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Global Regularity to the 3D Generalized MHD Equations with Nonlinear Damping Terms*

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Abstract In this paper, we consider the Cauchy problem of the 3D generalized MHD system with nonlinear damping terms. We establish the global existence of strong solutions with the help of damping terms. Furthermore, we consider the balance between damping terms.

Keywords MHD equations, nonlinear damping terms, global existence

MSC(2010) 35B08; 35Q35.

1. Introduction

In this paper, we consider the following magnetohydrodynamic(MHD) equations with damping terms:

$$u_t + (u\cdot\nabla)u + \nabla\pi + \mu\Lambda^{2\alpha}u + \nu|u|^{p-1}u = (b\cdot\nabla)b, \qquad (t,x)\in\mathbb{R}^+\times\mathbb{R}^3, \tag{1.1}$$

$$b_t + (u\cdot\nabla)b + \mu\Lambda^{2\alpha}b + \nu|b|^{q-1}b = (b\cdot\nabla)u, \qquad (t,x)\in\mathbb{R}^+\times\mathbb{R}^3, \tag{1.2}$$

$$\operatorname{div} u = \operatorname{div} b = 0, \qquad (t,x)\in\mathbb{R}^+\times\mathbb{R}^3, \tag{1.3}$$

$$(u,b)(x,0) = (u_0,b_0), \qquad x\in\mathbb{R}^3. \tag{1.4}$$

where $u = u(x,t)\in\mathbb{R}^3$, $b = b(x,t)\in\mathbb{R}^3$ and $\pi = \pi(x,t)\in\mathbb{R}$ represent the unknown velocity field, the magnetic field, and the pressure, respectively. $\alpha\geq 0$, $p,q\geq 1$, $\mu\geq 0$ and $\nu\geq 0$ are real parameters. $\Lambda := (-\Delta)^{\frac{1}{2}}$ is defined in terms of Fourier transform by

$$\widehat{\Lambda f}(\xi) = |\xi|\widehat{f}(\xi).$$

The damping describes the resistance to fluid motion, which describes many physical situations such as friction effects and dissipative mechanisms(see [1] for details). When $b = 0$, systems (1.1)-(1.4) become Navier-Stokes equations with damping terms. Cai and Jiu proved the global existence of the strong solution if $p\geq\frac{7}{2}$. Furthermore, if $\frac{7}{2}\leq p\leq 5$, then the strong solution is unique. Later, when $\alpha = 1$, it was improved by Zhang et al [2], who established the global existence if

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$p \geq 3$. The lower bound 3 is critical in some sense(see [3] for details). In [4], if $\frac{1}{2} + \frac{2}{p} \leq \alpha \leq \frac{5}{4}$, $p \geq \frac{8}{3}$ or $\alpha \geq \frac{5}{4}$, $p \geq 1$, then the global existence was established. Recently, it was proved in [5], if $1 \leq \alpha < \frac{5}{4}$ and $p \geq 1 + \frac{10}{4\alpha+1}$, the global strong solution exists. In [6], when $\alpha = 1$ and one of the following four conditions holds, our system has a unique global solution : (1) $3 \leq p \leq \frac{27}{8}$, $q \geq 4$, (2) $\frac{27}{8} < p \leq \frac{7}{2}$, $q \geq \frac{7}{2p-5}$, (3) $\frac{7}{2} < p < 4$, $q \geq \frac{5p+7}{2p}$, (4) $p \geq 4$, $q \geq 1$.

The purpose of this paper is to study the well-posedness of the incompressible MHD equations with damping terms. With the help of damping terms, we are devoted to establishing the global existence of the strong solutions. Furthermore, we consider the balance between $|u|^{p-1}u$ and $|b|^{q-1}b$. Actually, in Theorem 1.1, when p takes different values there are different requirements for q.

We give our main theorems as follows.

