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Existence and Uniqueness of Solutions of Nonlinear Integral Equations through Results in Fuzzy Bipolar Metric Spaces

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Abstract In this paper the concept of fuzzy bipolar metric spaces are investigated and their properties in relation with fixed points by the consideration of triangular property of fuzzy bipolar metric are explored. The study presents a series of established results under covariant and contravariant mappings supported by illustrative examples and discusses the implications of these findings. Furthermore, the established results are applied to demonstrate the existence and uniqueness of solutions to nonlinear integral equations. The exploration of fuzzy bipolar metric spaces and their application to integral equations provides valuable insights into the field of mathematical analysis and opens avenues for further research.

Keywords Triangular property, fuzzy metric space, fuzzy bipolar metric space, covariant mappings, contravariant mappings

MSC(2010) 47H10, 54H25.

1. Introduction

Fixed point theory has become a central focus across various disciplines, such as engineering, optimization, physics, economics, and mathematics. The foundation of this theory was significantly bolstered by the Banach fixed point theorem, introduced by Banach [2] which has catalyzed extensive research in both the mathematical and scientific spheres.

In 1975, Kramosil and Michalek [12] introduced a groundbreaking concept of fuzzy metric spaces. This innovation was built upon the groundwork of introduction of the continuous t-norm laid by Schweizer and Sklar [22] in 1960. The pivotal role played by L.A. Zadeh's fuzzy set theory, formulated in 1965 [28], cannot be overstated in this context. Subsequently, George and Veeramani [8] further extended the framework of fuzzy metric spaces by incorporating the Hausdorff topology and adapting established theorems from classical metric spaces. This extension yielded profound results in fuzzy metric space and its generalisations [1,5,9–11,16,20,23–27].

In a more recent mathematical advancement, Mutlu and Gurdal [17] introduced bipolar metric spaces in 2016. Unlike conventional metric spaces, which exclusively explore distances within a single set, bipolar metric spaces allow for the examination of distances between points drawn from two distinct sets. Following this

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development, subsequent researchers [6, 7, 13, 18, 19] delved into the realm of fixed point theorems within bipolar metric spaces, uncovering diverse applications across various contexts. Building upon this foundation, Bartwal et al. [3] introduced the notion of fuzzy bipolar metric spaces, extending the concept of fuzzy metric spaces to this new context by introducing a novel distance measurement scheme for points residing in different sets. This extension paved the way for the derivation of significant fixed point results within fuzzy bipolar metric spaces [4, 14, 15, 21]. The research gap addressed in our work lies in the establishment of new fixed point results in fuzzy bipolar metric spaces, specifically focusing on the generealising the constraint satisfied by self mappings by introducing the concept of control function, and taking under consideration the triangular property of induced fuzzy bipolar metric. While existing literature provides valuable insights into fixed point theory and fuzzy bipolar metric spaces, there is a need to exapand the implications of considering control function in fuzzy bipolar metric spaces. By bridging this research gap, our paper contributes to the theoretical development of generalisation of fuzzy metric spaces and expands the understanding of fixed point theory. With the consideration of control function and self-mappings with triangular property, the existing framework offers a versatile foundation that can be applied across various domains.

In this study, we embark on an exploration of the fundamental constructs of fuzzy bipolar metric spaces, introduced in Section 2. Subsequently, in Section 3, we establish some fixed point results within fuzzy bipolar metric spaces, incorporating the distinctive triangular property inherent in fuzzy bipolar metrics and implimenting the use of control function. Section 4 provides illuminating examples that provide empirical support for the established fixed point results, and Section 5 engages in a comprehensive discussion of the broader implications arising from these findings. Further, in Section 6, we showcase the practical application of these results by demonstrating their utility in establishing the existence and uniqueness of solutions for nonlinear integral equations. Ultimately, in Section 7, we summerize the findings and discussion in conclusion section.

2. Preliminaries

In this section, we provide some fundamental definitions and properties to establish the main results of this article for compact maps.

Definition 2.1. [22] Suppose there is a binary operation $*$ from $[0,1] \times [0,1] \to [0,1]$ is called a continuous t-norm if it satisfies the following conditions:

a) Commutativity and associativity of $*$;

b) Continuity of $*$;

c) $k * 1 = k$, for all $k \in [0,1]$;

d) $j * l \leqslant o * q$ whenever $j \leqslant o$, $l \leqslant q$ and $j, l, o, q \in [0,1]$.

Definition 2.2. [8] Consider a non-empty set M. Let $*$ be a t-norm which is continuous and S be a fuzzy set on $M \times M \times (0, \infty)$. Then, $(M, S, *)$ is called a fuzzy metric space if for all $m, i, j \in M$ and $r, s > 0$, the following conditions hold:

i) $S(m, i, r) > 0$;

ii) $S(m, i, r) = 1$ if and only if $m = i$;

iii) $S(m, i, r) * (i, j, s) \leqslant S(m, j, r + s)$;

v) $S(m, i, \_) : (0, \infty) \to (0, 1]$ is continuous.

Example 2.1. [24] Consider $M = \mathbb{R}^+$. We define

$$S(m,\ i,\ r) = \frac{min(m,\ i)\ +\ r}{max(m,\ i)\ +\ r},\ for\ all\ m,\ i \in M\ and\ r > 0.$$

Also, define $a * b = ab$.

Then, $(M, S, *)$ is a fuzzy metric space.

