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Stability and Bifurcation Analysis of the Nutrient-Microorganism Model*

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Abstract Stability analysis and bifurcation of the nutrient-microorganism model are presented in this paper. It is found that the model could experience the changes of the equilibrium points and the saddle-node, the Hopf and the codimension-2 Bogdanov-Takens bifurcations. The induced complex dynamics are also illustrated, by virtue of theory like the Sotomayor's theorem, the normal form and the universal unfolding. From the obtained results, some insights into interaction between the nutrient and the microorganism can be given. Further, numerical simulation is carried out to verify the theoretical results.

Keywords Nutrient-microorganism model, coexistence, Bogdanov-Takens bifurcation

MSC(2010) 34C23, 34D20, 37C20.

1. Introduction

The relationship in populations of nature has always been a hot topic in dynamics, in which the relationship between predator and prey has attracted researchers' attention. In the 1920s, a mathematical model [1, 2] between predator and prey was proposed by Lotka and Volterra. Since then, scientists have conducted more in-depth research on mathematical models of population relations. As a special predator-prey model, nutrient-microorganism model has also been considered extensively.

In 1996, Van Cappellen et al. [3] proposed that many microorganisms and chemical substances have taken their place in the circulation of substance. As we know, the interaction between microorganisms and nutrients is generally established through a decrease in nutrients and an increase in the number of microorganisms that feed on nutrients. For deeper exploration into the behavior of microorganisms and chemicals, Baurmann and Feudel established the model [4],

$$\begin{aligned} \frac{dB}{d\tau} &= \mu B^a N - mB, \\ \frac{dN}{d\tau} &= \phi(\hat{N} - N) - \varphi \frac{B}{B+L} BN, \end{aligned} \tag{1.1}$$

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where N represents the concentration of nutrients(mol/m^3), B is a bacterial population density that feeds on nutrients(kg biomass/m^3), a is the growth rate of bacteria, μ represents the conversion rate of biomass, m is the mortality rate of bacteria, φ is the capture rate of bacteria, and L is a semi saturated state constant. All parameters here are positive constants.

In the model (1.1), there are two ways to input nutrients. One is to input nutrients on the surface of the sediment, and the other is to input nutrients to the deeper sediment by bioirrigation technology. Because the depth of sediment affects the coverage rate of biological irrigation, the flux $\phi(\hat{N} - N)$ is used to deal with the diffusion term. Furthermore, it is assumed that only active portion of the bacteria is considered.

For model (1.1) with diffusion, Baurmann and Feudel [4] explore the turing patterns through a series of analysis and demonstration. In the same year, Baurmann et al. [5] proposed a new model by reconsidering the dormant bacteria and the activation of dormant bacteria into model (1.1). Then Turing instabilities and formation was found. Schmitz et al. [6] introduced a three species model in which nutrients are consumed by two competing populations of microbes in a marine sediment. Wetzel [7] investigated pattern formation of the benthic nutrient-microorganism model by Landau reductions and numerical methods and it is concluded that the system has an unstable state which means turing patterns start to exist for relatively small rates of food supply and ingestion. Furthermore, global bifurcation diagram for solutions over a bounded 2D domain is obtained. In Qian et al. [8], the local and the global bifurcations were considered in the diffusive nutrient-microorganism model by stability analysis, degree theory and bifurcation method. Moreover, the direction and the stability of the Hopf bifurcation was also obtained by considering the diffusive sediment model with no-flux boundary conditions in [9].

The delayed nutrient-microorganism model was put forwarded in [10] and its complex dynamical behavior was explored, including the codimension-2 bifurcation the Hopf-Hopf bifurcation. The resulting dynamical classification from bifurcation was also obtained by using the amplitude equations. Further results about the Hopf bifurcation in delayed nutrient-microorganism model with network structure were presented in [11] and dynamics of the diffusive nutrient-microorganism model with spatially heterogeneous environment was established in [12]. As further, the global existence and spatiotemporal pattern formation of the model with nutrient-taxis in the sediment [13] were given. The Hopf bifurcation of the diffusive nutrient-microorganism model with time delay was considered by applying the normal form theory and center manifold theorem [14]. For more detailed information about the nutrient-microorganism model, we refer to [15].

Although various dynamical results about the nutrient-microorganism model with the diffusion term, the delay and the taxis were reported, the bifurcation analysis for model (1.1) is also interesting and can be further explored in detail. There are few results in this area, then in this work qualitative analysis and bifurcation of model (1.1) will be carried out. To this end, make the following dimensionless transformation to system (1.1)

$$N = \frac{m}{\mu}v,\ \hat{N} = \frac{m}{\mu}\alpha,\ B = \frac{\phi}{\varphi}u,\ L = \frac{\phi}{\varphi}K,\ \tau = \frac{1}{\phi}t,\ \beta = \frac{m}{\phi},$$

then system (1.1) becomes

$$\dot{u} = \beta u(\frac{uv}{u+k} - 1),$$
$$\dot{v} = \alpha - v - \frac{u^2 v}{u+k}, \tag{1.2}$$

where α, β and k are positive constants. In order to explore the relationship between nutrients and microorganisms in more detail, we will analyze the dynamic behavior of the model (1.2) in this paper. We mainly focus on the analysis on existence and the stability of positive equilibrium points, and the bifurcation caused by the instability of positive equilibrium points, including saddle-node bifurcation, Hopf bifurcation and Bogdanov-Takens bifurcation.

The paper is organized as follows. In Section 2, we focus on the existence of equilibrium points, with the parameter conditions for the existence of different equilibria. In Section 3, the stability of equilibrium points is analyzed by stability analysis. In Section 4, the different bifurcations, such as the saddle-node bifurcation, the Hopf bifurcation and the Bogdanov-Takens bifurcation, induced by the instability of equilibria are described by employing bifurcation theory, such as the center manifold reduction and the normal form [16], the Sotomayor's theorem [17]. Numerical results are given to verify the theoretical analysis in Section 5. Finally, some conclusions are drawn in Section 6.

2. Existence of equilibria

In this section, we consider the existence of equilibria of system system (1.2). First note that the system has the boundary equilibria $E_0 = (0, \alpha)$. As for the positive equilibrium (u, v) of system (1.2), it satisfies

$$\frac{uv}{u+k} - 1 = 0,$$
$$\alpha - v - \frac{u^2 v}{u+k} = 0,$$

then

$$v = \frac{u+k}{u},$$

and u satisfies

$$g(u) = u^2 + (1-\alpha)u + k = 0. \tag{2.1}$$

The number of equilibrium points can be determined by judging the sign of discriminant of $g(u)$. Then, we have the following result about the existence of equilibrium points of system (1.2).

Theorem 2.1. *The existence of equilibrium points in model* (1.2) *is as follows:*

(i) If $0 < \alpha < 1 + 2\sqrt{k}$*, there is only one boundary equilibrium* $E_0(0, v_0) = (0, \alpha)$ *for system* (1.2)*.*

(ii) If $\alpha = 1 + 2\sqrt{k}$*, there is one boundary equilibrium* $E_0(0, v_0) = (0, \alpha)$ *and one positive equilibrium* $E_1(u_1, v_1) = (\sqrt{k}, 1 + \sqrt{k})$ *for system* (1.2)*.*

(iii) If $\alpha > 1 + 2\sqrt{k}$, there is one boundary equilibrium $E_0(0, v_0) = (0, \alpha)$ and two positive equilibriums $E_2(u_2, v_2)$ and $E_3(u_3, v_3)$ for system (1.2)*, where* $u_2 = \frac{\alpha-1+\sqrt{(1-\alpha)^2-4k}}{2}$, $u_3 = \frac{\alpha-1-\sqrt{(1-\alpha)^2-4k}}{2}$, $v_i = \frac{u_i+k}{u_i}$ *(i=2,3).*

