Journal of Nonlinear Modeling and Analysis
Volume 6, Number 3, September 2024, 825–840

http://jnma-online.com
DOI:10.12150/jnma.2024.825

New Fixed Point Results over Orthogonal $\mathcal{F}$-Metric Spaces and Application in Second-Order Differential Equations

Mohammed M.A. Taleb[1,2,†], Saeed A.A. Al-Salehi[1,3] and V.C. Borkar[4]

Abstract In this article, we introduce the notion of cyclic α-admissible mapping with respect to θ with its special cases, which are cyclic α-admissible mapping with respect to θ^* and cyclic α^*-admissible mapping with respect to θ. We present the notion of orthogonal $(\alpha\theta - \beta F)$-rational contraction and establish new fixed point results over orthogonal $\mathcal{F}$-metric space. The study includes illustrative examples to support our results. We apply our results to prove the existence and uniqueness of solutions for second-order differential equations.

Keywords Fixed point, orthogonal $(\alpha\theta - \beta F)$-rational contraction, cyclic α-admissible mapping with respect to θ, orthogonal $\mathcal{F}$-metric space, second-order differential equation

MSC(2010) 54H25, 47H10.

1. Introduction

In 1922, [9] the Polish mathematician Banach presented the most important of the fixed point theorems known as the Banach contraction principle, which proves the existence and uniqueness of the fixed point for any contraction mapping $\mathcal{H} : \Xi \to \Xi$, and (Ξ, D) should be a complete metric space, where $D : \Xi \times \Xi \to [0, \infty)$. In 2012, Wardowski [7] presented the concept of F-contraction, a generalization of the Banach contraction principle. This means that Banach contractions can be seen as a particular case of F-contractions. Therefore, many researchers used this concept to study the existence and uniqueness of the fixed point in a different way instead of using Banach's theorem. [10–16] is recommended for readers interested in fixed point findings obtained using the notion of F-contraction. Some authors modified or reformulated some of the conditions of this concept and studied some fixed point theorems (see [17–20]). Kanokwan et al. [27] introduced the notion of orthogonal F-contraction and established some fixed point results over orthogonal

†the corresponding author.
Email address: mohaayedtaleb@gmail.com(Mohammed M.A. Taleb), alsalehi.saeed72@gmail.com(Saeed A.A. Al-Salehi), borkarvc@gmail.com(V.C. Borkar)
1 Department of Mathematics, Science College, Swami Ramanand Teerth Marathwada University, Nanded-431606, India
2 Department of Mathematics, Hodeidah University, P.O. Box 3114, Al-Hodeidah, Yemen
3 Department of Mathematics, Aden University, P.O. Box 124, Aden, Yemen
4 Department of Mathematics, Yeshwant Mahavidyalaya, Swami Ramanand Teerth Marathwada University, Nanded-431606, India

metric space. Jleli and Samet [1] introduced the concept of $\mathcal{F}$-metric space in 2018. Researchers have since paid great attention to this space and have used it to study several fixed point theorems (see, e.g., [21–24]). Some even used F-contraction to study some fixed point theorems over $\mathcal{F}$-metric space (see [25, 26]). In 2020, T. Kanwal et al. [3] presented the notion of orthogonal $\mathcal{F}$-metric spaces and proved some fixed point theorems and [8,29,30] presented some new fixed point results over orthogonal $\mathcal{F}$-metric space. M. Taleb et al. studied some fixed point results and applied their results to study the existence and uniqueness of solutions of nonlinear neutral differential equations (see [23]) and in [28] also studied the existence and uniqueness of solutions to first-order differential equations.

In this paper, we present a modification of the cyclic (α, θ)-admissible mapping, which is done by introducing the concept of cyclic α-admissible mapping with respect to θ, presenting the notion of orthogonal $(\alpha\theta - \beta F)$-rational contraction. Additionally, we establish new fixed point results over orthogonal $\mathcal{F}$-metric space. The paper is organized as follows. In Sect.(3), the concept of cyclic α-admissible mapping with respect to θ is introduced (see Definition (3.1)) which is a modification to what was stated in definition (2.11) and we provide an illustrative example to support this result (see Example (3.1)). We deduce two special concepts from definition (3.1), namely cyclic α-admissible mapping with respect to θ^* and cyclic α^*-admissible mapping with respect to θ (see Remark (3.1)). We also introduce a concept of orthogonal $(\alpha\theta - \beta F)$-rational contraction (see Definition (3.2)) and we establish new fixed point results over orthogonal $\mathcal{F}$-metric space (see Theorem (3.1)) and (Corollaries (3.1), (3.2) and (3.3)). These results are supported by an example (3.2). In Sect. (4), we apply our results to show the existence and uniqueness of solutions for second-order differential equations (see Theorem (4.1)). Our results generalize and advance existing literature, such as [5], [23] and also present a novel approach to establish the existence and uniqueness of solutions for second-order differential equations.

2. Preliminaries

In 2012, Wardowski [7] introduced the notion of F- contraction as follows.

Definition 2.1 ([7]). Let $\Xi \neq \emptyset$, and (Ξ, D) be a metric space. A mapping $\mathcal{H} : \Xi \to \Xi$ is called F- contraction if $\exists\ \tau > 0,\ \forall\ \omega, \nu \in \Xi,\ D(\mathcal{H}\omega, \mathcal{H}\nu) > 0$, and we have

$$\tau + F\left(D(\mathcal{H}\omega, \mathcal{H}\nu)\right) \leq F\left(D(\omega, \nu)\right),$$

where $F : (0, \infty) \to \mathbb{R}$ satisfies the following conditions:

$(\mathcal{F}_1)$ $0 < \mathfrak{s} < \iota \quad \Rightarrow \quad F(s) \leq F(\iota)$.

$(\mathcal{F}_2)$ $\forall\ \{\iota_\mathfrak{n}\} \subset (0, +\infty)$, we have

$$\lim_{\mathfrak{n}\to+\infty} F(\iota_\mathfrak{n}) = -\infty \quad \Leftrightarrow \quad \lim_{\mathfrak{n}\to+\infty} \iota_\mathfrak{n} = 0.$$

$(\mathcal{F}_3)$ For some $r \in (0, 1)$, $\lim_{\iota\to 0^+} \iota^r F(\iota) = 0$.

The class of functions F is denoted by Φ.

Definition 2.2 ([1]). Let $D_\mathcal{F} : \Xi \times \Xi \to [0, \infty)$ be a given mapping. If $\exists\ a \in [0, \infty)$, such that

$(D_{\mathcal{F}1})$ $(\omega,\nu)\in\Xi\times\Xi$, $D_{\mathcal{F}}(\omega,\nu)=0 \quad\Leftrightarrow\quad \omega=\nu$.

$(D_{\mathcal{F}2})$ $D_{\mathcal{F}}(\omega,\nu)=D_{\mathcal{F}}(\nu,\omega)$, $\forall\ (\omega,\nu)\in\Xi\times\Xi$.

$(D_{\mathcal{F}3})$ For each $(\omega,\nu)\in\Xi\times\Xi$, $\forall\ N$ in $\mathbb{N}$, $N\geq 2$, $\forall\ (\omega_i)_{i=1}^{N}\subset\Xi$, with $(\omega_1,\omega_N)=(\omega,\nu)$, we have

$$D_{\mathcal{F}}(\omega,\nu)>0 \quad\Rightarrow\quad \xi\left(D_{\mathcal{F}}(\omega,\nu)\right)\leq\xi\left(\sum_{i=1}^{N-1}D_{\mathcal{F}}(\omega_i,\omega_{i+1})\right)+a,$$

where $\xi:(0,\infty)\to\mathbb{R}$ satisfies $\mathcal{F}_1$ and $\mathcal{F}_2$. Then $(\Xi,D_{\mathcal{F}})$ is called $\mathcal{F}$-metric space (briefly $\mathcal{F}$-MS).

Example 2.1 ([1]). Let $\Xi=\mathbb{N}$, $D_{\mathcal{F}}:\Xi\times\Xi\to[0,\infty)$ be a mapping define by

$$D_{\mathcal{F}}(\omega,\nu)=\begin{cases}e^{|\omega-\nu|}, & \text{if}\quad \omega\neq\nu,\\ 0, & \text{if}\quad \omega=\nu,\end{cases}$$

$\forall\ (\omega,\nu)\in\Xi\times\Xi$. Then $(\Xi,D_{\mathcal{F}})$ is an $\mathcal{F}$-MS with $\xi(\iota)=\frac{-1}{\iota}$, $\iota>0$ and $a=1$.

Definition 2.3 ([1]). Let $(\Xi,D_{\mathcal{F}})$ be an $\mathcal{F}$-MS.

1. $\{\omega_{\mathfrak{n}}\}$ is $\mathcal{F}$-convergent to $\omega\in\Xi$, if $\lim\limits_{\mathfrak{n}\to\infty}D_{\mathcal{F}}(\omega_{\mathfrak{n}},\omega)=0$.
2. $\{\omega_{\mathfrak{n}}\}$ is $\mathcal{F}$-Cauchy, if $\lim\limits_{\mathfrak{n},\mathfrak{m}\to+\infty}D_{\mathcal{F}}(\omega_{\mathfrak{n}},\omega_{\mathfrak{m}})=0$.
3. $(\Xi,D_{\mathcal{F}})$ is $\mathcal{F}$-complete, if each $\mathcal{F}$-Cauchy sequence in Ξ is $\mathcal{F}$-convergent to a point in Ξ.

Definition 2.4 ([2]). Let $\Xi\neq\emptyset$, and the binary relation $\perp\subset\Xi\times\Xi$ satisfies the following condition:

$$\exists\ \omega_0:(\forall\nu,\nu\perp\omega_0)\quad\text{or}\quad(\forall\nu,\omega_0\perp\nu).$$

Then it is called an orthogonal set (briefly O-set) and is denoted by $(\Xi,\perp)$.