Definition 2.3. [17] Let two sets $M \neq \phi$ and $I \neq \phi$ are taken into consideration and a function $g : M \times I \rightarrow \mathbb{R}^+ \cup \{0\}$. Then, g is known as a bipolar metric on (M, I) if the following conditions hold:

i) $g\ (m, i) = 0\ iff\ m = i,\ for\ all\ m \in M\ and\ i \in I.$

ii) $g\ (m, i) = g(i, m),\ for\ all\ m, i \in M\ \cap I.$

iii) $g(m_1, i_2) \leqslant g(m_1, i_1) + g(m_2, i_1) + g(m_2, i_2),\ for\ all\ m_1, m_2 \in M,\ i_1, i_2 \in I.$

Further, (M, I, g) is called the bipolar metric space. Here, M and I are called left pole and right pole of (M, I, g) respectively.

Example 2.2. [17] Consider $M = (1, \infty)$ and $I = (-1, 1)$. We define $g : M \times I \rightarrow \mathbb{R}^+ \cup \{0\}$ as $g(m,\ i) = |m^2 - i^2|,\ \forall\ (m,\ i) \in M \times I$. Then (M, I, g) is a bipolar metric space.

Definition 2.4. [3] Let two sets M$\neq \phi$ and I $\neq \phi$ be taken into consideration. Let $*$ be a t-norm(continuous) and consider a fuzzy set S on $M \times I \times (0, \infty)$. Then, the quadruple $(M, I, S, *)$ is said to be a fuzzy bipolar metric space if for all $r, s, t > 0$, the following conditions are satisfied:

i) $S(m, i, r) > 0$, for all $(m, i) \in M \times I$;

ii) $S(m, i, r) = 1 \iff m = i$, for all $m \in M$ and $i \in I$;

iii) $S(m, i, r) = S(i, m, r)$, for all $m \in M$ and $i \in I$;

iv) $S(m_1, i_2,\ r+s+t) \geqslant S(m_1, i_1,\ r) * S(m_2, i_1, s) * S(m_2, i_2, t)$, for all $m_1, m_2 \in M\ and\ i_1, i_2 \in I$;

v) $S(m, i, \_) : [0, \infty) \rightarrow (0, 1]$ is left continuous;

vi) $S(m, i, \_)$ is non decreasing for all $m \in M$ and $i \in I$.

Example 2.3. [3] Consider a bipolar metric space (M, I, g) and let $*$ be a continuous t-norm defined by $p*q = pq$. Now, for all $m \in M,\ i \in I$ and $r > 0$, we define:

$$S(m,\ i,\ r) = \frac{r}{r + g(m,\ i)}.$$

Then, $(M, I, S, *)$ is a fuzzy bipolar metric space.

Definition 2.5. [3] In a fuzzy bipolar metric space $(M, I, S, *)$, M is known as the set of all left points, I is known as the set of all right points and the points which belong to $M \cap I$ are known as central points.

Definition 2.6. [3] A sequence (k_n) on the set M is referred to as a left sequence in a fuzzy bipolar metric space $(M, I, S, *)$, whereas a sequence (l_n) on the set I is referred to as a right sequence.

A left sequence (k_n) converges to a right point ω_1 if for all $\delta > 0$, there exists a number $\gamma \in \mathbb{N}$ such that $S(k_n,\ \omega_1, r) > 1 - \delta$, for all $n > \gamma$ i.e., $\lim_{n\rightarrow\infty} S(k_n, \omega_1,\ r) \rightarrow 1$ for all $r > 0$.

A right sequence (l_n) converges to a left point ω_2 if for all $\delta > 0$, there exists a number $\gamma \in \mathbb{N}$ such that $S(\omega_2,\ l_n,\ r) > 1 - \delta$ for all $n > \gamma$ i.e., $\lim_{n\rightarrow\infty} S(\omega_2,\ l_n,\ r) \rightarrow 1$ for all $r > 0$.

Definition 2.7. [3] In a fuzzy bipolar metric space, a sequence (k_n, l_n) is called a bisequence on $M \times I$ and it is said to be convergent if both (k_n) and (l_n) are convergent. If both the sequence converges to a common point $\omega \in M \cap I$ then (k_n, l_n) is called biconvergent.

The bisequence (k_n, l_n) in a fuzzy bipolar metric space $(M, I, S, *)$ is called a Cauchy bisequence if for any $\delta > 0$, there exists a number $\gamma \in \mathbb{N}$ such that for all $e, f \geqslant \delta$ and $e, f \in \mathbb{N}$, we have

$$S(k_e, l_f,\ r) > \ 1 - \delta,\ \forall\ r > 0.$$

In other words, (k_n, l_n) is a Cauchy bisequence if

$$\lim_{\substack{e \to \infty \\ f \to \infty}} S(k_e, l_f,\ r) \to \ 1,\ \forall\ r > 0.$$

Lemma 2.1. *[3] In a fuzzy bipolar metric space, if a Cauchy bisequence is convergent then it is biconvergent.*

Definition 2.8. [3] A fuzzy bipolar metric space $(M, I, S, *)$ is purported to be complete if each Cauchy bisequence within $M \times I$ achieves convergence within it.

Definition 2.9. [3] In a fuzzy bipolar metric space $(M, I, S, *)$, a limit $u \in M \cap I$ of a bisequence is always unique.

Definition 2.10. Consider two fuzzy bipolar metric space $(M_1, I_1, S_1, *)$ and $(M_2, I_2, S_2, *)$ and a function $d : M_1 \cup I_1 \to M_2 \cup I_2$. Then,

i) If $d(M_1) \subseteq M_2$ and $d(I_1) \subseteq I_2$ then d is called a covariant map from $(M_1, I_1, S_1, *)$ to $(M_2, I_2, S_2, *)$ and it is denoted by $d\colon(M_1, I_1, S_1, *) \rightrightarrows (M_2, I_2, S_2, *)$.

ii) If $d(M_1) \subseteq I_2$ and $d(I_1) \subseteq M_2$ then d is called a contravariant map from $(M_1, I_1, S_1, *)$ to $(M_2, I_2, S_2, *)$ and it is denoted by $d\colon(M_1, I_1, S_1, *) \nearrow\!\!\!\!\searrow (M_2, I_2, S_2, *)$.