Proof. From the above analysis, note that there is only one boundary equilibrium $E_0 = (0, \alpha)$ in system (1.2). From equation (2.1), the discriminant of $g(u)$ is

$$\Delta = (1 - \alpha)^2 - 4k.$$

Regarding Δ as the function of α, then Δ has two different roots $\alpha_1 = 1 + 2\sqrt{k}$ and $\alpha_2 = 1 - 2\sqrt{k}$. If $\alpha = 1 + 2\sqrt{k}$, then $\Delta = 0$ and system (1.2) has one positive equilibrium $E_1(u_1, v_1) = (\sqrt{k}, 1 + \sqrt{k})$. If $\alpha = 1 - 2\sqrt{k}$, the obtained equilibrium is negative, which will have no practical sense and is not considered. If $0 < \alpha < 1 + 2\sqrt{k}$, $\Delta < 0$. The function $g(u) = u^2 + (1 - \alpha)u + k = 0$ does not have any positive real root, system (1.2) has no positive equilibrium. If $\alpha > 1 + 2\sqrt{k}$, $\Delta > 0$. The function $g(u) = u^2 + (1 - \alpha)u + k = 0$ has two positive real roots and system (1.2) has two positive equilibriums $E_2(u_2, v_2)$ and $E_3(u_3, v_3)$, where $u_2 = \frac{\alpha-1+\sqrt{(1-\alpha)^2-4k}}{2}$, $u_3 = \frac{\alpha-1-\sqrt{(1-\alpha)^2-4k}}{2}$, $v_i = \frac{u_i+k}{u_i} (i = 2, 3)$. □

Remark 2.1.

(i) It is not difficult to note that the first quadrant is invariant for system (1.2), that is, the orbits starting from the positive initial states will be always positive.

(ii) When $u = 0$ and $v = \alpha$, this is a critical state, where the microorganisms in the seawater disappear and the concentration of nutrients reaches the highest level.

3. Stability of equilibria

As for the stability of boundary equilibrium $E_0(0, \alpha)$, we have the following result.

Theorem 3.1. *The boundary equilibrium $E_0(0, \alpha)$ of system* (1.2) *is a hyperbolic stable node.*

Proof. The Jacobian matrices of system (1.2) at E_0 is

$$J_{E_0} = \begin{pmatrix} -\beta & 0 \\ 0 & -1 \end{pmatrix}.$$

It is obvious that the determinant value of Jacobian matrix of system (1.2) is β and the trace is $-(\beta + 1)$. Since $\beta > 0$ in the model, J_{E_0} has two negative eigenvalues $-\beta$ and -1. Hence E_0 is a hyperbolic stable node. □

Next, we prove the stability of positive equilibrium E_1.

Theorem 3.2. *When $\alpha = 1 + 2\sqrt{k}$, there is only one positive equilibrium E_1 in system* (1.2)*. Then it follows that:*

(i) If $\beta = 2 + \frac{1}{\sqrt{k}}$, E_1 is a cusp of codimension two.

(ii) If $\beta > 2 + \frac{1}{\sqrt{k}}$, E_1 is a saddle-node with a repelling parabolic sector.

(iii) If $0 < \beta < 2 + \frac{1}{\sqrt{k}}$, E_1 *is a saddle-node with an attracting parabolic sector.*

Proof. Translating E_1 to the origin by $(x, y) = (u - u_1, v - v_1)$, system (1.2) becomes

$$\begin{aligned}\dot{x} &= a_{10}x + a_{01}y + a_{20}x^2 + a_{11}xy + a_{30}x^3 + a_{21}x^2y + P_1(x, y),\\ \dot{y} &= b_{10}x + b_{01}y + b_{20}x^2 + b_{11}xy + b_{30}x^3 + b_{21}x^2y + Q_1(x, y),\end{aligned} \tag{3.1}$$

where $P_1(x, y)$ and $Q_1(x, y)$ are high order terms in (x, y) at least of three, and

$$\begin{aligned}&a_{10} = \frac{\beta k}{u_1 + k}, a_{01} = \frac{\beta u_1}{v_1} = \frac{\beta u_1^2}{u_1 + k}, a_{20} = \frac{\beta k^2 v_1}{(u_1 + k)^3}, a_{11} = \frac{\beta u_1}{u_1 + k}(1 + \frac{k}{u_1 + k}),\\ &b_{10} = -1 - \frac{k}{u_1 + k}, b_{01} = -1 - \frac{u_1^2}{u_1 + k}, b_{20} = -\frac{k^2 v_1}{(u_1 + k)^3}, b_{11} = -\frac{u_1(u_1 + 2k)}{(u_1 + k)^2},\\ &b_{21} = -\frac{k^2}{(u_1 + k)^3}.\end{aligned}$$

The Jacobian matrix of system (3.1) at origin $O = (0, 0)$ is

$$J_O = \begin{pmatrix} a_{10} & a_{01} \\ b_{10} & b_{01} \end{pmatrix} = \begin{pmatrix} \frac{\beta k}{u_1+k} & \frac{\beta u_1}{v_1} \\ -\frac{u_1+2k}{u_1+k} & -\frac{u_1+v_1}{v_1} \end{pmatrix}$$

After calculation, it follows that $det J_O = 0$. Considering the sign of the trace of J_O,

$$tr J_O = \frac{\beta\sqrt{k} - 1 - 2\sqrt{k}}{1 + \sqrt{k}}.$$

Hence, the relationship of sizes between β and $2 + \frac{1}{\sqrt{k}}$ determines the sign of J_O. We divide it into two cases.

(i) Assume $\beta = 2 + \frac{1}{\sqrt{k}}$. Then $\text{tr}J_O = 0$. It means J_O has two zero eigenvalues. Letting $x = u$, $y = \frac{-a_{10}u + v}{a_{01}}$, system (3.1) is changed into

$$\begin{aligned}\dot{u} &= v + c_{20}u^2 + c_{11}uv + P_2(u, v),\\ \dot{v} &= d_{20}u^2 + d_{11}uv + Q_2(u, v),\end{aligned} \tag{3.2}$$

where $P_2(u, v)$ and $Q_2(u, v)$ are power series at least of the third order in (u, v), and

$$\begin{aligned}&c_{20} = a_{20} - \frac{a_{11}a_{10}}{a_{01}}, && c_{11} = \frac{a_{11}}{a_{01}},\\ &d_{20} = a_{10}(a_{20} - \frac{a_{11}a_{10}}{a_{01}}) + a_{01}(b_{20} - \frac{b_{11}a_{10}}{a_{01}}), && d_{11} = \frac{a_{11}a_{10}}{a_{01}} + b_{11}.\end{aligned}$$

Then, we make the following transformation for system (3.2)

$$\begin{aligned}x &= u,\\ y &= v + c_{20}u^2 + c_{11}uv + P_2(u, v),\end{aligned}$$

then system (3.2) becomes

$$\begin{aligned}\dot{x} &= y,\\ \dot{y} &= e_{20}x^2 + e_{11}xy + e_{02}y^2 + Q_3(x,y),\end{aligned} \tag{3.3}$$

where $Q_3(x,y)$ are power series at least of the third order in (x,y), and

$$e_{20} = d_{20}, \qquad e_{11} = 2c_{20} + d_{11}, \qquad e_{02} = c_{11}.$$

Now, we introduce a new variable τ by $dt = (1 - e_{02}x)d\tau$ and get

$$\begin{aligned}\frac{dx}{d\tau} &= (1 - e_{02}x)y,\\ \frac{dy}{d\tau} &= (1 - e_{02}x)(e_{20}x^2 + e_{11}xy + e_{02}y^2 + Q_3(x,y)),\end{aligned} \tag{3.4}$$

where $Q_3(x,y)$ are power series at least of the third order in (x,y).

One more transformation $(u,v) = (x, 1 - (1 - e_{02}x)y)$ transforms system (3.4) into

$$\begin{aligned}\frac{du}{d\tau} &= v,\\ \frac{dv}{d\tau} &= e_{20}u^2 + e_{11}uv + Q_4(u,v),\end{aligned} \tag{3.5}$$

where $Q_4(u,v)$ are power series at least of the third order in (u,v), and

$$e_{20} = d_{20}, \qquad e_{11} = 2c_{20} + d_{11}.$$

After some calculation, one has

$$\begin{aligned}e_{20} &= -\frac{1 + 2\sqrt{k}}{\sqrt{k}(1 + \sqrt{k})} < 0,\\ e_{11} &= -\frac{2(1 + 2\sqrt{k})}{\sqrt{k}(1 + \sqrt{k})^2} < 0.\end{aligned}$$

Therefore $e_{20}e_{11} \neq 0$. Then E_1 is a cusp of codimension two. Item (i) is completed.