Definition 2.5 ([2]). Let $(\Xi,\perp)$ be an O-set. A sequence $\{\omega_{\mathfrak{n}}\}$ is called orthogonal sequence (O-sequence) if

$$(\omega_{\mathfrak{n}}\perp\omega_{\mathfrak{n}+1},\forall\ \mathfrak{n}\in\mathbb{N})\quad\vee\quad(\omega_{\mathfrak{n}+1}\perp\omega_{\mathfrak{n}},\forall\ \mathfrak{n}\in\mathbb{N}).$$

Definition 2.6 ([2]). Let $(\Xi,\perp)$ be an O-set. A mapping $\mathcal{H}:\Xi\to\Xi$ is said to be $\perp-$preserving if:

$$\omega\perp\nu\quad\Rightarrow\quad\mathcal{H}\omega\perp\mathcal{H}\nu.$$

Definition 2.7 ([3]). Let $(\Xi,\perp)$ be an O-set and $D_{O\mathcal{F}}$ be an $\mathcal{F}$-M on Ξ. The triplet $(\Xi,\perp,D_{O\mathcal{F}})$ is called orthogonal $\mathcal{F}$-metric space (briefly O-$\mathcal{F}$-MS).

Example 2.2 ([3]). Let $\Xi=[0,1]$ be an $\mathcal{F}$-MS with metric defined in Example (2.1), $\forall\ \omega,\nu\in\Xi$ with $\xi(\iota)=\frac{-1}{\iota}$, $\iota>0$ and $a=1$. Define $\omega\perp\nu$ if $\omega\nu\leq\omega$ or $\omega\nu\leq\nu$. Then, $\forall\ \omega\in\Xi$, $0\perp\omega$, so $(\Xi,\perp)$ is an O-set. Then $(\Xi,\perp,D_{O\mathcal{F}})$ is an O-$\mathcal{F}$-MS.

Definition 2.8 ([3]). Let $(\Xi,\perp,D_{O\mathcal{F}})$ be an O-$\mathcal{F}$-MS.

1. A mapping $\mathcal{H}:\Xi\to\Xi$ is called orthogonally continuous ($\perp$-continuous) at $\omega\in\Xi$ if for all O-sequence $\{\omega_{\mathfrak{n}}\}$ in Ξ with $\omega_{\mathfrak{n}}\to\omega$, then $\mathcal{H}\omega_{\mathfrak{n}}\to\mathcal{H}\omega$. And $\mathcal{H}$ is called $\perp$-continuous on Ξ if $\mathcal{H}$ is $\perp$-continuous in every $\omega\in\Xi$.

2. A set Ξ of $(\Xi, \perp, D_{O\mathcal{F}})$ is called orthogonally $\mathcal{F}$-complete (O-$\mathcal{F}$-complete) if every Cauchy O-sequence is $\mathcal{F}$- convergent in Ξ.

Let $\Im$ be the set of functions $\beta : [0, \infty) \to [0, 1)$ such that

$$\lim_{\mathfrak{n}\to\infty} \beta(\iota_{\mathfrak{n}}) = 1 \quad \Rightarrow \quad \lim_{\mathfrak{n}\to\infty} \iota_{\mathfrak{n}} = 0.$$

Definition 2.9 ([4]). Let $\mathcal{H} : \Xi \to \Xi$ and $\alpha : \Xi \times \Xi \to [0, \infty)$ be a mapping. Then $\mathcal{H}$ is called α-admissible mapping if:

$$\omega, \nu \in \Xi, \quad \alpha(\omega, \nu) \geq 1 \quad \Rightarrow \quad \alpha(\mathcal{H}\omega, \mathcal{H}\nu) \geq 1.$$

In 2013 Salimi et al. [6] extended the concept of α-admissible mapping as follows

Definition 2.10 ([6]). Let $\mathcal{H} : \Xi \to \Xi$ and $\alpha,\ \theta : \Xi \times \Xi \to [0, \infty)$. Then $\mathcal{H}$ is called α-admissible mapping with respect to θ if:

$$\omega, \nu \in \Xi, \quad \alpha(\omega, \nu) \geq \theta(\omega, \nu) \quad \Rightarrow \quad \alpha(\mathcal{H}\omega, \mathcal{H}\nu) \geq \theta(\mathcal{H}\omega, \mathcal{H}\nu).$$

Definition 2.11 ([5]). Let $\mathcal{H} : \Xi \to \Xi$ and $\alpha, \theta : \Xi \to [0, \infty)$. Then $\mathcal{H}$ is called cyclic (α, θ)-admissible mapping if:

1. $\alpha(\omega) \geq 1$ for some $\omega \in \Xi$, it implies that $\theta(\mathcal{H}\omega) \geq 1$,
2. $\theta(\omega) \geq 1$ for some $\omega \in \Xi$, it implies that $\alpha(\mathcal{H}\omega) \geq 1$.

3. Main results

In this section, we introduce the concepts of cyclic α-admissible mapping with respect to θ and orthogonal $(\alpha\theta - \beta F)$-rational contraction and we prove some new fixed point results in an O-$\mathcal{F}$-MS.

Definition 3.1. Let $\mathcal{H} : \Xi \to \Xi$ and $\alpha, \theta : \Xi \to [0, \infty)$. Then $\mathcal{H}$ is called cyclic α-admissible mapping with respect to θ if:

$$\alpha(\omega) \geq \theta(\omega) \quad \text{for some} \quad \omega \in \Xi \quad \Rightarrow \quad \theta(\mathcal{H}\omega) \geq \alpha(\mathcal{H}\omega).$$

Example 3.1. Let $\Xi = \mathbb{R}$. Define $\mathcal{H} : \Xi \to \Xi$ by

$$\mathcal{H}(\omega) = \begin{cases} \frac{1}{\omega}, & \text{if} \quad \omega \in [1, \infty), \\ 0, & \text{otherwies}, \end{cases}$$

and $\alpha, \theta : \Xi \to [0, \infty)$ by

$$\alpha(\omega) = \begin{cases} \omega, & \text{if} \quad \omega \in [1, \infty), \\ 0, & \text{otherwies}, \end{cases}$$

$$\theta(\omega) = \begin{cases} \frac{1}{\omega}, & \text{if} \quad \omega \in [1, \infty), \\ 0, & \text{otherwies}. \end{cases}$$

Clearly, $\alpha(\omega) \geq \theta(\omega)$ for all $\omega \in [1, \infty)$, we have, $\theta(\mathcal{H}\omega) = \frac{1}{\mathcal{H}\omega} = \omega$ and $\alpha(\mathcal{H}\omega) = \mathcal{H}\omega = \frac{1}{\omega}$. Then, we get $\theta(\mathcal{H}\omega) \geq \alpha(\mathcal{H}\omega)$ for all $\omega \in [1, \infty)$. Hence, $\mathcal{H}$ is a cyclic α-admissible mapping with respect to θ.

Remark 3.1. (1) If $\theta(\omega)=1$ in Definition (3.1) we get the following condition:

$$\alpha(\omega)\geq 1 \quad \text{for some} \quad \omega\in\Xi \quad \Rightarrow \quad \alpha(\mathcal{H}\omega)\leq 1.$$

In this case $\mathcal{H}$ is called cyclic α-admissible mapping with respect to θ^*.

(2) If $\alpha(\omega)=1$ in Definition (3.1) we get the following condition:

$$\theta(\omega)\leq 1 \quad \text{for some} \quad \omega\in\Xi \quad \Rightarrow \quad \theta(\mathcal{H}\omega)\geq 1.$$

In this case $\mathcal{H}$ is called cyclic α^*-admissible mapping with respect to θ.

In Example (3.1) if we take $\theta(\omega)=1$. Then, we get $\alpha(\omega)=\omega\geq 1$ for all $\omega\in[1,\infty)$. Since $\theta(\mathcal{H}\omega)=1$, then we get $\alpha(\mathcal{H}\omega)=\mathcal{H}\omega=\frac{1}{\omega}\leq 1$ for all $\omega\in[1,\infty)$. And if we take $\alpha(\omega)=1$. Then, we get $\theta(\omega)=\frac{1}{\omega}\leq 1$ for all $\omega\in[1,\infty)$. Since $\alpha(\mathcal{H}\omega)=1$, then we get $\theta(\mathcal{H}\omega)=\frac{1}{\mathcal{H}\omega}=\omega\geq 1$ for all $\omega\in[1,\infty)$.

Definition 3.2. Let $(\Xi,\perp,D_{O\mathcal{F}})$ be an O-$\mathcal{F}$-MS. A mapping $\mathcal{H}:\Xi\to\Xi$ is called an orthogonal $(\alpha\theta-\beta F)$- rational contraction if there exists $F\in\Phi$, $\alpha,\theta:\Xi\to(0,\infty)$, $\beta\in\Im$ and $\tau>0$ such that, $\forall\ \omega,\nu\in\Xi$ with $\omega\perp\nu$ and $D_{O\mathcal{F}}(\mathcal{H}\omega,\mathcal{H}\nu)>0$, the following condition is satisfied:

$$\alpha(\omega)\theta(\mathcal{H}\omega)\geq\theta(\omega)\alpha(\mathcal{H}\omega).$$

Also,

$$\alpha(\nu)\theta(\mathcal{H}\nu)\geq\theta(\nu)\alpha(\mathcal{H}\nu)$$

implies

$$\tau+F\left(D_{O\mathcal{F}}(\mathcal{H}\omega,\mathcal{H}\nu)\right)\leq F\left(\beta\left(\mathcal{M}(\omega,\nu)\right)\mathcal{M}(\omega,\nu)\right), \tag{3.1}$$

where

$$\mathcal{M}(\omega,\nu)=\max\left\{D_{O\mathcal{F}}(\omega,\nu),\ \min\left\{\frac{D_{O\mathcal{F}}(\omega,\mathcal{H}\omega)D_{O\mathcal{F}}(\nu,\mathcal{H}\nu)}{1+D_{O\mathcal{F}}(\omega,\nu)},\frac{D_{O\mathcal{F}}(\nu,\mathcal{H}\nu)[1+D_{O\mathcal{F}}(\omega,\mathcal{H}\omega)]}{1+D_{O\mathcal{F}}(\omega,\nu)}\right\}\right\}.$$

Theorem 3.1. *Let $(\Xi,\perp,D_{O\mathcal{F}})$ be an O-complete $\mathcal{F}$-MS and $\mathcal{H}:\Xi\to\Xi$ be an orthogonal $(\alpha\theta-\beta F)$- rational contraction. If the following conditions are satisfied:*

