad a_{44} = \theta_2. \end{aligned}$$

System (5.2) can be expressed as the following form:

$$dZ(t) = AZ(t)dt + QdB^*(t), \tag{5.3}$$

where

$$A = \begin{pmatrix} -a_{11} & -a_{12} & a_{13} & -a_{14} \\ a_{21} & -a_{22} & 0 & 0 \\ 0 & 0 & -a_{33} & 0 \\ 0 & 0 & 0 & -a_{44} \end{pmatrix},$$

and

$$Z(t) = (z_1(t), z_2(t), z_3(t), z_4(t))^T, Q = diag(0, 0, \sigma_1, \sigma_2), B^*(t) = (0, 0, B_1(t), B_2(t))^T.$$

Theorem 5.1. *If $R_0 > 1$, for any initial value $(z_1(0), z_2(0), z_3(0), z_4(0))$, the solution $Z(z_1, z_2, z_3, z_4)$ follows a normal density function $\Phi(z_1, z_2, z_3, z_4)$ around $(I^*, R^*, \bar{\beta}, \bar{\beta}_e)$ such that*

$$\Phi(z_1, z_2, z_3, z_4) = (2\pi)^{-2}|\Sigma|^{-\frac{1}{2}} e^{-\frac{1}{2}(z_1,z_2,z_3,z_4)\Sigma^{-1}(z_1,z_2,z_3,z_4)^T}, \tag{5.4}$$

where Σ is positive definite and the form is given as follows

$$\begin{aligned}\Sigma &= (R_1J_1)^{-1}\hat{\Sigma}_1^*[(R_1J_1)^{-1}]^T + (R_2J_2)^{-1}\hat{\Sigma}_2^*[(R_2J_2)^{-1}]^T\\ &:= \Sigma_1 + \Sigma_2,\end{aligned}$$

where

$$J_1 = \begin{pmatrix} 0&0&1&0\\ 1&0&0&0\\ 0&1&0&0\\ 0&0&0&1 \end{pmatrix}, J_2 = \begin{pmatrix} 0&0&0&1\\ 1&0&0&0\\ 0&1&0&0\\ 0&0&1&0 \end{pmatrix},$$

$$R_1 = \begin{pmatrix} a_{13}a_{21} & -(a_{11}+a_{22})a_{21} & a_{22}^2 - a_{12}a_{21} & -a_{14}a_{21}\\ 0 & a_{21} & -a_{22} & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$R_2 = \begin{pmatrix} -a_{14}a_{21} & -(a_{11}+a_{22})a_{21} & a_{22}^2 - a_{12}a_{21} & a_{13}a_{21}\\ 0 & a_{21} & -a_{22} & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\hat{\Sigma}_1^* = \begin{pmatrix} \frac{d_2(a_{13}a_{21}\sigma_1)^2}{2(d_1d_2-d_3)} & 0 & -\frac{(a_{13}a_{21}\sigma_1)^2}{2(d_1d_2-d_3)} & 0\\ 0 & \frac{(a_{13}a_{21}\sigma_1)^2}{2(d_1d_2-d_3)} & 0 & 0\\ -\frac{(a_{13}a_{21}\sigma_1)^2}{2(d_1d_2-d_3)} & 0 & \frac{d_1(a_{13}a_{21}\sigma_1)^2}{2d_3(d_1d_2-d_3)} & 0\\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\hat{\Sigma}_2^* = \begin{pmatrix} \frac{b_2(a_{14}a_{21}\sigma_2)^2}{2(b_1b_2-b_3)} & 0 & -\frac{(a_{14}a_{21}\sigma_2)^2}{2(b_1b_2-b_3)} & 0\\ 0 & \frac{(a_{14}a_{21}\sigma_2)^2}{2(b_1b_2-b_3)} & 0 & 0\\ -\frac{(a_{14}a_{21}\sigma_2)^2}{2(b_1b_2-b_3)} & 0 & \frac{b_1(a_{14}a_{21}\sigma_2)^2}{2b_3(b_1b_2-b_3)} & 0\\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Proof. System (5.3) has a unique explicit solution according to the results of Oksendal [20] and Mao [36]

$$Z(t) = e^{At}Z(0) + \int_0^t e^{A(t-\tau)}QdB(\tau).$$

Since the martingale $\int_0^t e^{A(t-\tau)}QdB(\tau)$ follows the Gaussian distribution $\mathbb{N}_4(0, \widetilde{\Sigma}(t))$ at time t, where $\widetilde{\Sigma}(t) = \int_0^t e^{A^T(t-\tau)}Q^2e^{A(t-\tau)}d\tau$, it is clear $Z(t)$ obeys a unique Gaussian distribution $\mathbb{N}_4(e^{At}Z(0), \widetilde{\Sigma}(t))$. The characteristic polynomial of A is

$$\begin{aligned}\varphi_A(\lambda) = |\lambda\mathbf{I}_4 - A| &= (\lambda + a_{33})(\lambda + a_{44})[\lambda^2 + (a_{11} + a_{22})\lambda + a_{11}a_{22} + a_{12}a_{21}] \\ &= \lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4, \end{aligned} \tag{5.5}$$

where

$$\begin{aligned} a_1 &= a_{11} + a_{22} + a_{33} + a_{44}, \\ a_2 &= a_{11}(a_{22} + a_{33} + a_{44}) + a_{22}(a_{33} + a_{44}) + a_{33}a_{44} + a_{12}a_{21}, \\ a_3 &= (a_{33} + a_{44})a_{11}a_{22} + (a_{11} + a_{22})a_{33}a_{44} + (a_{33} + a_{44})a_{12}a_{21}, \\ a_4 &= (a_{11}a_{22} + a_{12}a_{21})a_{33}a_{44}. \end{aligned}$$

Clearly, it has two eigenvalues $-a_{33} < 0$ and $-a_{44} < 0$. Notice that $a_{11}+a_{22} > 0$ and $a_{11}a_{22} + a_{12}a_{21} > 0$. With the help of Vieta's Theorem , we can get that $\varphi_A(\lambda) = 0$ has four negative real roots. By means of the stability theory of zero solution to the general linear equation [37], we derive $\lim_{t\to\infty} e^{At} = 0$, $\lim_{t\to\infty} e^{At}Z(0) = 0$ and

$$\Sigma := \lim_{t\to\infty} \widetilde{\Sigma}(t) = \lim_{t\to\infty} \int_0^t e^{A^T(t-\tau)}Q^2e^{A(t-\tau)}d\tau = \int_0^\infty e^{A^Tt}Q^2e^{At}dt,$$

where Σ is positive semi-definite due to Q is positive semi-definite. Then, because of the complexity of calculating $\int_0^\infty e^{A^Tt}Q^2e^{At}dt$, the property of Σ can be studied by matrix equation. Applying the continuity of matrix function $e^{A^Tt}Q^2e^{At}$, we have

$$\frac{d}{dt}\int_0^\infty (e^{A^Tt}Q^2e^{At}dt) = A\Sigma + \Sigma A^T, \qquad \int_0^\infty \frac{d}{dt}(e^{A^Tt}Q^2e^{At})dt = Q^2.$$

Thus Σ can be defined by the following algebraic equation:

$$Q^2 + A\Sigma + \Sigma A^T = 0. \tag{5.6}$$

According to the finite independent superposition principle, we can study the corresponding solution of the following two algebraic sub-equations:

$$Q_i^2 + A\Sigma_i + \Sigma_i A^T = 0, (i = 1, 2),$$

where $Q_1 = diag(0, 0, \sigma_1, 0), Q_2 = diag(0, 0, 0, \sigma_2)$. We obtain that $\Sigma = \Sigma_1 + \Sigma_2$ and $Q^2 = Q_1^2 + Q_2^2$. There are two steps to prove.

