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Quasi Statistical Convergence of Double Sequences in Neutrosophic Normed Spaces*

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Abstract In this paper, we introduce the notion of quasi statistical convergence of double sequences in the neutrosophic normed spaces mainly as a generalization of statistical convergence of double sequences. We investigate a few fundamental properties of the newly introduced notion and examine the relationship with statistical convergence of double sequences in the neutrosophic normed spaces. In the end, we introduce the concept of quasi statistical Cauchy sequence of double sequences and show that quasi statistical Cauchy sequences for double sequences are equivalent to quasi statistical convergent double sequences in the neutrosophic normed spaces.

Keywords Quasi-density, quasi statistical convergence of double sequences, neutrosophic normed space

MSC(2010) 46S40, 03E72, 40G15.

1. Introduction

In 1951, the concept of statistical convergence was developed independently by Fast [3] and Steinhaus [22] to provide deeper insights into summability theory. Later on, it was further investigated from the sequence space point of view by Fridy [6], Šalát [19], Tripathy [23], and many researchers [1, 7, 11, 13, 17]. In 2003, Mursaleen and Edely [18] extended this concept over double sequences and mainly studied the relationship between statistical convergence and statistical Cauchy double sequences, statistical convergence, and strong Cesàro summable double sequences. Besides this, in [23], Tripathy studied various properties of the sequence spaces formed by statistical convergent double sequences and proved a decomposition theorem. Statistical convergence has many applications in various branches of mathematics such as number theory, mathematical analysis, probability theory, etc.

In 2012, Özgüç and Yurdakadim [16], extended the notion of statistical convergence to quasi statistical convergence using quasi-density. They further investigated the connection between quasi statistical convergence and statistical convergence. Following their work, several works have been carried out so far. In 2016, Ganguly

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and Dafadar [4] introduced and investigated the notion of quasi statistical convergence of double sequences as a natural generalization of statistical convergence of double sequences and studied some fundamental properties. For more details on quasi statistical convergence and its related developments, one can see the work of Özgüç [15] and Turan et al. [24], where many more references can be found.

The concept of fuzzy sets was first introduced by Zadeh [25] in the year 1965 which was an extension of the classical set-theoretical concept. Nowadays it has wide applicability in different branches of science and engineering. The theory of fuzzy sets cannot always cope with the lack of knowledge of membership degrees. To overcome the drawbacks, in 1986, Atanassov [2] introduced intuitionistic fuzzy sets as an extension of fuzzy sets. Intuitionistic fuzzy sets have been widely used to solve various decision-making problems.

Many times, decision-makers face some hesitations besides going to direct approaches (i.e., yes or no) in a decision making. In addition, we can obtain a tri-component outcome in some real events like sports, the procedure for voting, etc. Considering all in 2005, Smarandache [21] introduced the notion of Neutrosophic set as a generalization of both fuzzy set and intuitionistic fuzzy set. An element belonging to a neutrosophic set consists of a triplet namely truth-membership function (T), falsity-membership function (I), and indeterminacy-membership function (F). A neutrosophic set is determined as a set where every component of the universe has a degree of T, I, and F.

Kirişçi and Simsek [12] introduced the notion of neutrosophic normed space and investigated the notion of statistical convergence therein. Following their work, several researchers such as Khan et al. [8–10] and Şengül et al. [20] investigated various notions of convergence of sequences in the neutrosophic normed space. Recently, Granados and Dhital [5] have extended the above notion to the statistical convergence of double sequences in neutrosophic normed spaces. In this paper, using the double quasi-density we further extend it to quasi statistical convergence of double sequences. We also examine the relationship between statistical convergence and quasi statistical convergence of double sequences in neutrosophic normed linear spaces. Research on the convergence of sequences in neutrosophic normed linear spaces has not yet gained much ground and it is still in its infant stage. The research carried out so far shows a strong analogy in the behavior of convergence of sequences in neutrosophic normed spaces.

2. Definitions and preliminaries

Definition 2.1. [18] Let $K \subseteq \mathbb{N} \times \mathbb{N}$ and let $K_{m,n}$ denote the set

$$\{(i,j) \in K : i \leq m, j \leq n\}.$$

The double natural density of K is denoted and defined by

$$\delta^2(K) = \lim_{m,n \to \infty} \frac{|K_{m,n}|}{mn}.$$

Here, $|K_{m,n}|$ denotes the cardinality of the set $K_{m,n}$.

Definition 2.2. [18] A double sequence (x_{ij}) is said to be statistical convergent to l if for each $\varepsilon > 0$,

$$\delta^2(A(\varepsilon)) = 0, \text{ where } A(\varepsilon) = \{(i,j) \in \mathbb{N} \times \mathbb{N} : |x_{ij} - l| \geq \varepsilon\}.$$

In this case, l is called the statistical limit of the double sequence (x_{ij}) and symbolically it is expressed as $x_{ij} \xrightarrow{st} l$.

Definition 2.3. [23] A double sequence (x_{ij}) is said to be statistical Cauchy if for each $\varepsilon > 0$, there exist two positive integers $M = M(\varepsilon)$ and $N = N(\varepsilon)$ such that

$$\delta^2(A(\varepsilon)) = 0, \text{ where } A(\varepsilon) = \{(i,j) \in \mathbb{N} \times \mathbb{N} : |x_{ij} - x_{MN}| \geq \varepsilon\}.$$

Definition 2.4. [4] Let $K \subseteq \mathbb{N} \times \mathbb{N}$ and $K_{m,n}$ denote the set

$$\{(i,j) \in K : i \leq m, j \leq n\}.$$

The double quasi-density of K is given by

$$\delta_q^{2,c}(K) = \lim_{m,n\to\infty} \frac{|K_{m,n}|}{c_{mn}},$$

where $c = (c_{mn})$ is a double sequence of real numbers satisfying the following properties

$$c_{mn} > 0 \,\forall (m,n) \in \mathbb{N} \times \mathbb{N}, \lim_{m,n\to\infty} c_{mn} = \infty \text{ and } \limsup_{m,n} \frac{c_{mn}}{mn} < \infty. \tag{2.1}$$

If $c_{mn} = mn$, then the above definition turns to the definition of double natural density. Throughout the paper, $c = (c_{mn})$ will be used to denote the double sequences that satisfy (2.1).

Definition 2.5. [4] A double sequence (x_{ij}) is said to be quasi statistical convergent to l if for each $\varepsilon > 0$,

$$\delta_q^{2,c}(A(\varepsilon)) = 0, \text{ where } A(\varepsilon) = \{(i,j) \in \mathbb{N} \times \mathbb{N} : |x_{ij} - l| \geq \varepsilon\}.$$

In this case, l is called the quasi statistical limit of the double sequence (x_{ij}) and symbolically it is expressed as $x_{ij} \xrightarrow{st_q^2} l$.

Definition 2.6. [14] A binary operation $\odot : [0,1] \times [0,1] \to [0,1]$, is said to be a continuous t-norm if the following conditions are satisfied:

i. $\odot$ is associative and commutative,

ii. $\odot$ is continuous,

iii. $s \odot 1 = s$, for all $s \in [0,1]$,

iv. $s \odot t \leq u \odot v$ whenever $s \leq u$ and $t \leq v$, for all $s,t,u,v \in [0,1]$.

Definition 2.7. [14] A binary operation $\circledast : [0,1] \times [0,1] \to [0,1]$, is said to be a continuous t-conorm if the following conditions are satisfied:

i. $\circledast$ is associative and commutative,

ii. $\circledast$ is continuous,

iii. $s \circledast 0 = s$, for all $s \in [0,1]$,

iv. $s \circledast t \leq u \circledast v$ whenever $s \leq u$ and $t \leq v$, for all $s,t,u,v \in [0,1]$.