Theorem 1.1. *If* $u_0(x) \in H^1(\mathbb{R}^3) \cap L^{p+1}(\mathbb{R}^3)$, $b_0(x) \in H^1(\mathbb{R}^3) \cap L^{q+1}(\mathbb{R}^3)$ *with* $1 \leq \alpha < \frac{\sqrt{5}}{2}$, $\operatorname{div} u_0 = \operatorname{div} b_0 = 0$, *and one of the following conditions holds*

$$
\begin{aligned}
&(1) 1 + \frac{10}{4\alpha+1} \leq p < \frac{4\alpha^2+8\alpha+15}{8\alpha}, q \geq \frac{4}{2\alpha-1},\\
&(2) \frac{4\alpha^2+8\alpha+15}{8\alpha} \leq p \leq \frac{2\alpha+5}{2\alpha}, q \geq \frac{2\alpha+5}{2\alpha p-5},\\
&(3) \frac{2\alpha+5}{2\alpha} < p < \frac{4}{2\alpha-1}, q \geq \frac{5p+2\alpha+5}{2\alpha p},\\
&(4) p \geq \frac{4}{2\alpha-1}, q \geq 1,
\end{aligned}
$$

then, for any $T > 0$, *the system* (1.1)-(1.4) *has a global strong solution* $(u,\, b)$ *satisfying*

$$
\begin{aligned}
u &\in L^\infty(0,T;H^1(\mathbb{R}^3)) \cap L^2(0,T;H^{1+\alpha}(\mathbb{R}^3)) \cap L^{p+1}(0,T;L^{p+1}(\mathbb{R}^3)),\\
b &\in L^\infty(0,T;H^1(\mathbb{R}^3)) \cap L^2(0,T;H^{1+\alpha}(\mathbb{R}^3)) \cap L^{q+1}(0,T;L^{q+1}(\mathbb{R}^3)).
\end{aligned}
$$

Remark 1.1. Actually, under the conditions (2), (3), we could further prove that

$$
\begin{aligned}
u &\in L^\infty(0,T;H^\alpha(\mathbb{R}^3)) \cap L^2(0,T;H^{1+\alpha}(\mathbb{R}^3)) \cap L^{p+1}(0,T;L^{p+1}(\mathbb{R}^3)),\\
b &\in L^\infty(0,T;H^\alpha(\mathbb{R}^3)) \cap L^2(0,T;H^{1+\alpha}(\mathbb{R}^3)) \cap L^{q+1}(0,T;L^{q+1}(\mathbb{R}^3)).
\end{aligned}
$$

Remark 1.2. When $\alpha = 1$, Theorem 1.1 is consistent with the results in [6].

Theorem 1.2. *If* $u_0(x) \in H^1(\mathbb{R}^3) \cap L^{p+1}(\mathbb{R}^3)$, $b_0(x) \in H^1(\mathbb{R}^3) \cap L^{q+1}(\mathbb{R}^3)$ *with* $\frac{\sqrt{5}}{2} \leq \alpha < \frac{5}{4}$, $\operatorname{div} u_0 = \operatorname{div} b_0 = 0$, *and one of the following conditions holds*

$$
\begin{aligned}
&(1) 1 + \frac{10}{4\alpha+1} \leq p < \frac{4}{2\alpha-1}, q \geq \frac{4}{2\alpha-1},\\
&(2) p \geq \frac{4}{2\alpha-1}, q \geq 1,
\end{aligned}
$$

then, for any $T > 0$, *the system* (1.1)-(1.4) *has a global strong solution* $(u,\, b)$ *satisfying*

$$
\begin{aligned}
u &\in L^\infty(0,T;H^1(\mathbb{R}^3)) \cap L^2(0,T;H^{1+\alpha}(\mathbb{R}^3)) \cap L^{p+1}(0,T;L^{p+1}(\mathbb{R}^3)),\\
b &\in L^\infty(0,T;H^1(\mathbb{R}^3)) \cap L^2(0,T;H^{1+\alpha}(\mathbb{R}^3)) \cap L^{q+1}(0,T;L^{q+1}(\mathbb{R}^3)).
\end{aligned}
$$

Remark 1.3. The proof of Theorem 1.2 is included in the proof of Theorem 1.1.

2. Proof of the Theorem 1.1

Proof. Multiplying (1.1) and (1.2) by u, b, after integration by parts and taking the divergence-free property into account, we have

$$\frac{1}{2}\frac{d}{dt}(\|u\|_{L^2}^2+\|b\|_{L^2}^2)+(\|\Lambda^\alpha u\|_{L^2}^2+\|\Lambda^\beta b\|_{L^2}^2)+(\|u\|_{L^{p+1}}^{p+1}+\|b\|_{L^{q+1}}^{q+1})=0.$$

Case 1. $1+\frac{10}{4\alpha+1}\le p<\frac{4\alpha^2+8\alpha+15}{8\alpha}$.