(ii) Assume $\beta \neq 2 + \frac{1}{\sqrt{k}}$. J_0 has two different eigenvalues 0 and $\frac{\beta\sqrt{k} - 1 - 2\sqrt{k}}{1 + \sqrt{k}}$. For system (3.1), we make the transformation: $x = u + v$, $y = -u + \frac{b_{10}}{a_{10}}v$. Then system (3.1) is rewritten as

$$\begin{aligned}\dot{u} &= \overline{c_{20}}u^2 + \overline{c_{11}}uv + \overline{c_{02}}v^2 + P_5(u,v),\\ \dot{v} &= \overline{d_{10}}v + \overline{d_{20}}u^2 + \overline{d_{11}}uv + \overline{d_{02}}v^2 + Q_5(u,v),\end{aligned} \tag{3.6}$$

where $P_5(u,v)$ and $Q_5(u,v)$ are power series at least of the third order in (u,v), and

$$\begin{aligned}\overline{c_{20}} &= \frac{b_{10}(a_{20} - a_{11}) - a_{10}(b_{20} - b_{11})}{a_{10} + b_{10}},\\ \overline{c_{11}} &= \frac{b_{10}(2a_{20} + a_{11}(\frac{b_{10}}{a_{10}} - 1)) - a_{10}(2b_{20} + b_{11}(\frac{b_{10}}{a_{10}} - 1))}{a_{10} + b_{10}},\end{aligned}$$

$$\overline{c_{02}} = \frac{b_{10}(a_{20} + a_{11}\frac{b_{10}}{a_{10}}) - a_{10}(b_{20} + b_{11}\frac{b_{10}}{a_{10}})}{a_{10} + b_{10}},$$
$$\overline{d_{20}} = \frac{a_{10}(a_{20} - a_{11}) + a_{10}(b_{20} - b_{11})}{a_{10} + b_{10}},$$
$$\overline{d_{11}} = \frac{a_{10}(2a_{20} + a_{11}(\frac{b_{10}}{a_{10}} - 1)) + a_{10}(2b_{20} + b_{11}(\frac{b_{10}}{a_{10}} - 1))}{a_{10} + b_{10}},$$
$$\overline{d_{02}} = \frac{a_{10}(a_{20} + a_{11}\frac{b_{10}}{a_{10}}) + a_{10}(b_{20} + b_{11}\frac{b_{10}}{a_{10}})}{a_{10} + b_{10}}.$$

Then, introducing a new variable τ by $\mathrm{d}\tau = (a_{10} + b_{10})dt$, we get

$$\begin{aligned} \frac{du}{d\tau} &= \overline{e_{20}}u^2 + \overline{e_{11}}uv + \overline{e_{20}}v^2 + P_6(u,v), \\ \frac{dv}{d\tau} &= v + \overline{f_{20}}u^2 + \overline{f_{11}}uv + \overline{f_{02}}v^2 + Q_6(u,v). \end{aligned} \tag{3.7}$$

where $P_6(u,v)$ and $Q_6(u,v)$ are power series at least of the third order in (u,v), and

$$\overline{e_{20}} = \frac{\overline{c_{20}}}{a_{10} + b_{10}}, \qquad \overline{e_{11}} = \frac{\overline{c_{11}}}{a_{10} + b_{10}}, \qquad \overline{e_{02}} = \frac{\overline{c_{02}}}{a_{10} + b_{10}},$$
$$\overline{f_{20}} = \frac{\overline{d_{20}}}{a_{10} + b_{10}}, \qquad \overline{f_{11}} = \frac{\overline{d_{11}}}{a_{10} + b_{10}}, \qquad \overline{f_{02}} = \frac{\overline{d_{02}}}{a_{10} + b_{10}}.$$

By some computation, the coefficient of u^2 of system (3.7) is

$$\overline{e_{20}} = \frac{(1+\sqrt{k})\beta}{(\beta\sqrt{k} - (1+2\sqrt{k}))^2} \neq 0.$$

Applying Theorem 7.1 in Chapter 2 in [18], $a_m = \overline{e_{20}} \neq 0, m = 2$. So E_1 is a saddle-node with a parabolic sector. Considering the time variable τ, if $a_{10} + b_{10} > 0$, i. e. $\beta > 2 + \frac{1}{\sqrt{k}}$, then the equilibrium E_1 is a saddle-node with a repelling parabolic sector; if $a_{10} + b_{10} < 0$, i.e. $\beta < 2 + \frac{1}{\sqrt{k}}$, then the equilibrium E_1 is a saddle-node with an attracting parabolic sector. This proves (b) and (c). □

Now the stability of E_2 and E_3 in system (1.2) is stated as follows.

Theorem 3.3. *Suppose $\alpha > 1 + 2\sqrt{k}$. There are two positive equilibria E_2 and E_3 in system* (1.2)*. E_3 is always a saddle point and E_2 is*

(i) a source if $\beta > \frac{u_2(u_2+v_2)}{k}$;

(ii) a sink if $0 < \beta < \frac{u_2(u_2+v_2)}{k}$;

(iii) a center or fine focus if $\beta = \frac{u_2(u_2+v_2)}{k}$.

Proof. The Jacobian matrix of system (1.2) at E_2 is

$$J_{E_2} = \begin{pmatrix} \frac{\beta k}{u_2+k} & \frac{\beta u_2}{v_2} \\ -1 - \frac{k}{u_2+k} & -\frac{u_2+v_2}{v_2} \end{pmatrix}.$$

After calculation,

$$\begin{aligned} detJ_{E_2} &= \frac{\beta v_2(u_2^2-k)}{u_2 v_2^2} \\ &= \frac{\beta\sqrt{\Delta}}{2u_2v_2}(\sqrt{\Delta}+(\alpha-1)) \\ trJ_{E_2} &= \frac{\beta k - u_2^2 - u_2v_2}{u_2v_2}, \end{aligned}$$

where $\Delta = (1-\alpha)^2-4k$. From the value range of α, it can be seen that $detJ_{E_2} > 0$. We consider the sign of trJ_{E_2}. If tr$J_{E_2} > 0$ ie. $\beta > \frac{u_2(u_2+v_2)}{k}$, E_2 is a source; if $trJ_{E_2} < 0$, i.e., $\beta < \frac{u_2(u_2+v_2)}{k}$, E_2 is a sink; if $trJ_{E_2} = 0$, i.e., $\beta = \frac{u_2(u_2+v_2)}{k}$, E_2 is a center or fine focus.

The Jacobian matrix of system (1.2) at E_3 is

$$J_{E_3} = \begin{pmatrix} \frac{\beta k}{u_3+k} & \frac{\beta u_3}{v_3} \\ -1-\frac{k}{u_3+k} & -\frac{u_2+v_3}{v_3} \end{pmatrix}.$$

Then

$$\begin{aligned} detJ_{E_3} &= \frac{\beta v_3(u_3^2-k)}{u_3v_3^2} \\ &= \frac{\beta\sqrt{\Delta}}{2u_3v_3}(\sqrt{(1-\alpha)^2-4k}-(\alpha-1)) \\ &< 0. \end{aligned}$$

Hence, E_3 is always a saddle point. □

4. Bifurcation

In view of the stability properties of equilibrium points of system (1.2) in Section 3, now we will discuss the bifurcations at points caused by their instability.

4.1. Saddle-node bifurcation

From Theorem2.1 in Section 2, the number of positive equilibria of system (1.2) changes with the value of α. When $0 < \alpha < 1+2\sqrt{k}$, there is no positive equilibrium; when α=$1+2\sqrt{k}$, there is only one positive equilibrium; When $\alpha > 1+2\sqrt{k}$, there are two different positive equilibria. Hence, when $\alpha = \alpha_{SN} = 1+2\sqrt{k}$, the saddle-node bifurcation may occur at positive equilibrium E_1. In the following we present the related results.