Step 1. Consider the algebraic equation

$$Q_1^2 + A\Sigma_1 + \Sigma_1 A^T = 0. \tag{5.7}$$

Let

$$J_1 = \begin{pmatrix} 0\,0\,1\,0 \\ 1\,0\,0\,0 \\ 0\,1\,0\,0 \\ 0\,0\,0\,1 \end{pmatrix}.$$

Calculation derives

$$A_1 = J_1 A J_1^{-1} = \begin{pmatrix} -a_{33} & 0 & 0 & 0 \\ a_{13} & -a_{11} & -a_{12} & -a_{14} \\ 0 & a_{21} & -a_{22} & 0 \\ 0 & 0 & 0 & -a_{44} \end{pmatrix}.$$

Hence, equation (5.7) can be obtained by similarity transformation

$$J_1 Q_1^2 J_1^T + J_1 A J_1^{-1} J_1 \Sigma_1 J_1^T + J_1 \Sigma_1 J_1^T (J_1 A J_1^{-1})^T = 0. \tag{5.8}$$

We denote

$$J_1 Q_1 J_1^T = diag(\sigma_1, 0, 0, 0) := H_1. \tag{5.9}$$

$$J_1 \Sigma_1 J_1^T := \hat{\Sigma}_1 = \begin{pmatrix} \hat{\Sigma}_1^{(3)} & 0 \\ 0 & 0 \end{pmatrix}. \tag{5.10}$$

Thus, equation (5.8) can be written as follows

$$H_1^2 + A_1 \hat{\Sigma}_1 + \hat{\Sigma}_1 A_1^T = 0. \tag{5.11}$$

There is a standard transform matrix

$$R_1 = \begin{pmatrix} a_{13}a_{21} & -(a_{11}+a_{22})a_{21} & a_{22}^2 - a_{12}a_{21} & -a_{14}a_{21} \\ 0 & a_{21} & -a_{22} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Here, R_1 can be got by the method in Appendix B. Let

$$B_1 = R_1 A_1 R_1^{-1} = \begin{pmatrix} -d_1 & -d_2 & -d_3 & -d_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -a_{44} \end{pmatrix},$$

where

$$\begin{aligned}
&d_1 = a_{11} + a_{22} + a_{33} > 0,\\
&d_2 = a_{11}a_{22} + a_{11}a_{33} + a_{22}a_{33} + a_{12}a_{21} > 0,\\
&d_3 = a_{11}a_{22}a_{33} + a_{12}a_{21}a_{33} > 0,\\
&d_4 = a_{14}a_{21}(a_{33} - a_{44}).
\end{aligned}$$

Furthermore, simple calculation yields

$$\begin{aligned}
d_1d_2 - d_3 =&(a_{11} + a_{22} + a_{33})(a_{11}a_{22} + a_{11}a_{33} + a_{22}a_{33} + a_{12}a_{21})\\
&- (a_{11}a_{22}a_{33} + a_{12}a_{21}a_{33})\\
=&(a_{11} + a_{22})(a_{11}a_{22} + a_{11}a_{33} + a_{22}a_{33} + a_{12}a_{21}) + a_{33}(a_{11}a_{33} + a_{22}a_{33})\\
>&0.
\end{aligned}$$

Hence, it follows from (5.11) that

$$R_1H_1^2R_1^T + R_1A_1R_1^{-1}R_1\hat{\Sigma}_1R_1^T + R_1\hat{\Sigma}_1R_1^T(R_1A_1R_1^{-1})^T = 0.$$

That is

$$G_1^2 + B_1\hat{\Sigma}_1^* + \hat{\Sigma}_1^*B_1^T = 0, \tag{5.12}$$

where

$$G_1 = diag(a_{13}a_{21}\sigma_1, 0, 0, 0), \tag{5.13}$$

$$\hat{\Sigma}_1^* = R_1\hat{\Sigma}_1R_1^T. \tag{5.14}$$

By substituting matrices G_1 and B_1 into equation (5.12), one gets

$$\hat{\Sigma}_1^* = \begin{pmatrix}
\frac{d_2(a_{13}a_{21}\sigma_1)^2}{2(d_1d_2 - d_3)} & 0 & -\frac{(a_{13}a_{21}\sigma_1)^2}{2(d_1d_2 - d_3)} & 0\\
0 & \frac{(a_{13}a_{21}\sigma_1)^2}{2(d_1d_2 - d_3)} & 0 & 0\\
-\frac{(a_{13}a_{21}\sigma_1)^2}{2(d_1d_2 - d_3)} & 0 & \frac{d_1(a_{13}a_{21}\sigma_1)^2}{2d_3(d_1d_2 - d_3)} & 0\\
0 & 0 & 0 & 0
\end{pmatrix}. \tag{5.15}$$

Since the elements of the first row of matrix $B_1^{(3)}$ satisfy $d_1 > 0$, $d_3 > 0$ and $d_1d_2 - d_3 > 0$, we get $\hat{\Sigma}_1^{*(3)}$ is positive definite according to Lemma 5.1. Thus, $\hat{\Sigma}_1^{(3)}$ is positive definite. Assume that l_1 is the minimum eigenvalue of $\hat{\Sigma}_1^{(3)}$. We obtain

$$\hat{\Sigma}_1^{(3)} \succeq l_1 diag(1,1,1) \ \textit{and} \ \hat{\Sigma}_1 \succeq l_1 diag(1,1,1,0).$$

It follows from (5.10) that

$$\Sigma_1 = J_1^{-1}\hat{\Sigma}_1(J_1^{-1})^T \succeq l_1J_1^{-1}diag(1,1,1,0)(J_1^{-1})^T = l_1 \begin{pmatrix}
1&0&0&0\\
0&1&0&0\\
0&0&1&0\\
0&0&0&0
\end{pmatrix}. \tag{5.16}$$

It is easy to see that

$$Z^T \Sigma_1 Z \succeq l_1(z_1^2 + z_2^2 + z_3^2). \tag{5.17}$$

Combining (5.10) and (5.14), one gets

$$\Sigma_1 = (R_1 J_1)^{-1} \hat{\Sigma}_1^* [(R_1 J_1)^{-1}]^T. \tag{5.18}$$

Step 2. Consider the algebraic equation

$$Q_2^2 + A\Sigma_2 + \Sigma_2 A^T = 0. \tag{5.19}$$

Let

$$J_2 = \begin{pmatrix} 0\,0\,0\,1 \\ 1\,0\,0\,0 \\ 0\,1\,0\,0 \\ 0\,0\,1\,0 \end{pmatrix},$$

such that

$$A_2 = J_2 A J_2^{-1} = \begin{pmatrix} -a_{44} & 0 & 0 & 0 \\ -a_{14} & -a_{11} & -a_{12} & a_{13} \\ 0 & a_{21} & -a_{22} & 0 \\ 0 & 0 & 0 & -a_{33} \end{pmatrix}.$$

By similarity transformation, equation (5.19) can be transformed into the following equation:

$$J_2 Q_2^2 J_2^T + J_2 A J_2^{-1} J_2 \Sigma_2 J_2^T + J_2 \Sigma_2 J_2^T (J_2 A J_2^{-1})^T = 0. \tag{5.20}$$

We denote

$$J_2 Q_2 J_2^T = diag(\sigma_2, 0, 0, 0) := H_2. \tag{5.21}$$

$$J_2 \Sigma_2 J_2^T := \hat{\Sigma}_2 = \begin{pmatrix} \hat{\Sigma}_2^{(3)} & 0 \\ 0 & 0 \end{pmatrix}. \tag{5.22}$$

Thus, equation (5.20) can be written as follows

$$H_2^2 + A_2 \hat{\Sigma}_2 + \hat{\Sigma}_2 A_2^T = 0. \tag{5.23}$$

There exists a standard transform matrix R_2, which can be obtained by the method in Appendix B.

