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Upper Bound of the Number of Zeros for Abelian Integrals in a Kind of Quadratic Reversible Centers of Genus One*

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Abstract By using the methods of Picard-Fuchs equation and Riccati equation, we study the upper bound of the number of zeros for Abelian integrals in a kind of quadratic reversible centers of genus one under polynomial perturbations of degree n. We obtain that the upper bound is $7[(n-3)/2]+5$ when $n \geq 5$, 8 when $n=4$, 5 when $n=3$, 4 when $n=2$, and 0 when $n=1$ or $n=0$, which linearly depends on n.

Keywords Abelian integral, quadratic reversible center, weakened Hilbert's 16th problem, Picard-Fuchs equation, Riccati equation

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1. Introduction

The last part of Hilbert's 16th problem is the discussion about the number and relative positions of limit cycles in planar dynamic systems. This problem is so complex and difficult that even the simplest nonlinear case, $n=2$, is not completely solved. In 1977, V.I. Arnold proposed a weakened form of this problem, which is to study maximum of the number of isolated zeros of the Abelian integral for the system. The problem is known as the weakened Hilbert's 16th problem [1], which is one of the hot topics in current research. So far, most of the results about determining the upper bound of the number of isolated zeros of Abelian integrals are related to perturbed Hamiltonian systems. The study of perturbed integrable non-Hamiltonian systems is more difficult than disturbance Hamiltonian systems. More details can be found in the review article [13] and the books [2,4].

The literature [12] lists all quadratic integrable non-Hamiltonian systems, because quadratic reversible systems have good functional properties. Therefore, they have been widely studied [15–17]. For quadratic reversible centers of genus one, there are essentially 22 cases in the classification in [3], namely $(r1)$–$(r22)$. The linear dependance of case $(r1)$ was studied in [18]; cases $(r3)$–$(r6)$ were studied in [14];

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cases $(r9), (r13), (r17)$ and $(r19)$ were studied in [10]; cases $(r11), (r16), (r18)$ and $(r20)$ were studied in [9]; cases $(r12)$ and $(r21)$ were studied in [8]; case $(r7)$ was studied in [7]; case $(r10)$ was studied in [6]; and $(r22)$ was studied in [5]. All of these upper bounds linearly depend on n. In order to thoroughly study these problems, we consider a special case in $(r5)$. Namely, when $(a,b) = (4,2)$, we can get the special case $(sr5)$ as follows:

$$(sr5) \quad \begin{aligned} &\dot{x} = -xy, \quad \dot{y} = -2y^2 + \frac{1}{2^4}x^2 - \frac{1}{2^4}x, \\ &H(x,y) = x^{-4}\left(\frac{1}{2}y^2 - \frac{1}{2^5}x^2 + \frac{1}{3\times 2^4}x\right) = h, \quad h \in \left(-\frac{1}{3\times 2^5}, 0\right), \end{aligned} \tag{1.1}$$

with the integrating factor $N(x,y) = x^{-5}$.

$(sr5)$ is an integrable non-Hamiltonian quadratic system, which has a center $(1,0)$, a family of periodic orbits $\{\Gamma_h\}$ $(-1/(3\times 2^5) < h < 0)$, and an integral curve $x = 0$ (see Figure 1).

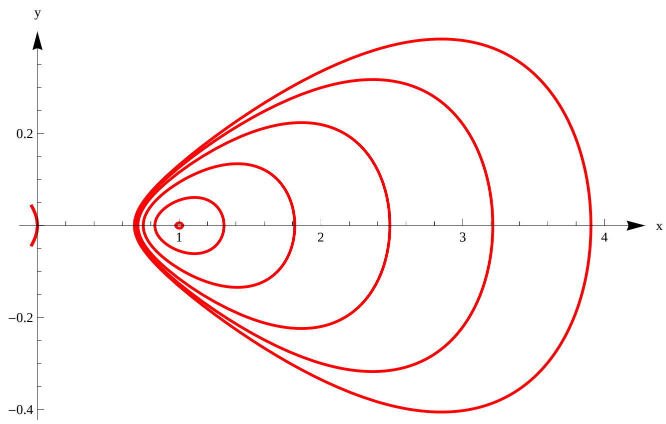

Figure 1. The periodic orbits of the system $(sr5)$

We consider the perturbation system of $(sr5)$, whose form is as follows:

$$\dot{x} = -xy + \varepsilon p(x,y), \quad \dot{y} = -2y^2 + \frac{1}{2^4}x^2 - \frac{1}{2^4}x + \varepsilon q(x,y), \tag{1.2}$$

where ε is a real parameter, and $0 < \varepsilon \ll 1$, $p(x,y)$ and $q(x,y)$ are polynomials of x and y, and $\max\{\deg(p(x,y)), \deg(q(x,y))\} = n$.

In this paper, we study the upper bound of the number of zeros for Abelian integrals of (1.2) by using the methods of Picard-Fuchs equation and Riccati equation [11]. Our main result is the following Theorem 1.1.

Theorem 1.1. *For the system (1.2), the upper bound for the number of zeros of the Abelian integral*

$$A(h) = \oint_{\Gamma_h} N(x,y)[q(x,y)dx - p(x,y)dy], \quad h \in \Omega, \tag{1.3}$$

where Γ_h are periodic orbits of system (1.1) and are defined on the maximum open interval $\Omega = (h_1, h_2)$, which linearly depends on n. Concretely, the upper bound is $7[(n-3)/2] + 5$ when $n \geq 5$, 8 when $n = 4$, 5 when $n = 3$, 4 when $n = 2$, and 0 when $n = 1$ or $n = 0$.

2. Abelian integral

Suppose that polynomials $p(x,y) = \sum_{0\le i+s\le n} a_{i,s}x^iy^s$, $q(x,y) = \sum_{0\le i+s\le n} b_{i,s}x^iy^s$. From (1.3), the Abelian integral $A(h)$ in Theorem 1.1 has the form

$$A(h) = \oint_{\Gamma_h} x^{-5}\left(\sum_{0\le i+s\le n} b_{i,s}x^iy^s dx - \sum_{0\le i+s\le n} a_{i,s}x^iy^s dy\right), \quad h \in \left(-\frac{1}{3\times 2^5}, 0\right), \tag{2.1}$$

where x^{-5} is the integrating factor.

For convenience, we introduce the integral functions $I_{r,s}(h)$ as follows:

$$I_{i,s}(h) = \oint_{\Gamma_h} x^{i-5}y^s dx,$$

where $i = -1, 0, 1, \cdots, n-1, n$; $s = 0, 1, 2, \cdots, n, n+1$. When $s = 1$, we use $J_i(h)$ instead of $I_{i,1}(h)$.

Note that

$$\oint_{\Gamma_h} x^{i-5}y^s dy = \frac{\oint_{\Gamma_h} x^{i-5}dy^{s+1}}{s+1} = \frac{5-i}{s+1}\oint_{\Gamma_h} x^{i-5-1}y^{s+1}dx = \frac{5-i}{s+1}I_{i-1,s+1}(h).$$

Thus, $A(h)$ can be written as

$$A(h) = \sum_{\substack{0\le i\le n,\\ 0\le s\le n,\\ 0\le i+s\le n}} b_{i,s}I_{i,s}(h) + \sum_{\substack{0\le i\le n,\\ 0\le s\le n,\\ 0\le i+s\le n}} a_{i,s}\frac{i-5}{s+1}I_{i-1,s+1}(h) = \sum_{\substack{-1\le i\le n,\\ 0\le s\le n+1,\\ 0\le i+s\le n}} \tilde{b}_{i,s}I_{i,s}(h). \tag{2.2}$$

For the Abelian integral $A(h)$, we have the following Proposition 2.1.