Theorem 4.1. *When α=α_{SN}, system* (1.2) *undergoes the saddle-node Bifurcation at positive equilibrium E_1, where the threshold of saddle-node bifurcation is α=α_{SN}=$1+2\sqrt{k}$.*

Proof. Applying Sotomayor's theorem in [17], the transversality conditions for the saddle-node bifurcation at positive equilibrium E_1 could be verified. In the proof of Theorem 3.2, when α=$1+2\sqrt{k}$, $\beta \neq 2+\frac{1}{\sqrt{k}}$, the determinant of Jacobian matrix of system (1.2) at E_1 is $detJ_{E_1} = 0$ and the trace of Jacobian matrix of system (1.2) at E_1 is $trJ_{E_1} = \frac{\beta\sqrt{k}-1-2\sqrt{k}}{1+\sqrt{k}}$. So J_{E_1} has a zero eigenvalue, denoted by λ_1. Let V and W represent eigenvectors of matrices J_{E_1} and $J_{E_1}^T$ corresponding to eigenvalues $\lambda_1 = 0$, respectively, where

$$V = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$
$$W = \begin{pmatrix} W_1 \\ W_2 \end{pmatrix} = \begin{pmatrix} 1+2\sqrt{k} \\ \beta\sqrt{k} \end{pmatrix}.$$

Then,

$$F_\alpha(E_1, \alpha_{SN}) = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

$$D^2F(E_1,\alpha_{SN})(V,V) = \begin{pmatrix} \frac{\partial F_1}{\partial x^2}V_1^2 + 2\frac{\partial F_1}{\partial x \partial y}V_1V_2 + \frac{\partial F_1}{\partial y^2}V_2^2 \\ \frac{\partial F_2}{\partial x^2}V_1^2 + 2\frac{\partial F_2}{\partial x \partial y}V_1V_2 + \frac{\partial F_2}{\partial y^2}V_2^2 \end{pmatrix}$$
$$= \begin{pmatrix} \frac{2\beta\sqrt{k}}{(1+\sqrt{k})^2} \\ -\frac{2\sqrt{k}}{(1+\sqrt{k})^2} \end{pmatrix},$$

then it is concluded that

$$W^TF_\alpha(E_1,\alpha_{SN}) = \beta\sqrt{k} \neq 0,$$
$$W^T[D^2F(\ E_1,\alpha_{SN})(V,V)] = (\frac{2\beta\sqrt{k}}{1+\sqrt{k}}) \neq 0,$$

which means eigenvectors V and W satisfy the conditions for the saddle-node bifurcation in Sotomayor's theorem. As a result, system (1.2) undergoes the saddle-node bifurcation at $\alpha = \alpha_{SN} = 1 + 2\sqrt{k}$. □

4.2. Hopf bifurcation

From Section 3, we know that E_3 is always a saddle, and the stability of positive equilibrium E_2 varies with the value of β, which means that the Hopf bifurcation may occur at E_2.

Theorem 4.2. *Suppose $\alpha > 1 + 2\sqrt{k}$ and $\beta = \beta_H = \frac{u_2(u_2+v_2)}{k}$. System (1.2) experiences the Hopf bifurcation at the positive equilibrium E_2, where the threshold of Hopf bifurcation is $\beta = \beta_H = \frac{u_2(u_2+v_2)}{k}$.*

Proof. From Theorem 3.3, the Jacobian matrix of system (1.2) at E_2 is

$$J_{E_2} = \begin{pmatrix} \frac{\beta k}{u_2+k} & \frac{\beta u_2}{v_2} \\ -1-\frac{k}{u_2+k} & -\frac{u_2+v_2}{v_2} \end{pmatrix} = \begin{pmatrix} \frac{\beta k}{u_2 v_2} & \frac{\beta u_2}{v_2} \\ -1-\frac{k}{u_2+k} & -\frac{u_2+v_2}{v_2} \end{pmatrix}.$$

Furthermore,

$$\begin{aligned} detJ_{E_2} &= \frac{\beta(u_2^2-k)}{u_2v_2} > 0, \\ trJ_{E_2} &= \frac{\beta k - u_2^2 - u_2v_2}{u_2v_2}. \end{aligned} \tag{4.1}$$

For J_{E_2}, the characteristic equation is

$$(\lambda - \frac{\beta k}{u_2v_2})(\lambda + \frac{u_2+v_2}{v_2}) + \frac{\beta u_2}{v_2}(1+\frac{k}{u_2+k}) = 0.$$

For simplification it is

$$\lambda^2 - \frac{\beta k - u_2^2 - u_2v_2}{u_2v_2}\lambda + \frac{\beta v_2(u_2^2-k)}{u_2v_2^2} = 0,$$

which means

$$\lambda^2 - trJ_{E_2}\lambda + detJ_{E_2} = 0. \tag{4.2}$$

From equation (4.2), the eigenvalue of J_{E_2} is

$$\lambda_{1,2} = \frac{tr(J_{E_2}) \pm \sqrt{tr(J_{E_2})^2 - 4det(J_{E_2})}}{2}.$$

Since $\beta = \beta_H = \frac{u_2(u_2+v_2)}{k}$, $trJ_{E_2} = \frac{\beta k - u_2^2 - u_2v_2}{u_2v_2} = 0$, $tr(J_{E_2})^2 - 4det(J_{E_2}) = -4det(J_{E_2}) < 0$, then $\lambda_{1,2}{=}\pm\sqrt{det(J_{1,2})}i$. The characteristic roots of J_{E_2} are a pair of pure imaginary ones. Moreover, when $\beta{=}\frac{u_2(u_2+v_2)}{k}$,

$$\frac{dRe\lambda_{1,2}}{d\beta} = \frac{k}{\alpha - 1 + \sqrt{(1-\alpha)^2 - 4k} + 2k} \neq 0.$$

So the conditions for the Hopf bifurcation hold immediately. Then the Hopf bifurcation occurs in system (1.2) at the positive equilibrium E_2 when $\beta = \frac{u_2(u_2+v_2)}{k}$.

Next we determine the direction of Hopf bifurcation of system (1.2) at E_2 in terms of the first Lyapunov coefficient.

Translating $E_2(u_2, v_2)$ to $O(0,0)$ with $(\hat{x}, \hat{y}) = (u - u_2, v - v_2)$, we have

$$\begin{aligned} \dot{\hat{x}} &= \hat{a}_{10}\hat{x} + \hat{a}_{01}\hat{y} + \hat{a}_{20}\hat{x^2} + \hat{a}_{11}\hat{x}\hat{y} + \hat{a}_{30}\hat{x^3} + \hat{a}_{21}\hat{x^2}\hat{y} + P_1(\hat{x},\hat{y}), \\ \dot{\hat{y}} &= \hat{b}_{10}\hat{x} + \hat{b}_{01}\hat{y} + \hat{b}_{20}\hat{x^2} + \hat{b}_{11}\hat{x}\hat{y} + \hat{b}_{30}\hat{x^3} + \hat{b}_{21}\hat{x^2}\hat{y} + Q_1(\hat{x},\hat{y}), \end{aligned} \tag{4.3}$$

where $\hat{P_1}(x,y)$ and $\hat{Q_1}(x,y)$ are power series at least of order four in $(\hat{x},\hat{y})$, and

$$\begin{aligned} &\hat{a_{10}} = \frac{\beta k}{u_2+k}, \hat{a_{01}} = \frac{\beta u_2}{v_2}, \hat{a_{20}} = \frac{\beta k^2 v_2}{(u_2+k)^3}, \hat{a_{11}} = \frac{\beta u_2}{u_2+k}(1+\frac{k}{u_2+k}), \\ &\hat{a_{30}} = -\frac{\beta k^2 v_2}{(u_2+k)^4}, \hat{a_{21}} = \frac{\beta k^2}{(u_2+k)^3}, \hat{b_{10}} = -1 - \frac{k}{u_2+k}, \hat{b_{01}} = -1 - \frac{u_2}{v_2}, \\ &\hat{b_{20}} = -\frac{k^2 v_2}{(u_2+k)^3}, \hat{b_{11}} = -\frac{u_2(u_2+2k)}{(u_2+k)^2}, \hat{b_{30}} = \frac{k^2 v_2}{(u_2+k)^4}, \hat{b_{21}} = -\frac{k^2}{(u_2+k)^3}. \end{aligned}$$