$$R_2 = \begin{pmatrix} -a_{14}a_{21} & -(a_{11}+a_{22})a_{21} & a_{22}^2 - a_{12}a_{21} & a_{13}a_{21} \\ 0 & a_{21} & -a_{22} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then direct calculation yields

$$B_2 = R_2 A_2 R_2^{-1} = \begin{pmatrix} -b_1 & -b_2 & -b_3 & -b_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -a_{33} \end{pmatrix},$$

where

$$\begin{aligned} b_1 &= a_{11} + a_{22} + a_{44} > 0, \\ b_2 &= a_{11}a_{22} + a_{11}a_{44} + a_{22}a_{44} + a_{12}a_{21} > 0, \\ b_3 &= a_{11}a_{22}a_{44} + a_{12}a_{21}a_{44} > 0, \\ b_4 &= a_{13}a_{21}(a_{33} - a_{44}). \end{aligned}$$

Moreover, calculation derives

$$\begin{aligned} b_1 b_2 - b_3 =& (a_{11} + a_{22} + a_{44})(a_{11}a_{22} + a_{11}a_{44} + a_{22}a_{44} + a_{12}a_{21}) \\ & - (a_{11}a_{22}a_{44} + a_{12}a_{21}a_{44}) \\ =& (a_{11} + a_{22})(a_{11}a_{22} + a_{11}a_{44} + a_{22}a_{44} + a_{12}a_{21}) + a_{44}(a_{11}a_{44} + a_{22}a_{44}) \\ >& 0. \end{aligned}$$

Therefore, from (5.23), it follows that

$$R_2 H_2^2 R_2^T + R_2 A_2 R_2^{-1} R_2 \hat{\Sigma}_2 R_2^T + R_2 \hat{\Sigma}_2 R_2^T (R_2 A_2 R_2^{-1})^T = 0.$$

That is

$$G_2^2 + B_2 \hat{\Sigma}_2^* + \hat{\Sigma}_2^* B_2^T = 0, \tag{5.24}$$

where

$$G_2 = diag(a_{14}a_{21}\sigma_2, 0, 0, 0), \tag{5.25}$$

$$\hat{\Sigma}_2^* = R_2 \hat{\Sigma}_2 R_2^T. \tag{5.26}$$

Substituting matrices G_2 and B_2 into equation (5.24) gets

$$\hat{\Sigma}_2^* = \begin{pmatrix} \frac{b_2(a_{14}a_{21}\sigma_2)^2}{2(b_1b_2 - b_3)} & 0 & -\frac{(a_{14}a_{21}\sigma_2)^2}{2(b_1b_2 - b_3)} & 0 \\ 0 & \frac{(a_{14}a_{21}\sigma_2)^2}{2(b_1b_2 - b_3)} & 0 & 0 \\ -\frac{(a_{14}a_{21}\sigma_2)^2}{2(b_1b_2 - b_3)} & 0 & \frac{b_1(a_{14}a_{21}\sigma_2)^2}{2b_3(b_1b_2 - b_3)} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{5.27}$$

Since the elements of the first row of matrix $B_2^{(3)}$ satisfy the conditions of $b_1 > 0$, $b_3 > 0$ and $b_1 b_2 - b_3 > 0$, one gets $\hat{\Sigma}_2^{*(3)}$ is positive definite from Lemma 5.1. Hence,

$\hat{\Sigma}_2^{(3)}$ is positive definite. Assume that l_2 is the minimum eigenvalue of $\hat{\Sigma}_2^{(3)}$. We obtain

$$\hat{\Sigma}_2^{(3)} \succeq l_2 diag(1,1,1) \ \ and \ \ \hat{\Sigma}_2 \succeq l_2 diag(1,1,1,0).$$

It follows from (5.22) that

$$\Sigma_2 = J_2^{-1}\hat{\Sigma}_2(J_2^{-1})^T \succeq l_2 J_2^{-1} diag(1,1,1,0)(J_2^{-1})^T = l_2 \begin{pmatrix} 1&0&0&0\\ 0&1&0&0\\ 0&0&0&0\\ 0&0&0&1 \end{pmatrix}. \tag{5.28}$$

Evidently

$$Z^T \Sigma_2 Z \succeq l_2(z_1^2 + z_3^2 + z_4^2). \tag{5.29}$$

Combining (5.22) and (5.26), one gains

$$\Sigma_2 = (R_2 J_2)^{-1}\hat{\Sigma}_2^*[(R_2 J_2)^{-1}]^T. \tag{5.30}$$

Thus, given the above two cases, from (5.17) and (5.29), we get

$$Z^T \Sigma Z \succeq \min\{l_1, l_2\}(2z_1^2 + z_2^2 + 2z_3^2 + z_4^2).$$

And from (5.18) and (5.30), one can easily see that

$$\Sigma = (R_1 J_1)^{-1}\hat{\Sigma}_1^*[(R_1 J_1)^{-1}]^T + (R_2 J_2)^{-1}\hat{\Sigma}_2^*[(R_2 J_2)^{-1}]^T.$$

It is clear that Σ is positive definite. Namely, the solution follows an exact normal density function around $(I^*, R^*, \bar{\beta}, \bar{\beta}_e)$. The proof has been completed. □

6. Simulations

In this section, we will give some examples and numerical simulations to validate the above theories. Using higher-order numerical methods of Milstein [38], we obtain the corresponding discretization equation of system (1.5)

$$\begin{cases} I^i = I^{i-1} + \left[\left(\beta^{i-1} - \dfrac{\beta_e^{i-1} I^{i-1}}{a + I^{i-1}}\right)(1 - I^{i-1} - R^{i-1})I^{i-1} - (\lambda+\mu)I^{i-1}\right]\Delta t, \\ R^i = R^{i-1} + \left[\lambda I^{i-1} - (\mu+\gamma)R^{i-1}\right]\Delta t, \\ \beta^i = \beta^{i-1} + \theta_1(\bar{\beta} - \beta^{i-1})\Delta t + \sigma_1\sqrt{\Delta t}\delta_{1,j}, \\ \beta_e^i = \beta_e^{i-1} + \theta_2(\bar{\beta}_e - \beta_e^{i-1})\Delta t + \sigma_2\sqrt{\Delta t}\delta_{1,j}, \end{cases} \tag{6.1}$$

where $(I^i, R^i, \beta^i, \beta_e^i)^T$ is the value of the i-th iteration of the discretization equation (6.1). The time increment $\Delta t > 0$. $\delta_{i,j}$ is a random variable which obeys the Gaussian distribution $\mathbb{N}(0,1)$ for $i = 1,2$; $j = 1,2,\ldots,n$. Let the initial value $(I(0), R(0), \beta(0), \beta_e(0)) = (0.2, 0.2, 0.8, 0.8)$.

6.1. The impact of $\bar{\beta}$

In this part, we study the impact of average transmission rate $\bar{\beta}$ on the long-term behavior of epidemic system (1.5). The parameter values for numerical simulations are shown as follows:

$$a = 1; \bar{\beta}_e = 0.1; \gamma = 0.05; \lambda = 0.2; \mu = 0.1; \sigma_1 = 0.2; \sigma_2 = 0.2; \theta_1 = 2; \theta_2 = 2.$$

Fig. 1 shows the variation trends of R_0, R_0^* and R_0^s with $\bar{\beta} \in [0.1, 0.5]$. We have the following conclusions:

- The disease of system (1.5) will go to extinction if $0 \leq \bar{\beta} < 0.2402$.
- There is at least one stationary distribution when $\bar{\beta} > 0.3598$.
- The solution follows a normal density function when $\bar{\beta} > 0.3000$.

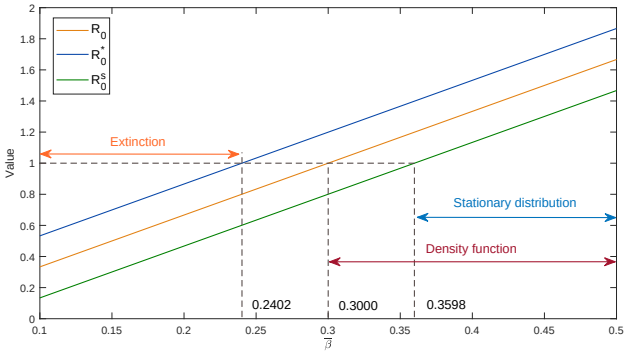

Figure 1. The variation trends of R_0, R_0^* and R_0^s with the variable $\bar{\beta} \in [0.1, 0.5]$.

