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Shallow-Water Models with the Weak Coriolis and Underlying Shear Flow Effects*

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Abstract In this paper, we are committed to deriving shallow-water model equations from the governing equations in the two-dimensional incompressible fluid with the effects of weak Coriolis force and underlying shear flow. These approximate models are established by working within a weakly nonlinear regime, introducing suitable far-field or near-field variables, and truncating the asymptotic expansions of the unknowns to an appropriate order. The obtained models generalize the classical KdV and Boussinesq equations, as well as KdV and Boussinesq equations with the Coriolis or shear flow effects.

Keywords KdV equation, Boussinesq equation, Coriolis effect, shear flow

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1. Introduction

The study of geophysical water waves is a fascinating subject in recent years. Within a certain large-scale geophysical water waves, fluid dynamics is mainly affected by the interaction between gravity and Coriolis force generated by the Earth's rotation. There are various shallow-water models with the Coriolis effect proposed as approximations to the governing equations for gravity water waves under different nonlinear regimes. In the Boussinesq scaling (weakly nonlinear regime), one can derive the geophysical Kortewe-de Vries (gKdV) equation [12]

$$2\eta_t - 2\omega_0\eta_x + 3\eta\eta_x + \frac{1}{3}\eta_{xxx} = 0,$$

where η is relevant to the free surface elevation and ω_0 is a constant related to the Coriolis effect. Recently, a modified version of the geophysical KdV of the above equation has been established in [1], which is called gpKdV equation. Additionally, the geophysical Boussinesq-type (gBouss) equation is also obtained in [12]

$$H_{tt} - 2\omega H_{tX} - H_{XX} + 3(H^2)_{XX} - H_{XXXX} = 0,$$

where H denotes the free surface elevation and ω is a constant related to the Coriolis effect. The new features of the Coriolis term introduced into a geophysical Boussinesq system in the long-wave assumption is considered in [14]. In the Camassa-Holm

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(CH) scaling (moderately nonlinear regime), the rotation-CH (R-CH) equation with the Coriolis effect is derived from incompressible and irrotational two-dimensional equatorial shallow water [13]. Recently, a highly nonlinear shallow-water model with the Coriolis effect has been proposed under a larger scaling than the CH one [19].

On the other hand, in order to represent another aspect of realistic observed flows, it requires us to study the propagation of various types of weakly nonlinear long waves in a flow moving according to some prescribed vorticity. Such a flow is often called a shear flow [8,18]. Due to the computational complexities for the case of an arbitrary shear flow, for simplicity (of course, non-trivial), most of the shallow-water models are derived with an underlying linear shear flow. It is known that the linear shear flow means constant vorticity. Recently, two-dimensional water-wave problem with a general non-zero vorticity field in a fluid volume with a flat bed and a free surface has been studied in [15]. The CH equation is relevant to water waves moving over a linear shear flow established by Johnson [18] via a double asymptotic expansion. In [21], a generalized CH equation in the shallow-water regime under the CH scaling with a linear shear is derived. Additionally, the Boussinesq equation with constant vorticity is given in [8]. For the case of general shear flow, an example is the KdV equation satisfied by the free surface elevation proposed in [10,18].

In this manuscript, we first derive a generalized KdV equation satisfied by the horizontal component of the velocity field with the effects of weak Coriolis and an arbitrary shear flow. It reads,

$$u_t + c_1 u_x + \alpha\varepsilon u u_x + \beta\mu u_{xxx} = 0, \tag{1.1}$$

where u is the fluid velocity in the horizontal direction. The coefficients in Eq. (1.1) depend on the right-going wave speed c, the constant rotational frequency Ω caused by the Coriolis effect, and the arbitrary underlying shear flow $U(z)$, given by

$$c_1 = c + \frac{c\Omega\gamma_1}{I_{31}}, \qquad \alpha = -\frac{3I_{41}\gamma_1}{2I_{31}}, \qquad \beta = -\frac{J_1}{2I_{31}},$$

where $\gamma_1 = -\frac{1}{((U(z)-c)I_2)'}$, $I_2 = \int_0^z \frac{1}{(U(z)-c)^2}\mathrm{d}z$, $I_{31} = \int_0^1 \frac{1}{(U(z)-c)^3}\mathrm{d}z$, $I_{41} = \int_0^1 \frac{1}{(U(z)-c)^4}\mathrm{d}z$, and $J_1 = \int_0^1\int_Z^1\int_0^\zeta \frac{(U(\zeta)-c)^2}{(U(Z)-c)^2(U(z)-c)^2}\mathrm{d}z\mathrm{d}\zeta\mathrm{d}Z$.

Particularly, in absence of the Coriolis effect and any shear flow ($\Omega \equiv 0$, $U(z) \equiv 0$), we can get the classical KdV equation from Eq. (1.1). When $U(z) \equiv 0$, in contrast to the gKdV equation of the free surface elevation, we obtain a KdV equation that the horizontal velocity satisfies only with the Coriolis effect. The reason why our equation is slightly different from gKdV is that we here use the far-field variables $\xi = \sqrt{\varepsilon}(x-ct), \tau = \varepsilon\sqrt{\varepsilon}t$, not $\xi = x - ct, \tau = \varepsilon t$. Moreover, as mentioned in Remark 2.1 of [13], it enables us to derive another KdV equation only with the Coriolis effect of the horizontal velocity under the Boussinesq scaling in the form

$$u_t + cu_x + 3\nu\varepsilon u u_x + \frac{\nu}{3}c\mu u_{xxx} = 0, \tag{1.2}$$

where $\nu = \frac{c^2}{c^2+1}, c = \sqrt{1+\Omega^2} - \Omega$ and Ω is the constant rotational frequency due to the Coriolis effect. Note that our equation is also different from Eq. (1.2), because we here use the weak Coriolis effect, not Coriolis effect, which leads to the difference in equations at the leading-order approximation. And we thus obtain $c = 1$ from the Burns condition when $U(z) \equiv 0$. This is really different from c appearing in Eq.

(1.2). When $\Omega \equiv 0$, we obtain a new KdV equation of the horizontal velocity in the presence of an arbitrary underlying shear flow. Moreover, in view of (3.17), we thus recover the resulting KdV equation of the free surface elevation for $\eta \sim \eta_{00} + \mu\eta_{01}$, given in [18].