Proposition 2.1. *The Abelian integral $A(h)$ can be expressed as*

$$A(h) = \begin{cases} \frac{1}{h^{[\frac{n-3}{2}]}}K(h), \quad K(h) = \alpha(h)J_0(h) + \beta(h)J_1(h) + \gamma(h)J_2(h), \quad (n\ge 5), \\ \frac{1}{h}K(h), \quad K(h) = \alpha(h)J_0(h) + \beta(h)J_1(h) + \gamma(h)J_2(h), \quad (n = 4), \\ K(h) = \alpha(h)J_0(h) + \beta(h)J_1(h) + \gamma(h)J_2(h), \quad (n = 3), \\ \alpha(h)J_0(h) + \beta(h)J_1(h), \quad (n = 2), \\ \alpha(h)J_0(h), \quad (n = 0, 1), \end{cases} \tag{2.3}$$

where $0 \le \deg(\alpha(h)) \le [(n-3)/2]$, $0 \le \deg(\beta(h)) \le [(n-3)/2]$, $0 \le \deg(\gamma(h)) \le [(n-3)/2]$, *when* $n \ge 5$; $0 \le \deg(\alpha(h)) \le 1$, $0 \le \deg(\beta(h)) \le 1$, $\deg(\gamma(h)) = 1$, *when* $n = 4$; $\deg(\alpha(h)) = 0$, $\deg(\beta(h)) = 0$, $\deg(\gamma(h)) = 0$, *when* $n = 3$; $\deg(\alpha(h)) = 0$, $\deg(\beta(h)) = 0$, *when* $n = 2$; *and* $\deg(\alpha(h)) = 0$ *when* $n = 1$ *or* $n = 0$.

Proof. Since $I_{i,s}(h) = 0$ as s is even, we only need to consider odd values for s.

For the system $(sr5)$, using C instead of $1/2^4$, it follows from (1.1) that

$$-2x^{-5}y^2 + x^{-4}y\frac{\partial y}{\partial x} + Cx^{-3} - Cx^{-4} = 0. \tag{2.4}$$

Multiplying both sides of the equality (2.4) by $x^i y^{s-2}$ and integrating it along the curve Γ_h, we obtain

$$\frac{i+2s-4}{s} I_{i,s}(h) = C[I_{i+2,s-2}(h) - I_{i+1,s-2}(h)], \tag{2.5}$$

where $s = 1, 3, 5, \cdots, 2[n/2]+1$. For the system $(sr5)$, we restrain $i = -1, 0, 1, 2, \cdots, n-1$, and $0 \le i+s \le n$.

(i) For $i+2s-4 = 0$, we get $(i,s) = (-2,3)$ or $(i,s) = (2,1)$. When $(i,s) = (-2,3)$, from (2.5), we get

$$J_0(h) = J_{-1}(h).$$

(ii) For $i+2s-4 \ne 0$, that is, $(i,s) \ne (-2,3)$ and $(i,s) \ne (2,1)$, from (2.5), we get

$$I_{i,s}(h) = \frac{Cs}{i+2s-4}[I_{i+2,s-2}(h) - I_{i+1,s-2}(h)], \tag{2.6}$$

which indicates that $I_{i,s}(h)$ can be expressed in terms of $I_{i+2,s-2}(h)$ and $I_{i+1,s-2}(h)$, and then step by step, $I_{i,s}(h)$ can be written as a linear combination of $J_i(h)(i = -1, 0, \cdots)$ with the form

$$I_{i,s}(h) = \sum_{k=0}^{\frac{s-1}{2}} c_{(i,s),k} J_{i+k+\frac{s-1}{2}}(h).$$

From equality (2.2), we obtain

$$A(h) = \sum_{\substack{0\le i+s\le n,\\ -1\le i\le n,\\ s\equiv 1\, mod\, 2.}} \tilde{b}_{i,s} \sum_{k=0}^{\frac{s-1}{2}} c_{(i,s),k} J_{i+k+\frac{s-1}{2}}(h).$$

The maximum number of $i+k+(s-1)/2$ is $n-1$, and the minimum number is -1, so $A(h)$ is a linear combination of $J_k(h)(k = -1, 0, \cdots, n-1)$. We have

$$A(h) = \sum_{k=-1}^{n-1} e_k J_k(h), \tag{2.7}$$

where $e_k \in \mathbb{R}$ $(k = -1, 0, \cdots, n-1)$.