Example 6.1. (Persistence). We focus on the persistence of the disease when $\bar{\beta} > 0.3598$. For the following four cases: (i) $\bar{\beta}$=0.5, (ii) $\bar{\beta}$=0.6, (iii) $\bar{\beta}$=0.7, (iv) $\bar{\beta}$=0.8, Fig. 2 presents the solution $(I(t),R(t))$ of system (1.5).

Then, choosing $\bar{\beta} = 0.5$, we can compute that $R_0 = 1.6667 > 1$ and $R_0^s = 1.4672 > 1$. If $R_0^s > 1$, system (1.5) has at least a stationary distribution. Besides, $R_0 = 1.6667 > 1$, the solution has a normal probability density function $\Phi \sim \mathbb{N}_4(E_1^*, \Sigma_1)$, where $E_1^* = (0.1640, 0.2186, 0.5, 0.1)$ and

$$\Sigma_1 = \begin{pmatrix} 0.000366 & 0.000226 & 0.000483 & -0.000068 \\ 0.000226 & 0.000302 & 0.000045 & -0.000006 \\ 0.000483 & 0.000045 & 0.01 & 0 \\ -0.000068 & -0.000006 & 0 & 0.01 \end{pmatrix}.$$

Furthermore, we can calculate the joint density function of (I, R):

$$\begin{aligned} &\Phi(I,R) \\ =&652.7130e^{-2539.6932(I-0.1640)^2+3801.1302(I-0.1640)(R-0.2186)-3077.9064(R-0.2186)^2}. \end{aligned}$$

The left of Fig. 3 reflects the numbers of I and R in both stochastic system (1.5) and deterministic system (1.4). The right of Fig. 3 shows the frequency distribution

histogram of the stochastic solution. Fig. 4 depicts the joint density function of (I, R).

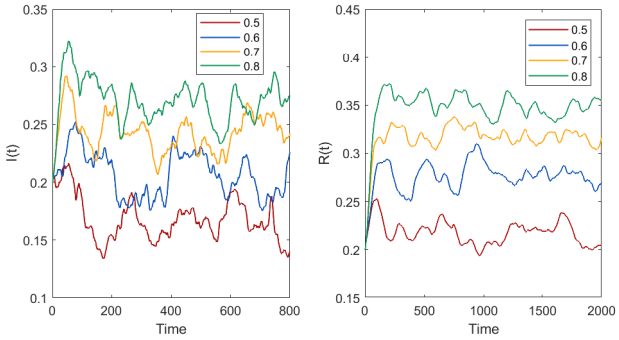

Figure 2. The stochastic solution $(I(t),R(t))$ under $\bar{\beta}$ =0.5, 0.6, 0.7, 0.8, respectively.

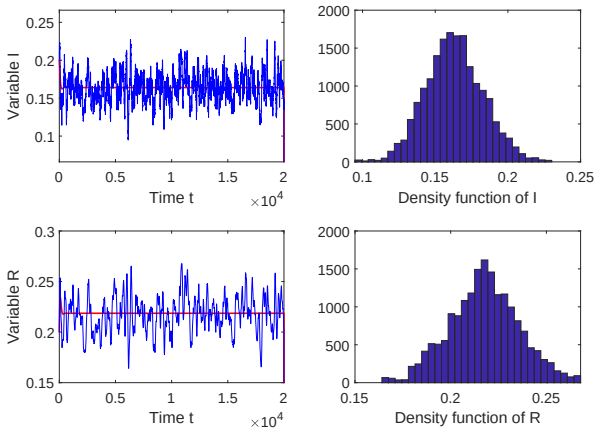

Figure 3. Left-hand column shows the numerical simulations of the solution $(I(t), R(t))$ in stochastic system (1.5) (blue) and its corresponding deterministic system (red) when $\bar{\beta} = 0.5$. Right-hand column reflects the frequency distribution histogram of I and R of system (1.5).

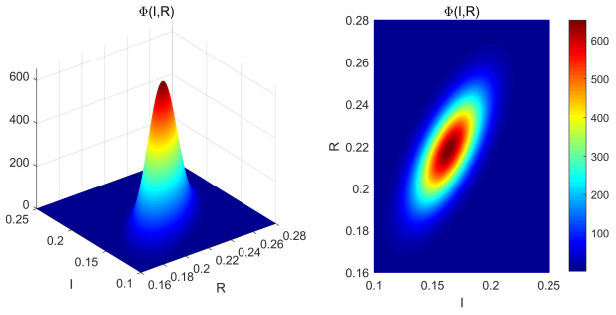

Figure 4. Computer simulations for $\Phi(I, R) \subseteq [0.1, 0.25] \times [0.16, 0.28]$.

Example 6.2. (Extinction). If $\bar{\beta} \in [0, 0.2402)$, the disease of system (1.5) will go to extinction almost surely. Hence, we choose $\bar{\beta}$=0.05, 0.1, 0.15, 0.2 to study the impact of $\bar{\beta}$ on disease extinction in Fig. 5. Fig. 6 shows simulations of the solution $(I(t),R(t))$ in deterministic model (1.4) and stochastic model (1.5) when $\bar{\beta}$=0.2.

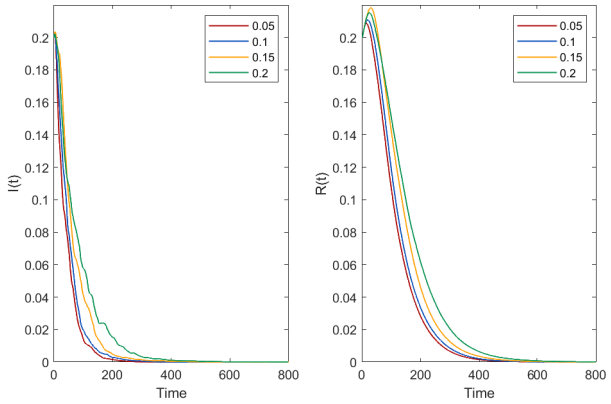

Figure 5. The stochastic solution $(I(t),R(t))$ under $\bar{\beta}$ =0.05, 0.1, 0.15, 0.2, respectively.

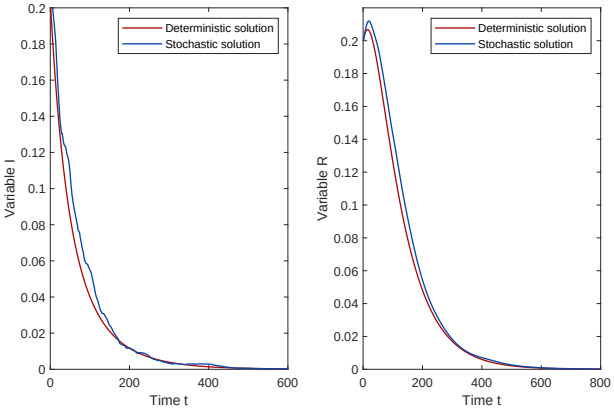

Figure 6. The numbers of infected individuals I and recovered individuals R in stochastic system (1.5) and its deterministic system (1.4) when $\bar{\beta}$=0.2.

Summing up, Fig. 3 and Fig. 6 show that the stochastic solution $(I(t), R(t))$ fluctuates around the deterministic solution regardless of the persistence or extinction. In other words, the long-time behavior of stochastic epidemic model (1.5) is consistent with that of the corresponding deterministic model (1.4). Obviously, the numbers of infectious individuals I and recovered individuals R will decrease and the rate of disease extinction will be faster when $\bar{\beta}$ decreases in Fig. 2 and Fig. 5. To put it another way, the small $\bar{\beta}$ plays a positive and effective role in preventing the spread of the disease. Therefore, during the COVID-19 pandemic, the public health department publishes some effective prevention measures, which can

reduce the contact rate $\bar{\beta}$. Measures include wearing a mask, putting in quarantine, keeping maximum social distancing and lockdown of the city.