Then we make the transformation $\hat{u} = -\hat{x}$, $\hat{v} = \frac{\hat{a}_{10}\hat{x}+\hat{a}_{01}\hat{y}}{\sqrt{D}}$, where $D = \hat{a}_{10}\hat{b}_{01} - \hat{a}_{01}\hat{b}_{10}$, and system (4.3) becomes

$$\begin{aligned}\dot{\hat{u}} &= -\sqrt{D}\hat{v} + \hat{c}_{20}\hat{u^2} + \hat{c}_{11}\hat{u}\hat{v} + \hat{c}_{30}\hat{u^3} + \hat{c}_{21}\hat{u^2}\hat{v} + P_2(\hat{u},\hat{v}),\\ \dot{\hat{v}} &= \sqrt{D}\hat{u} + \hat{d}_{20}\hat{u^2} + \hat{d}_{11}\hat{u}\hat{v} + \hat{d}_{30}\hat{u^3} + \hat{d}_{21}\hat{u^2}\hat{v} + Q_2(\hat{u},\hat{v}),\end{aligned} \tag{4.4}$$

where $\hat{P}_2(u,v)$ and $\hat{Q}_2(u,v)$ are power series at least of order four in $(\hat{u},\hat{v})$, and

$$\begin{aligned}&\hat{c}_{20} = \frac{\beta k}{(u_2+k)u_2}, \hat{c}_{11} = \frac{\sqrt{D}(u_2+2k)}{(u_2+k)u_2}, \hat{c}_{30} = -\frac{\beta k^2 v_2}{(u_2+k)^3 u_2}, \hat{c}_{21} = -\frac{\sqrt{D}k^2 v_2}{(u_2+k)^3 u_2},\\ &\hat{d}_{20} = -\frac{\beta k v_2}{\sqrt{D}(u_2+k)^2}, \hat{d}_{11} = -\frac{(u_2+2k)v_2}{(u_2+k)^2}, \hat{d}_{30} = \frac{\beta k^2 v_2}{\sqrt{D}(u_2+k)^3 u_2}, \hat{d}_{21} = \frac{k^2 v_2^2}{(u_2+k)^4}.\end{aligned}$$

Let

$$\begin{aligned}f(\hat{u},\hat{v}) &\triangleq \hat{c}_{20}\hat{u^2} + \hat{c}_{11}\hat{u}\hat{v} + \hat{c}_{30}\hat{u^3} + \hat{c}_{21}\hat{u^2}\hat{v} + P_2(\hat{u},\hat{v}),\\ g(\hat{u},\hat{v}) &\triangleq \hat{d}_{20}\hat{u^2} + \hat{d}_{11}\hat{u}\hat{v} + \hat{d}_{30}\hat{u^3} + \hat{d}_{21}\hat{u^2}\hat{v} + Q_2(\hat{u},\hat{v}).\end{aligned}$$

Applying the formula of the first Lyapunov coefficient [16], σ_1 is

$$\begin{aligned}\sigma_1 =& \frac{1}{16}(f_{\hat{u}\hat{u}\hat{u}} + g_{\hat{u}\hat{u}\hat{v}} + f_{\hat{u}\hat{v}\hat{v}} + g_{\hat{v}\hat{v}\hat{v}}) + \frac{1}{16\sqrt{D}}(f_{\hat{u}\hat{v}}(f_{\hat{u}\hat{u}} + f_{\hat{v}\hat{v}})\\ &- g_{\hat{u}\hat{v}}(g_{\hat{u}\hat{u}}g_{\hat{v}\hat{v}}) - f_{\hat{u}\hat{u}}g_{\hat{u}\hat{u}} + f_{\hat{v}\hat{v}}g_{\hat{v}\hat{v}})\\ =& \frac{1}{16}(6\hat{c}_{30} + 2\hat{d}_{21}) + \frac{1}{16\sqrt{D}}(2\hat{c}_{11}\hat{c}_{20} - 2\hat{d}_{11}\hat{d}_{20} - 4\hat{c}_{20}\hat{d}_{20})\\ =& \frac{1}{16}(\frac{-6\beta k^2 v_2}{(u_2+k)^3 u_2} + \frac{2k^2 v_2^2}{(u_2+k)^4}) + \frac{1}{16\sqrt{D}}(\frac{2\sqrt{D}(u_2+2k)}{(u_2+k)u_2}\frac{\beta k}{(u_2+k)u_2}\\ &- \frac{2(u_2+2k)v_2}{(u_2+k)^2}\frac{\beta k v_2}{\sqrt{D}(u_2+k)^2} + 4\frac{\beta k}{(u_2+k)u_2}\frac{\beta k v_2}{\sqrt{D}(u_2+k)^2})\end{aligned}$$

Hence, if $\sigma_1 < 0$, system (1.2) undergoes the supercritical Hopf bifurcation with a stable limit cycle; if $\sigma_1 > 0$, system (1.2) undergoes the subcritical Hopf bifurcation with an unstable limit cycle. □

4.3. Bogdanov-Takens bifurcation

Next we will focus on the Bogdanov-Takens bifurcation. From Theorem 3.2 in Section 3, when $\alpha = 1+2\sqrt{k}$, $\beta = 2+\frac{1}{\sqrt{k}}$, the positive equilibrium E_1 is a degenerate cusp of codimension 2 for system (1.2). Then system (1.2) may admit the Bogdanov-Takens bifurcation at positive equilibrium E_1. From the above discussions, choose α and β as the bifurcation parameters to investigate the Bogdanov-Takens bifurcation at the positive equilibrium E_1.

Theorem 4.3. *Let $\alpha_{BT} = 1 + 2\sqrt{k}$, $\beta_{BT} = 2 + \frac{1}{\sqrt{k}}$. If $(\alpha,\ \beta)$ changes in some neighborhood of $(\alpha_{BT},\ \beta_{BT})$, the Bogdanov-Takens bifurcation will occur in system (1.2) at the positive equilibrium E_1.*

Proof. Let $(\alpha,\beta)=(\alpha_{BT}+\varepsilon_1,\beta_{BT}+\varepsilon_2)$, where $\varepsilon=(\varepsilon_1,\varepsilon_2)$ varies slightly near the origin, and system (1.2) becomes

$$\begin{aligned}\dot{u}&=(\beta+\varepsilon_2)u(\frac{uv}{u+k}-1),\\ \dot{v}&=(\alpha+\varepsilon_1)-v-\frac{u^2v}{u+k}.\end{aligned}\tag{4.5}$$

After translating E_1 to the origin by the transformation $(x,y)=(u-u_1,v-v_1)$, system (4.5) is changed into

$$\begin{aligned}\dot{x}&=p_{00}+p_{10}x+p_{01}y+p_{20}x^2+p_{11}xy+P_7(x,y),\\ \dot{y}&=q_{00}+q_{10}x+q_{01}y+q_{20}x^2+q_{11}xy+Q_7(x,y),\end{aligned}\tag{4.6}$$

where $P_7(x,y)$ and $Q_7(x,y)$ are C^∞ functions whose coefficients smoothly depend on ε_1 and ε_2 and are not less than the third order in (x,y), and

$$\begin{aligned}&p_{00}=0,\quad p_{10}=\frac{(\beta_{BT}+\varepsilon_2)k}{u_1+k},\quad p_{01}=\frac{(\beta_{BT}+\varepsilon_2)u_1}{v_1},\quad p_{20}=\frac{(\beta_{BT}+\varepsilon_2)k^2v_1}{(u_1+k)^3},\\ &p_{11}=\frac{(\beta_{BT}+\varepsilon_2)u_1}{u_1+k}(1+\frac{k}{u_1+k}),\quad q_{00}=\varepsilon_1,\quad q_{10}=-\frac{u_1+2k}{u_1+k},\quad q_{01}=-\frac{\alpha_{BT}}{v_1},\\ &q_{20}=-\frac{k^2v_1}{(u_1+k)^3},\quad q_{11}=-\frac{u_1(u_1+2k)}{(u_1+k)^2}.\end{aligned}$$