Definition 2.11. [15] Let $(M, I, S, *)$ be a fuzzy bipolar metric space. The fuzzy bipolar metric S is said to be triangular if,

$$\frac{1}{S(m_1, i_2, r)} - 1 \leqslant \Big(\frac{1}{S(m_1, i_1, r)} - 1\Big) + \Big(\frac{1}{S(m_2, i_1, r)} - 1\Big) + \Big(\frac{1}{S(m_2, i_2, r)} - 1\Big),$$

for all $m_1, m_2 \in M$ and $i_1, i_2 \in I$.

Lemma 2.2. *Consider a fuzzy bipolar metric space $(M, I, S, *)$ where $*$ is a continuous t-norm and $S : M \times I \times (0, \infty) \longrightarrow [0, 1]$ is defined as,*

$$S(m,\ i,\ r) = \ \frac{r}{r + g(m,\ i)},$$

where $g(m, i)$ is a bipolar metric on $M \times I$. Then, the fuzzy bipolar metric S is triangular.

Proof. For any $m_1, m_2 \in M$ and $i_1, i_2 \in I$, we have

$$\begin{aligned}\frac{1}{S(m_1, i_2, r)} - 1 &= \frac{g(m_1, i_2)}{r} \\ &\leqslant \frac{g(m_1, i_1)}{r} + \frac{g(m_2, i_1)}{r} + \frac{g(m_2, i_2)}{r}\end{aligned}$$

$$= \Big(\frac{1}{S(m_1, i_1, r)} - 1\Big) + \Big(\frac{1}{S(m_2, i_1, r)} - 1\Big) + \Big(\frac{1}{S(m_2, i_2, r)} - 1\Big).$$

Therefore, S is triangular. □

Example 2.4. Consider a fuzzy bipolar metric space $(M, I, S, *)$ with $*$ as a continuous t-norm defined by $p * q = min\{p, q\}$ and $S : M \times I \times (0, \infty) \longrightarrow [0, 1]$ defined by,

$$S(m,\ i,\ r) = \frac{r}{r + |m - i|},$$

Then, by following the steps in the proof of Lemma 2.2, it is easy to verify that S is triangular.

3. Main results

In this section, we present fixed point theorems in complete fuzzy bipolar metric spaces for covariant mappings and contravariant mappings incorporating control functions and triangular property of fuzzy bipolar metric.

Definition 3.1. Assume $\eta : [0, \infty] \to [0, \infty]$ as a function which satisfies the below-stated conditions:
(P1) η is left continuous and strictly increasing;
(P2) $\eta(\lambda + \mu) \leqslant \eta(\lambda) + \eta(\mu)$;
(P3) $\eta(\lambda) = 0 \Leftrightarrow \lambda = 0$.

Theorem 3.1. *Consider a complete fuzzy bipolar metric space $(M, I, S, *)$ where S is triangular. Let $k : (0, \infty) \to (0, 1)$ be a function and $r > 0$. Then, a mapping $V : (M, I, S, *) \rightrightarrows (M, I, S, *)$ satisfying the condition:*

$$\eta\Big(\frac{1}{S(V(m), V(i), r)} - 1\Big) \leqslant k(r).\ \eta\Big(\frac{1}{S(m, i, r)} - 1\Big),\ \forall m \in M \text{ and } i \in I,\ m \neq i, \tag{3.1}$$

admits a unique fixed point.

Proof. Consider two points $m_0 \in M$ and $i_0 \in I$ and define $V(m_n) = m_{n+1}$ and $V(i_n) = i_{n+1}$, for all $n \in \mathbb{N} \cup \{0\}$. Then, $(\{m_n\}, \{i_n\})$ is a bisequence in $(M, I, S, *)$. For any $r > 0$, from (3.1) we get

$$\begin{aligned} \eta\Big(\frac{1}{S(m_{n+1}, i_{n+1}, r)} - 1\Big) &= \eta\Big(\frac{1}{S(V(m_n), V(i_n), r)} - 1\Big) \\ &\leqslant k(r).\ \eta\Big(\frac{1}{S(m_n, i_n, r)} - 1\Big) \\ &= k(r).\ \eta\Big(\frac{1}{S(V(m_{n-1}), V(i_{n-1}), r)} - 1\Big) \\ &\cdots \\ &\leqslant (k(r))^n.\ \eta\Big(\frac{1}{S(m_0, i_0, r)} - 1\Big). \end{aligned} \tag{3.2}$$

Now, since η is strictly increasing, leading $n \to \infty$, we have

$$\begin{aligned} &\lim_{n\to\infty} \eta\Big(\frac{1}{S(m_{n+1}, i_{n+1}, r)} - 1\Big) = 0,\ r > 0, \\ or,\ &\lim_{n\to\infty} S(m_n, i_n, r) = 1,\ r > 0. \end{aligned} \tag{3.3}$$

Also,

$$\begin{aligned} \eta\Big(\frac{1}{S(m_{n+1}, i_n, r)} - 1\Big) &= \eta\Big(\frac{1}{S(V(m_n), V(i_{n-1}), r)} - 1\Big) \\ &\leqslant k(r).\ \eta\Big(\frac{1}{S(m_n, i_{n-1}, r)} - 1\Big) \\ &... \\ &\leqslant (k(r))^n.\ \eta\Big(\frac{1}{S(m_1, i_0, r)} - 1\Big). \end{aligned} \tag{3.4}$$

