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Long-Time Asymptotics of Complex mKdV Equation with Weighted Sobolev Initial Data*

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Abstract In this paper, we apply $\bar{\partial}$-steepest descent method to analyze the long-time asymptotics of complex mKdV equation with the initial value belonging to weighted Sobolev spaces. Firstly, the Cauchy problem of the complex mKdV equation is transformed into the corresponding Riemann-Hilbert problem on the basis of the Lax pair and the scattering data. Then the long-time asymptotics of complex mKdV equation is obtained by studying the solution of the Riemann-Hilbert problem.

Keywords Riemann-Hilbert problem, complex mKdV equation, $\bar{\partial}$-steepest descent method, long-time asymptotics

MSC(2010) 35Q51, 35Q15.

1. Introduction

The study of nonlinear partial differential equations (NLPDEs) has played an important role in the development of science and technology. Until now, NLPDEs can be used to explain some complex physical phenomena, including mathematics, fluid mechanics, plasma physics, atmospheric oceans, aerodynamics, etc [2–9]. Nowadays, the inverse scattering transformation [10–13], Hirota bilinear method [14–16], Darboux transformation [17, 18] and so on are effective methods to solve NLPDEs. Especially, the inverse scattering transformation is the first method which was found and used to obtain the exact solution of the soliton equation. In the early 20th century, the solution of Riemann-Hilbert (RH) problem was developed and promoted [19, 20]. In 1993, Deift and Zhou proposed the famous nonlinear steepest descent method to analyze the long-time asymptotic behavior of integrable evolution equations. Deift and Zhou analyzed the long-time asymptotic behavior of the solution to the initial value problem of the famous mKdV equation and Schrödinger equation [21, 22]. Cuccagna studied the asymptotic stability of N-soliton solutions of the defocusing nonlinear schrödinger equation by $\bar{\partial}$-steepest descent method [23]. Robert analyzed the derivative nonlinear schrödinger equation via $\bar{\partial}$-steepest descent method [24]. In addition, Fan, Geng and Ma studied the soliton solutions and long-time asymptotic behavior of some integrable evolution equations based on

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RH problem [25–36]; among them, Ma has already done some work on nonlocal equations [35,36].

In this paper, we study the equation derived from the Lax pair given by Yishen Li [37]. The Lax pair is

$$\begin{aligned} &\psi_x = -i\lambda\sigma_3\psi + P\psi, \\ &\psi_t = (\zeta\lambda^3 + \eta\lambda^2 + \vartheta\lambda + \iota)\sigma_3\psi + Q\psi, \end{aligned} \tag{1.1}$$

where $\psi(x,t,\lambda)$ is a 2×2 matrix, σ_3=diag$(1,-1)$, and

$$P = \begin{pmatrix} 0 & u \\ v & 0 \end{pmatrix},$$

$$\begin{aligned} Q = {} & i\zeta\lambda^2 P - i\lambda \begin{pmatrix} \frac{i\zeta}{2}uv & -\frac{i\zeta}{2}u_x - \eta u \\ \frac{i\zeta}{2}v_x - \eta v & -\frac{i\zeta}{2}uv \end{pmatrix} \\ & - \begin{pmatrix} \frac{i\zeta}{4}(uv_x - vu_x) - \frac{\eta}{2}uv & -\frac{i\zeta}{4}(-u_{xx} + 2u^2v) + \frac{\eta}{2}u_x - i\vartheta u \\ \frac{-i\zeta}{4}(-v_{xx} + 2uv^2) - \frac{\eta}{2}v_x - i\vartheta v & -\frac{i\zeta}{4}(uv_x - vu_x) + \frac{\eta}{2}uv \end{pmatrix}. \end{aligned} \tag{1.2}$$

The Lax pair (1.1) derives the following system:

$$\begin{cases} u_t = -\frac{i\zeta}{4}(u_{xxx} - 6uvu_x) - \frac{\eta}{2}(u_{xx} - 2u^2v) + i\vartheta u_x + 2\iota u, \\ v_t = -\frac{i\zeta}{4}(v_{xxx} - 6uvv_x) + \frac{\eta}{2}(v_{xx} - 2v^2u) + i\vartheta v_x - 2\iota v. \end{cases} \tag{1.3}$$