Again, from (1.1), we have

$$\frac{1}{2}x^{-4}y^2 - \frac{1}{2}Cx^{-2} + \frac{1}{3}Cx^{-3} = h. \tag{2.8}$$

Multiplying both sides of equality (2.8) by $x^{i-1}y^{s-2}$ and integrating it along the curve Γ_h, we obtain

$$\frac{1}{2}I_{i,s}(h) = hI_{i+4,s-2}(h) + \frac{1}{2}CI_{i+2,s-2}(h) - \frac{1}{3}CI_{i+1,s-2}(h). \tag{2.9}$$

Letting $s = 3$, by (2.6), equality (2.9) can be written as

$$6h(i+2)J_{i+4}(h) = 3(1-i)CJ_{i+2}(h) + (2i-5)CJ_{i+1}(h). \tag{2.10}$$

A) When $i \geq 3$, suppose that $\hbar := 1/h$, and equality (2.10) can be written as

$$J_i(h) = \frac{(5-i)C}{2(i-2)}\hbar J_{i-2}(h) + \frac{(2i-13)C}{6(i-2)}\hbar J_{i-3}(h).$$

Therefore, $J_i(h)$ can be expressed in terms of $\hbar J_{i-2}(h)$ and $\hbar J_{i-3}(h)$. Then step by step, $J_i(h)$ can be written as a linear combination of $J_0(h)$, $J_1(h)$ and $J_2(h)$ with polynomial coefficients of $\hbar$:

$$J_i(h) = \alpha_i(\hbar)J_0(h) + \beta_i(\hbar)J_1(h) + \gamma_i(\hbar)J_2(h),$$

where $2 \leq \deg(\alpha_i(\hbar)) \leq [(i-2)/2]$, $2 \leq \deg(\beta_i(\hbar)) \leq [(i-2)/2]$, $2 \leq \deg(\gamma_i(\hbar)) \leq [(i-2)/2]$, when $i \geq 6$, $i \neq 7$; $\alpha_i(\hbar) = 0$, $\deg(\beta_i(\hbar)) = 2$, $\deg(\gamma_i(\hbar)) = 2$, when $i = 7$; $\alpha_i(\hbar) = 0$, $\beta_i(\hbar) = 0$, $\deg(\gamma_i(\hbar)) = 1$, when $i = 5$; $\alpha_i(\hbar) = 0$, $\deg(\beta_i(\hbar)) = 1$, $\deg(\gamma_i(\hbar)) = 1$, when $i = 4$; and $\deg(\alpha_i)(\hbar) = 1$, $\deg(\beta_i(\hbar)) = 1$, $\gamma_i(\hbar) = 0$, when $i = 3$.

B) When $i = 0, 1, 2$, we obtain $J_0(h) = J_0(h)$, $J_1(h) = J_1(h)$, and $J_2(h) = J_2(h)$.

C) When $i = -1$, based on the above results, we get

$$J_{-1}(h) = J_0(h). \tag{2.11}$$

As a consequence, all $J_i(h)$ $(i = -1, 0, \cdots, n-1)$ can be expressed in terms of $J_0(h)$, $J_1(h)$ and $J_2(h)$. From (2.7), we obtain

$$A(h) = \alpha(\hbar)J_0(h) + \beta(\hbar)J_1(h) + \gamma(\hbar)J_2(h),$$

where $0 \leq \deg(\alpha(\hbar)) \leq [(n-3)/2]$, $0 \leq \deg(\beta(\hbar)) \leq [(n-3)/2]$, $0 \leq \deg(\gamma(\hbar)) \leq [(n-3)/2]$, when $n \geq 5$; $0 \leq \deg(\alpha(\hbar)) \leq 1$, $0 \leq \deg(\beta(\hbar)) \leq 1$, $\deg(\gamma(\hbar)) = 0$, when $n = 4$; $\deg(\alpha(\hbar)) = 0$, $\deg(\beta(\hbar)) = 0$, $\deg(\gamma(\hbar)) = 0$, when $n = 3$; and $\deg(\alpha(\hbar)) = 0$, $\deg(\beta(\hbar)) = 0$, $\gamma(\hbar) = 0$, when $n = 2$.

(1) When $n \geq 5$, suppose that $K(h) := h^{[(n-3)/2]}A(h)$. We get

$$K(h) = h^{[\frac{n-3}{2}]}A(h) = \alpha(h)J_0(h) + \beta(h)J_1(h) + \gamma(h)J_2(h),$$

where $0 \leq \deg(\alpha(h)) \leq [(n-3)/2]$, $0 \leq \deg(\beta(h)) \leq [(n-3)/2]$, and $0 \leq \deg(\gamma(h)) \leq [(n-3)/2]$.