It is noticed that the applicability of KdV (as well as for gKdV and Eq. (1.1)) as a model for tsunami wave propagation is not appropriate [2, 5, 6, 9, 12, 20]. As analyzed in [12], the space scale can be estimated in original physical variables by $x = O(\lambda\sqrt{\mu}\varepsilon^{-2}) = O(a^{-2}h_0^3)$ for the far-field variables $\xi = \sqrt{\varepsilon}(x - ct) = O(1), \tau = \varepsilon\sqrt{\varepsilon}t = O(1)$. Hence we find that the distance where the balance between nonlinear and dispersive stemming from Eq. (1.1) occurs is approximately $2 \times 10^8 \times \Omega_0^2$, which is much larger than the size of the Earth. While the tsunami leading to time- and space-scales are orders of magnitude smaller than those required for KdV theory. Hence it is reasonable to derive shallow-water models for tsunami wave propagation in near-field variables $x = O(1)$ and $t = O(1)$, where the balance occurs after a distance of approximately $2 \times 10^2 \times \Omega_0^{1/2} h_0^{3/4}$ is more realistic. We then continue to derive a generalized Boussinesq-type equation with the effects of weak Coriolis and a linear shear flow in the near-field. It gives,

$$\eta_{tt} + (\gamma - 2\Omega)\eta_{tx} - \eta_{xx} = \frac{1}{3}\varepsilon((1+\gamma^2)\eta - \gamma\int_x^\infty \eta_t \mathrm{d}x')_{xxxx} + \varepsilon(\frac{1}{2}(1+\gamma^2)\eta^2 - \gamma\eta\int_x^\infty \eta_t \mathrm{d}x' + (\int_x^\infty \eta_t \mathrm{d}x')^2)_{xx}, \tag{1.3}$$

with an $O(\varepsilon^2)$ remainder term. Here η denotes the free surface elevation, γ is a constant vorticity and Ω is a constant related to the Coriolis force. Compared with derivation of Eq. (1.1), we need an additional assumption to eliminate the dependence of u on the vertical coordinate z at the leading order. Hence we here only derive Eq. (1.3) in the presence of constant vorticity. In particular, when the Coriolis effect and linear shear flow vanish ($\Omega \equiv 0, \gamma \equiv 0$), we deduce the classical Boussinesq equation from Eq. (1.3). When $\gamma \equiv 0$, using the transformation $X = x - \varepsilon\int_x^\infty \eta \mathrm{d}x', H = \eta - \varepsilon\eta^2$, and the scaling $H \to -\frac{2}{\varepsilon}H, (X, t) \to \sqrt{\frac{\varepsilon}{3}}(X, t)$, that is exactly the gBouss equation. When $\Omega \equiv 0$, we recover the Boussinesq equation only with constant vorticity in [8] (see equation (90) on page 171).

2. The governing equations

Assume that the two dimensional fluid is incompressible and inviscid with a constant density and no surface-tension effect. Let $z = 0$ denote the location of the flat bottom and h_0 be the mean depth, or the undisturbed depth of the water. Suppose that $D_t = \{(x, z) : 0 < z < h_0 + \eta(t, x)\}$, where $\eta(t, x)$ measures the deviation from the average level. Under the effects of the gravity and the Coriolis force caused by Earth's rotation [7, 11], the equations governing the motion of the fluid consist of Euler equations

$$\begin{cases} u_t + uu_x + wu_z + 2\Omega w = -\frac{1}{\rho}P_x, & \text{in } D_t, \\ w_t + uw_x + ww_z - 2\Omega u = -\frac{1}{\rho}P_z - g, & \text{in } D_t, \end{cases} \tag{2.1}$$

together with the equation of mass conservation

$$u_x + w_z = 0, \text{ in } D_t, \tag{2.2}$$

and the dynamic and kinematic boundary conditions on the free surface

$$P = P_a, \ w = \eta_t + u\eta_x, \ \text{on } z = h_0 + \eta(t,x), \tag{2.3}$$

and the boundary condition on the flat bottom

$$w = 0, \ \text{on } z = 0, \tag{2.4}$$

where $(u(t,x,z), w(t,x,z))$ is the two-dimensional velocity field, P_a is the constant atmospheric pressure, g is the constant Earth's gravity acceleration and ρ is the constant fluid density. The constant $\Omega \approx 7.29 \times 10^{-5} rad/s$ denotes the rotational speed of the Earth around the polar axis, thus the two Ω-terms in (2.1) capture the effects of the so-called Coriolis force.

As in [16], the pressure of the fluid is written as $P(t,x,z) = P_a + \rho g(h_0 - z) + p(t,x,z)$, where the variable p measures the deviation from the hydrostatic pressure distribution, thus on the surface $z = h_0 + \eta(t,x)$, the dynamic condition $P = P_a$ yields $p = \rho g \eta$. Therefore, the equations (2.1)-(2.4) can be summarized as the following form

$$\begin{cases} u_t + uu_x + wu_z + 2\Omega w = -\frac{1}{\rho} p_x, & \text{in } 0 < z < h_0 + \eta(t,x), \\ w_t + uw_x + ww_z - 2\Omega u = -\frac{1}{\rho} p_z, & \text{in } 0 < z < h_0 + \eta(t,x), \\ u_x + w_z = 0, & \text{in } 0 < z < h_0 + \eta(t,x), \\ p = \rho g \eta, & \text{on } z = h_0 + \eta(t,x), \\ w = \eta_t + u\eta_x, & \text{on } z = h_0 + \eta(t,x), \\ w = 0, & \text{on } z = 0. \end{cases} \tag{2.5}$$

Now we introduce the following standard dimensionless quantities as in [9,13,16,17], according to the magnitude of the physical quantities

$$x \to \lambda x, \quad z \to h_0 z, \quad \eta \to a\eta, \quad t \to \frac{\lambda}{\sqrt{gh_0}} t,$$

and

$$u \to \sqrt{gh_0} u, \quad w \to \sqrt{\mu g h_0} w, \quad p \to \rho g h_0 p, \quad \Omega \to \frac{\sqrt{gh_0}}{h_0} \Omega.$$

In terms of the above nondimensionalised variables, the governing equations (2.5) become

$$\begin{cases} u_t + uu_x + wu_z + 2\Omega w = -p_x, & \text{in } 0 < z < 1 + \varepsilon\eta(t,x), \\ \mu(w_t + uw_x + ww_z) - 2\Omega u = -p_z, & \text{in } 0 < z < 1 + \varepsilon\eta(t,x), \\ u_x + w_z = 0, & \text{in } 0 < z < 1 + \varepsilon\eta(t,x), \\ p = \varepsilon\eta, & \text{on } z = 1 + \varepsilon\eta(t,x), \\ w = \varepsilon(\eta_t + u\eta_x), & \text{on } z = 1 + \varepsilon\eta(t,x), \\ w = 0, & \text{on } z = 0, \end{cases} \tag{2.6}$$

where we use the following two fundamental small dimensionless parameters

$$\varepsilon = \frac{a}{h_0} \ll 1, \quad \mu = \frac{h_0^2}{\lambda^2} \ll 1,$$

referred to as the amplitude parameter and the shallowness parameter with the small amplitude a and the long wavelength λ.