- (h_1) *$\mathcal{H}$ is $\perp-$preserving, that is $\omega\perp\nu \Rightarrow \mathcal{H}\omega\perp\mathcal{H}\nu$.*
- (h_2) *There exists an orthogonal element $\omega_0\in\Xi$ such that $\omega_0\perp\mathcal{H}\omega_0$ or $\mathcal{H}\omega_0\perp\omega_0$ and $\alpha(\omega_0)\geq\theta(\omega_0)$.*
- (h_3) *$\mathcal{H}$ is $\perp-$continuous or,*
- (h_3') *if $\{\omega_\mathfrak{n}\}$ is a sequence in Ξ such that $\omega_\mathfrak{n}\to\omega\in\Xi$ and $\omega_\mathfrak{n}\perp\mathcal{H}\omega_\mathfrak{n}$ or $\mathcal{H}\omega_\mathfrak{n}\perp\omega_\mathfrak{n}$ and $\alpha(\omega_\mathfrak{n})\geq\theta(\omega_\mathfrak{n})$ then $\omega_\mathfrak{n}\perp\omega$ or $\omega\perp\omega_\mathfrak{n}$ and $\alpha(\omega)\geq\theta(\omega)$, $\forall\ \mathfrak{n}\in\mathbb{N}$.*
- (h_4) *$\mathcal{H}$ is cyclic α-admissible mapping with respect to θ.*
- (h_5) *For all $\omega,\nu\in\Xi$ $(\omega\neq\nu)$ fixed points of $\mathcal{H}$ with $\omega\perp\nu$ or $\nu\perp\omega$, then $\alpha(\omega)\geq\theta(\omega)$ and $\alpha(\nu)\geq\theta(\nu)$.*

Then $\mathcal{H}$ has a unique fixed point.

Proof. By (h_2), $\exists\ \omega_0 \in \Xi$ such that $\omega_0 \perp \mathcal{H}\omega_0$ or $\mathcal{H}\omega_0 \perp \omega_0$ and $\alpha(\omega_0) \geq \theta(\omega_0)$. Define a sequence $\{\omega_\mathfrak{n}\}$ by $\omega_{\mathfrak{n}+1} = \mathcal{H}\omega_\mathfrak{n}\ \forall\ \mathfrak{n} \in \mathbb{N}$. For some $\mathfrak{n}$, if $\omega_{\mathfrak{n}+1} = \omega_\mathfrak{n}$, then $\mathcal{H}\omega_\mathfrak{n} = \omega_\mathfrak{n}$. Thus, $\omega_\mathfrak{n}$ is a fixed point of $\mathcal{H}$, the proof is completed. So suppose that $\omega_{\mathfrak{n}+1} \neq \omega_\mathfrak{n}\ \forall\ \mathfrak{n} \in \mathbb{N}$. Then $D_{O_\mathcal{F}}(\omega_\mathfrak{n}, \omega_{\mathfrak{n}+1}) > 0$. As $\mathcal{H}$ is $\perp-$preserving, so we have

$$[\omega_\mathfrak{n} \perp \omega_{\mathfrak{n}+1}\ \forall\ \mathfrak{n} \in \mathbb{N}] \quad \text{or} \quad [\omega_{\mathfrak{n}+1} \perp \omega_\mathfrak{n}\ \forall\ \mathfrak{n} \in \mathbb{N}],$$

that is, the sequence $\{\omega_\mathfrak{n}\}$ is an $\perp$-sequence in Ξ. By (h_4), we have

$$\alpha(\omega_0) \geq \theta(\omega_0) \quad \Rightarrow \quad \theta(\mathcal{H}\omega_0) \geq \alpha(\mathcal{H}\omega_0).$$

Then, we get

$$\alpha(\omega_0)\theta(\mathcal{H}\omega_0) \geq \theta(\omega_0)\alpha(\mathcal{H}\omega_0).$$

Deductively, we have

$$\alpha(\omega_{\mathfrak{n}-1})\theta(\mathcal{H}\omega_{\mathfrak{n}-1}) \geq \theta(\omega_{\mathfrak{n}-1})\alpha(\mathcal{H}\omega_{\mathfrak{n}-1}), \tag{3.2}$$

and

$$\alpha(\omega_\mathfrak{n})\theta(\mathcal{H}\omega_\mathfrak{n}) \geq \theta(\omega_\mathfrak{n})\alpha(\mathcal{H}\omega_\mathfrak{n}), \quad \forall\ \mathfrak{n} \in \mathbb{N}. \tag{3.3}$$

By (3.2), (3.3) and using (3.1), we get

$$\begin{aligned} \tau + F\left(D_{O_\mathcal{F}}(\omega_\mathfrak{n}, \omega_{\mathfrak{n}+1})\right) &= \tau + F\left(D_{O_\mathcal{F}}(\mathcal{H}\omega_{\mathfrak{n}-1}, \mathcal{H}\omega_\mathfrak{n})\right) \\ &\leq F\left(\beta\left(\mathcal{M}(\omega_{\mathfrak{n}-1}, \omega_\mathfrak{n})\right)\mathcal{M}(\omega_{\mathfrak{n}-1}, \omega_\mathfrak{n})\right), \end{aligned} \tag{3.4}$$

where

$$\begin{aligned} \mathcal{M}(\omega_{\mathfrak{n}-1}, \omega_\mathfrak{n}) = \max\Bigg\{ & D_{O_\mathcal{F}}(\omega_{\mathfrak{n}-1}, \omega_\mathfrak{n}), \min\Bigg\{ \frac{D_{O_\mathcal{F}}(\omega_{\mathfrak{n}-1}, \mathcal{H}\omega_{\mathfrak{n}-1}) D_{O_\mathcal{F}}(\omega_\mathfrak{n}, \mathcal{H}\omega_\mathfrak{n})}{1 + D_{O_\mathcal{F}}(\omega_{\mathfrak{n}-1}, \omega_\mathfrak{n})}, \\ & \frac{D_{O_\mathcal{F}}(\omega_\mathfrak{n}, \mathcal{H}\omega_\mathfrak{n})[1 + D_{O_\mathcal{F}}(\omega_{\mathfrak{n}-1}, \mathcal{H}\omega_{\mathfrak{n}-1})]}{1 + D_{O_\mathcal{F}}(\omega_{\mathfrak{n}-1}, \omega_\mathfrak{n})} \Bigg\}\Bigg\} \\ = \max\Bigg\{ & D_{O_\mathcal{F}}(\omega_{\mathfrak{n}-1}, \omega_\mathfrak{n}), \min\Bigg\{ \frac{D_{O_\mathcal{F}}(\omega_{\mathfrak{n}-1}, \omega_\mathfrak{n}) D_{O_\mathcal{F}}(\omega_\mathfrak{n}, \omega_{\mathfrak{n}+1})}{1 + D_{O_\mathcal{F}}(\omega_{\mathfrak{n}-1}, \omega_\mathfrak{n})}, \\ & \frac{D_{O_\mathcal{F}}(\omega_\mathfrak{n}, \omega_{\mathfrak{n}+1})[1 + D_{O_\mathcal{F}}(\omega_{\mathfrak{n}-1}, \omega_\mathfrak{n})]}{1 + D_{O_\mathcal{F}}(\omega_{\mathfrak{n}-1}, \omega_\mathfrak{n})} \Bigg\}\Bigg\} \\ \leq \max & \left\{ D_{O_\mathcal{F}}(\omega_{\mathfrak{n}-1}, \omega_\mathfrak{n}), \min\left\{ D_{O_\mathcal{F}}(\omega_\mathfrak{n}, \omega_{\mathfrak{n}+1}), D_{O_\mathcal{F}}(\omega_\mathfrak{n}, \omega_{\mathfrak{n}+1}) \right\}\right\} \\ = \max & \left\{ D_{O_\mathcal{F}}(\omega_{\mathfrak{n}-1}, \omega_\mathfrak{n}), D_{O_\mathcal{F}}(\omega_\mathfrak{n}, \omega_{\mathfrak{n}+1}) \right\}. \end{aligned}$$

If $\max\{D_{O_\mathcal{F}}(\omega_{\mathfrak{n}-1}, \omega_\mathfrak{n}), D_{O_\mathcal{F}}(\omega_\mathfrak{n}, \omega_{\mathfrak{n}+1})\} = D_{O_\mathcal{F}}(\omega_\mathfrak{n}, \omega_{\mathfrak{n}+1})$, then by (3.4) and $\beta \in \Im$, we get

$$\begin{aligned} F\left(D_{O_\mathcal{F}}(\omega_\mathfrak{n}, \omega_{\mathfrak{n}+1})\right) &\leq F\left(\beta\left(D_{O_\mathcal{F}}(\omega_\mathfrak{n}, \omega_{\mathfrak{n}+1})\right) D_{O_\mathcal{F}}(\omega_\mathfrak{n}, \omega_{\mathfrak{n}+1})\right) - \tau \\ &\leq F\left(D_{O_\mathcal{F}}(\omega_\mathfrak{n}, \omega_{\mathfrak{n}+1})\right) - \tau, \quad \tau > 0, \end{aligned}$$

which is a contradiction, and hence,

$$\max\left\{D_{O_\mathcal{F}}(\omega_{\mathfrak{n}-1}, \omega_\mathfrak{n}), D_{O_\mathcal{F}}(\omega_\mathfrak{n}, \omega_{\mathfrak{n}+1})\right\} = D_{O_\mathcal{F}}(\omega_{\mathfrak{n}-1}, \omega_\mathfrak{n}),$$

then from (3.4), we get

$$F\left(D_{O_\mathcal{F}}(\omega_\mathfrak{n}, \omega_{\mathfrak{n}+1})\right)$$