Definition 2.8. [21] Let X be the universe of discourse. Then, the set $A_{NS} \subseteq X$ defined by

$$A_{NS} = \{< u, \mathcal{S}_A(u), \mathcal{T}_A(u), \mathcal{W}_A(u) >: u \in X\}$$

is called a neutrosophic set, where $\mathcal{S}_A(u), \mathcal{T}_A(u), \mathcal{W}_A(u) : X \to [0,1]$ represent the degree of truth-membership, degree of indeterminacy-membership, and degree of false-membership respectively, with $0 \leq \mathcal{S}_A(u) + \mathcal{T}_A(u) + \mathcal{W}_A(u) \leq 3$.

Definition 2.9. [12] Let F be a vector space and $\mathcal{N} = \{< u, \mathcal{S}(u), \mathcal{T}(u), \mathcal{W}(u) >: u \in F\}$ be a normed space (NS) such that $\mathcal{S}, \mathcal{T}, \mathcal{W} : F \times \mathbb{R}^+ \to [0,1]$. Let $\odot$ and $\circledast$ be the continuous t-norm and continuous t-conorm, respectively. Then the four tuple $V = (F, \mathcal{N}, \odot, \circledast)$ is called neutrosophic normed space (NNS) if the following conditions hold, for all $u, v \in F$ and $\eta, \nu > 0$ and for each $\sigma \neq 0$:

i. $0 \leq \mathcal{S}(u,\eta) \leq 1$, $0 \leq \mathcal{T}(u,\eta) \leq 1$, $0 \leq \mathcal{W}(u,\eta) \leq 1$,

ii. $\mathcal{S}(u,\eta) + \mathcal{T}(u,\eta) + \mathcal{W}(u,\eta) \leq 3$,

iii. $\mathcal{S}(u,\eta) = 1$ (for $\eta > 0$) if and only if $u = 0$,

iv. $\mathcal{S}(\sigma u,\eta) = \mathcal{S}(u, \frac{\eta}{|\sigma|})$,

v. $\mathcal{S}(u,\eta) \odot \mathcal{S}(v,\eta) \leq \mathcal{S}(u+v, \eta+\nu)$,

vi. $\mathcal{S}(u,.)$ is a continuous non-decreasing function,

vii. $\lim_{\eta\to\infty} \mathcal{S}(u,\eta) = 1$,

viii. $\mathcal{T}(u,\eta) = 0$ (for $\eta > 0$) iff $u = 0$,

ix. $\mathcal{T}(\sigma u,\eta) = \mathcal{T}(u, \frac{\eta}{|\sigma|})$,

x. $\mathcal{T}(u,\nu) \circledast \mathcal{T}(v,\eta) \geq \mathcal{T}(u+v, \eta+\nu)$,

xi. $\mathcal{T}(u,.)$ is a continuous and non-increasing function,

xii. $\lim_{\eta\to\infty} \mathcal{T}(u,\eta) = 0$,

xiii. $\mathcal{W}(u,\eta) = 0$ (for $\eta > 0$) if and only if $u = 0$,

xiv. $\mathcal{W}(\sigma u,\eta) = \mathcal{W}(u, \frac{\eta}{|\sigma|})$,

xv. $\mathcal{W}(u,\nu) \circledast \mathcal{W}(v,\eta) \geq \gamma(u+v, \eta+\nu)$,

xvi. $\mathcal{W}(u,.)$ is a continuous non-increasing function,

xvii. $\lim_{\eta\to\infty} \mathcal{W}(u,\eta) = 0$,

xviii. If $\eta \leq 0$, then $\mathcal{S}(u,\eta) = 0, \mathcal{T}(u,\eta) = 1$ and $\mathcal{W}(u,\eta) = 1$.

Then, $\mathcal{N} = (\mathcal{S}, \mathcal{T}, \mathcal{W})$ is called Neutrosophic norm (NN).

Example 2.1. [12] Suppose that $(F, ||\cdot||)$ is an NS. For $s, t \in [0,1]$, define the t-norm $\odot$ and the t-conorm $\circledast$ as $s \odot t = st$ and $s \circledast t = s + t - st$, respectively. For $\eta > ||u||$, let

$$\mathcal{S}(u,\eta) = \frac{\eta}{\eta + ||u||},\ \mathcal{T}(u,\eta) = \frac{||u||}{\eta + ||u||},\ \mathcal{W}(u,\eta) = \frac{||u||}{\eta}\ \ \forall u \in F \ and \ \eta > 0$$

and for $\eta \leq ||u||$, let $\mathcal{S}(u,\eta) = 0, \mathcal{T}(u,\eta) = 1$ and $\mathcal{W}(u,\eta) = 1$. Then, $(F, \mathcal{N}, \odot, \circledast)$ is an NNS.

Definition 2.10. [5] Let V be an NNS. A double sequence (x_{ij}) is said to be statistical convergent to l with respect to the neutrosophic norm (NN), if for every $0 < \varepsilon < 1$, $\delta^2(K(\varepsilon)) = 0$, where

$$K(\varepsilon) = \{(i,j) \in \mathbb{N} \times \mathbb{N} : \mathcal{S}(x_{ij} - l, \eta) \leq 1 - \varepsilon, \mathcal{T}(x_{ij} - l, \eta) \geq \varepsilon \\ \quad and\, \mathcal{W}(x_{ij} - l, \eta) \geq \varepsilon\}.$$

Symbolically it is denoted as $st^2 - \mathcal{N} - \lim x_{ij} = l$ or $x_{ij} \to l(st^2 - \mathcal{N})$.

Definition 2.11. [5] Let (x_{ij}) be a double sequence in an NNS V. Then, (x_{ij}) is said to be statistical Cauchy if for any $0 < \varepsilon < 1$, there exists $M = M(\varepsilon), N = N(\varepsilon)$ such that

$$\delta^2(KC(\varepsilon)) = 0, \text{ where } KC(\varepsilon) = \{(i,j) \in \mathbb{N} \times \mathbb{N} : \mathcal{S}(x_{ij} - x_{MN}, \eta) \leq 1 - \varepsilon \; or \\ \mathcal{T}(x_{ij} - x_{MN}, \eta) \geq \varepsilon, \mathcal{W}(x_{ij} - x_{MN}, \eta) \geq \varepsilon\}.$$

3. Main results

Definition 3.1. Let V be an NNS. A double sequence (x_{ij}) is said to be quasi statistical convergent to l with respect to neutrosophic norm (NN), if for every $0 < \varepsilon < 1$,

$$\delta_q^{2,c}(K(\varepsilon)) = 0, \text{ where } K(\varepsilon) = \{(i,j) \in \mathbb{N} \times \mathbb{N} : \mathcal{S}(x_{ij} - l, \eta) \leq 1 - \varepsilon, \\ \mathcal{T}(x_{ij} - l, \eta) \geq \varepsilon \; and \, \mathcal{W}(x_{ij} - l, \eta) \geq \varepsilon\}.$$

In this case, we write, $st_q^{2,c} - \mathcal{N} - \lim x_{ij} = l$ or $x_{ij} \to l(st_q^{2,c} - \mathcal{N})$.

In particular, if we take $c_{mn} = mn$ $(m, n \in \mathbb{N})$, then the above definition coincides with the definition of statistical convergence of double sequences in NNS, which was recently investigated by Granados and Dhital [5].