By imposing the scaling around a laminar flow

$$u \to u_1 + \varepsilon u, \quad w \to w_1 + \varepsilon w, \quad p \to p_1 + \varepsilon p, \tag{2.7}$$

where (u_1, w_1, p_1) is an exact solution to the system (2.6) of the form

$$u_1 = U(z), \quad w_1 \equiv 0, \quad p_1 = 2\Omega \int_1^z U(z)\mathrm{d}z, \tag{2.8}$$

characterized by the plane $\eta \equiv 0$, for any $U(z)$. This represents an arbitrary underlying shear flow. On the other hand, as is pointed out in [12], the Coriolis force parameter Ω and the amplitude parameter ε are of the same order of magnitude. It is thus reasonable to assume that $\Omega = O(\varepsilon)$, or what is the same,

$$\Omega = \varepsilon\Omega_0, \tag{2.9}$$

for some appropriate constant Ω_0. According to [12], as Ω_0 should not alter the order of magnitude in (2.9), it is required that $\frac{1}{2} < \Omega_0 < 5$.

In view of Eqs. (2.7)-(2.9), the governing equations (2.6) become

$$\begin{cases} u_t + Uu_x + wU' + \varepsilon(uu_x + wu_z + 2\Omega_0 w) = -p_x, & \text{in } 0 < z < 1 + \varepsilon\eta(t,x), \\ \mu(w_t + Uw_x + \varepsilon(uw_x + ww_z)) - 2\Omega_0\varepsilon u = -p_z, & \text{in } 0 < z < 1 + \varepsilon\eta(t,x), \\ u_x + w_z = 0, & \text{in } 0 < z < 1 + \varepsilon\eta(t,x), \\ p = \eta - 2\Omega_0 \int_1^z U\mathrm{d}z, & \text{on } z = 1 + \varepsilon\eta(t,x), \\ w = \eta_t + (U + \varepsilon u)\eta_x, & \text{on } z = 1 + \varepsilon\eta(t,x), \\ w = 0, & \text{on } z = 0, \end{cases} \tag{2.10}$$

where the prime in the first equation of (2.10) denotes the derivative with respect to z.

3. Generalized KdV equation

In this section, we pursue the derivation of a generalized KdV equation from the governing equations (2.10) with the weak Coriolis and an arbitrary shear flow effects. To proceed, we first introduce the following suitable variables in an appropriate far-field [17, 18],

$$\xi = \sqrt{\varepsilon}(x - ct), \quad \tau = \varepsilon\sqrt{\varepsilon}t,$$

where c is the constant speed for linear propagation. It is worth noting that Freeman, Johnson [10], and Geyer, Quirchmayr [12] applied the far-field variables $\xi = x - ct$, $\tau = \varepsilon t$ to deduce the KdV equation satisfied by the surface elevation with shear and weak Coriolis effects, respectively. The reason why the far-field variables we use here are different from theirs is mainly inspired by Johson's work [18] on recovering the KdV equation of the free surface elevation in the presence of an arbitrary shear. Then we replace w with $\sqrt{\varepsilon}W$ for the consistency of the condition of incompressibility. Hence, the governing equations (2.10) can be written as

$$\begin{cases} \varepsilon u_\tau - cu_\xi + Uu_\xi + WU' + \varepsilon(uu_\xi + Wu_z + 2\Omega_0 W) = -p_\xi, & \text{in } 0 < z < 1+\varepsilon\eta, \\ \varepsilon\mu(\varepsilon W_\tau - cW_\xi + UW_\xi + \varepsilon(uW_\xi + WW_z)) - 2\Omega_0\varepsilon u = -p_z, & \text{in } 0 < z < 1+\varepsilon\eta, \\ u_\xi + W_z = 0, & \text{in } 0 < z < 1+\varepsilon\eta, \\ p = \eta - 2\Omega_0 \int_1^z U\mathrm{d}z, & \text{on } z = 1+\varepsilon\eta, \\ W = \varepsilon\eta_\tau + (U-c)\eta_\xi + \varepsilon u\eta_\xi, & \text{on } z = 1+\varepsilon\eta, \\ W = 0, & \text{on } z = 0. \end{cases} \tag{3.1}$$

To search an asymptotic solution of Eq. (3.1) formally, it is assumed that the functions involved above can be written as double asymptotic expansions in ε and μ,

$$q \sim \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \varepsilon^n \mu^m q_{nm},$$

as $\varepsilon \to 0$ and $\mu \to 0$ independently. Here q (and correspondingly q_{nm}) stands for u, W, p and η. Note that the relations of u, W, p and η need to be satisfied on the free surface $1+\varepsilon\eta$, but itself is unknown. We deal with this difficulty by taking advantage of Taylor expansions of u, U, W, p on the surface about $z = 1$,

$$f(z) = f(1) + \sum_{n=1}^{\infty} \frac{(z-1)^n}{n!} f^{(n)}(1), \tag{3.2}$$

where f will be taken by the variables u, U, W, p.

In order to derive our equation (1.1), we perform calculations under the Boussinesq scaling: $\mu \ll 1$, $\varepsilon = O(\mu)$. Substituting the asymptotic expansions of u, W, p and η into Eq. (3.1), we check all the coefficients at each order $O(\varepsilon^0\mu^0), O(\varepsilon^0\mu^1), O(\varepsilon^1\mu^0), O(\varepsilon^2\mu^0), O(\varepsilon^1\mu^1)$, respectively.

For the leading-order $O(\varepsilon^0\mu^0)$ approximation, we obtain

$$\begin{cases} -cu_{00,\xi} + Uu_{00,\xi} + U'W_{00} = -p_{00,\xi}, & \text{in } 0 < z < 1, \\ p_{00,z} = 0, & \text{in } 0 < z < 1, \\ u_{00,\xi} + W_{00,z} = 0, & \text{in } 0 < z < 1, \\ p_{00} = \eta_{00}, & \text{on } z = 1, \\ W_{00} = (U_1 - c)\eta_{00,\xi}, & \text{on } z = 1, \\ W_{00} = 0, & \text{on } z = 0, \end{cases} \tag{3.3}$$

where $U_1 := U(1)$. To solve the above system, we first integrate the second equation of (3.3)

$$p_{00} = p_{00}|_{z=1} + \int_1^z p_{00,z'}\mathrm{d}z' = \eta_{00}.$$

Then substituting the third equation into the first one, and solving the differential equation of W_{00}, we obtain

$$W_{00} = \eta_{00,\xi}(U-c)\int_0^z \frac{1}{(U-c)^2}\mathrm{d}z.$$

Combining it with the fifth equation of (3.3), it follows

$$\int_0^1 \frac{1}{(U-c)^2}\mathrm{d}z = 1,$$

which is the familiar Burns condition [4]. For convenience, we introduce a compact notation for the various integrals