$$
\begin{aligned}
&\le F\left(\beta\left(D_{O_{\mathcal{F}}}(\omega_{\mathfrak{n}-1},\omega_{\mathfrak{n}})\right)D_{O_{\mathcal{F}}}(\omega_{\mathfrak{n}-1},\omega_{\mathfrak{n}})\right)-\tau\\
&= F\left(\beta\big(D_{O_{\mathcal{F}}}(\omega_{\mathfrak{n}-1},\omega_{\mathfrak{n}})\big)D_{O_{\mathcal{F}}}(\mathcal{H}\omega_{\mathfrak{n}-2},\mathcal{H}\omega_{\mathfrak{n}-1})\right)-\tau\\
&\le F\left(\beta\big(D_{O_{\mathcal{F}}}(\omega_{\mathfrak{n}-1},\omega_{\mathfrak{n}})\big)\beta\big(D_{O_{\mathcal{F}}}(\omega_{\mathfrak{n}-2},\omega_{\mathfrak{n}-1})\big)D_{O_{\mathcal{F}}}(\omega_{\mathfrak{n}-2},\omega_{\mathfrak{n}-1})\right)-2\tau\\
&\ \ \vdots\\
&\le F\left(\beta\big(D_{O_{\mathcal{F}}}(\omega_{\mathfrak{n}-1},\omega_{\mathfrak{n}})\big)\beta\big(D_{O_{\mathcal{F}}}(\omega_{\mathfrak{n}-2},\omega_{\mathfrak{n}-1})\big)\cdots\beta\big(D_{O_{\mathcal{F}}}(\omega_0,\omega_1)\big)D_{O_{\mathcal{F}}}(\omega_0,\omega_1)\right)\\
&\qquad -\mathfrak{n}\tau\\
&= F\left(\left(\prod_{i=1}^{\mathfrak{n}}\beta\left(D_{O_{\mathcal{F}}}(\omega_{\mathrm{i}-1},\omega_{\mathrm{i}})\right)\right)D_{O_{\mathcal{F}}}(\omega_0,\omega_1)\right)-\mathfrak{n}\tau\\
&\le F\left(D_{O_{\mathcal{F}}}(\omega_0,\omega_1)\right)-\mathfrak{n}\tau,\quad \forall\ \mathfrak{n}\in\mathbb{N}.
\end{aligned}
\tag{3.5}
$$

Let $\xi:(0,\infty)\to\mathbb{R}$ satisfy $(\mathcal{F}_1)$ and $(\mathcal{F}_2)$. Let $\varepsilon>0$ and $a\in[0,\infty)$ be such that $(D_{\mathcal{F}3})$ is satisfied. From $(\mathcal{F}_2)$, there exists $\delta>0$ such that

$$0<\iota<\delta \quad\Rightarrow\quad \xi(\iota)<\xi(\varepsilon)-a. \tag{3.6}$$

Now, applying $\mathfrak{n}\to\infty$ and by $(\mathcal{F}_2)$, we get

$$\lim_{\mathfrak{n}\to\infty}F\left(D_{O_{\mathcal{F}}}(\omega_{\mathfrak{n}},\omega_{\mathfrak{n}+1})\right)=-\infty \quad\Leftrightarrow\quad \lim_{\mathfrak{n}\to\infty}D_{O_{\mathcal{F}}}(\omega_{\mathfrak{n}},\omega_{\mathfrak{n}+1})=0. \tag{3.7}$$

By $(\mathcal{F}_3)$ $\exists\ r\in(0,1)$ such that

$$\lim_{\mathfrak{n}\to\infty}\big(D_{O_{\mathcal{F}}}(\omega_{\mathfrak{n}},\omega_{\mathfrak{n}+1})\big)^r F\big(D_{O_{\mathcal{F}}}(\omega_{\mathfrak{n}},\omega_{\mathfrak{n}+1})\big)=0. \tag{3.8}$$

Then, from (3.5), we get

$$\big(D_{O_{\mathcal{F}}}(\omega_{\mathfrak{n}},\omega_{\mathfrak{n}+1})\big)^r F\left(D_{O_{\mathcal{F}}}(\omega_{\mathfrak{n}},\omega_{\mathfrak{n}+1})\right)\le\big(D_{O_{\mathcal{F}}}(\omega_{\mathfrak{n}},\omega_{\mathfrak{n}+1})\big)^r\Big[F\left(D_{O_{\mathcal{F}}}(\omega_0,\omega_1)\right)-\mathfrak{n}\tau\Big].$$

We have

$$
\begin{aligned}
&\big(D_{O_{\mathcal{F}}}(\omega_{\mathfrak{n}},\omega_{\mathfrak{n}+1})\big)^r F\left(D_{O_{\mathcal{F}}}(\omega_{\mathfrak{n}},\omega_{\mathfrak{n}+1})\right)-\big(D_{O_{\mathcal{F}}}(\omega_{\mathfrak{n}},\omega_{\mathfrak{n}+1})\big)^r F\left(D_{O_{\mathcal{F}}}(\omega_0,\omega_1)\right)\\
&\le\big(D_{O_{\mathcal{F}}}(\omega_{\mathfrak{n}},\omega_{\mathfrak{n}+1})\big)^r\Big[F\left(D_{O_{\mathcal{F}}}(\omega_0,\omega_1)\right)-\mathfrak{n}\tau\Big]-\big(D_{O_{\mathcal{F}}}(\omega_{\mathfrak{n}},\omega_{\mathfrak{n}+1})\big)^r F\left(D_{O_{\mathcal{F}}}(\omega_0,\omega_1)\right)\\
&\le-\mathfrak{n}\tau\big(D_{O_{\mathcal{F}}}(\omega_{\mathfrak{n}},\omega_{\mathfrak{n}+1})\big)^r\\
&\le 0.
\end{aligned}
$$

Since

$$\lim_{\mathfrak{n}\to\infty}\mathfrak{n}\big(D_{O_{\mathcal{F}}}(\omega_{\mathfrak{n}},\omega_{\mathfrak{n}+1})\big)^r=0, \tag{3.9}$$

therefore, $\exists\ \mathfrak{n}_1\in\mathbb{N}$ such that

$$D_{O_{\mathcal{F}}}(\omega_{\mathfrak{n}},\omega_{\mathfrak{n}+1})\le\frac{1}{\mathfrak{n}^{\frac{1}{r}}},\quad \forall\ \mathfrak{n}\ge\mathfrak{n}_1, \tag{3.10}$$

which yields

$$\sum_{i=\mathfrak{n}}^{\mathfrak{m}-1}D_{O_{\mathcal{F}}}(\omega_i,\omega_{i+1})\le\sum_{i=\mathfrak{n}}^{\mathfrak{m}-1}\frac{1}{i^{\frac{1}{r}}},\quad \mathfrak{m}>\mathfrak{n}. \tag{3.11}$$

Since $\sum_{i=\mathfrak{n}}^{\mathfrak{m}-1} \frac{1}{i^{\frac{1}{r}}} < \infty$, then $\exists\ N \in \mathbb{N}$ such that

$$0 < \sum_{i=\mathfrak{n}}^{\mathfrak{m}-1} \frac{1}{i^{\frac{1}{r}}} < \sum_{i=\mathfrak{n}}^{\infty} \frac{1}{i^{\frac{1}{r}}} < \delta, \quad \mathfrak{m} > \mathfrak{n} \geq N. \tag{3.12}$$

By (3.6), (3.11), (3.12) and $(\mathcal{F}_1)$, we get

$$\xi\left(\sum_{i=\mathfrak{n}}^{\mathfrak{m}-1} D_{O_{\mathcal{F}}}(\omega_i, \omega_{i+1})\right) < \xi\left(\sum_{i=\mathfrak{n}}^{\infty} \frac{1}{i^{\frac{1}{r}}}\right) < \xi(\varepsilon) - a, \quad \mathfrak{m} > \mathfrak{n} \geq N. \tag{3.13}$$

Using $(D_{\mathcal{F}3})$ and (3.13), we get

$$\begin{aligned} D_{O_{\mathcal{F}}}(\omega_{\mathfrak{n}}, \omega_{\mathfrak{m}}) > 0, \quad \mathfrak{m} > \mathfrak{n} \geq N \Rightarrow \xi\left(D_{O_{\mathcal{F}}}(\omega_{\mathfrak{n}}, \omega_{\mathfrak{m}})\right) &\leq \xi\left(\sum_{i=\mathfrak{n}}^{\mathfrak{m}-1} D_{O_{\mathcal{F}}}(\omega_i, \omega_{i+1})\right) + a \\ &< \xi(\varepsilon). \end{aligned}$$

From $(\mathcal{F}_1)$, we have

$$D_{O_{\mathcal{F}}}(\omega_{\mathfrak{n}}, \omega_{\mathfrak{m}}) < \varepsilon, \quad \mathfrak{m} > \mathfrak{n} \geq N. \tag{3.14}$$

This means that $\omega_{\mathfrak{n}}$ is a Cauchy O-sequence. Since Ξ is O-complete, there exists $\omega^* \in \Xi$ such that

$$\lim_{\mathfrak{n}\to\infty} \omega_{\mathfrak{n}} = \omega^*.$$

Now, we prove that $\mathcal{H}\omega^* = \omega^*$. By (h_3), gives $\mathcal{H}\omega_{\mathfrak{n}} \to \mathcal{H}\omega^*$ as $\mathfrak{n} \to \infty$.