Multipling (1.1) and (1.2) by $-\triangle u$, $-\triangle b$, after integration by parts and taking the divergence-free property into account, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\nabla u\|_{L^2}^2+\|\nabla b\|_{L^2}^2)+(\|\Lambda^{1+\alpha}u\|_{L^2}^2+\|\Lambda^{1+\beta}b\|_{L^2}^2)\\
&+(\||u|^{\frac{p-1}{2}}\nabla u\|_{L^2}^2+\||b|^{\frac{q-1}{2}}\nabla b\|_{L^2}^2)+\frac{4(p-1)}{(p+1)^2}\|\nabla|u|^{\frac{p+1}{2}}\|_{L^2}^2+\frac{4(q-1)}{(q+1)^2}\|\nabla|b|^{\frac{q+1}{2}}\|_{L^2}^2\\
&=\int_{\mathbb{R}^3}(u\cdot\nabla)u\cdot\Delta u\,dx-\int_{\mathbb{R}^3}(b\cdot\nabla)b\cdot\Delta u\,dx\\
&+\int_{\mathbb{R}^3}(u\cdot\nabla)b\cdot\Delta b\,dx-\int_{\mathbb{R}^3}(b\cdot\nabla)u\cdot\Delta b\,dx\\
&:=M_1+M_2+M_3+M_4.
\end{aligned}$$

For M_1, when $p<3$, we have

$$\begin{aligned}
M_1&=\int_{\mathbb{R}^3}(u\cdot\nabla)u\cdot\Delta u\,dx\\
&\le\int_{\mathbb{R}^3}|u|\,|\nabla u|\,|\Delta u|dx\\
&\le\int_{\mathbb{R}^3}|u|^{\frac{p-1}{2}}\,|\nabla u|\,|u|^{\frac{3-p}{2}}\,|\triangle u|dx\\
&\le C\||u|^{\frac{p-1}{2}}\,|\nabla u|\|_{L^2}\|u\|_{L^{p+1}}^{\frac{3-p}{2}}\|\nabla u\|_{L^2}^{1-\theta_1}\|\Lambda^{1+\alpha}u\|_{L^2}^{\theta_1}\\
&\le\frac{1}{2}\||u|^{\frac{p-1}{2}}\,|\nabla u|\|_{L^2}^2+\frac{1}{8}\|\Lambda^{1+\alpha}u\|_{L^2}^2+C\|u\|_{L^{p+1}}^{\frac{3-p}{1-\theta_1}}\|\nabla u\|_{L^2}^2.
\end{aligned}$$

Here, we have used the Gagliardo-Nirenberg inequality:

$$\|\Delta u\|_{L^{\frac{p+1}{p-1}}}\le C\|\Lambda^{\alpha+1}u\|_{L^2}^{\theta_1}\|\nabla u\|_{L^2}^{1-\theta_1},$$

where

$$\frac{p-1}{p+1}=\frac{1}{3}+(\frac{1}{2}-\frac{\alpha}{3})\theta_1+\frac{1-\theta_1}{2}.$$

And conditions in Case 1 imply $\theta_1\in[\frac{1}{\alpha},1)$, $\frac{3-p}{1-\theta_1}\le p+1$.

When $3\le p<\frac{4\alpha^2+8\alpha+15}{8\alpha}$, we have

$$\begin{aligned}
|M_1|&\le\int_{\mathbb{R}^3}\frac{|u|}{1+|u|^{\frac{p-1}{2}}}(1+|u|^{\frac{p-1}{2}})|\nabla u||\Delta u|\,dx\\
&\le\|\nabla u\|_{L^2}\|\Delta u\|_{L^2}+\||u|^{\frac{p-1}{2}}\,|\nabla u|\|_{L^2}\|\Delta u\|_{L^2}
\end{aligned}$$

$$\leq C\|\nabla u\|_{L^2}^{2-\theta_2}\|\Lambda^{1+\alpha}u\|_{L^2}^{\theta_2}+C\||u|^{\frac{p-1}{2}}|\nabla u|\|_{L^2}\|\nabla u\|_{L^2}^{1-\theta_2}\|\Lambda^{1+\alpha}u\|_{L^2}^{\theta_2}$$
$$\leq \frac{1}{2}\||u|^{\frac{p-1}{2}}|\nabla u|\|_{L^2}^2+\frac{1}{8}\|\Lambda^{1+\alpha}u\|_{L^2}^2+C\|\nabla u\|_{L^2}^2$$

with $\theta_2=\frac{1}{\alpha}$.