$$I_n(z) = \int_0^z \frac{1}{(U(z)-c)^n}\mathrm{d}z.$$

Obviously, $I_2(1) = 1$, we denote it as I_{21}. Thus we have

$$W_{00} = (U-c)I_2\eta_{00,\xi},$$

and

$$u_{00} = -((U-c)I_2)'\eta_{00}. \tag{3.4}$$

For the order $O(\varepsilon^0\mu^1)$ terms of the governing equations (3.1), it yields

$$\begin{cases} -cu_{01,\xi} + Uu_{01,\xi} + U'W_{01} = -p_{01,\xi}, & \text{in } 0 < z < 1, \\ p_{01,z} = 0, & \text{in } 0 < z < 1, \\ u_{01,\xi} + W_{01,z} = 0, & \text{in } 0 < z < 1, \\ p_{01} = \eta_{01}, & \text{on } z = 1, \\ W_{01} = (U_1 - c)\eta_{01,\xi}, & \text{on } z = 1, \\ W_{01} = 0, & \text{on } z = 0. \end{cases} \tag{3.5}$$

Applying the same calculation to (3.5) as (3.3), we readily get

$$p_{01} = \eta_{01}, \quad W_{01} = (U-c)I_2\eta_{01,\xi}, \quad u_{01} = -((U-c)I_2)'\eta_{01}. \tag{3.6}$$

For the order $O(\varepsilon^1\mu^0)$ terms of the governing equations (3.1), we obtain from the Taylor expansion (3.2)

$$\begin{cases} u_{00,\tau} + (U-c)u_{10,\xi} + U'W_{10} + u_{00}u_{00,\xi} + W_{00}u_{00,z} & \\ \quad +2\Omega_0 W_{00} = -p_{10,\xi}, & \text{in } 0 < z < 1, \\ p_{10,z} = 2\Omega_0 u_{00}, & \text{in } 0 < z < 1, \\ u_{10,\xi} + W_{10,z} = 0, & \text{in } 0 < z < 1, \\ p_{10} + \eta_{00}p_{00,z} = \eta_{10} - 2\Omega_0 U_1 \eta_{00}, & \text{on } z = 1, \\ W_{10} + \eta_{00}W_{00,z} = \eta_{00,\tau} + (U_1 - c)\eta_{10,\xi} + U_1^{'}\eta_{00}\eta_{00,\xi} + u_{00}\eta_{00,\xi}, & \text{on } z = 1, \\ W_{10} = 0, & \text{on } z = 0. \end{cases} \tag{3.7}$$

Integrating the second equation of (3.7), along with the fourth one, and combining it with the third equation of (3.3), it implies that

$$p_{10,\xi} = p_{10,\xi}|_{z=1} + \int_1^z 2\Omega_0 u_{00,\xi}\mathrm{d}z$$

$$
\begin{aligned}
&= \eta_{10,\xi} - 2\Omega_0 U_1 \eta_{00,\xi} + \int_z^1 2\Omega_0 W_{00,z} \mathrm{d}z \\
&= \eta_{10,\xi} - 2\Omega_0 U_1 \eta_{00,\xi} + 2\Omega_0 W_{00}|_{z=1} - 2\Omega_0 W_{00}.
\end{aligned} \tag{3.8}
$$

Substituting (3.8) into the first equation of (3.7), and solving the differential equation of W_{10}, we have

$$
\begin{aligned}
W_{10} =&(U-c)(I_2\eta_{10,\xi} - 2c\Omega_0 I_2 \eta_{00,\xi} + (\frac{I_2}{U-c} - 2I_3)\eta_{00,\tau} \\
&+ (3I_4 - \frac{2I_2}{(U-c)^2} - \frac{U' I_2^2}{U-c})\eta_{00}\eta_{00,\xi}).
\end{aligned} \tag{3.9}
$$

It then follows from the fifth equation that

$$
-2I_{31}\eta_{00,\tau} + 3I_{41}\eta_{00}\eta_{00,\xi} - 2c\Omega_0 \eta_{00,\xi} = 0, \tag{3.10}
$$

which implies

$$
\eta_{00,\tau} = \frac{3I_{41}}{2I_{31}}\eta_{00}\eta_{00,\xi} - \frac{c\Omega_0}{I_{31}}\eta_{00,\xi}.
$$

Plugging it into (3.9), we deduce that

$$
\begin{aligned}
W_{10} =&(U-c)I_2\eta_{10,\xi} - (2c\Omega_0(U-c)I_2 + \frac{c\Omega_0 I_2}{I_{31}} - \frac{2c\Omega_0}{I_{31}}(U-c)I_3)\eta_{00,\xi} \\
&+ (\frac{3I_{41}}{2I_{31}}(I_2 - 2(U-c)I_3) + (3(U-c)I_4 - \frac{2I_2}{U-c} - U' I_2^2))\eta_{00}\eta_{00,\xi},
\end{aligned}
$$

and thus

$$
\begin{aligned}
u_{10} = &-((U-c)I_2)'\eta_{10} + (2c\Omega_0((U-c)I_2)' + \frac{c\Omega_0}{I_{41}}I_2' - \frac{2c\Omega_0}{I_{31}}((U-c)I_3)')\eta_{00} \\
&-(\frac{3I_{41}}{4I_{41}}(I_2 - 2(U-c)I_3)' + \frac{1}{2}(3(U-c)I_4 - \frac{2I_2}{U-c} - U' I_2^2)')\eta_{00}^2.
\end{aligned} \tag{3.11}
$$

For the order $O(\varepsilon^2\mu^0)$ terms of the governing equations (3.1), it is inferred that

$$
\begin{cases}
u_{10,\tau} + (U-c)u_{20,\xi} + U'W_{20} + u_{10}u_{00,\xi} + u_{00}u_{10,\xi} + W_{10}u_{00,z} & \\
\quad +W_{00}u_{10,z} + 2\Omega_0 W_{10} = -p_{20,\xi}, & \text{in } 0<z<1, \\
p_{20,z} = 2\Omega_0 u_{10}, & \text{in } 0<z<1, \\
u_{20,\xi} + W_{20,z} = 0, & \text{in } 0<z<1, \\
p_{20} + \eta_{00}p_{10,z} + \eta_{10}p_{00,z} + \frac{1}{2}\eta_{00}^2 p_{00,zz} = \eta_{20} & \\
\quad -2\Omega_0 U_1 \eta_{10} - \Omega_0 U_1' \eta_{00}^2, & \text{on } z=1, \\
W_{20} + \eta_{00}W_{10,z} + \eta_{10}W_{00,z} + \frac{1}{2}\eta_{00}^2 W_{00,zz} = \eta_{10,\tau} & \\
\quad +(U_1-c)\eta_{20,\xi} + U_1'\eta_{10}\eta_{00,\xi} + U_1'\eta_{00}\eta_{10,\xi} & \\
\quad +\frac{1}{2}U_1''\eta_{00}^2\eta_{00,\xi} + u_{10}\eta_{00,\xi} + u_{00}\eta_{10,\xi} + \eta_{00}u_{00,z}\eta_{00,\xi}, & \text{on } z=1, \\
W_{20} = 0, & \text{on } z=0.
\end{cases} \tag{3.12}
$$