Thus,

$$\mathcal{H}\omega^* = \lim_{\mathfrak{n}\to\infty} \mathcal{H}\omega_{\mathfrak{n}} = \lim_{\mathfrak{n}\to\infty} \omega_{\mathfrak{n}+1} = \omega^*.$$

Next, we will use (h_3') to prove that $\mathcal{H}\omega^* = \omega^*$. Since $\exists\ \omega^* \in \Xi$ such that $\omega_{\mathfrak{n}} \to \omega^*$ as $\mathfrak{n} \to \infty$. Put $\mathcal{A} = \{\mathfrak{n} \in \mathbb{N} : \mathcal{H}\omega_{\mathfrak{n}} = \mathcal{H}\omega^*\}$. Now, if $\mathcal{A}$ is not finite, then $\exists$ $\{\omega_{\mathfrak{n}(k)}\} \subset \{\omega_{\mathfrak{n}}\}$ such that $\omega_{\mathfrak{n}(k)+1} = \mathcal{H}\omega_{\mathfrak{n}(k)} = \mathcal{H}\omega^*$, $\forall\ \mathfrak{n} \in \mathbb{N}$. Since $\omega_{\mathfrak{n}} \to \omega^*$, then $\mathcal{H}\omega^* = \omega^*$. If $\mathcal{A}$ is finite, then $\exists\ \mathfrak{n}_0 \in \mathbb{N}$ such that $\mathcal{H}\omega_{\mathfrak{n}} \neq \mathcal{H}\omega^*$, $\forall\ \mathfrak{n} \geq \mathfrak{n}_0$, means that $D_{O_{\mathcal{F}}}(\mathcal{H}\omega^*, \mathcal{H}\omega_{\mathfrak{n}}) > 0$. Assume that $D_{O_{\mathcal{F}}}(\mathcal{H}\omega^*, \omega^*) > 0$. From (h_3'), we have $\omega_{\mathfrak{n}} \perp \omega^*$ or $\omega^* \perp \omega_{\mathfrak{n}}$ and $\alpha(\omega^*) \geq \theta(\omega^*)$. By (h_4), we get $\theta(\mathcal{H}\omega^*) \geq \alpha(\mathcal{H}\omega^*)$, then, we obtain

$$\alpha(\omega^*)\theta(\mathcal{H}\omega^*) \geq \theta(\omega^*)\alpha(\mathcal{H}\omega^*), \tag{3.15}$$

also we have

$$\alpha(\omega_{\mathfrak{n}})\theta(\mathcal{H}\omega_{\mathfrak{n}}) \geq \theta(\omega_{\mathfrak{n}})\alpha(\mathcal{H}\omega_{\mathfrak{n}}). \tag{3.16}$$

Using (3.15), (3.16) and applying (3.1), we get

$$\begin{aligned} \tau + F\left(D_{O_{\mathcal{F}}}(\mathcal{H}\omega^*, \mathcal{H}\omega_{\mathfrak{n}})\right) &\leq F\left(\beta\left(\mathcal{M}(\omega^*, \omega_{\mathfrak{n}})\right)\mathcal{M}(\omega^*, \omega_{\mathfrak{n}})\right) \\ &\leq F\left(\mathcal{M}(\omega^*, \omega_{\mathfrak{n}})\right), \ \forall\ \mathfrak{n} \geq \mathfrak{n}_0, \end{aligned}$$

where

$$\mathcal{M}(\omega^*, \omega_{\mathfrak{n}}) = \max\Bigg\{D_{O_{\mathcal{F}}}(\omega^*, \omega_{\mathfrak{n}}), \min\Bigg\{\frac{D_{O_{\mathcal{F}}}(\omega^*, \mathcal{H}\omega^*)D_{O_{\mathcal{F}}}(\omega_{\mathfrak{n}}, \mathcal{H}\omega_{\mathfrak{n}})}{1 + D_{O_{\mathcal{F}}}(\omega^*, \omega_{\mathfrak{n}})},$$

$$\left.\left.\frac{D_{O_{\mathcal{F}}}(\omega_{\mathfrak{n}},\mathcal{H}\omega_{\mathfrak{n}})[1+D_{O_{\mathcal{F}}}(\omega^*,\mathcal{H}\omega^*)]}{1+D_{O_{\mathcal{F}}}(\omega^*,\omega_{\mathfrak{n}})}\right\}\right\}$$

$$= \max\left\{D_{O_{\mathcal{F}}}(\omega^*,\omega_{\mathfrak{n}}), \min\left\{\frac{D_{O_{\mathcal{F}}}(\omega^*,\mathcal{H}\omega^*)D_{O_{\mathcal{F}}}(\omega_{\mathfrak{n}},\omega_{\mathfrak{n}+1})}{1+D_{O_{\mathcal{F}}}(\omega^*,\omega_{\mathfrak{n}})},\right.\right.$$

$$\left.\left.\frac{D_{O_{\mathcal{F}}}(\omega_{\mathfrak{n}},\omega_{\mathfrak{n}+1})[1+D_{O_{\mathcal{F}}}(\omega^*,\mathcal{H}\omega^*)]}{1+D_{O_{\mathcal{F}}}(\omega^*,\omega_{\mathfrak{n}})}\right\}\right\}.$$

For all $\mathcal{M}(\omega^*,\omega_{\mathfrak{n}})$, we get $\lim_{\mathfrak{n}\to\infty}\mathcal{M}(\omega^*,\omega_{\mathfrak{n}})=0$, then by $(\mathcal{F}_2)$, it implies

$$\lim_{\mathfrak{n}\to\infty} F(\mathcal{M}(\omega^*,\omega_{\mathfrak{n}})) = -\infty, \quad \text{then} \quad \lim_{\mathfrak{n}\to\infty} F\left(D_{O_{\mathcal{F}}}(\mathcal{H}\omega^*,\mathcal{H}\omega_{\mathfrak{n}})\right) = -\infty.$$

Also by $(\mathcal{F}_2)$, it implies

$$\lim_{\mathfrak{n}\to\infty} D_{O_{\mathcal{F}}}(\mathcal{H}\omega^*,\mathcal{H}\omega_{\mathfrak{n}}) = \lim_{\mathfrak{n}\to\infty} D_{O_{\mathcal{F}}}(\mathcal{H}\omega^*,\omega_{\mathfrak{n}+1}) = D_{O_{\mathcal{F}}}(\mathcal{H}\omega^*,\omega^*) = 0.$$

Therefore, our assumption is wrong and hence, $\mathcal{H}\omega^*=\omega^*$.

Uniqueness

Suppose that there are two fixed point $\omega^*,\nu^*\in\Xi$ such that $\omega^*\neq\nu^*$ and $\omega^*\perp\nu^*$ or $\nu^*\perp\omega^*$. Using (h_4) and (h_5), we get

$$\alpha(\omega^*)\geq\theta(\omega^*) \quad \Rightarrow \quad \theta(\mathcal{H}\omega^*)\geq\alpha(\mathcal{H}\omega^*),$$

then we get

$$\alpha(\omega^*)\theta(\mathcal{H}\omega^*)\geq\theta(\omega^*)\alpha(\mathcal{H}\omega^*). \tag{3.17}$$

And

$$\alpha(\nu^*)\geq\theta(\nu^*) \quad \Rightarrow \quad \theta(\mathcal{H}\nu^*)\geq\alpha(\mathcal{H}\nu^*),$$

then we get

$$\alpha(\nu^*)\theta(\mathcal{H}\nu^*)\geq\theta(\nu^*)\alpha(\mathcal{H}\nu^*). \tag{3.18}$$

From (3.17), (3.18), (3.1), and $\beta\in\Im$, we have

$$\begin{aligned}\tau+F\left(D_{O_{\mathcal{F}}}(\omega^*,\nu^*)\right) &= \tau+F\left(D_{O_{\mathcal{F}}}(\mathcal{H}\omega^*,\mathcal{H}\nu^*)\right)\\ &\leq F\left(\beta\left(\mathcal{M}(\omega^*,\nu^*)\right)\mathcal{M}(\omega^*,\nu^*)\right)\\ &\leq F\left(\mathcal{M}(\omega^*,\nu^*)\right),\end{aligned} \tag{3.19}$$

where

$$\begin{aligned}\mathcal{M}(\omega^*,\nu^*) &= \max\left\{D_{O_{\mathcal{F}}}(\omega^*,\nu^*), \min\left\{\frac{D_{O_{\mathcal{F}}}(\omega^*,\mathcal{H}\omega^*)D_{O_{\mathcal{F}}}(\nu^*,\mathcal{H}\nu^*)}{1+D_{O_{\mathcal{F}}}(\omega^*,\nu^*)},\right.\right.\\ &\left.\left.\qquad\frac{D_{O_{\mathcal{F}}}(\nu^*,\mathcal{H}\nu^*)[1+D_{O_{\mathcal{F}}}(\omega^*,\mathcal{H}\omega^*)]}{1+D_{O_{\mathcal{F}}}(\omega^*,\nu^*)}\right\}\right\}\\ &= D_{O_{\mathcal{F}}}(\omega^*,\nu^*).\end{aligned}$$

Then from (3.19), we get

$$\tau+F\left(D_{O_{\mathcal{F}}}(\omega^*,\nu^*)\right)\leq F\left(D_{O_{\mathcal{F}}}(\omega^*,\nu^*)\right), \quad \tau>0,$$

which is a contradiction. Hence, $\omega^*=\nu^*$. □

Example 3.2. Define the sequence $\{\eta_\mathfrak{n}\}$ by

$$\eta_\mathfrak{n} = \ln(1+4+7+\cdots+(3\mathfrak{n}-2)) = \ln\left(\frac{\mathfrak{n}(3\mathfrak{n}-1)}{2}\right), \ \forall\ \mathfrak{n} \in \mathbb{N}.$$

Let $\Xi = \{\eta_\mathfrak{n} : \mathfrak{n} \in \mathbb{N}\}$ equipped with the $\mathcal{F}$-metric as defined in Example (2.1). For all $\eta_\mathfrak{n}, \eta_\mathfrak{m} \in \Xi$, define $\eta_\mathfrak{n} \perp \eta_\mathfrak{m}$ iff $(\mathfrak{m} \geq 2 \wedge \mathfrak{n} = 1)$. Hence, $(\Xi, \perp, D_{O\mathcal{F}})$ is an O-complete $\mathcal{F}$-MS. Define $\mathcal{H} : \Xi \to \Xi$ by

$$\mathcal{H}(\eta_\mathfrak{n}) = \begin{cases} \eta_1, & \text{if} \quad \mathfrak{n} = 1, \\ \eta_{\mathfrak{n}-1}, & \text{if} \quad \mathfrak{n} > 1. \end{cases}$$

Then, $\mathcal{H}$ is $\perp$-continuous and $\perp$-preserving.
Now, define $\alpha,\ \theta : \Xi \to [1, \infty)$ by

$$\alpha(\eta_\mathfrak{n}) = \begin{cases} e^{\eta_1}, & \text{if} \quad \mathfrak{n} = 1, \\ e^{\eta_{\mathfrak{n}-1}}, & \text{if} \quad \mathfrak{n} > 1, \end{cases}$$

$$\theta(\eta_\mathfrak{n}) = \begin{cases} 1, & \text{if} \quad \mathfrak{n} = 1, \\ \alpha(\eta_\mathfrak{n}), & \text{if} \quad \mathfrak{n} > 1, \end{cases}$$

we get

$$\alpha(\eta_\mathfrak{n}) \geq \theta(\eta_\mathfrak{n}) \quad \Rightarrow \quad \theta(\mathcal{H}\eta_\mathfrak{n}) \geq \alpha(\mathcal{H}\eta_\mathfrak{n}), \quad \forall\ \eta_\mathfrak{n} \in \Xi,\ \mathfrak{n} \in \mathbb{N}.$$