(2) When $n = 4$, suppose that $K(h) := hA(h)$. We get

$$K(h) = hA(h) = \alpha(h)J_0(h) + \beta(h)J_1(h) + \gamma(h)J_2(h),$$

where $0 \leq \deg(\alpha(h)) \leq 1$, $0 \leq \deg(\beta(h)) \leq 1$, and $\deg(\gamma(h)) = 1$.

(3) When $n = 3$, suppose that $K(h) := A(h)$. We get

$$K(h) = A(h) = \alpha(h)J_0(h) + \beta(h)J_1(h) + \gamma(h)J_2(h),$$

where $\deg(\alpha(h)) = 0$, $\deg(\beta(h)) = 0$, and $\deg(\gamma(h)) = 0$.

(4) When $n = 2$, we get

$$A(h) = \alpha(h)J_0(h) + \beta(h)J_1(h),$$

where $\deg(\alpha(h)) = 0$, and $\deg(\beta(h)) = 0$.

(5) When $n = 0, 1$, from (2.2) and (2.11), we obtain

$$A(h) = \tilde{b}_{-1,1}J_{-1}(h) + \tilde{b}_{0,1}J_0(h) = (\tilde{b}_{-1,1} + \tilde{b}_{0,1})J_0(h) = \alpha(h)J_0(h),$$

where $\alpha(h) = \tilde{b}_{-1,1} + \tilde{b}_{0,1}$ and $\deg(\alpha(h)) = 0$. □

3. Picard-Fuchs equation and Riccati equation

For the relation among the functions $J_i(h)$ and their derivatives $J_i'(h)$ for $i = 0, 1, 2$, we obtain the following lemma.

Lemma 3.1. *The Abelian integrals $J_i(h)$ $(i = 0, 1, 2)$ satisfy the following Picard-Fuchs equation*

$$\begin{pmatrix} J_0(h) \\ J_1(h) \\ J_2(h) \end{pmatrix} = \begin{pmatrix} \frac{30h+6C}{35} & \frac{6h}{35} & 0 \\ \frac{C}{5} & \frac{6h}{5} & 0 \\ \frac{C}{3} & 0 & 2h \end{pmatrix} \begin{pmatrix} J_0'(h) \\ J_1'(h) \\ J_2'(h) \end{pmatrix}. \tag{3.1}$$

Proof. By (1.1), we obtain $y^2 = 2hx^4 + Cx^2 - 2Cx/3$, $\partial y/\partial h = x^4/y$, $ydy = (4hx^3 + Cx - C/3)dx$. Since $J_i(h) = \oint_{\Gamma_h} x^{i-5}ydx$, $J_i'(h) = \oint_{\Gamma_h} x^{i-1}/ydx$, we get

$$\begin{aligned} J_i(h) &= \oint_{\Gamma_h} \frac{x^{i-5}y^2}{y}dx = \oint_{\Gamma_h} \frac{x^{i-5}(2hx^4 + Cx^2 - \frac{2}{3}Cx)}{y}dx \\ &= 2hJ_i'(h) + CJ_{i-2}'(h) - \frac{2}{3}CJ_{i-3}'(h), \end{aligned} \tag{3.2}$$

$$\begin{aligned} (i-4)J_i(h) &= \oint_{\Gamma_h} (i-4)x^{i-5}ydx = \oint_{\Gamma_h} ydx^{i-4} = -\oint_{\Gamma_h} \frac{x^{i-4}(4hx^3 + Cx - \frac{1}{3}C)}{y}dx \\ &= -4hJ_i'(h) - CJ_{i-2}'(h) + \frac{1}{3}CJ_{i-3}'(h). \end{aligned} \tag{3.3}$$