M_2 can be estimated similarly if $q\geq 1+\frac{10}{4\alpha+1}$.

For M_3 and M_4, we have

$$\begin{aligned}|M_3|+|M_4| &\leq \|b\|_{L^{q+1}}\|\nabla u\|_{L^{\frac{2(q+1)}{q-1}}}\|\triangle b\|_{L^2}\\
&\leq C\|b\|_{L^{p+1}}\|\nabla u\|_{L^2}^{1-\theta_3}\|\Lambda^{1+\alpha}u\|_{L^2}^{\theta_3}\|\nabla b\|_{L^2}^{1-\theta_4}\|\Lambda^{1+\alpha}b\|_{L^2}^{\theta_4}\\
&\leq C\|b\|_{L^{p+1}}(\|\nabla u\|_{L^2}^{2-\theta_3-\theta_4}+\|\nabla b\|_{L^2}^{2-\theta_3-\theta_4})(\|\Lambda^{1+\alpha}u\|_{L^2}^{\theta_3+\theta_4}+\|\Lambda^{1+\alpha}b\|_{L^2}^{\theta_3+\theta_4})\\
&\leq \frac{1}{4}\|\Lambda^{1+\alpha}u\|_{L^2}^2+\frac{1}{4}\|\Lambda^{1+\alpha}b\|_{L^2}^2+C\|b\|_{L^{q+1}}^{\frac{2\alpha}{2\alpha-1-\frac{3}{q+1}}}(\|\nabla u\|_{L^2}^2+\|\nabla b\|_{L^2}^2)\end{aligned}$$

with $\theta_3=\frac{3}{\alpha(q+1)}$, $\theta_4=\frac{1}{\alpha}$. Note that $\theta_3\in[0,1)$, which implies that $q>\frac{3}{\alpha}-1$. If in addition, $\frac{2\alpha}{2\alpha-1-\frac{3}{q+1}}\leq q+1$, then $q\geq\frac{4}{2\alpha-1}$. We can verify that $1+\frac{10}{4\alpha+1}\leq\frac{4}{2\alpha-1}$.

Combining the above estimates, we obtain

$$\begin{aligned}&\frac{d}{dt}(\|\nabla u\|_{L^2}^2+\|\nabla b\|_{L^2}^2)+(\|\Lambda^{1+\alpha}u\|_{L^2}^2+\|\Lambda^{1+\beta}b\|_{L^2}^2)\\
&+(\||u|^{\frac{p-1}{2}}\nabla u\|_{L^2}^2+\||b|^{\frac{q-1}{2}}\nabla b\|_{L^2}^2)+\frac{4(p-1)}{(p+1)^2}\|\nabla|u|^{\frac{\beta+1}{2}}\|_{L^2}^2+\frac{4(q-1)}{(q+1)^2}\|\nabla|b|^{\frac{\beta+1}{2}}\|_{L^2}^2\\
&\leq C(\|u\|_{L^{p+1}}^{p+1}+\|b\|_{L^{q+1}}^{q+1}+1)(\|\nabla u\|_{L^2}^2+\|\nabla b\|_{L^2}^2).\end{aligned}$$

A standard Gronwall's inequality shows that

$$\begin{aligned}&\|\nabla u\|_{L^2}^2+\|\nabla b\|_{L^2}^2+\int_0^t(\|\Lambda^{\alpha+1}u\|_{L^2}^2+\|\Lambda^{\beta+1}b\|_{L^2}^2)\,ds\\
&+\int_0^t(\||u|^{\frac{p-1}{2}}\nabla u\|_{L^2}^2+\|\nabla|u|^{\frac{p+1}{2}}\|_{L^2}^2+\||b|^{\frac{q-1}{2}}\nabla b\|_{L^2}^2+\|\nabla|b|^{\frac{q+1}{2}}\|_{L^2}^2)\,ds\\
&\leq C(\,t,\|u_0\|_{H^1},\|b_0\|_{H^1}).\end{aligned}$$

Case 2. $\frac{4\alpha^2+8\alpha+15}{8\alpha}\leq p<\frac{4}{2\alpha-1}$.