Leading $n \to \infty$, we have

$$\lim_{n\to\infty} S(m_{n+1}, i_n, r) = 1,\ r > 0. \tag{3.5}$$

Let $n, q \in \mathbb{N}$ with $n < q$. Then, by properties of η, condition (3.2), (3.4) and triangularity of S, we have

$$\begin{aligned} \eta\Big(\frac{1}{S(m_n, i_q, r)} - 1\Big) &\leqslant \eta\Big(\frac{1}{S(m_n, i_n, r)} - 1\Big) + \eta\Big(\frac{1}{S(m_{n+1}, i_n, r)} - 1\Big) \\ &\quad + \eta\Big(\frac{1}{S(m_{n+1}, i_q, r)} - 1\Big) \\ &\quad ... \\ &\leqslant \eta\Big(\frac{1}{S(m_n, i_n, r)} - 1\Big) + \eta\Big(\frac{1}{S(m_{n+1}, i_n, r)} - 1\Big) + ... \\ &\quad + \eta\Big(\frac{1}{S(m_{q-1}, i_{q-1}, r)} - 1\Big) \\ &\quad + \eta\Big(\frac{1}{S(m_q, i_{q-1}, r)} - 1\Big) + \eta\Big(\frac{1}{S(m_q, i_q, r)} - 1\Big) \\ &\leqslant (k(r))^n \eta\Big(\frac{1}{S(m_0, i_0, r)} - 1\Big) + (k(r))^n \eta\Big(\frac{1}{S(m_1, i_0, r)} - 1\Big) \\ &\quad + ... + (k(r))^{q-1} \eta\Big(\frac{1}{S(m_0, i_0, r)} - 1\Big) \\ &\quad + (k(r))^{q-1} \eta\Big(\frac{1}{S(m_1, i_0, r)} - 1\Big) \\ &\quad + (k(r))^q \eta\Big(\frac{1}{S(m_0, i_0, r)} - 1\Big) \\ &\leqslant (k(r))^n (1 + k(r) + (k(r))^2 + ... \\ &\quad + (k(r))^{q-n})\ \eta\Big(\frac{1}{S(m_0, i_0, r)} - 1\Big) \\ &\quad + (k(r))^n (1 + k(r) + (k(r))^2 + ... \end{aligned}$$

$$
\begin{aligned}
&+ (k(r))^{q-n-1}\big)\ \eta\Big(\frac{1}{S(m_1,i_0,r)}-1\Big)\\
&\leqslant \Big[\frac{(k(r))^n}{1-k(t)}\Big)\Big]\ \eta\Big(\frac{1}{S(m_0,i_0,r)}-1\Big)\\
&+\Big[\frac{(k(r))^n}{1-k(t)}\Big]\ \ \eta\Big(\frac{1}{S(m_1,i_0,r)}-1\Big).
\end{aligned}
$$

Leading $n, q \to \infty$, we obtain,

$$\lim_{n\to\infty} S(m_n, i_q, r) = 1,\ \ r > 0.$$

Thus, the bisequence $(\{m_n\},\{i_n\})$ is Cauchy. By completeness of fuzzy bipolar metric space $(M, I, S, *)$, $(\{m_n\},\{i_n\})$ is a convergent bisequence and hence, through Lemma 2.1 it biconverges to a point $\zeta \in M \cap I$ i.e., $\{m_n\} \to \zeta$ and $\{i_n\} \to \zeta$.

Now, we show that ζ is a fixed point of V. By using properties of η and triangularity of S, we have

$$
\begin{aligned}
\eta\Big(\frac{1}{S(V(\zeta),\zeta,r)}-1\Big) &\leqslant \eta\Big(\frac{1}{S(V(\zeta),V(m_n),r)}-1\Big)+\eta\Big(\frac{1}{S(V(m_n),V(i_n),r)}-1\Big)\\
&+\eta\Big(\frac{1}{S(V(i_n),\zeta,r)}-1\Big)\\
&\leqslant k(r)\ \eta\Big(\frac{1}{S(\zeta,m_n,r)}-1\Big)+k(r)\ \eta\Big(\frac{1}{S(m_n,i_n,r)}-1\Big)\\
&+\eta\Big(\frac{1}{S(i_{n+1},\zeta,r)}-1\Big).
\end{aligned}
$$

By leading $n \to \infty$, we obtain

$$\lim_{n\to\infty} S(V(\zeta),\zeta,r) = 1,\ \ r > 0.$$

So, $V(\zeta) = \zeta$. Now, assume ℓ as another fixed point of the mapping V. Then,

$$
\begin{aligned}
\eta\Big(\frac{1}{S(\zeta,\ell,r)}-1\Big) &= \eta\Big(\frac{1}{S(V(\zeta),V(\ell),r)}-1\Big)\\
&\leqslant k(r)\ \eta\Big(\frac{1}{S(\zeta,\ell,r)}-1\Big).
\end{aligned}
$$

Since, $0 < k(r) < 1$, we get $S(\zeta,\ell,r) = 1$. Thus, $\zeta = \ell$. □

Theorem 3.2. *Consider a complete fuzzy bipolar metric space $(M, I, S, *)$ where S is triangular. Let $k : (0,\infty) \to (0,1)$ be a function and $r > 0$. Then, a mapping $V : (M, I, S, *) \rightrightarrows (M, I, S, *)$ satisfying the condition:*

$$\eta\Big(\frac{1}{S(V(m),V(i),r)}-1\Big) \leqslant k(r).\ \eta\Big(\frac{1}{S(m,i,r)}-1\Big),\ \forall m \in M \text{ and } i \in I,\ m \neq i, \tag{3.6}$$

admits a unique fixed point.