Further making the following changes into system (4.6),

$$\begin{aligned}u&=x,\\ v&=p_{10}x+p_{01}y+p_{20}x^2+p_{11}xy+Q_7(x,y),\end{aligned}\tag{4.7}$$

system (4.6) is reformulated as

$$\begin{aligned}\dot{u}&=v,\\ \dot{v}&=r_{00}+r_{10}u+r_{01}v+r_{20}u^2+r_{11}uv+r_{02}v^2+Q_8(u,v),\end{aligned}\tag{4.8}$$

where $Q_8(u,v)$ is a C^∞ function whose coefficients smoothly depend on ε_1 and ε_2 and is not less than the third order in (u,v), and

$$\begin{aligned}&r_{00}=p_{01}q_{00},\qquad r_{10}=p_{01}q_{10}+p_{11}q_{00}-p_{10}q_{01},\qquad r_{01}=p_{10}+q_{01},\\ &r_{20}=q_{01}(\frac{p_{10}p_{11}}{p_{01}})+p_{01}q_{20}-p_{10}q_{11}+p_{11}q_{10}-\frac{p_{11}p_{10}q_{01}}{p_{01}},\\ &r_{11}=2p_{20}-\frac{p_{11}q_{01}}{p_{01}}+q_{11}+\frac{p_{11}q_{01}}{p_{01}}-\frac{p_{11}p_{10}}{p_{01}},\qquad r_{02}=\frac{p_{11}}{p_{01}}.\end{aligned}$$

Next, the new time variable τ is introduced through $dt=(1-r_{02}u)d\tau$, and still denoting τ as t, then system (4.8) has the following form:

$$\begin{aligned}\dot{u}&=(1-r_{02}u)v,\\ \dot{v}&=(1-r_{02}u)(r_{00}+r_{10}u+r_{01}v+r_{20}u^2+r_{11}uv+r_{02}v^2+Q_8(u,v)).\end{aligned}\tag{4.9}$$

Let

$$(x,y)=(u,1-r_{02}u)v.$$

Then system (4.9) becomes

$$\begin{aligned}\dot{x} &= y,\\ \dot{y} &= d_1 + d_2x + d_3y + d_4x^2 + d_5xy + Q_9(x, y),\end{aligned} \tag{4.10}$$

where $Q_9(x, y)$ is a C^∞ function whose coefficients smoothly depend on ε_1 and ε_2 and is not less than the third order in (x, y), and

$$d_1 = r_{00}, \quad d_2 = r_{10} - 2r_{02}r_{00}, \quad d_3 = r_{01}, \quad d_4 = r_{20} - 2r_{02}r_{10} + {r_{02}}^2r_{00},$$
$$d_5 = r_{11} - r_{01}r_{02}.$$

Note that $d_i (i = 1, 2, 3, 4, 5)$ are functions about ε_1 and ε_2. When $\varepsilon_1 = \varepsilon_2 = 0$, it holds that

$$\begin{aligned}d_4 &= -\frac{\beta_{BT}}{1+\sqrt{k}} < 0,\\ d_5 &= -\frac{\beta_{BT} + 1 + 2\sqrt{k}}{(1+\sqrt{k})^2} < 0.\end{aligned} \tag{4.11}$$

Next, applying the transformation $(u, v) = (x, \frac{y}{\sqrt{-d_4}})$, and introducing a new variable by $\tau = \sqrt{-d_4}t$, still denoting τ as t, system (4.10) is written as

$$\begin{aligned}\dot{u} &= v,\\ \dot{v} &= e_1 + e_2u + e_3v - u^2 + e_4uv + Q_{10}(u, v),\end{aligned} \tag{4.12}$$

where $Q_{10}(u, v)$ is a C^∞ function whose coefficients smoothly depends on ε_1 and ε_2 and is not less than third order in (u, v), and

$$e_1 = -\frac{d_1}{d_4}, \quad e_2 = -\frac{d_2}{d_4}, \quad e_3 = \frac{d_3}{\sqrt{-d_4}}, \quad e_4 = \frac{d_5}{\sqrt{-d_4}}.$$

From the transformation $(x, y) = (u - \frac{e_2}{2}, v)$, system (4.12) is transformed into

$$\begin{aligned}\dot{x} &= y,\\ \dot{y} &= f_1 + f_2y - x^2 + f_3xy + Q_{11}(x, y),\end{aligned} \tag{4.13}$$

where $Q_{11}(x, y)$ is a C^∞ function whose coefficients smoothly depend on ε_1 and ε_2 and is not less than the third order in (x, y), and

$$f_1 = e_1 + \frac{{e_2}^2}{4}, \quad f_2 = e_3 + \frac{e_2e_4}{2}, \quad f_3 = e_4.$$

From equation (4.11), note that $f_3 = e_4 = \frac{d_5}{\sqrt{-d_4}} < 0$. Finally, applying $(u, v) = (-{f_3}^2x, {f_3}^3y)$, $\tau = -\frac{1}{f_3}t$, when τ is still denoted by t, we have

$$\begin{aligned}\dot{u} &= v,\\ \dot{v} &= g_1 + g_2v + u^2 + uv + Q_{12}(u, v),\end{aligned} \tag{4.14}$$

where $Q_{12}(u, v)$ is a C^∞ function whose coefficients smoothly depend on ε_1 and ε_2 and is not less than the third order in (u, v), and

$$g_1 = -f_1{f_3}^4, \qquad g_2 = -f_2f_3.$$

After calculation,

$$g_1 = \frac{\beta^2\sqrt{k}}{(1+\sqrt{k})^2}\varepsilon_1 + \frac{11\beta^2(1+2\sqrt{k})^2}{4(1+\sqrt{k})^4}\varepsilon_1^2 - \frac{6\beta\sqrt{k}}{(1+\sqrt{k})^2}\varepsilon_1\varepsilon_2 + o(\varepsilon^2),$$

$$g_2 = -\frac{\beta(1+2\sqrt{k})}{2(1+\sqrt{k})^2}\varepsilon_1 + \frac{\sqrt{k}}{1+\sqrt{k}}\varepsilon_2 + \frac{\beta(1+2\sqrt{k})^3}{\sqrt{k}(1+\sqrt{k})^4}\varepsilon_1^2 - \frac{(1+2\sqrt{k})(5+7\sqrt{k})}{2(1+\sqrt{k})^3}\varepsilon_1\varepsilon_2 + \frac{\sqrt{k}}{\beta(1+\sqrt{k})}\varepsilon_2^2 + o(\varepsilon^2), \tag{4.15}$$

$$\left|\frac{\partial(g_1g_2)}{\partial(\varepsilon_1\varepsilon_2)}\right|_{\varepsilon_1=\varepsilon_2=0} = -\frac{(1+2\sqrt{k})^2}{(1+\sqrt{k})^3} \neq 0. \tag{4.16}$$

Obviously, g_1 and g_2 are independent parameters. From (4.16), parameter transformation (4.15) is a homeomorphism in a small neighborhood of the origin. Since the coefficient of time transformation $\tau = -\frac{1}{f_3}t$, from equation (4.11), we know $-\frac{1}{f_3} = -\frac{1}{e_4} = -\frac{\sqrt{-d_4}}{d_5} > 0$. Hence, there is the subcritical Bogdanov-Takens bifurcation in system (1.2). From [19], system (4.14) is the universal unfolding of the Bogdanov-Takens bifurcation of codimension 2. Then local expressions for the bifurcation curves are given as follows.

(i) Saddle-node bifurcation curve $SN = \{(\varepsilon_1,\varepsilon_2) : g_1(\varepsilon_1,\varepsilon_2) = 0, g_2(\varepsilon_1,\varepsilon_2) \neq 0\}$;

(ii) Hopf bifurcation curve $H = \{(\varepsilon_1,\varepsilon_2) : g_2(\varepsilon_1,\varepsilon_2) = \sqrt{-g_1(\varepsilon_1,\varepsilon_2)}, g_1(\varepsilon_1,\varepsilon_2) < 0\}$;

(iii) homoclinic bifurcation curve $HL = \{(\varepsilon_1,\varepsilon_2) : g_2(\varepsilon_1,\varepsilon_2) = \frac{5}{7}\sqrt{-g_1(\varepsilon_1,\varepsilon_2)}, g_1(\varepsilon_1,\varepsilon_2) < 0\}$.