Then, (h_4) is satisfied. Let $F : (0, \infty) \to \mathbb{R}$ defined by $F(\iota) = \ln \iota + \iota,\ \iota > 0$. Then, $F \in \Phi$. Now, we prove that $\mathcal{H}$ is an orthogonal $(\alpha\theta - \beta F)$- rational contraction. Since $D_{O_\mathcal{F}}(\mathcal{H}\eta_\mathfrak{n}, \mathcal{H}\eta_\mathfrak{m}) > 0$, and we have

$$\alpha(\eta_\mathfrak{n})\theta(\mathcal{H}\eta_\mathfrak{n}) \geq \theta(\eta_\mathfrak{n})\alpha(\mathcal{H}\eta_\mathfrak{n}),$$

and

$$\alpha(\eta_\mathfrak{m})\theta(\mathcal{H}\eta_\mathfrak{m}) \geq \theta(\eta_\mathfrak{m})\alpha(\mathcal{H}\eta_\mathfrak{m}),$$

which implies

$$\begin{aligned} \tau + F\left(D_{O_\mathcal{F}}(\mathcal{H}\eta_\mathfrak{n}, \mathcal{H}\eta_\mathfrak{m})\right) &= \tau + F\left(D_{O_\mathcal{F}}(\eta_{\mathfrak{n}-1}, \eta_{\mathfrak{m}-1})\right) \\ &= \tau + F\left(e^{\eta_{\mathfrak{n}-1} - \eta_{\mathfrak{m}-1}}\right) \\ &\leq \tau + F\left(e^{3\eta_{\mathfrak{n}-1} + \eta_{\mathfrak{m}-1}}\right) \\ &= \tau + \ln\left(e^{3\eta_{\mathfrak{n}-1} + \eta_{\mathfrak{m}-1}}\right) + e^{3\eta_{\mathfrak{n}-1} + \eta_{\mathfrak{m}-1}} \\ &= \tau + (3\eta_{\mathfrak{n}-1} + \eta_{\mathfrak{m}-1}) + e^{3\eta_{\mathfrak{n}-1} + \eta_{\mathfrak{m}-1}} \\ &= \tau + (3\eta_{\mathfrak{n}-1} + 2\eta_{\mathfrak{m}-1} - \eta_{\mathfrak{m}-1}) + e^{3\eta_{\mathfrak{n}-1} + 2\eta_{\mathfrak{m}-1} - \eta_{\mathfrak{m}-1}}. \end{aligned}$$

Then, we get

$$\begin{aligned} e^{\tau + F\left(D_{O_\mathcal{F}}(\mathcal{H}\eta_\mathfrak{n}, \mathcal{H}\eta_\mathfrak{m})\right)} &\leq e^{\tau + (3\eta_{\mathfrak{n}-1} + 2\eta_{\mathfrak{m}-1} - \eta_{\mathfrak{m}-1}) + e^{3\eta_{\mathfrak{n}-1} + 2\eta_{\mathfrak{m}-1} - \eta_{\mathfrak{m}-1}}} \\ &\leq e^{\tau - \eta_{\mathfrak{m}-1}} e^{(3\eta_{\mathfrak{n}-1} + 2\eta_{\mathfrak{m}-1})} e^{e^{\tau - \eta_{\mathfrak{m}-1}} e^{(3\eta_{\mathfrak{n}-1} + 2\eta_{\mathfrak{m}-1})}} \\ &\leq \beta(\mathcal{M}(\eta_\mathfrak{n}, \eta_\mathfrak{m}))\mathcal{M}(\eta_\mathfrak{n}, \eta_\mathfrak{m}) e^{\beta(\mathcal{M}(\eta_\mathfrak{n}, \eta_\mathfrak{m}))\mathcal{M}(\eta_\mathfrak{n}, \eta_\mathfrak{m})}, \end{aligned}$$

which yields

$$\tau + F\left(D_{O_\mathcal{F}}(\mathcal{H}\eta_\mathfrak{n}, \mathcal{H}\eta_\mathfrak{m})\right) \leq \ln\left(\beta(\mathcal{M}(\eta_\mathfrak{n}, \eta_\mathfrak{m}))\mathcal{M}(\eta_\mathfrak{n}, \eta_\mathfrak{m})\right) + \beta(\mathcal{M}(\eta_\mathfrak{n}, \eta_\mathfrak{m}))\mathcal{M}(\eta_\mathfrak{n}, \eta_\mathfrak{m})$$

$$\leq F(\beta(\mathcal{M}(\eta_{\mathfrak{n}},\eta_{\mathfrak{m}}))\mathcal{M}(\eta_{\mathfrak{n}},\eta_{\mathfrak{m}})).$$

Then, $\mathcal{H}$ is an orthogonal $(\alpha\theta-\beta F)$- rational contraction where $\beta=e^{\tau-\eta_{\mathfrak{m}-1}}, \tau<\eta_{\mathfrak{m}-1}, \forall\ \mathfrak{m}>2$. So, from Theorem (3.1), it implies that $\eta=\ln(1)$ is a unique fixed point of $\mathcal{H}$.

We assume that $\mathcal{M}(\omega,\nu)$ is according to what was stated in Definition (3.2). Now, we will present some results.

Corollary 3.1. *Let $(\Xi,\perp,D_{O\mathcal{F}})$ be an O-complete $\mathcal{F}$-MS and $\mathcal{H}:\Xi\to\Xi$. Assume that the following conditions hold:*

- (h_1) *$\mathcal{H}$ is $\perp-$preserving, that is $\omega\perp\nu \quad\Rightarrow\quad \mathcal{H}\omega\perp\mathcal{H}\nu$.*
- (h_2) *If there exists $F\in\Phi$, $\alpha:\Xi\to[0,\infty)$, $\beta\in\Im$, $\tau>0$ and $\forall\ \omega,\nu\in\ \Xi$ with $\omega\perp\nu$, $D_{O_\mathcal{F}}(\mathcal{H}\omega,\mathcal{H}\nu)>0$, such that*

$$\alpha(\omega)\geq\alpha(\mathcal{H}\omega),\quad \textit{and}\quad \alpha(\nu)\geq\alpha(\mathcal{H}\nu)$$

implies

$$\tau+F\left(D_{O_\mathcal{F}}(\mathcal{H}\omega,\mathcal{H}\nu)\right)\leq F\left(\beta\left(\mathcal{M}(\omega,\nu)\right)\mathcal{M}(\omega,\nu)\right). \tag{3.20}$$

- (h_3) *There exists an orthogonal element $\omega_0\in\Xi$ such that $\omega_0\perp\mathcal{H}\omega_0$ or $\mathcal{H}\omega_0\perp\omega_0$ with $\alpha(\omega_0)\geq 1$.*
- (h_4) *$\mathcal{H}$ is cyclic α-admissible mapping with respect to θ^*.*
- (h_5) *Either $\mathcal{H}$ is $\perp-$continuous,*
- (h_5') *or if $\{\omega_{\mathfrak{n}}\}$ is a sequence in Ξ such that $\omega_{\mathfrak{n}}\to\omega\in\Xi$ and $\omega_{\mathfrak{n}}\perp\mathcal{H}\omega_{\mathfrak{n}}$ or $\mathcal{H}\omega_{\mathfrak{n}}\perp\omega_{\mathfrak{n}}$ with $\alpha(\omega_{\mathfrak{n}})\geq 1$, then $\omega_{\mathfrak{n}}\perp\omega$ or $\omega\perp\omega_{\mathfrak{n}}$ with $\alpha(\omega)\geq 1$, $\forall\ \mathfrak{n}\in\mathbb{N}$.*
- (h_6) *For all $\omega,\nu\in\Xi$ $(\omega\neq\nu)$ fixed points of $\mathcal{H}$ with $\omega\perp\nu$ or $\nu\perp\omega$, then $\alpha(\omega)\geq 1$ and $\alpha(\nu)\geq 1$.*

Then $\mathcal{H}$ has a unique fixed point.

Proof. Consider $\theta:\Xi\to[0,\infty)$ as $\theta(\omega)=1$, $\omega\in\Xi$ in Theorem (3.1). □

Corollary 3.2. *Let $(\Xi,\perp,D_{O\mathcal{F}})$ be an O-complete $\mathcal{F}$-MS and $\mathcal{H}:\Xi\to\Xi$. Assume that the following conditions hold:*

- (h_1) *$\mathcal{H}$ is $\perp-$preserving, that is $\omega\perp\nu \quad\Rightarrow\quad \mathcal{H}\omega\perp\mathcal{H}\nu$.*
- (h_2) *If there exists $F\in\Phi$, $\theta:\Xi\to[0,\infty)$, $\beta\in\Im$, $\tau>0$ and $\forall\ \omega,\nu\in\ \Xi$ with $\omega\perp\nu$, $D_{O_\mathcal{F}}(\mathcal{H}\omega,\mathcal{H}\nu)>0$, such that*

$$\theta(\mathcal{H}\omega)\geq\theta(\omega),\quad \textit{and}\quad \theta(\mathcal{H}\nu)\geq\theta(\nu)$$

implies

$$\tau+F\left(D_{O_\mathcal{F}}(\mathcal{H}\omega,\mathcal{H}\nu)\right)\leq F\left(\beta\left(\mathcal{M}(\omega,\nu)\right)\mathcal{M}(\omega,\nu)\right). \tag{3.21}$$

- (h_3) *There exists an orthogonal element $\omega_0\in\Xi$ such that $\omega_0\perp\mathcal{H}\omega_0$ or $\mathcal{H}\omega_0\perp\omega_0$ with $\theta(\omega_0)\leq 1$.*
- (h_4) *$\mathcal{H}$ is cyclic α^*-admissible mapping with respect to θ.*
- (h_5) *Either $\mathcal{H}$ is $\perp-$continuous,*
- (h_5') *or if $\{\omega_{\mathfrak{n}}\}$ is a sequence in Ξ s.t $\omega_{\mathfrak{n}}\to\omega\in\Xi$ and $\omega_{\mathfrak{n}}\perp\mathcal{H}\omega_{\mathfrak{n}}$ or $\mathcal{H}\omega_{\mathfrak{n}}\perp\omega_{\mathfrak{n}}$ with $\theta(\omega_{\mathfrak{n}})\leq 1$, then $\omega_{\mathfrak{n}}\perp\omega$ or $\omega\perp\omega_{\mathfrak{n}}$ with $\theta(\omega)\leq 1$, $\forall\ \mathfrak{n}\in\mathbb{N}$.*

(h_6) *For all* $\omega, \nu \in \Xi$ $(\omega \neq \nu)$ *fixed points of* $\mathcal{H}$ *with* $\omega \perp \nu$ *or* $\nu \perp \omega$, *then* $\theta(\omega) \leq 1$ *and* $\theta(\nu) \leq 1$.