From (3.2) and (3.3), we have

$$(2i-7)J_i(h) = -6hJ_i'(h) - CJ_{i-2}'(h). \tag{3.4}$$

From (3.4), letting $i = 0, 1, 2$, we get

$$J_0(h) = \frac{6}{7}hJ_0'(h) + \frac{1}{7}CJ_{-2}'(h), \tag{3.5}$$

$$J_1(h) = \frac{6}{5}hJ_1'(h) + \frac{1}{5}CJ_{-1}'(h), \tag{3.6}$$

$$J_2(h) = 2hJ_2'(h) + \frac{1}{3}CJ_0'(h). \tag{3.7}$$

From (2.11), we obtain

$$J_{-1}'(h) = J_0'(h). \tag{3.8}$$

From (2.10), letting $i = -3$, by (3.6) and (3.8), we obtain

$$J_{-2}'(h) = \frac{6}{5}J_0'(h) + \frac{6h}{5C}J_1'(h). \tag{3.9}$$

By (3.5)-(3.9), we obtain equation (3.1). □

For the relation among the functions $J_i'(h)$ and the second derivatives $J_i''(h)$ for $i = 0, 1, 2$, we get the following lemma.

Lemma 3.2. *The Abelian integrals $J_i'(h)$ $(i = 0, 1, 2)$ satisfy the following Picard-Fuchs equation*

$$\begin{pmatrix} J_0''(h) \\ J_1''(h) \\ J_2''(h) \end{pmatrix} = \frac{1}{B(h)} \begin{pmatrix} 6h & -6h & 0 \\ -C & -6h & 0 \\ -C & C & -3(6h+C) \end{pmatrix} \begin{pmatrix} J_0'(h) \\ J_1'(h) \\ J_2'(h) \end{pmatrix}, \tag{3.10}$$

where $B(h) = 6h(6h + C)$.

Proof. By differentiating both sides of equation (3.1) with respect to h, we can get

$$-J_2'(h) = 2hJ_2''(h) + \frac{1}{3}CJ_0''(h), \tag{3.11}$$

$$-\frac{1}{5}J_1'(h) = \frac{6}{5}hJ_1''(h) + \frac{1}{5}CJ_0''(h), \tag{3.12}$$

$$\frac{1}{7}J_0'(h) - \frac{6}{35}J_1'(h) = \frac{30h + 6C}{35}J_0''(h) + \frac{6}{35}hJ_1''(h). \tag{3.13}$$

By (3.11)-(3.13), we obtain equation (3.10). □

Lemma 3.3. $J_i(-1/(3\times 2^5)) = 0$ $(i = 0, 1, 2)$*; when* $h \in (-1/(3\times 2^5), 0)$, $J_i(h) < 0$, *and* $J_i'(h) > 0$ $(i = 0, 1, 2)$.

Since $J_i(h) = \oint_{\Gamma_h} x^{i-5}ydx$, $J_i'(h) = \oint_{\Gamma_h} x^{i-1}/ydx$. The proof only requires some simple calculations, so it is omitted.

For the relation between $J_0'(h)$ and $J_1'(h)$, we have the following corollary.

Corollary 3.1. *Suppose that $D(h) := J_0'(h)/J_1'(h)$. Then the function $D(h)$ satisfies the following Riccati equation*

$$B(h)D'(h) = CD^2(h) + 12hD(h) - 6h, \tag{3.14}$$

where $B(h) = 6h(6h + C)$.

Proof. Using equation (3.10), and by differentiating both sides of $D(h)$ with respect to h, we obtain equation (3.14). □

4. The upper bound of the number of zeros

(1) When $n \geq 3$, from (2.3) and (3.1), we obtain

$$K(h) = \alpha_1(h)J_0'(h) + \beta_1(h)J_1'(h) + \gamma_1(h)J_2'(h), \tag{4.1}$$

where $\alpha_1(h) = (30h + 6C)\alpha(h)/35 + C\beta(h)/5 + C\gamma(h)/3$, $\beta_1(h) = 6h\alpha(h)/35 + 6h\beta(h)/5$, and $\gamma_1(h) = 2h\gamma(h)$. Thus, when $n \geq 5$, $0 \leq \deg(\alpha_1(h)) \leq [(n-3)/2]+1$, $1 \leq \deg(\beta_1(h)) \leq [(n-3)/2] + 1$, $1 \leq \deg(\gamma_1(h)) \leq [(n-3)/2] + 1$; when $n = 4$, $0 \leq \deg(\alpha_1(h)) \leq 2$, $1 \leq \deg(\beta_1(h)) \leq 2$, $\deg(\gamma_1(h)) = 2$; and when $n = 3$, $0 \leq \deg(\alpha_1(h)) \leq 1$, $\deg(\beta_1(h)) = 1$, $\deg(\gamma_1(h)) = 1$.