6.2. The impact of $\bar{\beta}_e$

In this part, for the dynamics of system (1.5), we will focus on the impact of the maximum reduced contact rate $\bar{\beta}_e$ due to coverage media. The parameters are shown as follows:

$$a = 1; \bar{\beta} = 0.8; \gamma = 0.05; \lambda = 0.3; \mu = 0.1; \sigma_1 = 0.1; \sigma_2 = 0.1; \theta_1 = 3; \theta_2 = 3.$$

Example 6.3. We consider four cases: (i) $\bar{\beta}_e = 0$, (ii) $\bar{\beta}_e = 0.2$, (iii) $\bar{\beta}_e = 0.6$, (iv) $\bar{\beta}_e = 0.8$. Fig. 7 presents paths of $I(t)$ and $R(t)$ for stochastic system (1.5) and deterministic system (1.4). Looking at Fig. 7 from left to right, we can find that the numbers of infectious individuals and recovered individuals decrease as $\bar{\beta}_e$ increases and the stochastic solution $(I(t), R(t))$ fluctuates around the endemic equilibrium of system (1.4). It is easy to conclude that the spread of the infectious diseases can be effectively controlled by media coverage.

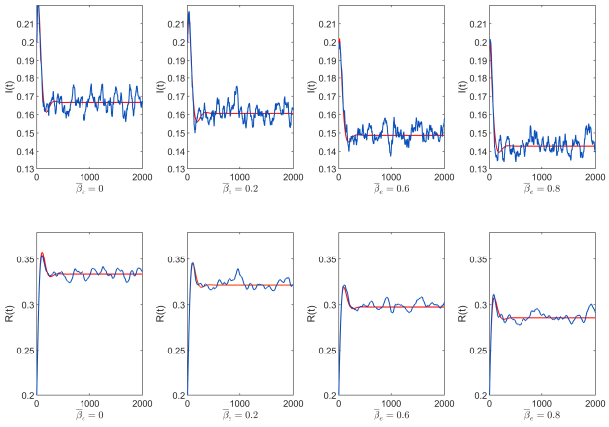

Figure 7. The deterministic solution (red) and stochastic solution (blue) for $I(t)$ and $R(t)$ when $\bar{\beta}_e$=0, 0.2, 0.6, 0.8, respectively.

6.3. The impact of μ

In this part, we concentrate on the impact of the natural birth and death rate μ on long-time behavior of model (1.5). Fig. 8 depicts the variation tendency of R_0, R_0^* and R_0^s with variable $\mu \in [0, 0.4]$. Fig. 8 indicates that the existence of stationary distribution of system (1.5) when $\mu \in [0, 0.0522)$. And the stationary distribution obeys a normal density function when $\mu \in [0, 0.1)$. Besides, the disease of system (1.5) will die out if $\mu > 0.2572$. The numerical simulation parameters are shown below:

$$a = 1; \bar{\beta}_e = 0.1; \gamma = 0.05; \lambda = 0.2; \sigma_1 = 0.2; \sigma_2 = 0.2; \theta_1 = 2; \theta_2 = 2; \bar{\beta} = 0.3.$$

Example 6.4. For the following four cases of μ: (i) μ=0.01, (ii) μ=0.05, (iii) μ=0.3, (iv) μ= 0.4, Figure. 9 gives information about the stochastic dynamics of

$I(t)$ and $R(t)$. Clearly, the numbers of infected individuals and recovery individuals will decline as μ rises. The numerical simulations verify a conclusion that a big μ can result in disease extinction.

Then, choosing $\mu = 0.05$, we can compute that $R_0 = 1.2 > 1$, which means that the solution has a normal probability density function $\Phi \sim \mathbb{N}_4(E_2^*, \Sigma_2)$, where $E_2^* = (0.0510, 0.1020, 0.3, 0.1)$ and

$$\Sigma_2 = \begin{pmatrix} 0.000234 & 0.000320 & 0.000214 & -0.000010 \\ 0.000320 & 0.000640 & 0.000020 & -0.000001 \\ 0.000214 & 0.000020 & 0.01 & 0 \\ -0.000010 & -0.000001 & 0 & 0.01 \end{pmatrix}.$$

In addition, we can calculate the joint density function of (I, R):

$$\begin{aligned} &\Phi(I, R) \\ =&731.3315e^{-6756.7568(I-0.0510)^2+6756.7568(I-0.0510)(R-0.1020)-2470.4392(R-0.1020)^2}. \end{aligned}$$

Computer simulations for the joint density function of $\Phi(I, R)$ are given in Fig. 10.

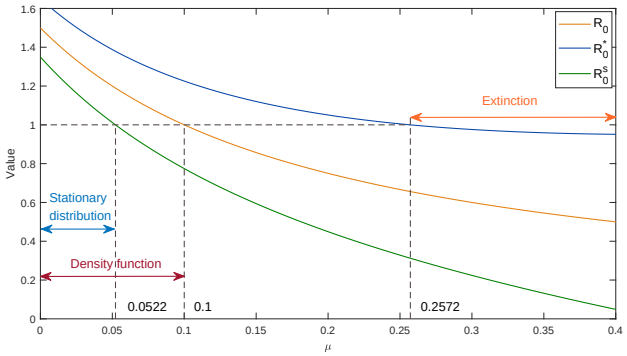

Figure 8. The variation trends of R_0, R_0^* and R_0^s with the variable $\mu \in [0, 0.4]$.

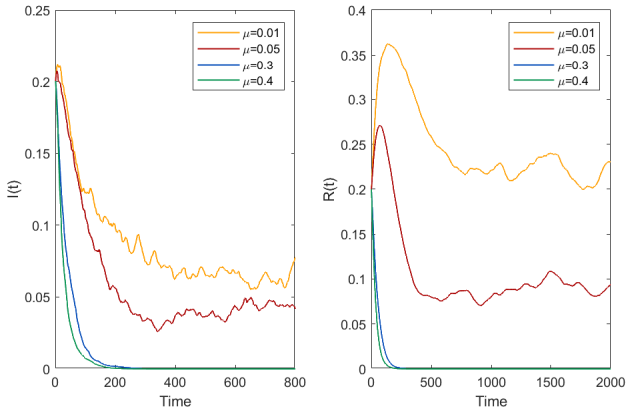

Figure 9. The stochastic solution $(I(t),R(t))$ under μ =0.01, 0.05, 0.3, 0.4, respectively.

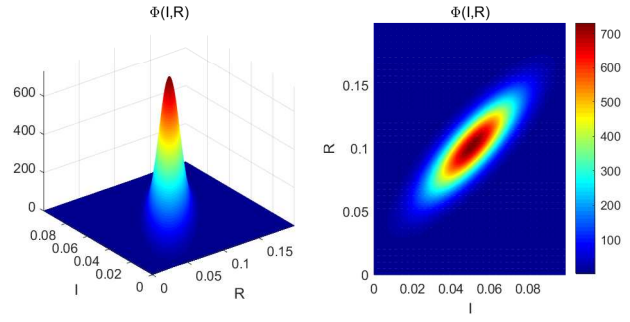

Figure 10. Computer simulations for $\Phi(I,R) \subseteq [0,0.1] \times [0,0.2]$.

6.4. The impact of speed of reversions

In this part, we separately study the speed of reversions θ_1 and θ_2 on the dynamical behavior of model (1.5) by the method of controlling variables. Assume that the parameters of epidemic model (1.5) are given by:

$$a = 1, \gamma = 0.05, \lambda = 0.2, \mu = 0.2, \bar{\beta}_e = 0.1, \sigma_1 = 0.2, \sigma_2 = 0.2.$$

Example 6.5. (Persistence). We choose $\bar{\beta}$=0.6 to guarantee disease persistence of the deterministic system. We separately consider two cases: (i) fixing $\theta_2 = 3$ to study the effect of θ_1; (ii) fixing $\theta_1 = 3$ to study the impact of θ_2. For case (i), we can directly see that $R_0^s > 1$ when $\theta_1 \in [0, 12.8757)$ in the left of Fig. 11. The specific changes of I and R with θ_1=1, 2 and 3 can be seen in Fig. 12. In a similar way, the right of Fig. 11 exhibits the variation trends of R_0, R_0^* and R_0^s with $\theta_2 \in [0, 10]$ under case (ii). Fig. 13 illustrates the stochastic solution $(I(t), R(t))$ with θ_2=1, 2 and 3. In the above cases, the condition of Theorem 4.1 is satisfied. Thus, the disease will persist in a long time.