Then $\mathcal{H}$ *has a unique fixed point.*

Proof. Consider $\alpha : \Xi \to [0, \infty)$ as $\alpha(\omega) = 1$, $\omega \in \Xi$ in Theorem (3.1). □

Corollary 3.3. *Let* $(\Xi, \perp, D_{O\mathcal{F}})$ *be an O-complete* $\mathcal{F}$*-MS and* $\mathcal{H} : \Xi \to \Xi$. *Assume that the following conditions hold:*

(h_1) $\mathcal{H}$ *is* $\perp-$*preserving, that is* $\omega \perp \nu \quad \Rightarrow \quad \mathcal{H}\omega \perp \mathcal{H}\nu$.

(h_2) *If there exists* $F \in \Phi$, $\alpha, \theta : \Xi \to [0, \infty)$, $\tau > 0$ *and* $\forall\ \omega, \nu \in \Xi$ *with* $\omega \perp \nu$, $D_{O_{\mathcal{F}}}(\mathcal{H}\omega, \mathcal{H}\nu) > 0$, *such that*

$$\alpha(\omega)\theta(\mathcal{H}\omega) \geq \theta(\omega)\alpha(\mathcal{H}\omega),$$

and

$$\alpha(\nu)\theta(\mathcal{H}\nu) \geq \theta(\nu)\alpha(\mathcal{H}\nu)$$

implies

$$\tau + F\left(D_{O_{\mathcal{F}}}(\mathcal{H}\omega, \mathcal{H}\nu)\right) \leq F\left(k\mathcal{M}(\omega, \nu)\right). \tag{3.22}$$

(h_3) *There exists an orthogonal element* $\omega_0 \in \Xi$ *such that* $\omega_0 \perp \mathcal{H}\omega_0$ *or* $\mathcal{H}\omega_0 \perp \omega_0$ *and* $\alpha(\omega_0) \geq \theta(\omega_0)$.

(h_4) $\mathcal{H}$ *is cyclic* α*-admissible mapping with respect to* θ.

(h_5) *Either* $\mathcal{H}$ *is* $\perp-$*continuous,*

(h'_5) *or if* $\{\omega_\mathfrak{n}\}$ *is a sequence in* Ξ *such that* $\omega_\mathfrak{n} \to \omega \in \Xi$ *and* $\omega_\mathfrak{n} \perp \mathcal{H}\omega_\mathfrak{n}$ *or* $\mathcal{H}\omega_\mathfrak{n} \perp \omega_\mathfrak{n}$ *and* $\alpha(\omega_\mathfrak{n}) \geq \theta(\omega_\mathfrak{n})$, *then* $\omega_\mathfrak{n} \perp \omega$ *or* $\omega \perp \omega_\mathfrak{n}$ *and* $\alpha(\omega) \geq \theta(\omega)$, $\forall\ \mathfrak{n} \in \mathbb{N}$.

(h_6) *For all* $\omega, \nu \in \Xi$ $(\omega \neq \nu)$ *fixed points of* $\mathcal{H}$ *with* $\omega \perp \nu$ *or* $\nu \perp \omega$, *then* $\alpha(\omega) \geq \theta(\omega)$ *and* $\alpha(\nu) \geq \theta(\nu)$.

Then $\mathcal{H}$ *has a unique fixed point.*

Proof. Taking $\beta(\iota) = k$, for all $\iota \geq 0$ and $k \in [0, 1)$ in Theorem (3.1). □

4. Application

In this section, we use Theorem (3.1) to prove the existence and uniqueness of solutions for the following differential equation:

$$\begin{cases} \frac{d^2\omega}{d\iota^2} = -\xi(\iota, \omega(\iota)), & \iota \in I = [0, 1], \\ \omega(0) = 0 = \omega(1), \end{cases} \tag{4.1}$$

where $\xi : I \times \mathbb{R} \to \mathbb{R}$ is a continuous function on I.
Problem (4.1) is equivalent to the following integral equation

$$\omega(\iota) = \int_0^1 G(\iota, \mathfrak{s})\xi(\mathfrak{s}, \omega(\mathfrak{s}))d\mathfrak{s}, \quad \forall\ \iota \in I, \tag{4.2}$$

where G is given by

$$G(\iota, \mathfrak{s}) = \begin{cases} \iota(1 - \mathfrak{s}), & 0 \leq \iota \leq \mathfrak{s} \leq 1, \\ \mathfrak{s}(1 - \iota), & 0 \leq \mathfrak{s} \leq \iota \leq 1. \end{cases}$$

It is clear that

$$\int_0^1 G(\iota, \mathfrak{s}) d\mathfrak{s} = \frac{\iota}{2} - \frac{\iota^2}{2}, \quad \forall\ \iota \in [0,1] \quad \text{and} \quad \sup_{\iota \in I} \int_0^1 G(\iota, \mathfrak{s}) d\mathfrak{s} = \frac{1}{8}.$$

Let $\Xi = \{\omega | \omega \in C(I, \mathbb{R})\}$ with the supremum norm $\|\omega\|_\infty = \sup_{\iota \in I} |\omega(\iota)|$. Then, $(\Xi, \|\omega\|_\infty)$ is a Banach space.

Lemma 4.1 ([3]). *The Banach space $(\Xi, \| \,.\, \|_\infty)$ endowed with the metric D defined by*

$$D(\omega, \nu) = \|\omega - \nu\|_\infty = \sup_{\iota \in I} |\omega(\iota) - \nu(\iota)|,$$

and orthogonal relation $\omega \perp \nu \quad \Leftrightarrow \quad \omega\nu \geq 0$, where $\omega, \nu \in \Xi$, is an O-$\mathcal{F}$-MS.

Theorem 4.1. *Consider the function $\xi : I \times \mathbb{R} \to \mathbb{R}$ is continuous and satisfies the follwing condition:*

$$|\xi(\mathfrak{s}, \omega(\mathfrak{s})) - \xi(\mathfrak{s}, \nu(\mathfrak{s}))| \leq e^{-\tau} \mathcal{M}(\omega, \nu), \tag{4.3}$$

where

$$\mathcal{M}(\omega, \nu) = \max \Bigg\{ D_{O_\mathcal{F}}(\omega, \nu), \min \left\{ \frac{D_{O_\mathcal{F}}(\omega, \mathcal{H}\omega) D_{O_\mathcal{F}}(\nu, \mathcal{H}\nu)}{1 + D_{O_\mathcal{F}}(\omega, \nu)}, \frac{D_{O_\mathcal{F}}(\nu, \mathcal{H}\nu)[1 + D_{O_\mathcal{F}}(\omega, \mathcal{H}\omega)]}{1 + D_{O_\mathcal{F}}(\omega, \nu)} \right\} \Bigg\}.$$

$\forall\ \omega, \nu \in \Xi$ s.t $\omega(\mathfrak{s})\nu(\mathfrak{s}) \geq 0$, $\mathfrak{s} \in I$ and $\tau > 0$.
Then, the differential equation (4.1) has a unique solution in Ξ.

Proof. Suppose that the orthogonality relation on Ξ by $\omega \perp \nu$ if $\omega(\mathfrak{s})\nu(\mathfrak{s}) \geq 0$, $\forall\ \mathfrak{s} \in I$.

Now, for $\omega \in \Xi$, $\exists\ \nu(\mathfrak{s}) = 0$, $\forall\ \mathfrak{s} \in I$ such that $\omega(\mathfrak{s})\nu(\mathfrak{s}) = 0$. Then, Ξ is an orthogonal. Define $D_{O_\mathcal{F}} : \Xi \times \Xi \to [0, \infty)$ by

$$D_{O_\mathcal{F}}(\omega, \nu) = \sup_{\mathfrak{s} \in I} |\omega(\mathfrak{s}) - \nu(\mathfrak{s})|, \text{ for all } \omega, \nu \in \Xi \text{ and } \mathfrak{s} \in I.$$

Then, $(\Xi, \perp, D_{O_\mathcal{F}})$ is a complete O-$\mathcal{F}$-MS.

Define a mapping $\mathcal{H} : \Xi \to \Xi$ by

$$\mathcal{H}\omega(\iota) = \int_0^1 G(\iota, \mathfrak{s}) \xi(\mathfrak{s}, \omega(\mathfrak{s})) d\mathfrak{s}, \quad \forall\ \iota \in I. \tag{4.4}$$

It is clear that $\mathcal{H}$ is $\perp$-continuous. If $\mathcal{H}\omega(\iota) = \omega(\iota)$ then, $\omega(\iota)$ is a solution of differential equation (4.1). So, we are now trying to prove that $\mathcal{H}$ has a unique fixed point, by satisfying the conditions of the theorem (3.1). Frist we prove that $\mathcal{H}$ is $\perp$-preserving. Let $\omega(\iota) \perp \nu(\iota)$ for all $\iota \in [0,1]$, we have

$$\mathcal{H}\omega(\iota) = \int_0^1 G(\iota, \mathfrak{s}) \xi(\mathfrak{s}, \omega(\mathfrak{s})) d\mathfrak{s} \geq 0,$$

which implies that $\mathcal{H}\omega(\iota)\perp\mathcal{H}\nu(\iota)$. This means, $\mathcal{H}$ is $\perp$-preserving.