Case 2 covers the conditions (2) and (3) in Theorem 1.1.

Multipling (1.1) and (1.2) by $-\triangle u$, $-\triangle b$, after integration by parts and taking the divergence-free property into account, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\|\nabla u\|_{L^2}^2+\|\nabla b\|_{L^2}^2)+(\|\Lambda^{1+\alpha}u\|_{L^2}^2+\|\Lambda^{1+\beta}b\|_{L^2}^2)\\
&+(\||u|^{\frac{p-1}{2}}\nabla u\|_{L^2}^2+\||b|^{\frac{q-1}{2}}\nabla b\|_{L^2}^2)+\frac{4(p-1)}{(p+1)^2}\|\nabla|u|^{\frac{p+1}{2}}\|_{L^2}^2+\frac{4(q-1)}{(q+1)^2}\|\nabla|b|^{\frac{q+1}{2}}\|_{L^2}^2\\
&=\int_{\mathbb{R}^3}(u\cdot\nabla)u\cdot\Delta u\,dx-\int_{\mathbb{R}^3}(b\cdot\nabla)b\cdot\Delta u\,dx\\
&+\int_{\mathbb{R}^3}(u\cdot\nabla)b\cdot\Delta b\,dx-\int_{\mathbb{R}^3}(b\cdot\nabla)u\cdot\Delta b\,dx.\end{aligned}$$

Multipling (1.1) and (1.2) by $\frac{1}{2}u_t$, $\frac{1}{2}b_t$, after integration by parts and taking the divergence-free property into account, we have

$$\frac{1}{4}\frac{d}{dt}(\|\Lambda^{\alpha}u\|_{L^2}^2+\|\Lambda^{\alpha}b\|_{L^2}^2)+\frac{1}{2}\frac{d}{dt}(\frac{1}{p+1}\|u\|_{L^{p+1}}^{p+1}+\frac{1}{q+1}\|b\|_{L^{q+1}}^{q+1})+\frac{1}{2}(\|u_t\|_{L^2}^2+\|b_t\|_{L^2}^2)$$

$$
\begin{aligned}
&= -\frac{1}{2}\int_{\mathbb{R}^3}(u\cdot\nabla)u\cdot u_t\,dx + \frac{1}{2}\int_{\mathbb{R}^3}(b\cdot\nabla)b\cdot u_t\,dx \\
&\quad -\frac{1}{2}\int_{\mathbb{R}^3}(u\cdot\nabla)b\cdot b_t\,dx + \frac{1}{2}\int_{\mathbb{R}^3}(b\cdot\nabla)u\cdot b_t\,dx.
\end{aligned}
$$

Add the two equations above, we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2 + \frac{1}{p+1}\|u\|_{L^{p+1}}^{p+1} + \frac{1}{q+1}\|b\|_{L^{q+1}}^{q+1}) + \frac{1}{4}\frac{d}{dt}(\|\Lambda^\alpha u\|_{L^2}^2 \\
&+ \|\Lambda^\alpha b\|_{L^2}^2) + (\|\Lambda^{1+\alpha}u\|_{L^2}^2 + \|\Lambda^{1+\beta}b\|_{L^2}^2) + (\||u|^{\frac{p-1}{2}}\nabla u\|_{L^2}^2 + \||b|^{\frac{q-1}{2}}\nabla b\|_{L^2}^2) \\
&+ \frac{4(p-1)}{(p+1)^2}\|\nabla|u|^{\frac{p+1}{2}}\|_{L^2}^2 + \frac{4(q-1)}{(q+1)^2}\|\nabla|b|^{\frac{q+1}{2}}\|_{L^2}^2 + \frac{1}{2}(\|u_t\|_{L^2}^2 + \|b_t\|_{L^2}^2) \\
=&\int_{\mathbb{R}^3}(u\cdot\nabla)u\cdot\Delta u\,dx - \int_{\mathbb{R}^3}(b\cdot\nabla)b\cdot\Delta u\,dx + \int_{\mathbb{R}^3}(u\cdot\nabla)b\cdot\Delta b\,dx \\
&- \int_{\mathbb{R}^3}(b\cdot\nabla)u\cdot\Delta b\,dx \ - \frac{1}{2}\int_{\mathbb{R}^3}(u\cdot\nabla)u\cdot u_t\,dx \\
&+ \frac{1}{2}\int_{\mathbb{R}^3}(b\cdot\nabla)b\cdot u_t\,dx - \frac{1}{2}\int_{\mathbb{R}^3}(u\cdot\nabla)b\cdot b_t\,dx + \frac{1}{2}\int_{\mathbb{R}^3}(b\cdot\nabla)u\cdot b_t\,dx \\
\le&\|u\cdot\nabla u\|_{L^2}^2 + \|b\cdot\nabla b\|_{L^2}^2 + \|u\cdot\nabla b\|_{L^2}^2 + \|b\cdot\nabla u\|_{L^2}^2 + \frac{2}{3}\|\triangle u\|_{L^2}^2 \\
&+ \frac{2}{3}\|\triangle b\|_{L^2}^2 + \frac{1}{2}\|u_t\|_{L^2}^2 + \frac{1}{2}\|b_t\|_{L^2}^2.
\end{aligned}
$$