Proof. Consider two points $m_0 \in M$ and $i_0 \in I$ and define $V(m_n) = i_n$ and $V(i_n) = m_{n+1}$, for all $n \in \mathbb{N} \cup \{0\}$. Then, $(\{m_n\},\{i_n\})$ is a bisequence in $(M, I, S, *)$. For any $r > 0$, we have

$$\eta\Big(\frac{1}{S(m_n,i_n,r)}-1\Big) = \eta\Big(\frac{1}{S(V(i_{n-1}),V(m_n),r)}-1\Big)$$

$$
\begin{aligned}
&\leqslant k(r).\ \eta\Big(\frac{1}{S(m_n, i_{n-1}, r)} - 1\Big)\\
&= k(r).\ \eta\Big(\frac{1}{S(V(i_{n-1}), V(m_{n-1}), r)} - 1\Big)\\
&\leqslant (k(r))^2.\ \eta\Big(\frac{1}{S(m_n, i_{n-1}, r)} - 1\Big)\\
&...\\
&\leqslant (k(r))^{2n}.\ \eta\Big(\frac{1}{S(m_0, i_0, r)} - 1\Big). \qquad (3.7)
\end{aligned}
$$

Now, since η is strictly increasing, leading $n \to \infty$, we can derive

$$
\lim_{n\to\infty} S(m_n, i_n, r) = 1,\ \ r > 0. \qquad (3.8)
$$

Also,

$$
\begin{aligned}
\eta\Big(\frac{1}{S(m_{n+1}, i_n, r)} - 1\Big) &= \eta\Big(\frac{1}{S(V(i_n), V(m_n), r)} - 1\Big)\\
&\leqslant k(r).\ \eta\Big(\frac{1}{S(m_n, i_n, r)} - 1\Big)\\
&= k(r).\ \eta\Big(\frac{1}{S(V(i_{n-1}), V(m_n), r)} - 1\Big)\\
&\leqslant (k(r))^2.\ \eta\Big(\frac{1}{S(m_n, i_{n-1}, r)} - 1\Big)\\
&...\\
&\leqslant (k(r))^{2n+1}.\ \eta\Big(\frac{1}{S(m_0, i_0, r)} - 1\Big). \qquad (3.9)
\end{aligned}
$$

Leading $n \to \infty$, we have

$$
\lim_{n\to\infty} S(m_{n+1}, i_n, r) = 1,\ \ r > 0. \qquad (3.10)
$$

Let $n, q \in \mathbb{N}$ with $n < q$. Then, by properties of η, condition (3.8), (3.9) and triangularity of S, we have

$$
\begin{aligned}
&\eta\Big(\frac{1}{S(m_n, i_q, r)} - 1\Big)\\
\leqslant&\eta\Big(\frac{1}{S(m_n, i_n, r)} - 1\Big) + \eta\Big(\frac{1}{S(m_{n+1}, i_n, r)} - 1\Big) + \eta\Big(\frac{1}{S(m_{n+1}, i_q, r)} - 1\Big)\\
&...\\
\leqslant&\eta\Big(\frac{1}{S(m_n, i_n, r)} - 1\Big) + \eta\Big(\frac{1}{S(m_{n+1}, i_n, r)} - 1\Big) + ... + \eta\Big(\frac{1}{S(m_{q-1}, i_{q-1}, r)} - 1\Big)\\
&+ \eta\Big(\frac{1}{S(m_q, i_{q-1}, r)} - 1\Big) + \eta\Big(\frac{1}{S(m_q, i_q, r)} - 1\Big)\\
\leqslant&(k(r))^{2n}\ \eta\Big(\frac{1}{S(m_0, i_0, r)} - 1\Big) + (k(r))^{2n+1}\ \eta\Big(\frac{1}{S(m_0, i_0, r)} - 1\Big) + ...\\
&+ (k(r))^{2q-2}\ \eta\Big(\frac{1}{S(m_0, i_0, r)} - 1\Big) + (k(r))^{2q-1}\eta\Big(\frac{1}{S(m_0, i_0, r)} - 1\Big)
\end{aligned}
$$

$$+ (k(r))^{2q} \ \eta\Big(\frac{1}{S(m_0,i_0,r)} - 1\Big)$$
$$\leqslant (k(r))^{2n}(1 + k(r) + (k(r))^2 + ... + (k(r))^{2(q-n)}) \ \eta\Big(\frac{1}{S(m_0,i_0,r)} - 1\Big)$$
$$\leqslant \Big[\frac{(k(r))^{2n}}{1-k(t)}\Big)\Big] \ \eta\Big(\frac{1}{S(m_0,i_0,r)} - 1\Big).$$

Leading $n, q \to \infty$, we obtain,

$$\lim_{n\to\infty} S(m_n, i_q, r) = 1, \ \ r > 0.$$

Thus, the bisequence $(\{m_n\}, \{i_n\})$ is Cauchy. By completeness of fuzzy bipolar metric space $(M, I, S, *)$, $(\{m_n\}, \{i_n\})$ is a convergent bisequence and hence, through Lemma 2.1 it biconverges to a point $\zeta \in M \cap I$ i.e., $\{m_n\} \to \zeta$ and $\{i_n\} \to \zeta$.