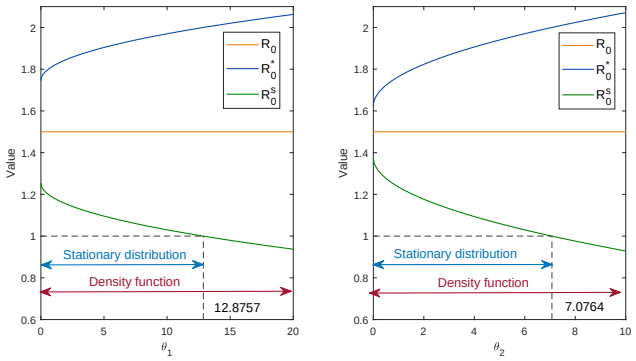

Figure 11. The variation trends of R_0, R_0^* and R_0^s with the variable $\theta_1 \in [0, 20]$ when $\theta_2 = 3$ (left). The variation trends of R_0, R_0^* and R_0^s with the variable $\theta_2 \in [0, 10]$ when θ_1=3 (right).

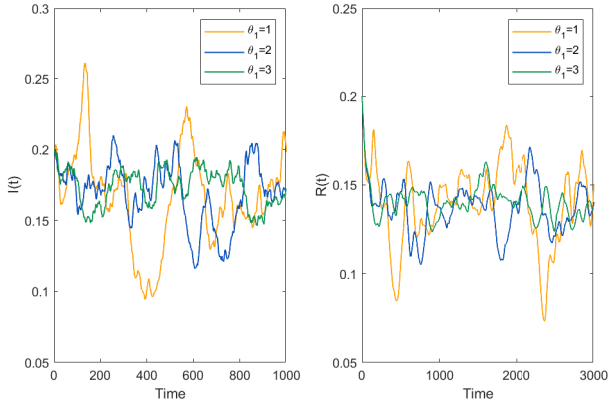

Figure 12. The solutions of I and R under $\theta_1 = 1, 2, 3$ when $\theta_2 = 3$.

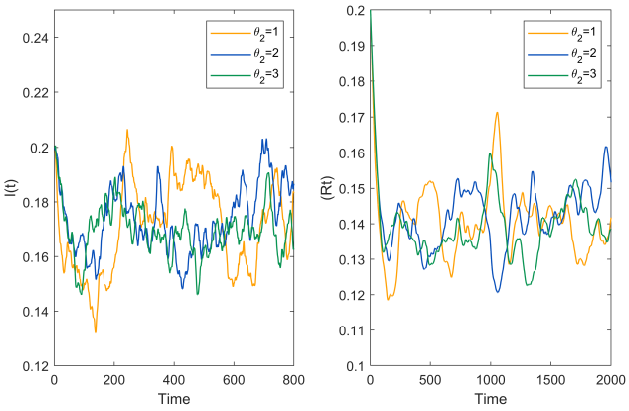

Figure 13. The solutions of I and R under $\theta_2 = 1, 2, 3$ when $\theta_1 = 3$.

Example 6.6. (Extinction). To investigate the extinction of the disease, we choose other parameters $\bar{\beta}$=0.3 to guarantee $R_0 = 0.75 < 1$. We think about two cases: (i) fixing $\theta_2 = 1$ to focus on the effect of θ_1; (ii) fixing $\theta_1 = 1$ to explore the impact of θ_2. The left of Fig. 14 shows the variation trends of R_0, R_0^* and R_0^s with $\theta_1 \in [0, 3.5]$ under case (i) and the disease will die out when $\theta_1 \in [0, 2.3292)$. Fig. 15 explicitly reveals the numbers of $I(t)$ and $R(t)$ with different θ_1= 0.5, 1 and 1.5. Similarly, for case (ii), the right of Fig. 14 depicts the variation trends when $\theta_2 \in [0, 3]$. If θ_2=0.5, 1 and 1.5, we can get $R_0^* < 1$, which means that the disease will eventually go extinct (see Fig. 16).

From Figs. 12-13 and 15-16, we can observe that a small speed of reversions has an unfavorable effect on the stabilization whether persistence or extinction.

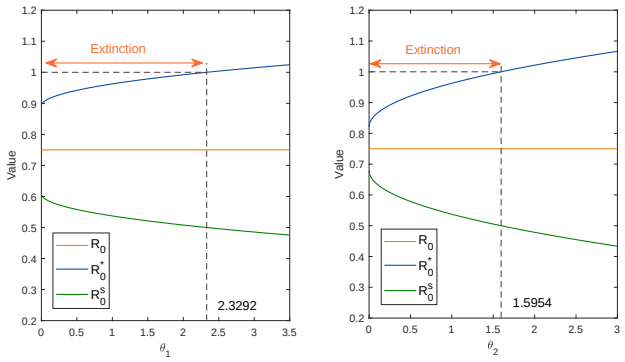

Figure 14. The variation trends of R_0, R_0^* and R_0^s with the variable $\theta_1 \in [0, 3.5]$ when $\theta_2 = 1$ (left). The variation trends of R_0, R_0^* and R_0^s with the variable $\theta_2 \in [0, 3]$ when θ_1=1 (right).

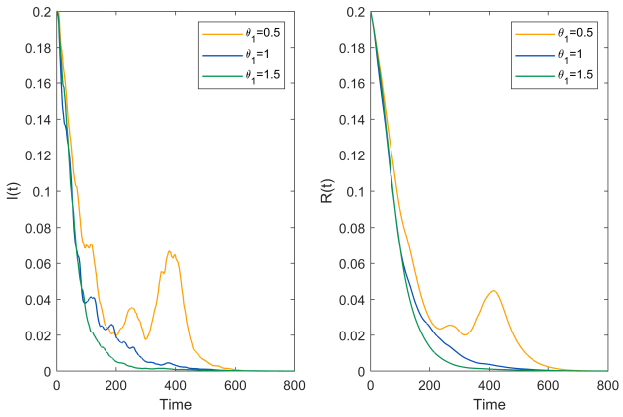

Figure 15. The solutions of I and R under $\theta_1 = 0.5, 1, 1.5$ when $\theta_2 = 1$.

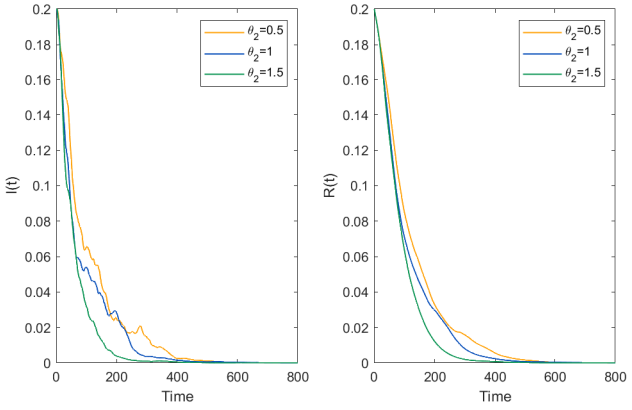

Figure 16. The solutions of I and R under $\theta_2 = 0.5, 1, 1.5$ when $\theta_1 = 1$.