Example 3.1. Let $(F, ||\cdot||)$ be an NS. For all $s, t \in [0,1]$, define the continuous t-norm $s \odot t = st$ and the continuous t-conorm $s \circledast t = \min\{s + t, 1\}$. We take $\mathcal{S}, \mathcal{T}, \mathcal{W}$ from Example 2.1, for all $\eta > 0$. Then, V is an NNS. Let $c_{mn} = mn$ for all $m, n \in \mathbb{N}$. Define the double sequence (x_{ij}) as

$$x_{ij} = \begin{cases} 1, & i \, is \, a \, perfect \, cube, \; for \, all \; j \in \mathbb{N} \\ 0, & \text{otherwise.} \end{cases}$$

Then, $x_{ij} \to 0(st_q^{2,c} - \mathcal{N})$.
Justification: For every $0 < \varepsilon < 1$, we have

$$K_{m,n}(\varepsilon) = \{i \leq m, j \leq n : \mathcal{S}(x_{ij} - 0, \eta) \leq 1 - \varepsilon, \; \mathcal{T}(x_{ij} - 0, \eta) \geq \varepsilon \\ and \, \mathcal{W}(x_{ij} - 0, \eta) \geq \varepsilon\}.$$

This implies that,

$$\begin{aligned} K_{m,n}(\varepsilon) &= \{i \leq m, j \leq n : \frac{\eta}{\eta + ||x_{ij}||} \leq 1 - \varepsilon, \; \frac{||x_{ij}||}{\eta + ||x_{ij}||} \geq \varepsilon \; and \; \frac{||x_{ij}||}{\eta} \geq \varepsilon\} \\ &= \{i \leq m, j \leq n : ||x_{ij}|| \geq \frac{\eta\varepsilon}{1 - \varepsilon} \; and \; ||x_{ij}|| \geq \eta\varepsilon\} \end{aligned}$$

$= \{i \leq m, j \leq n : x_{ij} = 1\}$.

Then, we have, $\delta_q^{2,c}(K(\varepsilon)) = \lim_{n\to\infty} \frac{|K_{m,n}(\varepsilon)|}{c_{mn}} \leq \lim_{n\to\infty} \frac{\sqrt[3]{mn}}{mn} = 0$.

Hence, $x_{ij} \to 0(st_q^{2,c} - \mathcal{N})$.

Lemma 3.1. *Let V be an NNS. Then, for any $0 < \varepsilon < 1$, the following statements are equivalent:*

i. $x_{ij} \to l(st_q^{2,c} - \mathcal{N})$;

ii. $\delta_q^{2,c}(\{(i,j) \in \mathbb{N}\times\mathbb{N} : \mathcal{S}(x_{ij}-l,\eta) \leq 1-\varepsilon\}) = \delta_q^{2,c}(\{(i,j) \in \mathbb{N}\times\mathbb{N} : \mathcal{T}(x_{ij}-l,\eta) \geq \varepsilon\}) = \delta_q^{2,c}(\{(i,j) \in \mathbb{N}\times\mathbb{N} : \mathcal{W}(x_{ij}-l,\eta) \geq \varepsilon\}) = 0$;

iii. $\delta_q^{2,c}(\{(i,j) \in \mathbb{N}\times\mathbb{N} : \mathcal{S}(x_{ij}-l,\eta) > 1-\varepsilon,\ \mathcal{T}(x_{ij}-l,\eta) < \varepsilon \text{ and } \mathcal{W}(x_{ij}-l,\eta) < \varepsilon\}) = 1$;

iv. $\delta_q^{2,c}(\{(i,j) \in \mathbb{N}\times\mathbb{N} : \mathcal{S}(x_{ij}-l,\eta) > 1-\varepsilon\}) = \delta_q^{2,c}(\{(i,j) \in \mathbb{N}\times\mathbb{N} : \mathcal{T}(x_{ij}-l,\eta) < \varepsilon\}) = \delta_q^{2,c}(\{(i,j) \in \mathbb{N}\times\mathbb{N} : \mathcal{W}(x_{ij}-l,\eta) < \varepsilon\}) = 1$;

v. $\mathcal{S}(x_{ij}-l,\eta) \to 1(st_q^{2,c} - \mathcal{N})$, $\mathcal{T}(x_{ij}-l,\eta) \to 0(st_q^{2,c} - \mathcal{N})$ *and* $\mathcal{W}(x_{ij}-l,\eta) \to 0(st_q^{2,c} - \mathcal{N})$.

Proof. The proof is easy, so it is omitted. □

Theorem 3.1. *Let V be an NNS and let (x_{ij}) be a double sequence such that $x_{ij} \to l(st_q^{2,c} - \mathcal{N})$. Then, l is unique.*

Proof. If possible, let $x_{ij} \to l_1(st_q^{2,c} - \mathcal{N})$ and $x_{ij} \to l_2(st_q^{2,c} - \mathcal{N})$ for $l_1 \neq l_2$. Then, for a given $0 < \varepsilon < 1$, we can choose $\nu > 0$ satisfying $(1-\varepsilon) \odot (1-\varepsilon) > 1-\nu$ and $\varepsilon \circledast \varepsilon < \nu$. Now, for any $\eta > 0$ we define the following sets:

$$K_{\mathcal{S}_1}(\varepsilon,\eta) = \left\{(i,j) \in \mathbb{N}\times\mathbb{N} : \mathcal{S}(x_{ij}-l_1, \frac{\eta}{2}) \leq 1-\varepsilon\right\},$$
$$K_{\mathcal{S}_2}(\varepsilon,\eta) = \left\{(i,j) \in \mathbb{N}\times\mathbb{N} : \mathcal{S}(x_{ij}-l_2, \frac{\eta}{2}) \leq 1-\varepsilon\right\},$$
$$K_{\mathcal{T}_1}(\varepsilon,\eta) = \left\{(i,j) \in \mathbb{N}\times\mathbb{N} : \mathcal{T}(x_{ij}-l_1, \frac{\eta}{2}) \geq \varepsilon\right\},$$
$$K_{\mathcal{T}_2}(\varepsilon,\eta) = \left\{(i,j) \in \mathbb{N}\times\mathbb{N} : \mathcal{T}(x_{ij}-l_2, \frac{\eta}{2}) \geq \varepsilon\right\},$$
$$K_{\mathcal{W}_1}(\varepsilon,\eta) = \left\{(i,j) \in \mathbb{N}\times\mathbb{N} : \mathcal{W}(x_{ij}-l_1, \frac{\eta}{2}) \geq \varepsilon\right\},$$
$$K_{\mathcal{W}_2}(\varepsilon,\eta) = \left\{(i,j) \in \mathbb{N}\times\mathbb{N} : \mathcal{W}(x_{ij}-l_2, \frac{\eta}{2}) \geq \varepsilon\right\}.$$

Since $x_{ij} \to l_1(st_q^{2,c} - \mathcal{N})$, by Lemma 3.1, for any $\eta > 0$ we have,

$$\delta_q^{2,c}(K_{\mathcal{S}_1}(\varepsilon,\eta)) = \delta_q^{2,c}(K_{\mathcal{T}_1}(\varepsilon,\eta)) = \delta_q^{2,c}(K_{\mathcal{W}_1}(\varepsilon,\eta)) = 0.$$

Again since $x_{ij} \to l_2(st_q^{2,c} - \mathcal{N})$, by Lemma 3.1, for any $\eta > 0$ we have,

$$\delta_q^{2,c}(K_{\mathcal{S}_2}(\varepsilon,\eta)) = \delta_q^{2,c}(K_{\mathcal{T}_2}(\varepsilon,\eta)) = \delta_q^{2,c}(K_{\mathcal{W}_2}(\varepsilon,\eta)) = 0.$$