For $\|u\cdot\nabla u\|_{L^2}^2$, we have

$$
\begin{aligned}
\|u\cdot\nabla u\|_{L^2}^2 &\le \int_{\mathbb{R}^3}\frac{|u|^2}{|u|^{p-1}+1}(|u|^{p-1}+1)|\nabla u|^2\,dx \\
&\le \||u|^{\frac{p-1}{2}}\nabla u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2.
\end{aligned}
$$

Similarly, we have $\|b\cdot\nabla b\|_{L^2}^2 \le \||b|^{\frac{q-1}{2}}\nabla b\|_{L^2}^2 + \|\nabla b\|_{L^2}^2$.
For $\|u\cdot\nabla b\|_{L^2}^2$, we have

$$
\begin{aligned}
\|u\cdot\nabla b\|_{L^2}^2 &\le \|u\|_{L^m}^2\|\nabla b\|_{L^n}^2 \\
&\le C\|u\|_{L^2}^{2(1-\delta_1)}\|u\|_{L^{p+1}}^{2\delta_1}\|b\|_{L^{q+1}}^{2(1-\delta_2)}\|\Lambda^{1+\alpha}b\|_{L^2}^{2\delta_2} \\
&\le C\|u\|_{L^{p+1}}^{2(p+1)} + C\|b\|_{L^{q+1}}^{2(q+1)} + \frac{1}{6}\|\Lambda^{1+\alpha}b\|_{L^2}^2,
\end{aligned}
$$

which satisfies the following conditions

$$
\begin{cases}
\frac{2}{m} + \frac{2}{n} = 1, \\
\frac{1}{m} = \frac{1-\delta_1}{2} + \frac{\delta_1}{p+1}, \delta_1 \in [0,1], \\
\frac{1}{n} = \frac{1}{3} + \delta_2(\frac{1}{2} - \frac{1+\alpha}{3}) + \frac{1-\delta_2}{q+1}, \delta_2 \in [\frac{1}{1+\alpha}, 1], \\
\frac{\delta_1}{p+1} + \frac{1-\delta_2}{q+1} + \delta_2 = 1.
\end{cases}
$$

By direct calculation, we have

$$
m = \frac{2[3pq - 2(\alpha+1)q - 2\alpha - 5]}{2\alpha pq - (4\alpha-1)q - 2\alpha - 5},
$$

$$n = \frac{2[3pq - 2(\alpha+1)q - 2\alpha - 5]}{(3-2\alpha)pq + (2\alpha-3)q},$$
$$\delta_1 = \frac{(3-2\alpha)(p+1)q}{3pq - 2(\alpha+1)q - 2\alpha - 5},$$
$$\delta_2 = \frac{3pq - 5q - 8}{3pq - 2(\alpha+1)q - 2\alpha - 5}.$$

Note $\delta_1 \in [0,1]$ and $\delta_2 \in [\frac{1}{1+\alpha}, 1]$, which implies $q \ge \frac{2\alpha+5}{3p-2\alpha-2}$, $q \ge \frac{2\alpha+5}{2\alpha p-5}$ and $q \ge \frac{2\alpha+1}{\alpha p-1-\alpha}$. We can verify that $\frac{2\alpha+5}{2\alpha p-5} \ge \frac{2\alpha+5}{3p-2\alpha-2}$ and $\frac{2\alpha+5}{2\alpha p-5} \ge \frac{2\alpha+1}{\alpha p-1-\alpha}$.