(I)Taking $\zeta = -4i$, $\eta = \vartheta = \iota = 0$, and $v = -1$, system (1.3) reduces to the KdV equation:

$$u_t + 6uu_x + u_{xxx} = 0. \tag{1.4}$$

(II)Taking $\zeta = -4i$, $\eta = \vartheta = \iota = 0$, and $v = -u$, system (1.3) reduces to the mKdV equation:

$$u_t + 6u^2u_x + u_{xxx} = 0. \tag{1.5}$$

(III)Taking $\eta = -2i$, $\zeta = \vartheta = \iota = 0$, and $v = \mp\overline{u}$, system (1.3) reduces to the nonlinear Schrödinger equation:

$$iu_t + u_{xx} \pm 2u^2\overline{u} = 0, \tag{1.6}$$

where superscript bar denotes complex conjugate.

(IV)Taking $\iota = -2$, $\zeta = \vartheta = \iota = 0$, and $q_x = uv = \left(\frac{u_x}{u}\right)_x$, system (1.3) reduces to the Burger equation

$$q_t = 2qq_x - q_{xx}. \tag{1.7}$$

In addition, taking $\zeta = -i\alpha$ $(\alpha > 0)$, $\eta = \vartheta = \iota = 0$ and $v = \overline{u}$, system (1.3) reduces to the complex mKdV equation:

$$u_t = \frac{\alpha}{4}(-u_{xxx} + 6|u|^2u_x), \tag{1.8}$$

where $u(x,t)$ is complex-valued function of variate (x,t). In [38], Chen and Liu obtained the long-time asymptotics of the mKdV equation in weighted Sobolev spaces. However, the long-time asymptotics of the complex mKdV equation have not been studied. In this paper, we apply $\bar{\partial}$-steepest descent method to analyze long-time asymptotics of the complex mKdV equation with weighted Sobolev initial data $u(t=0,x)=u_0(x)\in H^{1,1}(\mathbb{R})=\{f(x): f^{'}(x), xf(x)\in L^2(\mathbb{R})\}$. The significance of our work is that it gives a referenceable example for later generalization of the real equation to complexified equations in the study of the dynamical behaviour of the solutions.

The layout of the paper is as follows. In Section 2, we analyze eigenfunction and spectral function of equation (1.8) to construct the original Riemann-Hilbert problem. In Section 3, by deforming the jump matrix of the original Riemann-Hilbert problem and extending the region, the original Riemann-Hilbert problem is transformed into a model Riemann-Hilbert problem. Then the solution of the model Riemann-Hilbert problem can be expressed by the solution of Weber equation. Finally, we obtain the long-time asymptotics of the Cauchy problem for complex mKdV equation.

2. Spectral analysis

In this section, by analyzing the Lax pair, the matrix Jost solutions of complex mKdV equation (1.8) are constructed. Then the Cauchy problem of complex mKdV equation (1.8) turns into the corresponding Riemann-Hilbert problem. The Lax pair of complex mKdV equation is

$$\psi_x = -i\lambda\sigma_3\psi + M\psi, \tag{2.1}$$

$$\psi_t = -i\alpha\lambda^3\sigma_3\psi + N\psi, \tag{2.2}$$

where

$$M = \begin{pmatrix} 0 & u \\ v & 0 \end{pmatrix}, \tag{2.3}$$

$$N = \alpha\lambda^2 M - i\lambda \begin{pmatrix} \frac{\alpha}{2}uv & -\frac{\alpha}{2}u_x \\ \frac{\alpha}{2}v_x & -\frac{\alpha}{2}uv \end{pmatrix} - \begin{pmatrix} \frac{\alpha}{4}(uv_x - vu_x) & -\frac{\alpha}{4}(-u_{xx}+2u^2v) \\ -\frac{\alpha}{4}(-v_{xx}+2uv^2) & -\frac{\alpha}{4}(uv_x - vu_x) \end{pmatrix}. \tag{2.4}$$