By differentiating both sides of equality (2.3) with respect to h and using (3.1), we get

$$K'(h) = \alpha_2(h)J_0'(h) + \beta_2(h)J_1'(h) + \gamma_2(h)J_2'(h), \tag{4.2}$$

where $\alpha_2(h) = (30h+6C)\alpha'(h)/35+\alpha(h)+C\beta'(h)/5+C\gamma'(h)/3$, $\beta_2(h) = 6h\alpha'(h)/35 + 6h\beta'(h)/5+\beta(h)$, and $\gamma_2(h) = 2h\gamma'(h)+\gamma(h)$. Thus, when $n \geq 5$, $0 \leq \deg(\alpha_2(h)) \leq [(n-3)/2]$, $0 \leq \deg(\beta_2(h)) \leq [(n-3)/2]$, $0 \leq \deg(\gamma_2(h)) \leq [(n-3)/2]$; when $n = 4$, $0 \leq \deg(\alpha_2(h)) \leq 1$, $0 \leq \deg(\beta_2(h)) \leq 1$, $\deg(\gamma_2(h)) = 1$; when $n = 3$, $\deg(\alpha_2(h)) = 0$, $\deg(\beta_2(h)) = 0$, $\deg(\gamma_2(h)) = 0$.

By equalities (4.1) and (4.2), we obtain

$$\gamma_1(h)K'(h) = \gamma_2(h)K(h) + I(h), \tag{4.3}$$

$$I(h) = E(h)J_0'(h) + F(h)J_1'(h), \tag{4.4}$$

where $E(h) = \alpha_2(h)\gamma_1(h)-\alpha_1(h)\gamma_2(h)$, and $F(h) = \beta_2(h)\gamma_1(h)-\beta_1(h)\gamma_2(h)$. Thus, when $n \geq 5$, $0 \leq \deg(E(h)) \leq 2[(n-3)/2]+1$, $1 \leq \deg(F(h)) \leq 2[(n-3)/2]+1$; when $n = 4$, $1 \leq \deg(E(h)) \leq 3$, $2 \leq \deg(F(h)) \leq 3$; when $n = 3$, $0 \leq \deg(E(h)) \leq 1$, $\deg(F(h)) = 1$.

(2) When $n = 2$, from (2.3) and (3.1), we obtain

$$A(h) = E(h)J_0'(h) + F(h)J_1'(h), \tag{4.5}$$

where $E(h) = (30h + 6C)\alpha(h)/35 + C\beta(h)/5$, and $F(h) = 6h\alpha(h)/35 + 6h\beta(h)/5$. Thus, $0 \leq \deg(E(h)) \leq 1$, and $\deg(F(h)) = 1$.

Lemma 4.1. *When $n \geq 3$, suppose that $W(h) := I(h)/J_1'(h)$; when $n = 2$, suppose that $W(h) := A(h)/J_1'(h)$. Then the function $W(h)$ satisfies the following Riccati equation*

$$B(h)E(h)W'(h) = CW^2(h) + M(h)W(h) + G(h), \tag{4.6}$$

where $M(h) = B(h)E'(h)+12hE(h)-2CF(h)$, $G(h) = CF^2(h)+E(h)(B(h)F'(h)-6hE(h)) - F(h)(B(h)E'(h) + 12hE(h))$. Thus, when $n \geq 5$, $1 \leq \deg(M(h)) \leq 2[(n-3)/2]+2$, $1 \leq \deg(G(h)) \leq 4[(n-3)/2]+3$; when $n = 4$, $1 \leq \deg(M(h)) \leq 4$, $3 \leq \deg(G(h)) \leq 7$; when $n = 3$, $1 \leq \deg(M(h)) \leq 2$, $1 \leq \deg(G(h)) \leq 3$; when $n = 2$, $1 \leq \deg(M(h)) \leq 2$, $1 \leq \deg(G(h)) \leq 3$.