Suppose $K(\varepsilon,\eta) = (K_{\mathcal{S}_1}(\varepsilon,\eta) \cup K_{\mathcal{S}_2}(\varepsilon,\eta)) \cap (K_{\mathcal{T}_1}(\varepsilon,\eta) \cup K_{\mathcal{T}_2}(\varepsilon,\eta)) \cap (K_{\mathcal{W}_1}(\varepsilon,\eta) \cup K_{\mathcal{W}_2}(\varepsilon,\eta))$. Then, we have $\delta_q^{2,c}(K(\varepsilon,\eta)) = 0$ and consequently $\delta_q^{2,c}((\mathbb{N}\times\mathbb{N}) \setminus K(\varepsilon,\eta)) = 1$. Thus, the set $(\mathbb{N}\times\mathbb{N}) \setminus K(\varepsilon,\eta)$ is non-empty. Choose $(s,t) \in \mathbb{N} \setminus K(\varepsilon,\eta)$. Then, there are three possibilities:

i. $(s,t) \in ((\mathbb{N}\times\mathbb{N}) \setminus K_{\mathcal{S}_1}(\varepsilon,\eta)) \cap ((\mathbb{N}\times\mathbb{N}) \setminus K_{\mathcal{S}_2}(\varepsilon,\eta))$;

ii. $(s,t) \in ((\mathbb{N}\times\mathbb{N}) \setminus K_{\mathcal{T}_1}(\varepsilon,\eta)) \cap ((\mathbb{N}\times\mathbb{N}) \setminus K_{\mathcal{T}_2}(\varepsilon,\eta))$;

iii. $(s,t) \in ((\mathbb{N}\times\mathbb{N}) \setminus K_{\mathcal{W}_1}(\varepsilon,\eta)) \cap ((\mathbb{N}\times\mathbb{N}) \setminus K_{\mathcal{W}_2}(\varepsilon,\eta))$.

If we consider (i), then we have the following

$$\mathcal{S}(l_1 - l_2,\eta) \geq \mathcal{S}(x_{st} - l_1, \frac{\eta}{2}) \odot \mathcal{S}(x_{st} - l_2, \frac{\eta}{2}) > (1-\varepsilon) \odot (1-\varepsilon) > 1-\nu. \quad (3.1)$$

Now since ν is arbitrary from Equation (3.1), for any $\eta > 0$, we obtain $\mathcal{S}(l_1-l_2,\eta) = 1$ i.e., $l_1 = l_2$.

Again, if we consider (ii), then we have the following

$$\mathcal{T}(l_1 - l_2,\eta) \leq \mathcal{T}(x_{st} - l_1, \frac{\eta}{2}) \circledast \mathcal{T}(x_{st} - l_2, \frac{\eta}{2}) < \varepsilon \circledast \varepsilon < \nu. \quad (3.2)$$

Now, since ν is arbitrary from Equation (3.2), for any $\eta > 0$, we obtain $\mathcal{T}(l_1-l_2,\eta) = 0$ i.e., $l_1 = l_2$.

Finally, if we consider (iii), then we have the following

$$\mathcal{W}(l_1 - l_2,\eta) \leq \mathcal{W}(x_{st} - l_1, \frac{\eta}{2}) \circledast \mathcal{W}(x_{st} - l_2, \frac{\eta}{2}) < \varepsilon \circledast \varepsilon < \nu. \quad (3.3)$$

Now, since ν is arbitrary from Equation (3.3), for any $\eta > 0$, we obtain $\mathcal{W}(l_1 - l_2,\eta) = 0$ i.e., $l_1 = l_2$. Thus in all cases, we obtain $l_1 = l_2$ and this completes the proof. □

Theorem 3.2. *Let (x_{ij}) and (y_{ij}) be two double sequences in the NNS V such that $x_{ij} \to l_1(st_q^{2,c} - \mathcal{N})$ and $y_{ij} \to l_2(st_q^{2,c} - \mathcal{N})$. Then,*
(i) $x_{ij} + y_{ij} \to l_1 + l_2(st_q^{2,c} - \mathcal{N})$ and (ii) $\alpha x_{ij} \to \alpha l_1(st_q^{2,c} - \mathcal{N})$ where $\alpha \in \mathbb{R}$.

Proof. (i) Suppose $x_{ij} \to l_1(st_q^{2,c}-\mathcal{N})$ and $y_{ij} \to l_2(st_q^{2,c}-\mathcal{N})$. Then, by definition for any $0 < \varepsilon < 1$,

$$\delta_q^{2,c}(K(\varepsilon)) = \delta_q^{2,c}(K'(\varepsilon)) = 0, \text{ where}$$

$$K(\varepsilon) = \{(i,j) \in \mathbb{N}\times\mathbb{N} : \mathcal{S}(x_{ij} - l_1, \frac{\eta}{2}) \leq 1-\varepsilon, \mathcal{T}(x_{ij} - l_1, \frac{\eta}{2}) \geq \varepsilon \ \textit{and}\ \mathcal{W}(x_{ij} - l_1, \frac{\eta}{2}) \geq \varepsilon\}$$

and

$$K'(\varepsilon) = \{(i,j) \in \mathbb{N}\times\mathbb{N} : \mathcal{S}(y_{ij} - l_2, \frac{\eta}{2}) \leq 1-\varepsilon,\ \mathcal{T}(y_{ij} - l_2, \frac{\eta}{2}) \geq \varepsilon \ \textit{and}\ \mathcal{W}(y_{ij} - l_2, \frac{\eta}{2}) \geq \varepsilon\}.$$

Now consider the inclusion

$$((\mathbb{N}\times\mathbb{N}) \setminus K(\varepsilon)) \cap ((\mathbb{N}\times\mathbb{N}) \setminus K'(\varepsilon)) \subseteq \{(i,j) \in \mathbb{N}\times\mathbb{N} : \mathcal{S}(x_{ij} + y_{ij} - l_1 - l_2, \eta) > 1-\varepsilon, \mathcal{T}(x_{ij} + y_{ij} - l_1 - l_2, \eta) < \varepsilon, \mathcal{W}(x_{ij} + y_{ij} - l_1 - l_2, \eta) < \varepsilon\}$$

holds, so we must have

$$K''(\varepsilon) = \{(i,j) \in \mathbb{N}\times\mathbb{N} : \mathcal{S}(x_{ij}+y_{ij}-l_1-l_2,\eta) \leq 1-\varepsilon,\, \mathcal{T}(x_{ij}+y_{ij}-l_1-l_2,\eta) \geq \varepsilon \\ and\, \mathcal{W}(x_{ij}+y_{ij}-l_1-l_2,\eta) \geq \varepsilon\} \subseteq K(\varepsilon) \cup K'(\varepsilon)$$

and consequently, $\delta_q^{2,c}(K''(\varepsilon)) = 0$ i.e., $x_{ij} + y_{ij} \to l_1 + l_2(st_q^{2,c} - \mathcal{N})$.