Integrating the second equation of (3.12), together with the fourth one, it gives

$$p_{20,\xi} = \eta_{20,\xi} - 2\Omega_0 U_1 \eta_{10,\xi} - \Omega_0 U_1'(\eta_{00}^2)_\xi - 2\Omega_0(\eta_{00}u_{00})_\xi + 2\Omega_0 W_{10}|_{z=1} - 2\Omega_0 W_{10}. \tag{3.13}$$

Substituting (3.13) into the first equation of (3.12), and solving the differential equation of W_{20}, we have

$$\begin{aligned} W_{20} =&(U-c)((\frac{I_2}{U-c} - 2I_3)\eta_{10,\tau} + I_2\eta_{20,\xi} + (3I_4 - \frac{2I_2}{(U-c)^2} - \frac{U'I_2^2}{U-c})(\eta_{10}\eta_{00,\xi} \\ &+ \eta_{00}\eta_{10,\xi}) - 2c\Omega_0 I_2 \eta_{10,\xi} + (\frac{3c^2\Omega_0^2}{I_{31}^2}I_4 - \frac{4c^2\Omega_0^2}{I_{31}}I_3 - \frac{2c^2\Omega_0^2}{I_{31}^2}\frac{I_3}{U-c} \\ &+ \frac{2c^2\Omega_0^2}{I_{31}}\frac{I_2}{U-c} - \frac{2c\Omega_0^2}{I_{31}}I_2)\eta_{00,\xi} + (\frac{12c\Omega_0}{I_{31}}I_5 - (\frac{9c\Omega_0 I_{41}}{I_{31}^2} + 12c\Omega_0)I_4 \\ &- \frac{3c\Omega_0}{I_{31}}\frac{I_4}{U-c} + (\frac{3c\Omega_0 I_{41}}{I_{31}} + 4\Omega_0)I_3 + \frac{6c\Omega_0 I_{41}}{I_{31}^2}\frac{I_3}{U-c} - \frac{4c\Omega_0}{I_{31}}\frac{I_3}{(U-c)^2} \\ &+ (\frac{2\Omega_0 I_{41}}{I_{31}} - \frac{4\Omega_0}{U_1 - c} - 4\Omega_0 U')I_2 - (\frac{c\Omega_0 I_{41}}{I_{31}} + 2\Omega_0)\frac{I_2}{U-c} + 8c\Omega_0\frac{I_2}{(U-c)^2} \\ &- \frac{2c\Omega_0}{I_{31}}\frac{I_2}{(U-c)^3} + 4c\Omega_0\frac{U'I_2^2}{U-c} - \frac{4c\Omega_0}{I_{31}}\frac{U'I_2I_3}{U-c})\eta_{00}\eta_{00,\xi} \\ &+ (\frac{9I_{41}^2}{4I_{31}^2}(3I_4 - \frac{2I_3}{U-c}) - \frac{I_{41}}{I_{31}}(\frac{43}{4}I_5 - \frac{9}{2}\frac{I_4}{U-c} - \frac{9}{2}\frac{I_3}{(U-c)^2} \\ &- \frac{17}{4}\frac{I_2}{(U-c)^3} - \frac{9}{2}\frac{U'I_2I_3}{U-c}) + \frac{15}{2}I_6 - \frac{9}{4}I_4 - \frac{9}{4}\frac{I_4}{(U-c)^2} - \frac{3}{2}\frac{I_2}{(U-c)^4} \\ &- \frac{9}{2}\frac{U'I_2I_4}{U-c} + \frac{1}{2}\frac{U''I_2^3}{U-c})\eta_{00}^2\eta_{00,\xi}). \end{aligned}$$

Hence, according to the fifth equation of (3.12), we obtain

$$\begin{aligned} -2I_{31}\eta_{10,\tau} + 3I_{41}(\eta_{10}\eta_{00,\xi} + \eta_{00}\eta_{10,\xi}) - 2c\Omega_0\eta_{10,\xi} + A_1\eta_{00,\xi} \\ + A_2\eta_{00}\eta_{00,\xi} + A_3\eta_{00}^2\eta_{00,\xi} = 0, \end{aligned} \tag{3.14}$$

where

$$\begin{aligned} A_1 &= \frac{3c^2\Omega_0^2 I_{41}}{I_{31}^2} - 4c^2\Omega_0^2 - \frac{2c\Omega_0^2}{I_{31}}, \\ A_2 &= \frac{12c\Omega_0 I_{51}}{I_{31}} - \frac{9c\Omega_0 I_{41}^2}{I_{31}^2} - 9c\Omega_0 I_{41} + 4c\Omega_0 I_{31} + \frac{3c\Omega_0 I_{41}}{I_{31}(U_1 - c)} + \frac{2\Omega_0 I_{41}}{I_{31}} \\ &\quad - \frac{6\Omega_0}{U_1 - c} - 4\Omega_0 U_1', \\ A_3 &= \frac{27I_{41}^3}{4I_{31}^2} - \frac{43}{4}\frac{I_{41}I_{51}}{I_{31}} + \frac{9}{4}\frac{I_{41}}{(U_1 - c)^2} + \frac{17}{4}\frac{I_{41}}{I_{31}(U_1 - c)^3} - \frac{15}{2}I_{61} - \frac{9}{4}I_{41}. \end{aligned}$$

For the order $O(\varepsilon^1\mu^1)$ terms of the governing equations (3.1), we derive from the Taylor expansion (3.2)

$$\begin{cases} u_{10,\tau} - cu_{11,\xi} + Uu_{11,\xi} + U'W_{11} + u_{01}u_{00,\xi} + u_{00}u_{01,\xi} & \\ +W_{00}u_{01,z} + W_{01}u_{00,z} + 2\Omega_0 W_{01} = -p_{11,\xi}, & \text{in } 0 < z < 1, \\ -cW_{00,\xi} + UW_{00,\xi} - 2\Omega_0 u_{01} = -p_{11,z}, & \text{in } 0 < z < 1, \\ u_{11,\xi} + W_{11,z} = 0, & \text{in } 0 < z < 1, \\ p_{11} + \eta_{01}p_{00,z} + \eta_{00}p_{01,z} = \eta_{11} - 2\Omega_0 U\eta_{01}, & \text{on } z = 1, \\ W_{11} + \eta_{01}W_{00,z} + \eta_{00}W_{01,z} = \eta_{01,\tau} + (U-c)\eta_{11,\xi} + U'\eta_{00}\eta_{01,\xi} & \\ +U'\eta_{01}\eta_{00,\xi} + u_{00}\eta_{01,\xi} + u_{01}\eta_{00,\xi}, & \text{on } z = 1, \\ W_{11} = 0, & \text{on } z = 0. \end{cases} \tag{3.15}$$