□

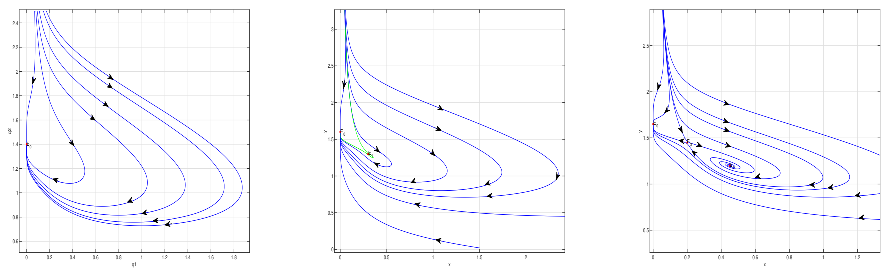
5. Numerical simulation

In the above sections, theoretical analysis about the existence of equilibria in system (1.2), their stability and bifurcations is presented. In order to validate the effectiveness of analysis, we will carry out some numerical simulation.

Example 5.1. Parameters k and β are fixed, α is varying. Now take $k = 0.09, \beta = 5.33333$.

From Fig. 1, we could find the equilibrium points and their stability in system (1.2) with different values of parameter α. System (1.2) has a boundary equilibrium $E_0 = (0,\alpha)$ which is a stable node. When $\alpha = 1.4 < 1+2\sqrt{k}$, system (1.2) has no positive equilibrium, as shown in Fig. 1(a); when $\alpha = 1+2\sqrt{k} = 1.6$, system (1.2) has one positive equilibrium $E_1 = (\sqrt{k}, 1+\sqrt{k}) = (0.3, 1.3)$ where the saddle-node bifurcation may occur, as shown in Fig. 1(b); when $\alpha = 1.65 > 1+2\sqrt{k}$, system (1.2) has two positive equilibria $E_2 = (0.45, 1.0667)$ and $E_3 = (0.2, 1.5)$, where E_2 is a stable focus and E_3 is a saddle, as shown in Fig. 1(c).

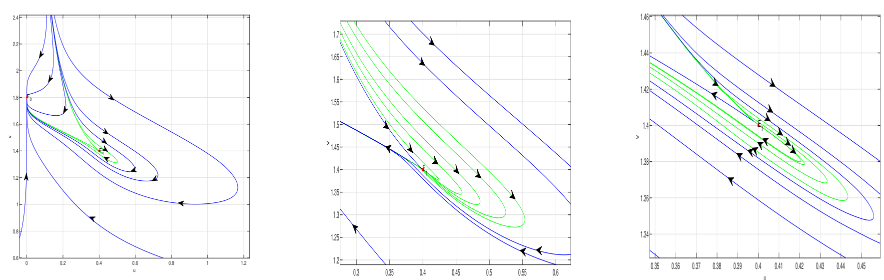
Example 5.2. In this example, we can find the saddle-node bifurcation with varying β and fixed k and α. The parameter is taken to be $k = 0.16,\ \alpha = 1.8$.



(a) (b) (c)

Figure 1. Phase portraits of (1.2). (a) $\alpha = 1.4$; (b) $\alpha = 1.6$; (c) $\alpha = 1.65$.

System (1.2) has a boundary equilibrium $E_0 = (0, \alpha)$, which is a stable node. When $\alpha = 1 + 2\sqrt{k} = 1.8$, system (1.2) has a positive equilibrium $E_1 = (\sqrt{k}, 1 + \sqrt{k}) = (0.4, 1.4)$. When $\beta = \beta_{BT} = 4.5$, E_1 is a cusp of codimension two and system (1.2) may undergo the Bogdanov-Takens bifurcation near E_1, as shown in Fig. 2(b); when $\beta = 3 < \beta_{BT}$, positive equilibrium E_1 is a saddle-node with an attracting parabolic sector, as shown in Fig. 2(a); when $\beta = 5.5 > \beta_{BT}$, positive equilibrium E_1 is a saddle-node with a repelling parabolic sector, as shown in Fig. 2(c).



(a) (b) (c)

Figure 2. Phase portraits of (1.2). (a) $\beta = 3$; (b) $\beta = 4.5$; (c) $\beta = 5.5$.

Example 5.3. Here the Hopf bifurcation will be illustrated. The parameter is taken to be $k = 0.16, \alpha = 2$.

With the fixed parameters k and α and varying β, system (1.2) has a boundary equilibrium $E_0 = (0, \alpha)$ which is a stable node. Now $\alpha = 2 > 1 + 2\sqrt{k}$, there are two positive equilibrium points $E_2 = (0.8, 1.2)$ and $E_3 = (0.2, 1.8)$, where E_3 is always a saddle. When $\beta = \beta_H = 10$, E_2 is a fine focus, as shown in Fig. 3(b); when $\beta = 10.6006 > \beta_H$, E_2 is an unstable focus as shown in Fig. 3(a); when $\beta = 9.9 < \beta_H$, E_2 is a stable focus, as shown in Fig. 3(c); when β crosses from the right side of β_H to the left, the stability of the equilibrium point E_2 is changed and the Hopf bifurcation occurs near E_2, with a periodic orbit appearing, as shown in Fig. 3(d).

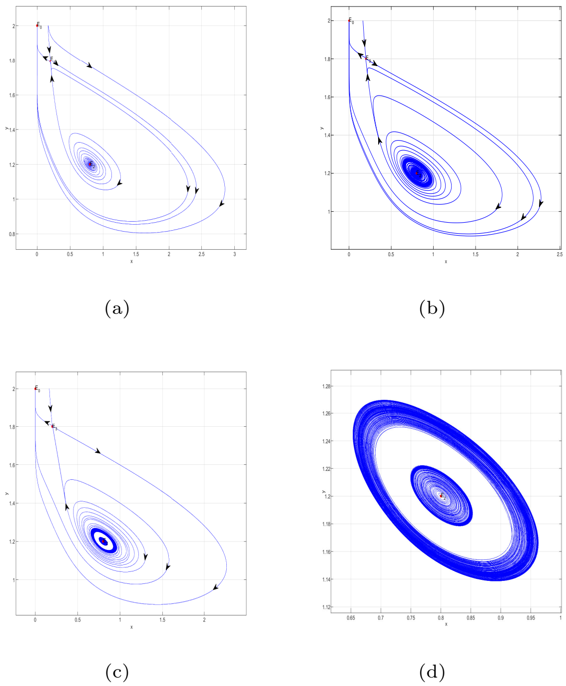


Figure 3. Phase portraits of (1.2).(a) $\beta = 10.6006$; (b) $\beta = 10$; (c) $\beta = 9.9$; (d)$\beta = 9.9$.