Now, we show that ζ is a fixed point of V. By using properties of η and triangularity of S, we have

$$\begin{aligned}\eta\Big(\frac{1}{S(V(\zeta),\zeta,r)} - 1\Big) \leqslant & \eta\Big(\frac{1}{S(V(\zeta),V(m_n),r)} - 1\Big) + \eta\Big(\frac{1}{S(V(m_n),V(i_n),r)} - 1\Big) \\ & + \eta\Big(\frac{1}{S(V(i_n),\zeta,r)} - 1\Big) \\ \leqslant & k(r).\ \eta\Big(\frac{1}{S(\zeta,m_n,r)} - 1\Big) + k(r).\ \eta\Big(\frac{1}{S(m_n,i_n,r)} - 1\Big) \\ & + \eta\Big(\frac{1}{S(m_{n+1},\zeta,r)} - 1\Big).\end{aligned}$$

By leading $n \to \infty$, we obtain

$$\lim_{n\to\infty} S(V(\zeta), \zeta, r) = 1, \quad r > 0.$$

So, $V(\zeta) = \zeta$. Now, assume ℓ as another fixed point of the contravariant mapping V. Then,

$$\begin{aligned}\eta\Big(\frac{1}{S(\zeta,\ell,r)} - 1\Big) &= \eta\Big(\frac{1}{S(V(\zeta),V(\ell),r)} - 1\Big) \\ &\leqslant k(r).\ \eta\Big(\frac{1}{S(\zeta,\ell,r)} - 1\Big).\end{aligned}$$

Since $0 < k(r) < 1$, we get $S(\zeta, \ell, r) = 1$. Thus, $\zeta = \ell$.

Therefore, ζ is a unique fixed point of the mapping V in $M \cap I$. □

4. Examples

Example 4.1. Let $M, T \subseteq \mathbb{R}$, $M = [1, 2] \cup 0$ and $I = \{\frac{3}{2}\} \cup \mathbb{N}$. Define the t-norm as $c * d = cd$. Consider $\eta(\lambda) = \sqrt{\lambda}$, for all $\lambda \in [0, \infty)$ and $S(m,\ i,\ p) = \frac{p}{p\ +\ g(m,\ i)}$, $p > 0$, where $g(m, i)$ denotes the Euclidean distance between two points in $\mathbb{R}$. Then, $(M, I, S, *)$ is a complete fuzzy bipolar metric space. Suppose that

$V : M \cup I \;\to\; M \cup I$ is defined by

$$V(u) = \begin{cases} 1 + \frac{u}{3} & , u \in [1, 2) \\ 1 & , u = 0, 2 \\ 1.5 & , u \in \mathbb{N}. \end{cases}$$

Define a function $k : (0, \infty) \to (0, 1)$ as

$$k(p) = \begin{cases} 1 - e^{\frac{-x+0.25}{4}} & , 0 < p \leqslant 2 \\ \frac{1}{2} & , p > 2. \end{cases}$$

It can be verified that η is a left continuous and a strictly decreasing function satisfying the property that $\eta(\lambda) = 0$ if and only if $\lambda = 0$.

$\forall m \in M \; and \; i \in I, \; m \neq i,$

$$\begin{aligned} \eta\Big(\frac{1}{S(V(m), V(i), r)} - 1\Big) &= \sqrt{\frac{|V(m) - V(i)|}{r}} \\ &= \sqrt{\frac{|1 + \frac{m}{3} - 1 - \frac{i}{3}|}{r}} \\ &= \frac{1}{\sqrt{3}}\sqrt{\frac{|m - i|}{r}} \\ &\leqslant k(r).\; \eta\Big(\frac{1}{S(m, i, r)} - 1\Big). \end{aligned}$$

Thus, $V : (M, I, S, *) \rightrightarrows (M, I, S, *)$ satisfies the following:

$$\eta\Big(\frac{1}{S(V(m), V(i), r)} - 1\Big) \leqslant k(r).\; \eta\Big(\frac{1}{S(m, i, r)} - 1\Big), \;\forall\; m \in M \; and \; i \in I, \; m \neq i.$$

So, all the requirements of Theorem 3.1 are satisfied and hence V has a unique fixed point of V, i.e., $\frac{3}{2}$.

Example 4.2. Let $M, T \subseteq \mathbb{R}$, $M = \{1, 2, 3, 4\}$ and $I = \{1, 2, 5, 10\}$. Define the t-norm as $c * d = cd$. Consider $\eta(\lambda) = 2\lambda$, for all $\lambda \in [0, \infty)$ and $S(m,\; i,\; p) = \frac{p}{p \,+\, g(m,\; i)}$, $p > 0$, where $g(m, i)$ denotes the Euclidean distance between two points in $\mathbb{R}$. Then, $(M, I, S, *)$ is a complete fuzzy bipolar metric space. Suppose that $V : M \cup I \;\to\; M \cup I$ is defined by

$$V(1) = V(2) = V(3) = 2,\; V(4) = V(5) = 1,\; V(10) = 3.$$

Define a function $k : (0, \infty) \to (0, 1)$ as

$$k(p) = \begin{cases} e^{-\frac{x}{x+5}} & , 0 < p \leqslant 1.869, \\ 1 - e^{-\frac{x+1}{2}} & , p > 1.869. \end{cases}$$

It can be verified that η is a left continuous and a strictly decreasing function satisfying the property that $\eta(\lambda) = 0$ if and only if $\lambda = 0$. Then, $V : (M, I, S, *) \rightleftarrows (M, I, S, *)$ satisfies the following:

$$\eta\Big(\frac{1}{S(V(m), V(i), r)} - 1\Big) \leqslant k(r).\; \eta\Big(\frac{1}{S(m, i, r)} - 1\Big), \;\forall m \in M \; and \; i \in I, \; m \neq i.$$

Therefore, all the conditions of Theorem 3.2 are fulfilled and hence there is a unique fixed point of V, i.e., 2.

Remark 4.1. Both Example 4.1 and Example 4.2 also satisfy for continuous t-norm $*$ defined as $c * d = min\{c, d\}$.