In view of the second and the fourth equations of (3.15), we get

$$p_{11} = \eta_{11} - 2c\Omega_0\eta_{01} + \eta_{00,\xi\xi}\int_z^1 (U-c)^2 I_2 \mathrm{d}z - 2\Omega_0(U-c)I_2\eta_{01}.$$

Substituting it into the first equation of (3.15) and together with the third equation of (3.15), we obtain

$$\begin{aligned} W_{11} =& (U-c)(I_2\eta_{11,\xi} + (\frac{I_2}{U-c} - 2I_3)\eta_{01,\tau} + (3I_4 - \frac{2I_2}{(U-c)^2} - \frac{U'I_2^2}{U-c})(\eta_{01}\eta_{00})_\xi \\ & - 2c\Omega_0 I_2\eta_{01,\xi} + J\eta_{00,\xi\xi\xi}), \end{aligned}$$

where $J = J(z) = \int_0^z \int_Z^1 \int_0^\zeta \frac{(U(\zeta)-c)^2}{(U(Z)-c)^2(U(z)-c)^2}\mathrm{d}z\mathrm{d}\zeta\mathrm{d}Z$. Thus it follows from the fifth equation of (3.15) that

$$-2I_{31}\eta_{01,\tau} + 3I_{41}(\eta_{01}\eta_{00,\xi} + \eta_{00}\eta_{01,\xi}) - 2c\Omega_0\eta_{01,\xi} + J_1\eta_{00,\xi\xi\xi} = 0, \tag{3.16}$$

where $J_1 = J(1)$.

According to the double asymptotic expansion of η, we take $\eta := \eta_{00} + \varepsilon\eta_{10} + \mu\eta_{01} + O(\varepsilon^2, \varepsilon\mu, \mu^2)$. Multiplying each of the three equations of (3.10), (3.14), (3.16) by 1, ε, μ, respectively, and then summating these results, we get the equation of η up to the order $O(\varepsilon^2, \varepsilon\mu, \mu^2)$

$$\begin{aligned} &-2I_{31}\eta_\tau + 3I_{41}\eta\eta_\xi - 2c\Omega_0\eta_\xi + \mu J_1\eta_{\xi\xi\xi} \\ &\quad + \varepsilon(A_1\eta_\xi + A_2\eta\eta_\xi + A_3\eta^2\eta_\xi) = O(\varepsilon^2, \varepsilon\mu, \mu^2). \end{aligned} \tag{3.17}$$

On the other hand, it follows from (3.4), (3.6) and (3.11) that

$$\eta_{00} = \gamma_1 u_{00}, \quad \eta_{01} = \gamma_1 u_{01}, \quad \eta_{10} = \gamma_1 u_{10} + \gamma_2 u_{00} + \gamma_3 u_{00}^2,$$

where $\gamma_1 = -\frac{1}{((U-c)I_2)'}$, $\gamma_2 = -\frac{1}{(((U-c)I_2)')^2}(2c\Omega_0((U-c)I_2)' + \frac{c\Omega_0}{I_{31}}(I_2 - 2(U-c)I_3)')$, $\gamma_3 = -\frac{1}{(((U-c)I_2)')^3}(\frac{3I_{41}}{4I_{31}}(I_2 - 2(U-c)I_3)' + \frac{1}{2}(3(U-c)I_4 - \frac{2I_2}{U-c} - U'I_2^2)')$. Substituting the above three equalities into the double asymptotic expansions of η, and noticing that $u := u_{00} + \varepsilon u_{10} + \mu u_{01} + O(\varepsilon^2, \varepsilon\mu, \mu^2)$, we obtain

$$\eta = \gamma_1 u + \varepsilon\gamma_2 u + \varepsilon\gamma_3 u^2 + O(\varepsilon^2, \varepsilon\mu, \mu^2). \tag{3.18}$$

To proceed, substituting (3.18) into (3.17) gives rise to

$$
\begin{aligned}
&u_\tau + \frac{\gamma_2}{\gamma_1}\varepsilon u_\tau + \frac{2\gamma_3}{\gamma_1}\varepsilon u u_\tau + \frac{c\Omega_0\gamma_1}{I_{31}}u_\xi - \frac{A_1\gamma_1 - 2c\Omega_0\gamma_2}{2I_{31}\gamma_1}\varepsilon u_\xi - \frac{3I_{41}\gamma_1}{2I_{31}}uu_\xi \\
&- \frac{6I_{41}\gamma_1\gamma_2 - 4c\Omega_0\gamma_3 + A_2\gamma_1^2}{2I_{31}\gamma_1}\varepsilon uu_\xi - \frac{9I_{41}\gamma_3 + A_3\gamma_1^2}{2I_{31}}\varepsilon u^2 u_\xi - \frac{J_1}{2I_{31}} = O(\varepsilon^2, \varepsilon\mu, \mu^2),
\end{aligned} \tag{3.19}
$$

which implies that

$$
\varepsilon u_\tau = -\frac{c\Omega_0\gamma_1}{I_{31}}\varepsilon u_\xi + \frac{3I_{41}\gamma_1}{2I_{31}}\varepsilon uu_\xi + O(\varepsilon^2, \varepsilon\mu, \mu^2),
$$

and

$$
\varepsilon uu_\tau = -\frac{c\Omega_0\gamma_1}{I_{31}}\varepsilon uu_\xi + \frac{3I_{41}\gamma_1}{2I_{31}}\varepsilon u^2 u_\xi + O(\varepsilon^2, \varepsilon\mu, \mu^2).
$$

Hence, the equation (3.19) becomes

$$
\begin{aligned}
u_\tau + \frac{c\Omega_0\gamma_1}{I_{31}}u_\xi - \frac{3I_{41}\gamma_1}{2I_{31}}uu_\xi - B_1\varepsilon u_\xi - B_2\varepsilon uu_\xi - B_3\varepsilon u^2 u_\xi \\
- \frac{J_1}{2I_{31}}\mu u_{\xi\xi\xi} = O(\varepsilon^2, \varepsilon\mu, \mu^2),
\end{aligned} \tag{3.20}
$$

where $B_1 = \frac{A_1\gamma_1 - 2c\Omega_0\gamma_2 + 2c\Omega_0\gamma_1\gamma_2}{2I_{31}\gamma_1}$, $B_2 = \frac{3I_{41}\gamma_1\gamma_2 + A_2\gamma_1^2 - 4c\Omega_0\gamma_3 + 4c\Omega_0\gamma_1\gamma_3}{2I_{31}\gamma_1}$, $B_3 = \frac{3I_{41}\gamma_3 + A_3\gamma_1^2}{2I_{31}}$.