6.5. The impact of stochastic noises

In this part, we separately discuss the impact of environmental noises σ_1 and σ_2 on long-time behavior of epidemic model (1.5). The numerical simulations use the following parameters:

$$a = 1, \gamma = 0.05, \lambda = 0.2, \mu = 0.2, \bar{\beta}_e = 0.1.$$

Example 6.7. (Persistence). For the prevalence of the disease in deterministic system (1.4), we choose ($\bar{\beta}$, θ_1, θ_2)=(0.55, 2, 2) to guarantee $R_0 = 1.375 > 1$. The left of Fig. 17 shows the trends R_0, R_0^* and R_0^s with σ_1 in the interval $[0, 1]$ when $\sigma_2 = 0$. There exists a stationary distribution if $\sigma_1 \in [0, 0.7480)$. Fig. 18 visualizes the trends of I and R with σ_1= 0, 0.1, 0.2 and 0.3. While, if $\sigma_1 = 0$, the trends R_0, R_0^* and R_0^s with different $\sigma_2 \in [0, 0.6]$ are presented in the right of Fig. 17, which shows the existence of the stationary distribution of system (1.5) when $\sigma_2 \in [0, 0.3740)$. Similar data simulations are demonstrated in Fig. 19.

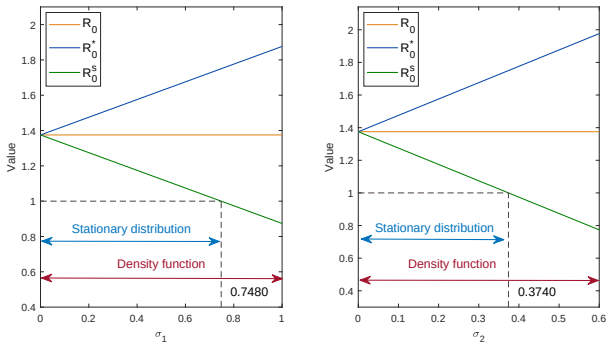

Figure 17. The variation trends of R_0, R_0^* and R_0^s with the variable $\sigma_1 \in [0, 1]$ when $\sigma_2 = 0$ (left). The variation trends of R_0, R_0^* and R_0^s with the variable $\sigma_2 \in [0, 0.6]$ when $\sigma_1 = 0$ (right). Other parameter values ($\bar{\beta}$, θ_1, θ_2)=(0.55, 2, 2).

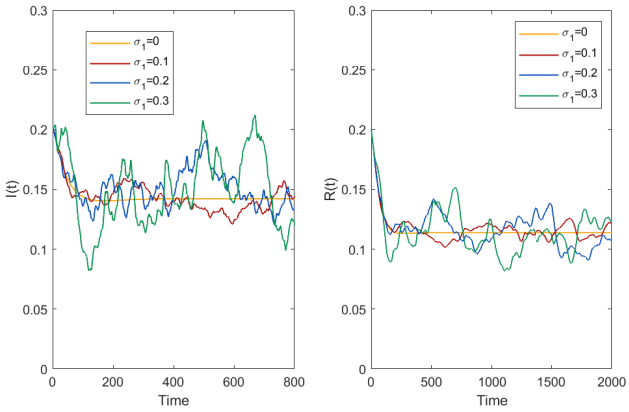

Figure 18. The solutions of I and R under σ_1=0, 0.1, 0.2, 0.3 when $\sigma_2 = 0$.

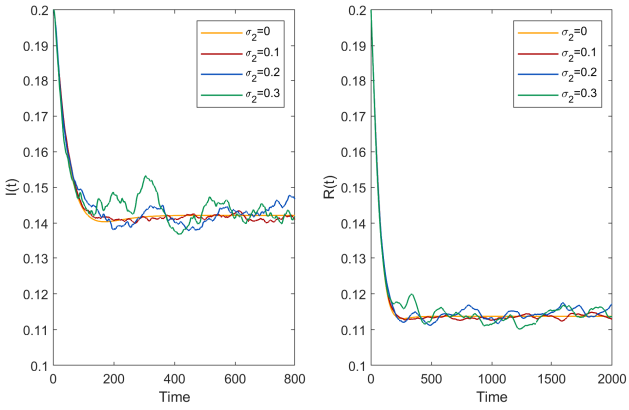

Figure 19. The solutions of I and R under σ_2=0, 0.1, 0.2, 0.3 when $\sigma_1 = 0$.

Example 6.8. (Extinction). We choose ($\bar{\beta}$, θ_1, θ_2)=(0.35, 0.6, 0.6) to guarantee $R_0 = 0.875 < 1$, which implies that the infectious disease will not exist for deterministic system (1.4). The left of Fig. 20 illustrates the trends of R_0, R_0^* and R_0^s with $\sigma_1 \in [0, 1]$ when $\sigma_2 = 0$ and shows disease extinction of system (1.5) when $\sigma_1 \in [0, 0.4552)$. The right of Fig. 20 depicts the similar trends when $\sigma_2 \in [0, 0.5]$ and $\sigma_1 = 0$. If $\sigma_2 \in [0, 0.2276)$, the disease of system (1.5) will be extinct. When σ_2= 0, Fig. 21 describes disease extinction with different σ_1= 0, 0.06, 0.12, 0.18. If $\sigma_1 = 0$, Fig. 22 shows the analogous data simulations.

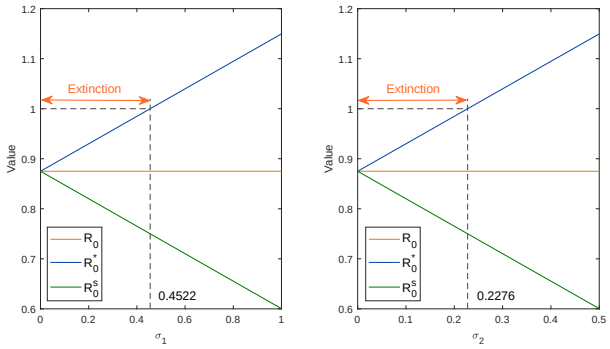

Figure 20. The variation trends of R_0, R_0^* and R_0^s with the variable $\sigma_1 \in [0, 1]$ when $\sigma_2 = 0$ (left). The variation trends of R_0, R_0^* and R_0^s with the variable $\sigma_2 \in [0, 0.5]$ when $\sigma_1 = 0$ (right). Other parameter values ($\bar{\beta}$, θ_1,θ_2)=(0.35, 0.6, 0.6).

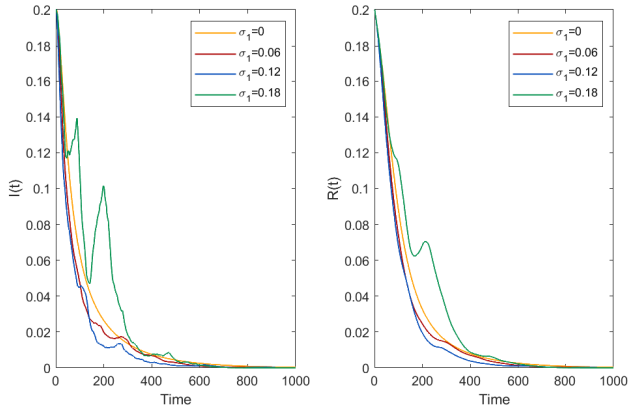

Figure 21. The solutions of I and R under σ_1=0, 0.06, 0.12, 0.18 when $\sigma_2 = 0$.

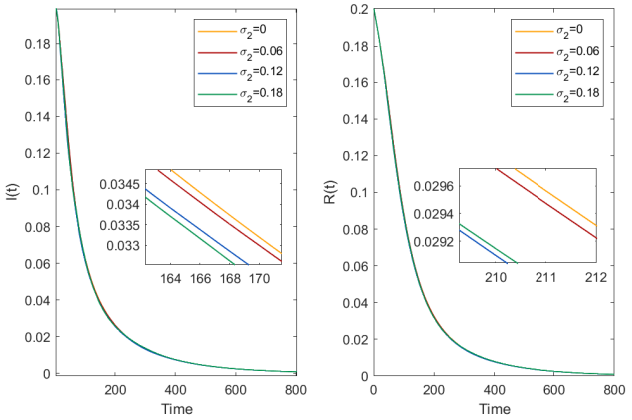

Figure 22. The solutions of I and R under σ_2=0, 0.06, 0.12, 0.18 when $\sigma_1 = 0$.