Similarly, we have

$$\begin{aligned}\|b\cdot\nabla u\|_{L^2}^2 &\le \|b\|_{L^r}^2\|\nabla u\|_{L^s}^2 \\ &\le C\|b\|_{L^2}^{2(1-\delta_3)}\|b\|_{L^{q+1}}^{2\delta_3}\|u\|_{L^{p+1}}^{2(1-\delta_4)}\|\Lambda^{1+\alpha}u\|_{L^2}^{2\delta_4} \\ &\le C\|u\|_{L^{p+1}}^{2(p+1)} + C\|b\|_{L^{q+1}}^{2(q+1)} + \frac{1}{6}\|\Lambda^{1+\alpha}u\|_{L^2}^2,\end{aligned}$$

where satisfies the following conditions

$$\begin{cases}\frac{2}{r} + \frac{2}{s} = 1, \\ \frac{1}{r} = \frac{1-\delta_3}{2} + \frac{\delta_3}{q+1}, \delta_3 \in [0,1], \\ \frac{1}{s} = \frac{1}{3} + \delta_4(\frac{1}{2} - \frac{1+\alpha}{3}) + \frac{1-\delta_4}{p+1}, \delta_4 \in [\frac{1}{1+\alpha}, 1], \\ \frac{\delta_3}{q+1} + \frac{1-\delta_4}{p+1} + \delta_4 = 1.\end{cases}$$

By direct calculation, we have

$$r = \frac{2[3pq - 2(\alpha+1)p - 2\alpha - 5]}{2\alpha pq - (4\alpha-1)p - 2\alpha - 5},$$
$$s = \frac{2[3pq - 2(\alpha+1)p - 2\alpha - 5]}{(3-2\alpha)pq + (2\alpha-3)p},$$
$$\delta_3 = \frac{(3-2\alpha)(q+1)p}{3pq - 2(\alpha+1)p - 2\alpha - 5},$$
$$\delta_4 = \frac{3pq - 5p - 8}{3pq - 2(\alpha+1)p - 2\alpha - 5}.$$

Note $\delta_3 \in [0,1]$ and $\delta_4 \in [\frac{1}{1+\alpha}, 1]$, which implies $q \ge \frac{2(\alpha+1)p+2\alpha+5}{3p}$, $q \ge \frac{5p+2\alpha+5}{2\alpha p}$ and $q \ge \frac{(1+\alpha)p+2\alpha+1}{\alpha p}$. We can verify that $\frac{5p+2\alpha+5}{2\alpha p} \ge \frac{2(\alpha+1)p+2\alpha+5}{3p}$ and $\frac{5p+2\alpha+5}{2\alpha p} \ge \frac{(1+\alpha)p+2\alpha+1}{\alpha p}$.

Combining the above estimates, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2 + \frac{1}{p+1}\|u\|_{L^{p+1}}^{p+1} + \frac{1}{q+1}\|b\|_{L^{q+1}}^{q+1}) + \frac{1}{4}\frac{d}{dt}(\|\Lambda^\alpha u\| + \|\Lambda^\alpha b\|) \\ &+ \frac{1}{6}(\|\Lambda^{1+\alpha}u\|_{L^2}^2 + \|\Lambda^{1+\alpha}b\|_{L^2}^2) + (\||u|^{\frac{p-1}{2}}\nabla u\|_{L^2}^2 + \||b|^{\frac{q-1}{2}}\nabla b\|_{L^2}^2) \\ &+ \frac{4(p-1)}{(p+1)^2}\|\nabla |u|^{\frac{\beta+1}{2}}\|_{L^2}^2 + \frac{4(q-1)}{(q+1)^2}\|\nabla |b|^{\frac{\beta+1}{2}}\|_{L^2}^2 \\ &\le C(\|u\|_{L^{p+1}}^{p+1} + \|b\|_{L^{q+1}}^{q+1} + 1)(\|u\|_{L^{p+1}}^{p+1} + \|b\|_{L^{q+1}}^{q+1} + \|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2).\end{aligned}$$