(ii) If $\alpha = 0$, then there is nothing to prove. So we assume $\alpha \neq 0$. Since, $x_{ij} \to l_1(st_q^{2,c} - \mathcal{N})$, so for any $0 < \varepsilon < 1$, $\delta_q^{2,c}(K(\varepsilon)) = 0$, where

$$K(\varepsilon) = \{(i,j) \in \mathbb{N}\times\mathbb{N} : \mathcal{S}(x_{ij} - l, \frac{\eta}{|\alpha|}) \leq 1-\varepsilon,\, \mathcal{T}(x_{ij} - l, \frac{\eta}{|\alpha|}) \geq \varepsilon \\ and\, \mathcal{W}(x_{ij} - l, \frac{\eta}{|\alpha|}) \geq \varepsilon\}.$$

Now let $K'(\varepsilon)$ denote the set

$$\{(i,j) \in \mathbb{N}\times\mathbb{N} : \mathcal{S}(\alpha x_{ij} - \alpha l, \eta) \leq 1-\varepsilon,\, \mathcal{T}(\alpha x_{ij} - \alpha l, \eta) \geq \varepsilon \\ and\, \mathcal{W}(\alpha x_{ij} - \alpha l, \eta) \geq \varepsilon\}.$$

Then, the inclusion $K'(\varepsilon) \subseteq K(\varepsilon)$ holds and eventually, $\delta_q^{2,c}(K'(\varepsilon)) = 0$. Hence, $\alpha x_{ij} \to \alpha l_1(st_q^{2,c} - \mathcal{N})$. □

Theorem 3.3. *Let (x_{ij}) be a double sequence in an NNS V. If $x_{ij} \to l(st_q^{2,c} - \mathcal{N})$ holds, then $x_{ij} \to l(st^2 - \mathcal{N})$.*

Proof. By our assumption, for any $0 < \varepsilon < 1$,

$$\delta_q^{2,c}(\{(i,j) \in \mathbb{N}\times\mathbb{N} : \mathcal{S}(x_{ij} - l, \eta) \leq 1-\varepsilon,\, \mathcal{T}(x_{ij} - l, \eta) \geq \varepsilon \\ and\, \mathcal{W}(x_{ij} - l, \eta) \geq \varepsilon\}) = 0$$

$$i.e., \lim_{n\to\infty} \frac{1}{c_{mn}} |\{i \leq m, j \leq n : \mathcal{S}(x_{ij} - l, \eta) \leq 1-\varepsilon,\, \mathcal{T}(x_{ij} - l, \eta) \geq \varepsilon \\ and\, \mathcal{W}(x_{ij} - l, \eta) \geq \varepsilon\}| = 0. \quad (3.4)$$

Let $C = \sup\limits_{m,n} \frac{c_{mn}}{mn}$. Then, we have

$$\frac{1}{mn} |\{i \leq m, j \leq n : \mathcal{S}(x_{ij} - l, \eta) \leq 1-\varepsilon,\, \mathcal{T}(x_{ij} - l, \eta) \geq \varepsilon \\ and\, \mathcal{W}(x_{ij} - l, \eta) \geq \varepsilon\}| \\ \leq \frac{C}{c_{mn}} |\{i \leq m, j \leq n : \mathcal{S}(x_{ij} - l, \eta) \leq 1-\varepsilon,\, \mathcal{T}(x_{ij} - l, \eta) \geq \varepsilon \\ and\, \mathcal{W}(x_{ij} - l, \eta) \geq \varepsilon\}|.$$

Taking $m, n \to \infty$ on both sides of the above inequality and using the equality (3.4) we obtain $x_{ij} \to l(st^2 - \mathcal{N})$. □

Remark 3.1. The converse of Theorem 3.3 is not necessarily true. We present the following counterexample to illustrate the fact.

Example 3.2. Let $(F, ||\cdot||)$ be a NS. For all $u, v \in [0,1]$, define the TN $u \odot v = uv$ and the TC $u \circledast v = \min\{u+v, 1\}$. We take $\mathcal{S}, \mathcal{T}, \mathcal{W}$ in Example 2.1, for all $\eta > 0$. Then V is an NNS. Let (c_{mn}) be a double sequence satisfying $\lim_{m,n\to\infty} c_{mn} = \infty$ and $\lim_{m,n\to\infty} \frac{\sqrt[3]{mn}}{c_{mn}} = \infty$. We can choose a double subsequence $(c_{m_s n_t})$ such that $c_{m_s n_t} > 1$ for all $s, t \in \mathbb{N}$. Define the double sequence (x_{ij}) as

$$x_{ij} = \begin{cases} c_{ij}, & if\ i,j\ are\ perfect\ cube\ and\ c_{ij} \in \{c_{m_s n_t} : s,t \in \mathbb{N}\} \\ 3, & if\ i,j\ are\ perfect\ cube\ and\ c_{ij} \notin \{c_{m_s n_t} : s,t \in \mathbb{N}\} \\ 0, & otherwise \end{cases}.$$

Then, it is easy to verify that $x_{ij} \to 0(st^2 - \mathcal{N})$. But considering $\varepsilon = 1$, we obtain $x_{ij} \nrightarrow 0(st_q^{2,c} - \mathcal{N})$.

From the above example, the question naturally arises: under what conditions does the converse of Theorem 3.3 hold? The following theorem gives the answer.

Theorem 3.4. *Let (x_{ij}) be a double sequence in an NNS V such that $x_{ij} \to l(st^2 - \mathcal{N})$. Then, $x_{ij} \to l(st_q^{2,c} - \mathcal{N})$ holds if $\inf_{m,n} \frac{c_{mn}}{mn} > 0$.*

Proof. The proof follows directly from the following inequation:

$$\begin{aligned} &\frac{1}{mn}|\{i \le m, j \le n : \mathcal{S}(x_{ij} - l, \eta) \le 1 - \varepsilon,\ \mathcal{T}(x_{ij} - l, \eta) \ge \varepsilon\ and\ \mathcal{W}(x_{ij} - l, \eta) \ge \varepsilon\} \\ &\quad \ge (\inf_{m,n} \frac{c_{mn}}{mn}) \frac{1}{c_{mn}} |\{i \le m, j \le n : \mathcal{S}(x_{ij} - l, \eta) \le 1 - \varepsilon,\ \mathcal{T}(x_{ij} - l, \eta) \ge \varepsilon \\ &\qquad\qquad and\ \mathcal{W}(x_{ij} - l, \eta) \ge \varepsilon\}. \end{aligned}$$

□

Theorem 3.5. *Let (x_{ij}) be a double sequence in an NNS V such that $x_{ij} \to l(\mathcal{N})$. Then $x_{ij} \to l(st_q^{2,c} - \mathcal{N})$.*

Proof. By our assumption, for any $0 < \varepsilon < 1$, there exists $i_0, j_0 \in \mathbb{N}$ such that for all $i \ge i_0, j \ge j_0$,

$$\mathcal{S}(x_{ij} - l, \eta) > 1 - \varepsilon,\ \mathcal{T}(x_{ij} - l, \eta) < \varepsilon\ and\ \mathcal{W}(x_{ij} - l, \eta) < \varepsilon.$$

Eventually, the inclusion

$$\begin{aligned} K(\varepsilon) = \{(i,j) \in \mathbb{N} \times \mathbb{N} : \mathcal{S}(x_{ij} - l, \eta) \le 1 - \varepsilon,\ \mathcal{T}(x_{ij} - l, \eta) \ge \varepsilon \\ and\ \mathcal{W}(x_{ij} - l, \eta) \ge \varepsilon\} \\ \subseteq \{1, 2, ..., i_0 - 1\} \times \{1, 2, ..., j_0 - 1\} \end{aligned}$$

holds. Now since the double quasi-density of a finite set is zero, so the double quasi-density of the set $K(\varepsilon)$ is also zero. Hence, $x_{ij} \to l(st_q^{2,c} - \mathcal{N})$. □

Theorem 3.6. *Let (x_{ij}) and (y_{ij}) be two double sequences in the NNS V such that $y_{ij} \to l(\mathcal{N})$ and $\delta_q^{2,c}(\{(i,j) \in \mathbb{N} \times \mathbb{N} : x_{ij} \ne y_{ij}\}) = 0$. Then, $x_{ij} \to l(st_q^{2,c} - \mathcal{N})$.*

Proof. Suppose that $\delta_q^{2,c}(\{(i,j) \in \mathbb{N} \times \mathbb{N} : x_{ij} \ne y_{ij}\}) = 0$ holds and $y_{ij} \to l(\mathcal{N})$. Then, by definition for every $0 < \varepsilon < 1$, the set $K(\varepsilon) = \{(i,j) \in \mathbb{N} \times \mathbb{N} : \mathcal{S}(y_{ij} -$