Proof. Using Corollary 3.1, and by the equalities (4.4) and (4.5), we obtain Lemma 4.1. □

We use $\natural A(h)$ to denote the number of zeros of $A(h)$ in Ω, and we need the following Lemma 4.2.

Lemma 4.2 ([14]). *The smooth functions $V(h)$, $\phi(h)$, $\varphi(h)$, $\xi(h)$ and $\eta(h)$ satisfy the following Riccati equation*

$$\eta(h)V'(h) = \phi(h)V^2(h) + \varphi(h)V(h) + \xi(h), \tag{4.7}$$

then

$$\natural V(h) \leq \natural\eta(h) + \natural\xi(h) + 1.$$

Lemma 4.2 is the Lemma 5.3 in [14], and the proof can be found in [14], so it is omitted.

Finally, we conclude the proof of Theorem 1.1.

Proof. (1) When $n \geq 3$, using Lemma 4.2, from (2.3) and (4.3), we get

$$\natural A(h) = \natural K(h) \leq \natural B(h) + \natural\gamma_1(h) + \natural I(h) + 1. \tag{4.8}$$

From (4.6), we have

$$\natural I(h) = \natural W(h) \leq \natural B(h) + \natural E(h) + \natural G(h) + 1. \tag{4.9}$$

From (4.8) and (4.9), we obtain

$$\natural A(h) \le 2\natural B(h) + \natural\gamma_1(h) + \natural E(h) + \natural G(h) + 2.$$

When $n \ge 5$, since $1 \le \deg(\gamma_1(h)) \le [(n-3)/2] + 1$, $0 \le \deg(E(h)) \le 2[(n-3)/2] + 1$, $1 \le \deg(G(h)) \le 4[(n-3)/2] + 3$, noticing that $B(h) = 6h(6h + C)$ and there is no zero in $(-1/(3 \times 2^6), 0)$, so $\natural B(h) = 0$, we obtain

$$\natural A(h) \le 0 + \left[\frac{n-3}{2}\right] + \left(2\left[\frac{n-3}{2}\right] + 1\right) + \left(4\left[\frac{n-3}{2}\right] + 2\right) + 2 = 7\left[\frac{n-3}{2}\right] + 5.$$

When $n = 4$, since $\deg(\gamma_1(h)) = 2$, $1 \le \deg(E(h)) \le 3$, and $3 \le \deg(G(h)) \le 7$, we obtain

$$\natural A(h) \le 0 + 0 + 2 + 4 + 2 = 8.$$

When $n = 3$, since $\deg(\gamma_1(h)) = 1$, $0 \le \deg(E(h)) \le 1$, and $1 \le \deg(G(h)) \le 3$, we obtain

$$\natural A(h) \le 0 + 0 + 1 + 2 + 2 = 5.$$

(2) When $n = 2$, using Lemma 4.2, from (2.3) and (4.6), we get

$$\natural A(h) = \natural W(h) \le \natural B(h) + \natural E(h) + \natural G(h) + 1. \tag{4.10}$$

Since $0 \le \deg(E(h)) \le 1$, $1 \le \deg(G(h)) \le 3$, and $\natural B(h) = 0$, we obtain

$$\natural A(h) \le 0 + 1 + 2 + 1 = 4.$$

(3) When $n = 0, 1$, since $A(h) = \alpha(h)J_0(h)$, $\deg(\alpha(h)) = 0$, $J_0(h) < 0$, we obtain

$$\natural A(h) = 0.$$

□

5. Conclusion

In this paper, according to the methods of Picard-Fuchs equation and Riccati equation, we study upper bound for the number of zeros of Abelian integrals in the quadratic reversible system ($sr5$) under any polynomial perturbations of degree n. We obtain that the upper bound of the number is $7[(n-3)/2] + 2$ ($n \ge 5$), which linearly depends on n. Moreover, from the above results, it can be seen that for the system ($r5$), when $F = 0$, the upper bound of the number of zeros of the Abelian integrals has been greatly reduced, which is very helpful for the comprehensive study of the limit cycle of the perturbed system ($r5$).

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