5. Consequences

Corollary 5.1. *Consider a complete fuzzy bipolar metric space* $(M, I, S, *)$ *where* S *is triangular, a function* $k : (0, \infty) \to (0, 1)$ *and* $r > 0$*. Then, a mapping* $V : (M, I, S, *) \rightrightarrows (M, I, S, *)$ *satisfying the condition:*

$$\frac{1}{S(V(m), V(i), r)} - 1 \leqslant k(r). \left(\frac{1}{S(m, i, r)} - 1\right), \ \forall m \in M \text{ and } i \in I, \ m \neq i,$$

enables the existence of a unique fixed point.

Proof. This can be proved by the same method as was employed in Theorem 3.1 by considering η as an identity map on $[0, \infty)$. □

Corollary 5.2. *Consider a complete fuzzy bipolar metric space* $(M, I, S, *)$ *where* S *is triangular, a function* $k : (0, \infty) \to (0, 1)$ *and* $r > 0$*. Then, a mapping* $V : (M, I, S, *) \nearrow\!\!\!\!\searrow (M, I, S, *)$ *satisfying the condition:*

$$\frac{1}{S(V(m), V(i), r)} - 1 \leqslant k(r). \left(\frac{1}{S(m, i, r)} - 1\right), \ \forall m \in M \text{ and } i \in I, \ m \neq i,$$

enables the existence of a unique fixed point.

Proof. This can be proved by the same method as was employed in Theorem 3.2 by considering $\eta : [0, \infty) \to [0, \infty)$ as $\eta(c) = c$. □

Corollary 5.3. *Consider a complete fuzzy bipolar metric space* $(M, I, S, *)$ *where* S *is triangular. Then, for a mapping* $V : (M, I, S, *) \rightrightarrows (M, I, S, *)$ *there exists* $\rho \in (0, 1)$ *satisfying the condition:*

$$\frac{1}{S(V(m), V(i), r)} - 1 \leqslant \rho. \left(\frac{1}{S(m, i, r)} - 1\right), \ \forall m \in M \text{ and } i \in I, \ m \neq i,$$

enables the existence of a unique fixed point.

Proof. This assertion can be verified using a similar approach as that applied in the proof of Theorem 3.1, by considering $\eta(\lambda) = \lambda$ and selecting an appropriate $\rho \in (0, 1)$. □

Remark 5.1. Corollary 5.3 coincides with Theorem 2.1 in [15].

Corollary 5.4. *Consider a complete fuzzy bipolar metric space* $(M, I, S, *)$ *where* S *is triangular. Then, for a mapping* $V : (M, I, S, *) \nearrow\!\!\!\!\searrow (M, I, S, *)$ *there exists* $\rho \in (0, 1)$ *satisfying the condition:*

$$\frac{1}{S(V(m), V(i), r)} - 1 \leqslant \rho. \left(\frac{1}{S(m, i, r)} - 1\right), \ \forall m \in M \text{ and } i \in I, \ m \neq i,$$

enables the existence of a unique fixed point.

Proof. This assertion can be verified using a similar approach to that applied in the proof of Theorem 3.2, by considering $\eta(\lambda) = \lambda$ and selecting an appropriate $\rho \in (0,1)$. $\square$

Remark 5.2. Corollary 5.4 coincides with Theorem 2.3 in [15].

6. Application

This section is dedicated to demonstrating the existence and uniqueness of a solution of nonlinear integral equations through the application of established results pertaining to covariant mappings. The problem is framed in the subsequent manner:

$$\mathbb{W}(\eth) = \mathbb{F}(\eth) + \zeta \int_0^{\mathcal{K}} \mathcal{I}_K(\eth,\theta)\mathbb{W}(\theta)d\theta \ , \tag{6.1}$$

where $\zeta > 0$, $\mathbb{F}(\eth)$ is a fuzzy function of $\eth \in [0,\mathcal{K}]$ and $\mathcal{I}_K : [0,\mathcal{K}] \times [0,\mathcal{K}] \times \mathbb{R} \longrightarrow \mathbb{R}$ is an integral kernel. Our objective is to demonstrate the existence and uniqueness of the solution to equation (6.1) by utilizing the principles outlined in Theorem 3.1. We consider the set $\mathfrak{H} = C([0,\mathcal{K}],\mathbb{R})$ as a collection of all real-valued continuous functions defined on the set $[0,\mathcal{K}]$. The induced metric $\mathfrak{G} : \mathfrak{H} \times \mathfrak{H} \longrightarrow \mathbb{R}^+$ is defined as $\mathfrak{G}(\mathbb{M},\mathbb{I}) = ||\mathbb{M} - \mathbb{I}||,\ \mathbb{M},\mathbb{I} \in \mathfrak{H}$.

Now, define a binary relation $*$ as a continuous t-norm and $S : \mathfrak{H}\times\mathfrak{H}\times(0,\infty) \longrightarrow [0,1]$ as follows.

For $\mathbb{W}(\eth), \mathbb{A}(\eth) \in \mathfrak{H}$ and $\rho > 0$, $S(\mathbb{W}(\eth),\mathbb{A}(\eth),\rho) = \frac{\rho}{\rho+\mathfrak{G}(\mathbb{W}(\eth),\mathbb{A}(\eth))}$. Then, S is triangular and the quadruple $(\mathfrak{H},\mathfrak{H},S,*)$ forms a complete fuzzy bipolar metric space.

Theorem 6.1. *Suppose there exists a function* $k : (0,\infty) \to (0,1)$ *such that for all* $\mathbb{W},\mathbb{A} \in \mathfrak{H}$*, the inequality holds*

$$||V(\mathbb{W}) - V(\mathbb{A})|| \leqslant k(\rho) \cdot ||\mathbb{W} - \mathbb{A}||, \tag{6.2}$$

where ρ *is a positive constant. Under these conditions, the solution to Problem (6.1) is both unique and found within the set* $\mathfrak{H}$.