$l,\eta) \leq 1-\varepsilon$, $\mathcal{T}(y_{ij}-l,\eta) \geq \varepsilon \, and \mathcal{W}(y_{ij}-l,\eta) \geq \varepsilon\}$ contains at most a finite number of elements and consequently, $\delta_q^{2,c}(K(\varepsilon)) = 0$. Now, since the inclusion

$$\begin{aligned} K'(\varepsilon) = \{(i,j) \in \mathbb{N}\times\mathbb{N} : \mathcal{S}(x_{ij}-l,\eta) \leq 1-\varepsilon,\ \mathcal{T}(x_{ij}-l,\eta) \geq \varepsilon \\ and\, \mathcal{W}(x_{ij}-l,\eta) \geq \varepsilon\} \\ \subseteq K(\varepsilon) \cap \{(i,j) \in \mathbb{N}\times\mathbb{N} : x_{ij} \neq y_{ij}\} \end{aligned}$$

holds, we must have, $\delta_q^{2,c}(K'(\varepsilon)) = 0$. Hence, $x_{ij} \to l(st_q^{2,c} - \mathcal{N})$. □

Theorem 3.7. *Let $c = (c_{mn})$ and $d = (d_{mn})$ be two double sequences both satisfying Equation (2.1) and $c_{mn} \leq d_{mn}$ $\forall m,n \in \mathbb{N}$. If (x_{ij}) is a double sequence in an NNS V such that $x_{ij} \to l(st_q^{2,c} - \mathcal{N})$, then, $x_{ij} \to l(st_q^{2,d} - \mathcal{N})$.*

Proof. By our assumption, for any $0 < \varepsilon < 1$,

$$\begin{aligned} \delta_q^{2,c}(\{(i,j) \in \mathbb{N}\times\mathbb{N} : \mathcal{S}(x_{ij}-l,\eta) \leq 1-\varepsilon,\ \mathcal{T}(x_{ij}-l,\eta) \geq \varepsilon \\ and\, \mathcal{W}(x_{ij}-l,\eta) \geq \varepsilon\}) = 0 \\ i.e., \lim_{m,n\to\infty} \frac{1}{c_{mn}} |\{i \leq m, j \leq n : \mathcal{S}(x_{ij}-l,\eta) \leq 1-\varepsilon,\ \mathcal{T}(x_{ij}-l,\eta) \geq \varepsilon \\ and\, \mathcal{W}(x_{ij}-l,\eta) \geq \varepsilon\}| = 0. \end{aligned}$$

Now, since $c_{mn} \leq d_{mn}$ holds for all $m,n \in \mathbb{N}$, we must have,

$$\begin{aligned} \frac{1}{d_{mn}} |\{i \leq m, j \leq n : \mathcal{S}(x_{ij}-l,\eta) \leq 1-\varepsilon,\ \mathcal{T}(x_{ij}-l,\eta) \geq \varepsilon \\ and\, \mathcal{W}(x_{ij}-l,\eta) \geq \varepsilon\} \\ \leq \frac{1}{c_{mn}} |\{i \leq m, j \leq n : \mathcal{S}(x_{ij}-l,\eta) \leq 1-\varepsilon,\ \mathcal{T}(x_{ij}-l,\eta) \geq \varepsilon \\ and\, \mathcal{W}(x_{ij}-l,\eta) \geq \varepsilon\}. \end{aligned}$$

Letting $m,n \to \infty$ on both sides of the above inequation, we obtain $x_{ij} \to l(st_q^{2,d} - \mathcal{N})$. □

Remark 3.2. The converse of the Theorem 3.7 is not necessarily true. For the justification, consider Example 3.2. If we take $c_{mn} = (mn)^{\frac{1}{6}}$ and $d_{mn} = mn$, then $x_{ij} \to 0(st_q^{d} - \mathcal{N})$ but $x_{ij} \nrightarrow 0(st_q^{2,c} - \mathcal{N})$.

Definition 3.2. Let (x_{ij}) be a double sequence in an NNS V. Then (x_{ij}) is said to be quasi statistical Cauchy if for any $0 < \varepsilon < 1$, there exist two positive integers $M = M(\varepsilon), N = N(\varepsilon)$ such that $\delta_q^{2,c}(KC(\varepsilon)) = 0$, where

$$\begin{aligned} KC(\varepsilon) = \{(i,j) \in \mathbb{N}\times\mathbb{N} : \mathcal{S}(x_{ij}-x_{MN},\eta) \leq 1-\varepsilon \\ or\, \mathcal{T}(x_{ij}-x_{MN},\eta) \geq \varepsilon, \mathcal{W}(x_{ij}-x_{MN},\eta) \geq \varepsilon\}. \end{aligned}$$

Theorem 3.8. *Let (x_{ij}) be a double sequence in an NNS V. Then, (x_{ij}) is a quasi statistical convergent sequence if and only if it is quasi statistical Cauchy sequence.*

Proof. Suppose $x_{ij} \to l(st_q^{2,c} - \mathcal{N})$. For a given $0 < \varepsilon < 1$, we choose $\nu > 0$ such that $(1-\varepsilon) \odot (1-\varepsilon) > 1-\nu$ and $\varepsilon \circledast \varepsilon < \nu$. Then by definition, for any $0 < \varepsilon < 1$, $\delta_q^{2,c}((\mathbb{N}\times\mathbb{N}) \setminus K(\varepsilon)) = 1$, where

$$K(\varepsilon)=\{(i,j)\in\mathbb{N}\times\mathbb{N}:\mathcal{S}(x_{ij}-l,\frac{\eta}{2})\leq 1-\varepsilon,\ \mathcal{T}(x_{ij}-l,\frac{\eta}{2})\geq\varepsilon \\ and\,\mathcal{W}(x_{ij}-l,\frac{\eta}{2})\geq\varepsilon\}.$$

Thus, the set $(\mathbb{N}\times\mathbb{N})\setminus K(\varepsilon)$ is non-empty. Let $(M,N)\in(\mathbb{N}\times\mathbb{N})\setminus K(\varepsilon)$. Then we have,

$$\mathcal{S}(x_{MN}-l,\frac{\eta}{2})>1-\varepsilon,\ \mathcal{T}(x_{MN}-l,\frac{\eta}{2})<\varepsilon\,and\,\mathcal{W}(x_{MN}-l,\frac{\eta}{2})<\varepsilon.$$

Now suppose $KC(\varepsilon)=\{(i,j)\in\mathbb{N}\times\mathbb{N}:\mathcal{S}(x_{ij}-x_{MN},\eta)\leq 1-\nu,\ \mathcal{T}(x_{ij}-x_{MN},\eta)\geq\nu\,and\,\mathcal{W}(x_{ij}-x_{MN},\eta)\geq\nu\}$. We claim that $KC(\varepsilon)\subseteq K(\varepsilon)$ because if the inclusion does not hold then we must have some $(M_0,N_0)\in KC(\varepsilon)\setminus K(\varepsilon)$ which immediately yields $\mathcal{S}(x_{M_0N_0}-x_{MN},\eta)\leq 1-\nu$ and $\mathcal{S}(x_{M_0N_0}-l,\frac{\eta}{2})>1-\varepsilon$. In particular, $\mathcal{S}(x_{MN}-l,\frac{\eta}{2})>1-\varepsilon$. But then,

$$1-\nu\geq\mathcal{S}(x_{M_0N_0}-x_{MN},\eta)\geq\mathcal{S}(x_{M_0N_0}-l,\frac{\eta}{2})\odot\mathcal{S}(x_{MN}-l,\frac{\eta}{2})>(1-\varepsilon)\odot(1-\varepsilon)>1-\nu,$$