Example 5.4. The Bogdanov-Takens bifurcation will be illustrated in this example. According to the above discussions, when parameter $k = 0.16$, $\alpha = 1.8$, $\beta = 4.5$, the Bogdanov-Takens bifurcation will occur at E_1. Choosing α and β as bifurcation parameters, their thresholds are $\alpha = 1.8$, $\beta = 4.5$. By calculation,

$$\begin{aligned} g_1 &= -4.1327\varepsilon_1 + 46.9668\varepsilon_1^2 - 5.5102\varepsilon_1\varepsilon_2, \\ g_2 &= -17.0788\varepsilon_1^2 + 0.1968\varepsilon_1\varepsilon_2 + 2.0663\varepsilon_1 + 0.0635\varepsilon_2^2 + 0.2857\varepsilon_2, \end{aligned} \tag{5.1}$$

$$\left|\frac{\partial(g_1 g_2)}{\partial(\varepsilon_1 \varepsilon_2)}\right|_{\varepsilon_1=\varepsilon_2=0} = -\frac{(1+2\sqrt{k})^2}{(1+\sqrt{k})^3} = -1.18076 \neq 0. \tag{5.2}$$

Hence, parameter transformation (5.1) is a homeomorphism in a small neighborhood of the origin. Meanwhile, when ε_1 and ε_2 is small enough, we have

$$\begin{aligned} d_3 &= -13.2835\varepsilon_1 - 0.7143\varepsilon_2 - 2.9519\varepsilon_1\varepsilon_2 - 3.2143 < 0, \\ d_4 &= -1.4286\varepsilon_2 - 3.2143 < 0, \end{aligned}$$

and the local representation of the second order approximation of the bifurcation curve is as follows:

(i) The saddle-node bifurcation curve $SN = \{(\varepsilon_1, \varepsilon_2) : \varepsilon_1 = 0, \varepsilon_2 \neq 0\}$;

(ii) the Hopf bifurcation curve $H = \{(\varepsilon_1, \varepsilon_2) : 51.2365\varepsilon_1^2 - 4.3294\varepsilon_1\varepsilon_2 - 4.1327\varepsilon_1 + 0.0816\varepsilon_2^2 = 0, \varepsilon_1 > 0\}$;

(iii) the homoclinic bifurcation curve $HL = \{(\varepsilon_1, \varepsilon_2) : 28.2323\varepsilon_1^2 - 1.6306\varepsilon_1\varepsilon_2 - 2.1085\varepsilon_1 + 0.0816\varepsilon_2^2 = 0, \varepsilon_1 > 0\}$.

According to the local representations of the bifurcation curves, the bifurcation diagram and phase diagram of system (4.5) can be obtained from the values of ε_1 and ε_2 in the small neighborhood of the origin. As shown in Fig. 4(a), the two-dimensional plane $(\varepsilon_1, \varepsilon_2)$ is divided into four parts by bifurcation curves. With $(0, 0)$ as the center, the bifurcation diagram is described in clockwise rotation and the corresponding dynamical behaviors of system (4.5) can be found.

(i) When $(\varepsilon_1,\varepsilon_2)$=(0,0), system (4.5) has a boundary equilibrium and a unique positive equilibrium where the positive equilibrium point is the cusp of codimension 2 (as shown in Figure. 2(b)).

(ii) When $(\varepsilon_1,\varepsilon_2)$ is located in region I, system (4.5) has a boundary equilibrium and no positive equilibrium (as shown in Figure. 4(b)).

(iii) When $(\varepsilon_1,\varepsilon_2)$ is on SN curve, system (4.5) has a boundary equilibrium and one positive equilibrium where the positive equilibrium point is a saddle.

(iv) When $(\varepsilon_1,\varepsilon_2)$ crosses the SN curve and reaches region II, system (4.5) has a boundary equilibrium and two positive equilibria where one is a saddle and the other equilibrium point of the two is an unstable focus (as shown in Figure. 4(c)).

(v) When $(\varepsilon_1,\varepsilon_2)$ is on H curve, system (4.5) has a boundary equilibrium and two positive equilibria where one is a saddle and the other equilibrium point of the two is an unstable weak focus.

(vi) When $(\varepsilon_1,\varepsilon_2)$ crosses the H curve and reaches region III, system (4.5) undergoes the subcritical Hopf bifurcation, where an unstable limit cycle appears. There are two positive equilibria for system (4.5), where one is the saddle and the other is a stable focus (as shown in Figure. 4(d)).

(vii) When $(\varepsilon_1,\varepsilon_2)$ is on HL cruve, an unstable homoclinic orbit will appear. There are two positive equilibria for system (4.5), where one is saddle and the other is a stable focus (as shown in Figure. 4(e)).

(viii) When $(\varepsilon_1,\varepsilon_2)$ crosses the HL curve and reaches region IV, the homoclinic orbit is broken. There are two positive equilibria for system (4.5), where one is saddle and the other is a stable focus (as shown in Figure. 4(f)).

6. Conclusion

By employing the stability analysis and bifurcation theory, stability of equilibrium points and their bifurcation results about the nutrient-microorganism model are obtained, such as the saddle-node bifurcation, the Hopf bifurcation and the Bogdanov-Takens bifurcation, and the induced dynamics are also found, such as the appearance and disappearance of equilibrium points, periodic orbits, and homoclinic orbits. These complex behaviors are interesting parts of the dynamics in the model, and few results in this area are found to be reported. From the obtained results, we can have some more insights into the interaction between the nutrient and the microorganism. In the presence of stable positive equilibrium points, the nutrients and the microorganism can keep the stable coexistence state for a long

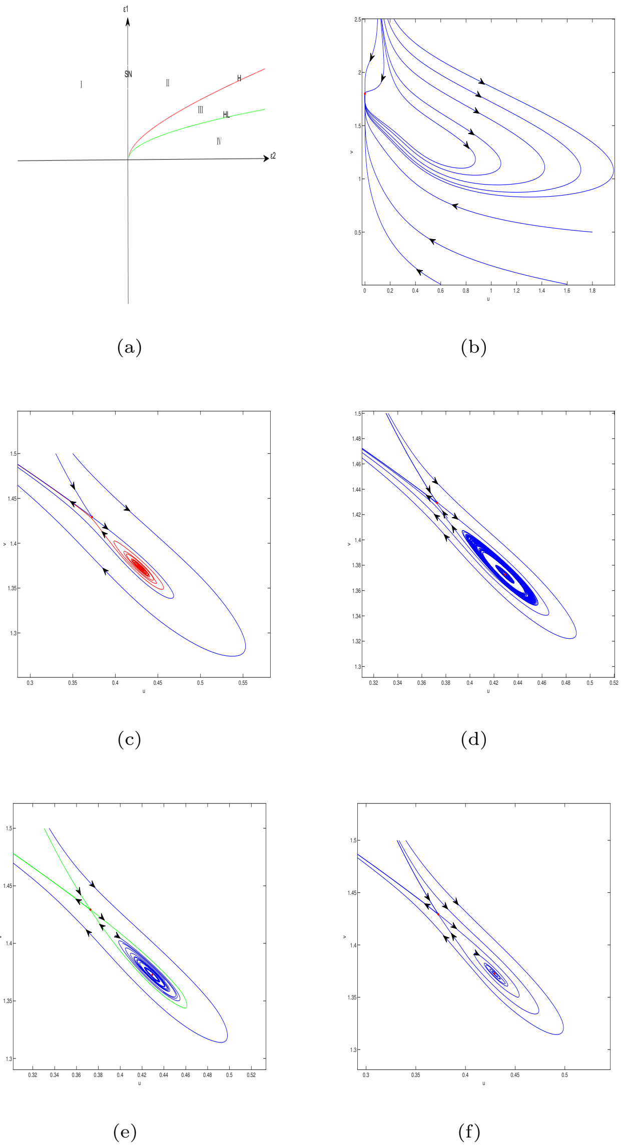


Figure 4. (a)Subcritical Bogdanov-Takens bifurcation diagram; (b)$(\varepsilon_1,\varepsilon_2)$=(-0.002, 0.2) is located in region I, system(4.5) has no positive equilibrium; (c)$(\varepsilon_1,\varepsilon_2)$=(0.002, 0.4) is located in region II, system(4.5) has an unstable focus; (d)$(\varepsilon_1,\varepsilon_2)$=(0.002, 0.29) is located in region III, system(4.5) has an unstable limit cycle; (e)$(\varepsilon_1,\varepsilon_2)$=(0.002, 0.2450828170) is located at curve HL, system(4.5) has an unstable homoclinic orbit; (f)$(\varepsilon_1,\varepsilon_2)$=(0.002, 0.00102) is located region IV, system(4.5) has a stable focus.

time. In the case of stable boundary points or unstable states, one of them will go extinct, the coexistence could not be kept, which is usually undesirable. In light of the theoretical and numerical analysis, we can get the designated interaction states by adapting the system parameters. In this model, we assume that the part of bacteria is dormant, and the activation mechanism of bacteria is not well represented. For nutrient-microorganism models, further exploration can be conducted to investigate whether different activation mechanisms can induce more complex dynamical behaviors.

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