Summing up, from Figs.18-19 and 21-22, we can find that small stochastic noises make the solution of the stochastic system closer to that of the deterministic system. In other words, the obvious conclusion is that the fluctuation of the solution $(I(t), R(t))$ will become larger as stochastic noises increase.

7. Conclusions

We propose a stochastic SIRS epidemic model with media coverage and two Ornstein-Uhlenbeck processes. We first show the existence and uniqueness of the solution with any initial value. Then, we provide sufficient conditions for extinction and persistence, which gives theoretical support for gaining insight into the complex dynamics of disease transmission. Our results reveal that the disease will be extinct with probability one when $R_0^* := R_0 + \frac{1}{2\sqrt{\pi}(\lambda+\mu)}\left(\frac{\sigma_1}{\sqrt{\theta_1}} + \frac{\sigma_2}{(a+1)\sqrt{\theta_2}}\right) < 1$, and the disease will be persistent in a long term when $R_0^s := R_0 - \frac{1}{2\sqrt{\pi}(\lambda+\mu)}\left(\frac{\sigma_1}{\sqrt{\theta_1}} + \frac{\sigma_2}{(a+1)\sqrt{\theta_2}}\right) > 1$. Besides, if $R_0 = \frac{\bar{\beta}}{\lambda+\mu} > 1$, the solution follows a density function, which provides us a more comprehensive of distribution character near the positive equilibrium point of corresponding deterministic system. To further analyze the effect of parameters in detail, Figs. 1-22 show the impact of $\bar{\beta}$, $\bar{\beta}_e$, μ, the speed of reversions (θ_1, θ_2) and stochastic noises (σ_1, σ_2) on stochastic SIRS model. It is possible to visually see the specific impact of these parameters on the development of epidemic and validate our findings. As a result of our simulations, we conclude that small $\bar{\beta}$ will effectively keep down the spread of the disease. Larger μ and suitable $\bar{\beta}_e$ play a positive role in preventing the spread of the disease. The speed of reversions and the stochastic noises are the main factors affecting the stability of stochastic epidemic model.

Some valuable research issues are worth in-depth study. Firstly, due to the limitation of our mathematical approaches to stochastic epidemic model, it is hard to find a threshold for disease persistence and extinction. Secondly, pulse vaccination is an important policy to control and prevent the spread of the infectious disease [39–41]. Therefore, we expect to learn a stochastic SIRS epidemic model with pulse vaccination and mean-reverting Ornstein-Uhlenbeck process. In addition, in order to analyze and deal with the probability density function more numerically, we need to build a more complete and systematic theory. Finally, it is also a significant question to investigate whether the method used in this paper can be applied to other stochastic models.

Acknowledgements

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Appendix A

We give the specific proof of Lemma 5.1.

For the algebraic equation $\mathcal{K}^2 + \hat{\mathcal{A}}\bar{\Sigma} + \bar{\Sigma}\hat{\mathcal{A}}^T = 0$, where $\mathcal{K} = diag(\tilde{\sigma}, 0, 0)$, $\bar{\Sigma}$ is

a real symmetric matrix, and

$$\hat{A} = \begin{pmatrix} -a_1 & -a_2 & -a_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Solving the algebraic equation obtains

$$\bar{\Sigma} = \begin{pmatrix} \dfrac{a_2\tilde{\sigma}^2}{2(a_1a_2-a_3)} & 0 & -\dfrac{\tilde{\sigma}^2}{2(a_1a_2-a_3)} \\ 0 & \dfrac{\tilde{\sigma}^2}{2(a_1a_2-a_3)} & 0 \\ -\dfrac{\tilde{\sigma}^2}{2(a_1a_2-a_3)} & 0 & \dfrac{a_1\tilde{\sigma}^2}{2a_3(a_1a_2-a_3)} \end{pmatrix} := (\theta_{ij})_{(3\times 3)},$$

where $\theta_{12} = \theta_{21} = \theta_{23} = \theta_{32} = 0$. If $a_1 > 0$, $a_3 > 0$ and $a_1a_2 - a_3 > 0$, then we can get

$$\theta_{11} = a_2\frac{\tilde{\sigma}^2}{2(a_1a_2-a_3)} > 0,\ \theta_{11}\theta_{22} = a_2\frac{\tilde{\sigma}^4}{4(a_1a_2-a_3)^2} > 0,$$
$$\theta_{22}(\theta_{11}\theta_{33} - \theta_{13}^2) = \theta_{22}(\frac{a_1a_2}{a_3} - 1)\frac{\tilde{\sigma}^2}{4(a_1a_2-a_3)^2} > 0,$$

which implies that all leading principal minors of the matrix $\bar{\Sigma}$ are positive. Hence $\bar{\Sigma}$ is positive definite. The proof of the Lemma 5.1 is confirmed. □

Appendix B

The method of transforming standard is provided.

Consider the algebraic equation $\tilde{G}^2 + \tilde{\mathcal{A}}\breve{\Sigma} + \breve{\Sigma}\tilde{\mathcal{A}}^T = 0$, where $\tilde{G} = (\sigma, 0, 0, 0)$ and

$$\tilde{A} = \begin{pmatrix} \tilde{a}_{11} & \tilde{a}_{12} & \tilde{a}_{13} & \tilde{a}_{14} \\ \tilde{a}_{21} & \tilde{a}_{22} & \tilde{a}_{23} & \tilde{a}_{24} \\ 0 & \tilde{a}_{32} & \tilde{a}_{33} & \tilde{a}_{34} \\ 0 & 0 & 0 & \tilde{a}_{44} \end{pmatrix}.$$

Assume that $\tilde{a}_{21} \neq 0$, $\tilde{a}_{32} \neq 0$ and $\tilde{a}_{44} \neq 0$. Using the linear transformation of ordinary differential equations, we set $dX = \tilde{A}Xdt$, $X = (x_1, x_2, x_3, x_4)^T$, $Y = (y_1, y_2, y_3, y_4)^T$ and let

$$\begin{cases} y_4 = x_4, \\ y_3 = x_3, \\ y_2 = y_3' = dx_3 = \tilde{a}_{32}x_2 + \tilde{a}_{33}x_3 + \tilde{a}_{34}x_4, \\ y_1 = y_2' = \tilde{a}_{32}dx_2 + \tilde{a}_{33}dx_3 + \tilde{a}_{34}dx_4 \\ \quad = \tilde{a}_{21}\tilde{a}_{32}x_1 + (\tilde{a}_{22}+\tilde{a}_{33})\tilde{a}_{32}x_2 + (\tilde{a}_{23}\tilde{a}_{32}+\tilde{a}_{33}^2)x_3 + (\tilde{a}_{24}\tilde{a}_{32}+\tilde{a}_{33}\tilde{a}_{34}+\tilde{a}_{34}\tilde{a}_{44})x_4. \end{cases}$$

We get

$$\bar{M} = \begin{pmatrix} \tilde{a}_{21}\tilde{a}_{32} & (\tilde{a}_{22}+\tilde{a}_{33})\tilde{a}_{32} & \tilde{a}_{23}\tilde{a}_{32}+\tilde{a}_{33}^2 & \tilde{a}_{24}\tilde{a}_{32}+\tilde{a}_{34}(\tilde{a}_{33}+\tilde{a}_{44}) \\ 0 & \tilde{a}_{32} & \tilde{a}_{33} & \tilde{a}_{34} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Here, $\bar{M}$ is the standard transformation matrix. Then one has

$$dY = \bar{M}dX = \bar{M}\tilde{A}Xdt = \bar{M}\tilde{A}\bar{M}^{-1}Ydt.$$

That is

$$dY = d\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} -e_1 & -e_2 & -e_3 & -e_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -\tilde{a}_{44} \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} dt, \tag{7.1}$$

where

$$\bar{M}\tilde{A}\bar{M}^{-1} = \begin{pmatrix} -e_1 & -e_2 & -e_3 & -e_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -\tilde{a}_{44} \end{pmatrix}.$$

Hence, the standard transform matrix $\bar{M}$ is obtained.

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