which is a contradiction. Further, we have, $\mathcal{T}(x_{M_0N_0}-x_{MN},\eta)\geq\nu$ and $\mathcal{T}(x_{M_0N_0}-l,\frac{\eta}{2})<\varepsilon$. In particular, $\mathcal{T}(x_{MN}-l,\frac{\eta}{2})<\varepsilon$. But then,

$$\nu\leq\mathcal{T}(x_{M_0N_0}-x_{MN},\eta)\leq\mathcal{T}(x_{M_0N_0}-l,\frac{\eta}{2})\circledast\mathcal{T}(x_{MN}-l,\frac{\eta}{2})<\varepsilon\circledast\varepsilon<\nu,$$

which is a contradiction. Finally, we have, $\mathcal{W}(x_{M_0N_0}-x_{MN},\eta)\geq\nu$ and $\mathcal{W}(x_{M_0N_0}-l,\frac{\eta}{2})<\varepsilon$. In particular, $\mathcal{W}(x_{MN}-l,\frac{\eta}{2})<\varepsilon$. But then,

$$\nu\leq\mathcal{W}(x_{M_0N_0}-x_{MN},\eta)\leq\mathcal{W}(x_{M_0N_0}-l,\frac{\eta}{2})\circledast\mathcal{W}(x_{MN}-l,\frac{\eta}{2})<\varepsilon\circledast\varepsilon<\nu,$$

which is a contradiction. Thus all possibilities contradict the existence of an element $(M_0,N_0)\in KC(\varepsilon)\setminus K(\varepsilon)$. Therefore, we must have $KC(\varepsilon)\subseteq K(\varepsilon)$ and as a consequence $\delta_q^{2,c}(KC(\varepsilon))=0$. Hence (x_{ij}) is quasi statistical Cauchy.

To prove the converse part, we assume that (x_{ij}) is a quasi statistical Cauchy sequence but not quasi statistical convergent. For a given $0<\varepsilon<1$, we choose $\nu>0$ such that $(1-\varepsilon)\odot(1-\varepsilon)>1-\nu$ and $\varepsilon\circledast\varepsilon<\nu$. Then, since (x_{ij}) is not quasi statistical convergent,

$$\mathcal{S}(x_{ij}-x_{MN},\eta)\geq\mathcal{S}(x_{ij}-l,\frac{\eta}{2})\odot\mathcal{S}(x_{MN}-l,\frac{\eta}{2})>(1-\varepsilon)\odot(1-\varepsilon)>1-\nu,$$
$$\mathcal{T}(x_{ij}-x_{MN},\eta)\leq\mathcal{T}(x_{ij}-l,\frac{\eta}{2})\circledast\mathcal{T}(x_{MN}-l,\frac{\eta}{2})<\varepsilon\circledast\varepsilon<\nu,$$
$$\mathcal{W}(x_{ij}-x_{MN},\eta)\leq\mathcal{W}(x_{ij}-l,\frac{\eta}{2})\circledast\mathcal{W}(x_{MN}-l,\frac{\eta}{2})<\varepsilon\circledast\varepsilon<\nu$$

hold for

$$P(\varepsilon,\nu)=\{i\leq M,j\leq N:\mathcal{T}(x_{ij}-x_{MN},\eta)\leq 1-\nu\}.$$

Therefore, $\delta_q^{2,c}(P(\varepsilon,\nu))=1$, which is a contradiction to the fact that (x_{ij}) is quasi statistical Cauchy. Hence, (x_{ij}) must be a quasi statistical convergent sequence. This completes the proof. □

Next, we present the notion of strongly quasi-summability in neutrosophic normed spaces and show some of its basic properties.

Definition 3.3. Let V be an NNS. A double sequence (x_{ij}) is said to be strongly quasi-summable to l with respect to NN (SQS-NN), if for every $0 < \varepsilon < 1$ and $\eta > 0$, $\delta_q^{2,c}(U(\varepsilon)) = 0$, where

$$U(\varepsilon) = \left\{ (i,j) \in \mathbb{N} \times \mathbb{N} : \sum_i \sum_j \mathcal{S}(x_{ij} - l, \eta) \leq 1 - \varepsilon, \sum_i \sum_j \mathcal{T}(x_{ij} - l, \eta) \geq \varepsilon \right. \\ \left. \text{and } \sum_i \sum_j \mathcal{W}(x_{ij} - l, \eta) \geq \varepsilon \right\}.$$

We will denote it as $S\text{-}st_q^{2,c} - \mathcal{N} - \lim x_{ij} = l$.

Lemma 3.2. *For every $\varepsilon > 0$ and $\eta > 0$, the following statements are equivalent:*

i. $S\text{-}st_q^{2,c} - \mathcal{N} - \lim x_{ij} = l$,

ii.

$$\begin{aligned} &\delta_q^{2,c}\left(\left\{(i,j) \in \mathbb{N} \times \mathbb{N} : \sum_i \sum_j \mathcal{S}(x_{ij} - l, \eta) \leq 1 - \varepsilon\right\}\right) \\ =&\delta_q^{2,c}\left(\left\{(i,j) \in \mathbb{N} \times \mathbb{N} : \sum_i \sum_j \mathcal{T}(x_{ij} - l, \eta) \geq \varepsilon\right\}\right) \\ =&\delta_q^{2,c}\left(\left\{(i,j) \in \mathbb{N} \times \mathbb{N} : \sum_i \sum_j \mathcal{W}(x_{ij} - l, \eta) \geq \varepsilon\right\}\right) = 0, \end{aligned}$$

iii.

$$\delta_q^{2,c}\left(\left\{(i,j) \in \mathbb{N} \times \mathbb{N} : \sum_i \sum_j \mathcal{S}(x_{ij} - l, \eta) > 1 - \varepsilon, \right.\right. \\ \left.\left. \sum_i \sum_j \mathcal{T}(x_{ij} - l, \eta) < \varepsilon \text{ and } \sum_i \sum_j \mathcal{W}(x_{ij} - l, \eta) < \varepsilon \right\}\right) = 1.$$

Proof. The proof is easy, so it is omitted. □

Theorem 3.9. *Let V be an NNS and let (x_{ij}) be a double sequence such that $S\text{-}st_q^{2,c} - \mathcal{N} - \lim x_{ij} = l$. Then, l is uniquely determined.*

Proof. If possible, consider $S\text{-}st_q^{2,c} - \mathcal{N} - \lim x_{ij} = l_1$ and $S\text{-}st_q^{2,c} - \mathcal{N} - \lim x_{ij} = l_2$ for $l_1 \neq l_2$. Then, for a given $0 < \varepsilon < 1$, we can choose $\nu > 0$ such that $(1-\varepsilon) \odot (1-\varepsilon) > 1 - \nu$ and $\varepsilon \circledast \varepsilon < \nu$. Now, for any $\eta > 0$ we define the following sets:

$$K_{\mathcal{S}_1}(\varepsilon, \eta) = \left\{(i,j) \in \mathbb{N} \times \mathbb{N} : \sum_i \sum_j \mathcal{S}(x_{ij} - l_1, \frac{\eta}{2}) \leq 1 - \varepsilon\right\},$$

$$K_{\mathcal{S}_2}(\varepsilon, \eta) = \left\{(i,j) \in \mathbb{N} \times \mathbb{N} : \sum_i \sum_j \mathcal{S}(x_{ij} - l_2, \frac{\eta}{2}) \leq 1 - \varepsilon\right\},$$

$$K_{\mathcal{T}_1}(\varepsilon,\eta)=\left\{(i,j)\in\mathbb{N}\times\mathbb{N}:\sum_i\sum_j\mathcal{T}(x_{ij}-l_1,\frac{\eta}{2})\geq\varepsilon\right\},$$

$$K_{\mathcal{T}_2}(\varepsilon,\eta)=\left\{(i,j)\in\mathbb{N}\times\mathbb{N}:\sum_i\sum_j\mathcal{T}(x_{ij}-l_2,\frac{\eta}{2})\geq\varepsilon\right\},$$

$$K_{\mathcal{W}_1}(\varepsilon,\eta)=\left\{(i,j)\in\mathbb{N}\times\mathbb{N}:\sum_i\sum_j\mathcal{W}(x_{ij}-l_1,\frac{\eta}{2})\geq\varepsilon\right\},$$

$$K_{\mathcal{W}_2}(\varepsilon,\eta)=\left\{(i,j)\in\mathbb{N}\times\mathbb{N}:\sum_i\sum_j\mathcal{W}(x_{ij}-l_2,\frac{\eta}{2})\geq\varepsilon\right\}.$$