A standard Gronwall's inequality shows that

$$\begin{aligned}
&\frac{1}{2}(\|\nabla u\|_{L^2}^2+\|\nabla b\|_{L^2}^2+\frac{1}{p+1}\|u\|_{L^{p+1}}^{p+1}+\frac{1}{q+1}\|b\|_{L^{q+1}}^{q+1})+\frac{1}{4}(\|\Lambda^\alpha u\|+\|\Lambda^\alpha b\|)\\
&+\frac{1}{6}\int_{\mathbb{R}^3}(\|\Lambda^{1+\alpha}u\|_{L^2}^2+\|\Lambda^{1+\alpha}b\|_{L^2}^2)dx\\
&\le C(\ t,\|u_0\|_{H^1},\|u_0\|_{L^{p+1}},\|b_0\|_{H^1},\|b_0\|_{L^{p+1}}).
\end{aligned}$$

Case 3. $p\ge\frac{4}{2\alpha-1}$.

$$\begin{aligned}
|M_1|&\le\|u\|_{L^{p+1}}\|\nabla u\|_{L^{\frac{2(p+1)}{p-1}}}\|\triangle u\|_{L^2}\\
&\le C\|u\|_{L^{p+1}}\|\nabla u\|_{L^2}^{1-\theta_3}\|\Lambda^{1+\alpha}u\|_{L^2}^{\theta_3}\|\nabla u\|_{L^2}^{1-\theta_4}\|\Lambda^{1+\alpha}u\|_{L^2}^{\theta_4}\\
&\le C\|u\|_{L^{p+1}}\|\nabla u\|_{L^2}^{2-\theta_3-\theta_4}\|\Lambda^{1+\alpha}u\|_{L^2}^{\theta_3+\theta_4}\\
&\le\frac{1}{4}\|\Lambda^{1+\alpha}u\|_{L^2}^2+C\|u\|_{L^{p+1}}^{\frac{2\alpha}{2\alpha-1-\frac{3}{p+1}}}\|\nabla u\|_{L^2}^2
\end{aligned}$$

with $\theta_3=\frac{3}{\alpha(p+1)}$, $\theta_4=\frac{1}{\alpha}$.

And conditions in Case 3 imply θ_3, $\theta_4\in[0,1)$, $\frac{2\alpha}{2\alpha-1-\frac{3}{p+1}}\le p+1$.

Similarly, one can obtain

$$|M_2|+|M_3|+|M_4|\le\frac{1}{4}\|\Lambda^{1+\alpha}b\|_{L^2}^2+C\|u\|_{L^{p+1}}^{\frac{2\alpha}{2\alpha-1-\frac{3}{p+1}}}\|\nabla b\|_{L^2}^2.$$

Combining the above estimates, we obtain

$$\begin{aligned}
&\frac{d}{dt}(\|\nabla u\|_{L^2}^2+\|\nabla b\|_{L^2}^2)+(\|\Lambda^{1+\alpha}u\|_{L^2}^2+\|\Lambda^{1+\alpha}b\|_{L^2}^2)\\
&+(\||u|^{\frac{p-1}{2}}\nabla u\|_{L^2}^2+\||b|^{\frac{q-1}{2}}\nabla b\|_{L^2}^2)+\frac{4(p-1)}{(p+1)^2}\|\nabla|u|^{\frac{\beta+1}{2}}\|_{L^2}^2+\frac{4(q-1)}{(q+1)^2}\|\nabla|b|^{\frac{\beta+1}{2}}\|_{L^2}^2\\
&\le C(\|u\|_{L^{p+1}}^{p+1}+1)(\|\nabla u\|_{L^2}^2+\|\nabla b\|_{L^2}^2).
\end{aligned}$$

A standard Gronwall's inequality shows that

$$\begin{aligned}
&\|\nabla u\|_{L^2}^2+\|\nabla b\|_{L^2}^2+\int_0^t(\|\Lambda^{\alpha+1}u\|_{L^2}^2+\|\Lambda^{\beta+1}b\|_{L^2}^2)\,ds\\
&+\int_0^t(\||u|^{\frac{p-1}{2}}\nabla u\|_{L^2}^2+\|\nabla|u|^{\frac{p+1}{2}}\|_{L^2}^2+\||b|^{\frac{q-1}{2}}\nabla b\|_{L^2}^2+\|\nabla|b|^{\frac{q+1}{2}}\|_{L^2}^2)\,ds\\
&\le C(\ t,\|u_0\|_{H^1},\|b_0\|_{H^1}).
\end{aligned}$$

This completes the proof of the Theorem 1.1. □

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