Now, define $\alpha,\ \theta : \Xi \to [0,\infty)$ by

$$\alpha(\omega) = \theta(\omega) = 1, \quad \text{for all} \quad \omega \in \Xi.$$

Clearly (h_4) in theorem (3.1) is satisfied. Next, we show that $\mathcal{H}$ is an orthogonal $(\alpha\theta - \beta F)$-rational contraction. For all $\iota \in [0,1]$, $\omega(\iota)\perp\nu(\iota)$, then we have

$$\alpha(\omega(\iota))\theta(\mathcal{H}\omega(\iota)) \geq \theta(\omega(\iota))\alpha(\mathcal{H}\omega(\iota)), \tag{4.5}$$

and

$$\alpha(\nu(\iota))\theta(\mathcal{H}\nu(\iota)) \geq \theta(\nu(\iota))\alpha(\mathcal{H}\nu(\iota)). \tag{4.6}$$

Using (3.1), we get

$$\begin{aligned} D_{O_{\mathcal{F}}}(\mathcal{H}\omega(\iota), \mathcal{H}\nu(\iota)) &= \|\mathcal{H}\omega(\iota) - \mathcal{H}\nu(\iota)\|_\infty \\ &= \sup_{\iota\in I} |\mathcal{H}\omega(\iota) - \mathcal{H}\nu(\iota)| \\ &= \sup_{\iota\in I} \left| \int_0^1 G(\iota,\mathfrak{s}) \left[\xi(\mathfrak{s},\omega(\mathfrak{s})) - \xi(\mathfrak{s},\nu(\mathfrak{s}))\right] d\mathfrak{s} \right| \\ &\leq \left(\sup_{\iota\in I} \int_0^1 G(\iota,\mathfrak{s}) d\mathfrak{s} \right) |\xi(\mathfrak{s},\omega(\mathfrak{s})) - \xi(\mathfrak{s},\nu(\mathfrak{s}))| \\ &\leq \frac{1}{8} e^{-\tau} \mathcal{M}(\omega,\nu). \end{aligned}$$

Then we get

$$\begin{aligned} \ln\left(D_{O_{\mathcal{F}}}(\mathcal{H}\omega(\iota), \mathcal{H}\nu(\iota))\right) &\leq \ln\left(\frac{1}{8} e^{-\tau} \mathcal{M}(\omega,\nu)\right) \\ &\leq (-\tau) + \ln\left(\frac{1}{8}\mathcal{M}(\omega,\nu)\right), \end{aligned}$$

thus, we have

$$\tau + \ln\left(D_{O_{\mathcal{F}}}(\mathcal{H}\omega(\iota), \mathcal{H}\nu(\iota))\right) \leq \ln\left(\frac{1}{8}\mathcal{M}(\omega,\nu)\right). \tag{4.7}$$

Consider $F : (0,\infty) \to \mathbb{R}$ given by $F(\iota) = \ln(\iota)$ and $\beta(\iota) = \frac{1}{8} \in [0,1)$ for all $\iota > 0$. Then we get

$$\tau + F\left(D_{O_{\mathcal{F}}}(\mathcal{H}\omega(\iota), \mathcal{H}\nu(\iota))\right) \leq F\left(\beta\left(\mathcal{M}(\omega,\nu)\right)\mathcal{M}(\omega,\nu)\right).$$

Thus, $\mathcal{H}$ is an orthogonal $(\alpha\theta - \beta F)$- rational contraction. Hence, by Theorem (3.1), $\mathcal{H}$ has a unique fixed point, which is a unique solution to the equation (4.1). This completes the proof. □

5. Conclusions

In this paper, we introduced the concept of cyclic α-admissible mapping with respect to θ and proved some new fixed point theorems in an orthogonal $\mathcal{F}$-metric space using orthogonal $(\alpha\theta-\beta F)$- rational contraction which we also defined in this article. Additionally, we provided illustrative examples to bolster our results and applied the results that we obtained to prove the existence and uniqueness of solutions for second-order differential equations. Our results extend and enhance existing literature, while also introducing a fresh approach to verifying the existence and uniqueness of solutions for second-order differential equations.

References

[1] M. Jleli and B. Samet, *On a new generalization of metric spaces*, Journal of Fixed Point Theory and Applications, 2018, 20(3), Article ID 128, 20 pages.

[2] M.E. Gordji, M. Ramezani, M. De La Sen and Y. J. Cho, *On orthogonal sets and Banach fixed point theorem*, Fixed Point Theory, 2017, 18(2), 569–578.

[3] T. Kanwal, A. Hussain, H. Baghani and M. De La Sen, *New fixed point theorems in orthogonal F-metric spaces with application to fractional differential equation*, Symmetry, 2020, 12(5), Article ID 832, 15 pages.

[4] B. Samet, C. Vetro and P. Vetro, *Fixed point theorems for α–ψ-contractive type mappings*, Nonlinear Analysis, 2012, 75(4), 2154–2165.

[5] S.A. Alizadeh, F. Moradlou and P. Salimi, *Some fixed point results for $(\alpha, \beta)-(\psi, \phi)$-contractive mappings*, Filomat, 2014, 28(3), 635–647.

[6] P. Salimi, A. Latif and N. Hussain, *Modified α-ψ-contractive mappings with applications*, Fixed Point Theory and Applications, 2013, Article ID 151, 19 pages.

[7] D. Wardowski, *Fixed points of a new type of contractive mappings in complete metric spaces*, Fixed Point Theory and Applications, 2012, Article ID 94, 6 pages.

[8] J. Ahmad, A.S. Al-Rawashdeh and A.E. Al-Mazrooei, *Fixed Point Results for $(\alpha, \perp_{\mathcal{F}})$-Contractions in Orthogonal $\mathcal{F}$-Metric Spaces with Applications*, Journal of Function Spaces, 2022, Article ID 8532797, 10 pages.

[9] S. Banach, *Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales*, Fundamenta Mathematicae, 1922, 3(1), 133–181.

[10] R. Batra and S. Vashistha, *Fixed points of an F-contraction on metric spaces with a graph*, International Journal of Computer Mathematics, 2014, 91(12), 2483–2490.

[11] H. Huang, B. Carić, T. Došenović, D. Rakić and M. Brdar, *Fixed-point theorems in fuzzy metric spaces via fuzzy F-contraction*, Mathematics, 2021, 9(6), Article ID 641, 10 pages.

[12] G. Mınak, A. Helvacı and I. Altun, *Ćirić type generalized F-contractions on complete metric spaces and fixed point results*, Filomat, 2014, 28(6), 1143–1151.

[13] S. Luambano, S. Kumar and G. Kakiko, *Fixed point theorem for F-contraction mappings, in partial metric spaces*, Lobachevskii Journal of Mathematics, 2019, 40, 183–188.

[14] H. Afshari and S.M.A. Aleomraninejad, *Some fixed point results of F-contraction mapping in D-metric spaces by Samet's method*, Journal of Mathematical Analysis and Modeling, 2021, 2(3), 1–8.

[15] H. Piri, S. Rahrovi, H. Marasi and P. Kumam, *F-contraction on asymmetric metric spaces*, Journal of Mathematics and Computer Science, 2017, 17, 32–40.

[16] D. Derouiche and H. Ramoul, *New fixed point results for F-contractions of Hardy–Rogers type in b-metric spaces with applications*, Journal of Fixed Point Theory and Applications, 2020, 22, 1–44.

[17] J. Vujaković, S. Mitrović, M. Pavlović and S. Radenović, *On recent results concerning F-contraction in generalized metric spaces*, Mathematics, 2020, 8(5), Article ID 767, 12 pages.

[18] H. Piri and P. Kumam, *Some fixed point theorems concerning F-contraction in complete metric spaces*, Fixed Point Theory and Applications, 2014, Article ID 210, 11 pages.

[19] O. Popescu and G. Stan, *Two fixed point theorems concerning F-contraction in complete metric spaces*, Symmetry, 2019, 12(1), Article ID 58, 10 pages.

[20] N.A. Secelean, *Iterated function systems consisting of F-contractions*, Fixed Point Theory and Applications, 2013, Article ID 277, 13 pages.

[21] S.A. Al-Mezel, J. Ahmad and G. Marino, *Fixed point theorems for generalized* $(\alpha\beta-\psi)$*-contractions in F-metric spaces with applications*, Mathematics, 2020, 8(4), Article ID 584, 14 pages.

[22] A. Hussain and T. Kanwal, *Existence and uniqueness for a neutral differential problem with unbounded delay via fixed point results*, Transactions of A. Razmadze Mathematical Institute, 2018, 172(3), 481–490.

[23] M. Taleb and V.C. Borkar, *Some rational contraction and applications of fixed point theorems to F-metric space in differential equations*, Journal of Mathematical and Computational Science, 2022, Article ID 133, 13 pages.

[24] L.A. Alnaser, J. Ahmad, D. Lateef and H.A. Fouad, *New fixed point theorems with applications to non-linear neutral differential equations*, Symmetry, 2019, 11(5), Article ID 602, 11 pages.

[25] A. Asif, M. Nazam, M. Arshad and S.O. Kim, *F-metric, F-contraction and common fixed-point theorems with applications*, Mathematics, 2019, 7(7), Article ID 586, 13 pages.

[26] A.H. Albargi , *Some new results in F-metric spaces with applications*, AIMS Mathematics, 2023, 8(5), 10420–10434.

[27] K. Sawangsup, W. Sintunavarat and Y.J. Cho, *Fixed point theorems for orthogonal F-contraction mappings on O-complete metric spaces*, Journal of Fixed Point Theory and Applications, 2020, 22, 1–14.

[28] M. Taleb and V.C. Borkar, *New Fixed Point Results for Some Rational Contraction on* (ϕ, ψ)*-Metric Spaces*, Sahand Communications in Mathematical Analysis, 2024, 21(1), 47–66.

[29] A.E. Al-Mazrooei and J. Ahmad , *Fixed point approach to solve nonlinear fractional differential equations in orthogonal F-metric spaces*, AIMS Mathematics, 2023, 8(3), 5080–5098.

[30] M.H. Alharbi and J. Ahmad , *Solution of fractional differential equation by fixed point results in orthogonal* $\mathcal{F}$*-metric spaces*, AIMS Mathematics, 2023, 8(11), 27347–27362.