Back to the original transformation $x = \varepsilon^{-\frac{1}{2}}\xi + c\varepsilon^{-\frac{3}{2}}\tau,\ t = \varepsilon^{-\frac{3}{2}}\tau$, we have $\frac{\partial}{\partial\xi} = \varepsilon^{-\frac{1}{2}}\partial_x,\ \frac{\partial}{\partial\tau} = \varepsilon^{-\frac{3}{2}}(c\partial_x + \partial_t)$. Making use of this transformation, the equation (3.20) can be written as

$$
u_t + c_1 u_x + \alpha\varepsilon uu_x + \beta\mu u_{xxx} = 0, \tag{3.21}
$$

with $c_1 = c + \frac{c\Omega\gamma_1}{I_{31}}$, $\alpha = -\frac{3I_{41}\gamma_1}{2I_{31}}$, $\beta = -\frac{J_1}{2I_{31}}$, which give the generalized KdV equation (1.1) with both weak Coriolis and an arbitrary shear flow effects. Significantly, the coefficients satisfy $c_1 \to 1$, $\alpha \to \frac{3}{2}$, $\beta \to \frac{1}{6}$, when $\Omega \to 0$ and $U(z) \equiv 0$, which implies

$$
u_t + u_x + \frac{3}{2}\varepsilon uu_x + \frac{1}{6}\mu u_{xxx} = 0.
$$

Applying the transformation $u_{\varepsilon,\mu}(t,x) = \varepsilon u(\sqrt{\mu}t, \sqrt{\mu}x)$ to the above equation, we find that $u_{\varepsilon,\mu}(t,x)$ solves the classical KdV equation.

4. Generalized Boussinesq equation

In this section, we turn to the derivation of a generalized Boussinesq-type equation in the region where the non-dimensional scaled near-field variables satisfy $x = O(1)$ and $t = O(1)$. Indeed, the derivation of the Boussinesq equation is considerably more involved than the corresponding problem for the KdV equation [8]. In view of this, we consider a constant vorticity instead of an arbitrary vorticity, which implies $U(z) = \gamma z$, for constant γ. Correspondingly, the governing equations (2.10) become

$$\begin{cases} u_t + \gamma z u_x + \gamma w + \varepsilon(u u_x + w u_z + 2\Omega_0 w) = -p_x, & \text{in } 0 < z < 1 + \varepsilon\eta(t,x), \\ \mu(w_t + \gamma z w_x + \varepsilon(u w_x + w w_z)) - 2\Omega_0 \varepsilon u = -p_z, & \text{in } 0 < z < 1 + \varepsilon\eta(t,x), \\ u_x + w_z = 0, & \text{in } 0 < z < 1 + \varepsilon\eta(t,x), \\ p = \eta - 2\Omega_0\gamma\varepsilon\eta - \Omega_0\gamma\varepsilon^2\eta^2, & \text{on } z = 1 + \varepsilon\eta(t,x), \\ w = \eta_t + \varepsilon u \eta_x + \varepsilon\gamma\eta\eta_x + \gamma\eta_x, & \text{on } z = 1 + \varepsilon\eta(t,x), \\ w = 0, & \text{on } z = 0. \end{cases} \tag{4.1}$$

It is worth noting that before the nondimensionalisation and scaling, the vorticity $\omega = U' + u_z - w_x$. Using the nondimensionalised variables and the scalings of u, w as before, with the vorticity scaling $\omega \to \sqrt{g/h_0}\omega$, we get $\omega = \gamma + \varepsilon(u_z - \mu w_x)$. Thus, in order to seek a solution with constant vorticity, it is required that

$$u_z - \mu w_x = 0, \tag{4.2}$$

which implies that the vorticity $\omega \equiv \gamma$. On the other hand, we are working in the regime $\mu \ll 1$, $\varepsilon = O(\mu)$, so the parameter μ appearing in Eqs. (4.1)- (4.2) can be replaced with ε. Therefore, we obtain the governing equations as follows

$$\begin{cases} u_t + \gamma z u_x + \gamma w + \varepsilon(u u_x + w u_z + 2\Omega_0 w) = -p_x, & \text{in } 0 < z < 1 + \varepsilon\eta(t,x), \\ \varepsilon(w_t + \gamma z w_x + \varepsilon(u w_x + w w_z)) - 2\Omega_0 \varepsilon u = -p_z, & \text{in } 0 < z < 1 + \varepsilon\eta(t,x), \\ u_x + w_z = 0, & \text{in } 0 < z < 1 + \varepsilon\eta(t,x), \\ u_z - \varepsilon w_x = 0, & \text{in } 0 < z < 1 + \varepsilon\eta(t,x), \\ p = \eta - 2\Omega_0\gamma\varepsilon\eta - \Omega_0\gamma\varepsilon^2\eta^2, & \text{on } z = 1 + \varepsilon\eta(t,x), \\ w = \eta_t + \varepsilon u \eta_x + \varepsilon\gamma\eta\eta_x + \gamma\eta_x, & \text{on } z = 1 + \varepsilon\eta(t,x), \\ w = 0, & \text{on } z = 0. \end{cases} \tag{4.3}$$

As in [8, 12], we formally expand the respective variables u, w, p and η in the form

$$q \sim \sum_{n=0}^{\infty} \varepsilon^n q_n,$$

and rewrite the boundary conditions at the free surface by means of the Taylor expansions (3.2) of the involved variables u, w and p about $z = 1$. Hence, for the order $O(\varepsilon^0)$ terms of the (4.3), we obtain

$$\begin{cases} u_{0,t} + \gamma z u_{0,x} + \gamma w_0 = -p_{0,x}, & \text{in } 0 < z < 1, \\ p_{0,z} = 0, & \text{in } 0 < z < 1, \\ u_{0,x} + w_{0,z} = 0, & \text{in } 0 < z < 1, \\ u_{0,z} = 0, & \text{in } 0 < z < 1, \\ p_0 = \eta_0, & \text{on } z = 1, \\ w_0 = \eta_{0,t} + \gamma\eta_{0,x}, & \text{on } z = 1, \\ w_0 = 0, & \text{on } z = 0. \end{cases}$$