Since $S\text{-}st_q^{2,c}-\mathcal{N}-\lim x_{ij}=l_1$, by Lemma 3.2, we have for any $\eta>0$,

$$\delta_q^{2,c}(K_{\mathcal{S}_1}(\varepsilon,\eta))=\delta_q^{2,c}(K_{\mathcal{T}_1}(\varepsilon,\eta))=\delta_q^{2,c}(K_{\mathcal{W}_1}(\varepsilon,\eta))=0.$$

Again since $S\text{-}st_q^{2,c}-\mathcal{N}-\lim x_{ij}=l_2$, by Lemma 3.2, we have for any $\eta>0$,

$$\delta_q^{2,c}(K_{\mathcal{S}_2}(\varepsilon,\eta))=\delta_q^{2,c}(K_{\mathcal{T}_2}(\varepsilon,\eta))=\delta_q^{2,c}(K_{\mathcal{W}_2}(\varepsilon,\eta))=0.$$

Let $K(\varepsilon,\eta)=(K_{\mathcal{S}_1}(\varepsilon,\eta)\cup K_{\mathcal{S}_2}(\varepsilon,\eta))\cap(K_{\mathcal{T}_1}(\varepsilon,\eta)\cup K_{\mathcal{T}_2}(\varepsilon,\eta))\cap(K_{\mathcal{W}_1}(\varepsilon,\eta)\cup K_{\mathcal{W}_2}(\varepsilon,\eta))$. Then, we have $\delta_q^{2,c}(K(\varepsilon,\eta))=0$ and eventually $\delta_q^{2,c}(\mathbb{N}\setminus K(\varepsilon,\eta))=1$ and therefore $\mathbb{N}\setminus K(\varepsilon,\eta)$ is non-empty. Choose $p\in\mathbb{N}\setminus K(\varepsilon,\eta)$. Then, there are three possibilities:

i. $p\in(\mathbb{N}\setminus(K_{\mathcal{S}_1}(\varepsilon,\eta))\cap(\mathbb{N}\setminus(K_{\mathcal{S}_2}(\varepsilon,\eta))$;
ii. $p\in(\mathbb{N}\setminus(K_{\mathcal{T}_1}(\varepsilon,\eta))\cap(\mathbb{N}\setminus(K_{\mathcal{T}_2}(\varepsilon,\eta))$;
iii. $p\in(\mathbb{N}\setminus(K_{\mathcal{W}_1}(\varepsilon,\eta))\cap(\mathbb{N}\setminus(K_{\mathcal{W}_2}(\varepsilon,\eta))$.

If we consider (i), then we have the following

$$\mathcal{S}(l_1-l_2,\eta)\geq\sum_i\sum_j\mathcal{S}(x_{ij}-l_1,\frac{\eta}{2})\odot\sum_i\sum_j\mathcal{S}(x_{ij}-l_2,\frac{\eta}{2})>(1-\varepsilon)\odot(1-\varepsilon)>1-\nu. \tag{3.5}$$

Now since ν is arbitrary, from Equation (3.5), for any $\eta>0$, we obtain $\mathcal{S}(l_1-l_2,\eta)=1$ i.e., $l_1=l_2$.

Again, if we consider (ii), then we have the following

$$\mathcal{T}(l_1-l_2,\eta)\leq\sum_i\sum_j\mathcal{T}(x_{ij}-l_1,\frac{\eta}{2})\circledast\sum_i\sum_j\mathcal{T}(x_{ij}-l_2,\frac{\eta}{2})<\varepsilon\circledast\varepsilon<\nu. \tag{3.6}$$

Now, since ν is arbitrary, from Equation (3.6), for any $\eta>0$, we obtain $\mathcal{T}(l_1-l_2,\eta)=0$ i.e., $l_1=l_2$.

Finally, if we consider (iii), then we have the following

$$\mathcal{W}(l_1-l_2,\eta)\leq\sum_i\sum_j\mathcal{W}(x_{ij}-l_1,\frac{\eta}{2})\circledast\sum_i\sum_j\mathcal{W}(x_{ij}-l_2,\frac{\eta}{2})<\varepsilon\circledast\varepsilon<\nu. \tag{3.7}$$

Now, since ν is arbitrary, from Equation (3.7), for any $\eta>0$, we obtain $\mathcal{W}(l_1-l_2,\eta)=0$ i.e., $l_1=l_2$.

Thus in all cases, we obtain $l_1=l_2$ and this completes the proof. □

Theorem 3.10. *If $S\text{-}st_q^{2,c} - \mathcal{N} - \lim x_{ij} = l$, then there is a double subsequence $(x_{p_i h_j})$ of x such that $S\text{-}st_q^{2,c} - \mathcal{N} - \lim x_{p_i h_j} = l$.*

Proof. Let $S\text{-}st_q^{2,c} - \mathcal{N} - \lim x_{ij} = l$, then for every $\eta > 0$ and $0 < \varepsilon < 1$, we have

$$\sum_i \sum_j \mathcal{S}(x_{ij} - l, \eta) \leq 1 - \varepsilon, \sum_i \sum_j \mathcal{T}(x_{ij} - l, \eta) \geq \varepsilon \text{ and } \sum_i \sum_j \mathcal{W}(x_{ij} - l, \eta) \geq \varepsilon.$$

Obviously, if we take $(p_i, h_j) \in \mathbb{N} \times \mathbb{N}$ such that

$$\sum_i \sum_j \mathcal{S}(x_{p_i h_j} - l, \eta) > \sum_i \sum_j \mathcal{S}(x_{ij} - l, \eta) > 1 - \varepsilon,$$
$$\sum_i \sum_j \mathcal{T}(x_{p_i h_j} - l, \eta) < \sum_i \sum_j \mathcal{T}(x_{ij} - l, \eta) < \varepsilon,$$
$$\sum_i \sum_j \mathcal{W}(x_{p_i h_j} - l, \eta) < \sum_i \sum_j \mathcal{W}(x_{ij} - l, \eta) < \varepsilon.$$

It follows that $S\text{-}st_q^{2,c} - \mathcal{N} - \lim x_{p_i h_j} = l$. □

4. Conclusion

In this paper, we mainly investigated various fundamental properties of quasi statistical convergence of double sequences in neutrosophic normed spaces. We established the relationship between this convergence method and the recently introduced statistical convergence of double sequences in neutrosophic normed spaces by Granados and Dhital [5]. We further investigated the notion of quasi statistical Cauchy sequences and strong quasi-summability. As a continuation of this work, one may study various properties such as solidity, symmetricity and monotonicity of the sequence spaces formed by the collection of all quasi statistical convergent double sequences in the neutrosophic normed spaces.

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