Proof. We define $V : \mathfrak{H} \longrightarrow \mathfrak{H}$ by

$$[V\mathbb{W}](\eth) = \mathbb{F}(\eth) + \zeta \int_0^{\mathcal{K}} \mathcal{I}_K(\eth,\theta)\mathbb{W}(\theta)d\theta. \tag{6.3}$$

Then, V is well defined. Note that, V has a unique fixed point in $\mathfrak{H}$ if and only if Problem (6.1) has a unique solution. Also, $\eta(\sigma) = \frac{45}{77}\sigma,\ \forall\ \sigma \in [0,\infty)$. Then, it can be verified that η is a left continuous and a strictly decreasing function satisfying the property that $\eta(\sigma) = 0$ if and only if $\sigma = 0$.

By utilising (6.2) and (6.3), we have for $\mathbb{W},\mathbb{A} \in \mathfrak{H}$,

$$\begin{aligned}\eta\Big(\frac{1}{S(V(\mathbb{W}),V(\mathbb{A}),\rho)} - 1\Big) &= \eta\Big(\frac{\mathfrak{G}(V(\mathbb{W}),V(\mathbb{A}))}{\rho}\Big)\\ &= \frac{45}{77}\Big(\frac{|\|V(\mathbb{W}) - V(\mathbb{A})\|}{\rho}\Big)\\ &\leqslant k(\rho).\frac{45}{77}\Big(\frac{\|\mathbb{W} - \mathbb{A}\|}{\rho}\Big)\end{aligned}$$

$$= k(\rho).\ \eta\Big(\frac{\mathfrak{G}(\mathbb{W},\mathbb{A})}{\rho}\Big)$$
$$= k(\rho).\ \eta\Big(\frac{1}{S(\mathbb{W},\mathbb{A},\rho)}-1\Big).$$

So, we obtain

$$\eta\Big(\frac{1}{S(V(\mathbb{W}),V(\mathbb{A}),\rho)}-1\Big) \leqslant k(\rho).\ \eta\Big(\frac{1}{S(\mathbb{W},\mathbb{A},\rho)}-1\Big), \quad \forall\ \mathbb{W},\mathbb{A}\in\mathfrak{H}.$$

Hence, the integral operator V fulfills all the requirements outlined in Theorem 3.1. Thus, Theorem 3.1 guarantees the existence of a unique fixed point in $\mathfrak{H}$ for the operator V. This ensures a unique solution of the Problem (6.1) in $\mathfrak{H}$. □

Example 6.1. Let $\mathfrak{H} = C([0,1],\mathbb{R})$. Consider the problem:

$$\mathbb{W}(\eth) = 2\eth - \zeta\int_0^1 \sin\log\eth\ \mathbb{W}(\theta)d\theta\ , \tag{6.4}$$

where $0 \leqslant \zeta \leqslant \frac{1}{2}$. Then, for $\mathbb{W},\mathbb{A}\in\mathfrak{H}$, we have

$$\begin{aligned}
\|[V\mathbb{W}](\eth) - [V(\mathbb{A})](\eth)\| &= \|2\eth - \zeta\int_0^1 \sin\log\eth\ \mathbb{W}(\theta)d\theta - [2\eth - \zeta\int_0^1 \sin\log\eth\ \mathbb{A}(\theta)d\theta]\| \\
&= \|\zeta\int_0^1 \sin\log\eth\ \mathbb{A}(\theta)d\theta - \zeta\int_0^1 \sin\log\eth\ \mathbb{W}(\theta)d\theta\| \\
&= \zeta\|\int_0^1 \sin\log\eth\ [\mathbb{A}(\theta)-\mathbb{W}(\theta)]d\theta\| \\
&\leqslant \zeta\int_0^1 \sin\log\eth\ \|\mathbb{A}(\theta)-\mathbb{W}(\theta)\|d\theta \\
&\leqslant -\frac{1}{4}\|\mathbb{W}(\eth)-\mathbb{A}(\eth)\| \\
&\leqslant k(\rho).\|\mathbb{W}(\eth)-\mathbb{A}(\eth)\|,
\end{aligned}$$

where

$$k(\rho) = \begin{cases} e^{-\frac{x}{2x+5}} & , 0 < \rho \leqslant 1.316, \\ 1 - e^{-\frac{7x}{5}} & , \rho > 1.316. \end{cases}$$

Note that, $0 < k(\rho) < 1$. As a result, all the prerequisites of Theorem 6.1 are fulfilled. Consequently, the unique solution to the nonlinear integral problem 6.4 is ensured within the space $\mathfrak{H} = C([0,1],\mathbb{R})$.

7. Conclusion

In this study, the exploration of fuzzy bipolar metric spaces has yielded significant insights into their properties, particularly concerning fixed points and the triangular property of fuzzy bipolar metrics. The study has presented a comprehensive analysis supported by covariant and contravariant mappings, elucidated through illustrative examples, establishing fundamental results in this domain.

Moreover, the application of these findings to nonlinear integral equations has showcased their practical significance by demonstrating the existence and uniqueness of solutions. This not only solidifies the theoretical framework but also extends its applicability to real-world problem-solving scenarios.

The implications of this research are far-reaching, contributing substantially to the field of mathematical analysis. The insights gained not only enrich the understanding of fuzzy bipolar metric spaces but also pave the way for future investigations, offering fertile ground for further exploration and advancements in this area. The integration of these concepts into integral equations marks a promising direction for continued research, opening avenues for innovative approaches and applications in diverse mathematical contexts.

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