We thus find that, for $z \in [0,1]$,

$$p_0 = \eta_0, \quad w_0 = z(\eta_{0,t} + \gamma\eta_{0,x}), \quad u_{0,t} = -\eta_{0,x}, \quad u_{0,x} = -(\eta_{0,t} + \gamma\eta_{0,x}), \tag{4.4}$$

which implies

$$\eta_{0,tt} + \gamma\eta_{0,xt} - \eta_{0,xx} = 0. \tag{4.5}$$

For the order $O(\varepsilon^1)$ terms of the (4.3), we get

$$\begin{cases} u_{1,t} + \gamma z u_{1,x} + \gamma w_1 + u_0 u_{0,x} + w_0 u_{0,z} + 2\Omega_0 w_0 = -p_{1,x}, & \text{in } 0 < z < 1, \\ w_{0,t} + \gamma z w_{0,x} - 2\Omega_0 u_0 = -p_{1,z}, & \text{in } 0 < z < 1, \\ u_{1,x} + w_{1,z} = 0, & \text{in } 0 < z < 1, \\ u_{1,z} - w_{0,x} = 0, & \text{in } 0 < z < 1, \\ p_1 = \eta_1 - 2\Omega_0\gamma\eta_0, & \text{on } z = 1, \\ w_1 + \eta_0 w_{0,z} = \eta_{1,t} + u_0\eta_{0,x} + \gamma\eta_0\eta_{0,x} + \gamma\eta_{1,x}, & \text{on } z = 1, \\ w_1 = 0, & \text{on } z = 0. \end{cases} \tag{4.6}$$

From the second and the fifth equations of (4.6), we find that

$$p_{1,x} = \eta_{1,x} - 2\Omega_0 w_0 + 2\Omega_0\eta_{0,t} - \frac{1}{2}(z^2 - 1)Q_{tx} - \frac{\gamma}{3}(z^3 - 1)Q_{xx},$$

where $Q(t,x) = \eta_{0,t} + \gamma\eta_{0,x}$. Combining the third and the fourth equations of (4.6) with (4.4), we achieve

$$w_{1,zz} = -zQ_{xx},$$

which implies

$$w_1 = -\frac{1}{6}z^3 Q_{xx} + z\Phi(t,x),$$

for some arbitrary smooth function $\Phi(t,x)$ independent of z. Taking account of the sixth equation of (4.6), we have

$$\Phi(t,x) = \eta_{1,t} + u_0\eta_{0,x} + \gamma\eta_0\eta_{0,x} + \gamma\eta_{1,x} + \frac{1}{6}Q_{xx} - \eta_0 Q.$$

Substituting all the above known quantities into the first equation of (4.6), it yields

$$u_{1,t} = -\eta_{1,x} - 2\Omega_0\eta_{0,t} + \frac{1}{2}(z^2 - 1)Q_{tx} - \frac{\gamma}{3}Q_{xx} - u_0 u_{0,x},$$

which together with the third equation of (4.6) implies that

$$w_{1,zt} = -u_{1,xt} = \eta_{1,xx} + 2\Omega_0\eta_{0,tx} - \frac{1}{2}(z^2 - 1)Q_{txx} + \frac{\gamma}{3}Q_{xxx} + (u_0 u_{0,x})_x.$$

Integrating the above equation, we deduce that

$$w_{1,t} = z(\eta_{1,xx} + 2\Omega_0\eta_{0,tx} + \frac{1}{2}Q_{txx} + \frac{\gamma}{3}Q_{xxx} + (u_0 u_{0,x})_x - \frac{z^3}{6}Q_{txx}. \tag{4.7}$$

Finally, differentiating the sixth equation of (4.6) with respect to t, and substituting (4.7) into it, we get

$$\eta_{1,tt} + \gamma\eta_{1,tx} - \eta_{1,xx} = 2\Omega_0\eta_{0,tx} + \frac{1}{3}Q_{txx} + \frac{\gamma}{3}Q_{xxx} + (u_0u_{0,x})_x + (\eta_0 Q)_t - (u_0\eta_{0,x})_t - \gamma(\eta\eta_{0,x})_t,$$

that is

$$\eta_{1,tt} + \gamma\eta_{1,tx} - \eta_{1,xx} = 2\Omega_0\eta_{0,tx} + \frac{1}{3}(\eta_0 - \gamma u_0)_{xxxx} + (u_0^2 + \frac{1}{2}\eta_0^2 + \gamma u_0\eta_0 + \frac{1}{2}\gamma^2\eta_0^2)_{xx}, \tag{4.8}$$

with $u_0 = -\gamma\eta_0 + \int_x^\infty \eta_{0,t}\mathrm{d}x'$ under the assumption of the decay conditions ahead ($x \to \infty$) of any right-running wave.

According to the asymptotic expansion of η, we take $\eta := \eta_0 + \varepsilon\eta_1 + O(\varepsilon^2)$. Multiplying equation (4.5), (4.8) by 1, ε, respectively, and then summating these results, we get the equation of η up to the order $O(\varepsilon^2)$

$$\eta_{tt} + (\gamma - 2\Omega)\eta_{tx} - \eta_{xx} = \frac{1}{3}\varepsilon((1+\gamma^2)\eta - \gamma\int_x^\infty \eta_t\mathrm{d}x')_{xxxx} + \varepsilon(\frac{1}{2}(1+\gamma^2)\eta^2 - \gamma\eta\int_x^\infty \eta_t\mathrm{d}x' + (\int_x^\infty \eta_t\mathrm{d}x')^2)_{xx} + O(\varepsilon^2). \tag{4.9}$$

Note that Eq. (4.9) cannot be transformed into anything equivalent to the classical Boussinesq equation by any transformation [8]. Hence we obtain a new Boussinesq-type equation with both weak Coriolis force and constant vorticity.

5. Conclusion

In this paper, we are mainly motivated by the works [12,18] to derive shallow-water model equations from the governing equations of two-dimensional incompressible fluid, accounting for the effects of weak Coriolis force and underlying shear flow. More precisely, on the one hand, when Johnson [18] applied the double asymptotic expansion to explore the relevance of the Camassa-Holm equation to water waves moving over a shear flow, he recovered the KdV equation obtained earlier in the presence of an arbitrary shear [10]. On the other hand, Geyer and Quirchmayr [12] derived the gKdV and gBouss equations only with the weak Coriolis effect, and considered their application as models for tsunami wave propagation. This inspires us to encompass both the Coriolis and shear flow effects. We first follow the procedure employed in [18] (thus using the same far-field variables) to establish the KdV equation satisfied by not only the free surface, but also the horizontal component of the velocity. We then continue with the near-field variables in [12] to derive the Boussinesq equation with the weak Coriolis effect and linear shear flow. It should be noted that the travelling wave solutions of the KdV and Boussinesq equations with only the Coriolis, and the KdV only with shear were obtained in [3,8,12,16]. What about our Eqs. (1.1) and Eqs. (1.3)? As pointed out in [8], the Boussinesq equation only with linear shear flow cannot be transformed into anything equivalent to the classical Boussinesq equation by any transformation. Moreover, there is an arbitrary function $U(z)$ appearing in the coefficients in Eq. (1.1). These difficulties make the problem of travelling wave solutions still open. We expect the future study may give some new phenomena for the equations.

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