2.1. Asymptotics

Lax pair (2.1)-(2.2) has a Jost solution of the following asymptotic form

$$\psi(x,t,\lambda) = e^{-i(\lambda\sigma_3 x + \alpha\lambda^3\sigma_3 t)}, \quad |x|\to\infty. \tag{2.5}$$

Therefore, we make the transformation

$$\mu(x,t,\lambda) = \psi(x,t,\lambda)e^{i(\lambda\sigma_3 x + \alpha\lambda^3\sigma_3 t)}, \tag{2.6}$$

where $\mu(x,t,\lambda)$ satisfies the following Lax pair

$$\mu_x + i\lambda[\sigma_3,\mu] = M\mu, \tag{2.7}$$

$$\mu_t + i\alpha\lambda^3[\sigma_3,\mu] = N\mu, \tag{2.8}$$

which can be written in the full derivative form

$$d\left(e^{i(\lambda x+\alpha\lambda^3 t)\hat{\sigma}_3}\mu\right)=e^{i(\lambda x+\alpha\lambda^3 t)\hat{\sigma}_3}[(Mdx+Ndt)\mu]. \tag{2.9}$$

Considering the asymptotic expansion

$$\mu=\mu_0+\frac{\mu_1}{\lambda}+\frac{\mu_2}{\lambda^2}+\frac{\mu_3}{\lambda^3}+o\left(\frac{1}{\lambda^4}\right),\quad \lambda\to\infty, \tag{2.10}$$

where μ_0, μ_1, μ_2 and μ_3 are independent of λ. Substituting (2.10) into (2.7) and comparing the coefficients of λ, we obtain that μ_0 is a diagonal matrix and

$$\mu_{0,x}+i\sigma_3\mu_1-i\mu_1\sigma_3=M\mu_0, \tag{2.11}$$

$$i\sigma_3\mu_0-i\mu_0\sigma_3=0. \tag{2.12}$$

In the same way, substituting (2.10) into (2.8) and comparing the coefficients of λ, we get

$$\mu_{0,t}+i\alpha\sigma_3\mu_3-i\alpha\mu_3\sigma_3-\alpha M\mu_2+\frac{i\alpha}{2}u\bar{u}\sigma_3\mu_1+\frac{i\alpha}{2}M_x\sigma_3\mu_1+\frac{\alpha}{4}(u\bar{u}_x-\bar{u}u_x)\sigma_3\mu_0 \tag{2.13}$$

$$+\frac{\alpha}{4}M_{xx}\mu_0-\frac{\alpha}{2}M^3\mu_0=0,$$

$$i\alpha\sigma_3\mu_2-i\alpha\mu_2\sigma_3=\alpha M\mu_1-\frac{i\alpha}{2}u\bar{u}\sigma_3\mu_0-\frac{i\alpha}{2}M_x\sigma_3\mu_0, \tag{2.14}$$

$$i\alpha\sigma_3\mu_1-i\alpha\mu_1\sigma_3=\alpha M\mu_0, \tag{2.15}$$

$$i\alpha\sigma_3\mu_0-i\alpha\mu_0\sigma_3=0. \tag{2.16}$$

Through a series of calculations, we obtain

$$\mu(x,t,\lambda)\to I,\qquad \lambda\to\infty, \tag{2.17}$$

$$u(x,t)=2i\lim_{\lambda\to\infty}(\lambda\mu)_{12}=2i(\mu_1)_{12}. \tag{2.18}$$

2.2. Analyticity and symmetry

To analyze the eigenfunction $\mu(x,t,\lambda)$, we choose two special integral paths

$$(-\infty,t)\to(x,t)\ \ and\ \ (\infty,t)\to(x,t) \tag{2.19}$$

and acquire two Volterra type integral equations

$$\mu_1(x,t,\lambda)=I+\int_{-\infty}^{x}e^{i(\lambda y-\lambda x)\hat{\sigma}_3}M(y,t,\lambda)\mu_1(y,t,\lambda)dy, \tag{2.20}$$

$$\mu_2(x,t,\lambda)=I-\int_{x}^{\infty}e^{i(\lambda y-\lambda x)\hat{\sigma}_3}M(y,t,\lambda)\mu_2(y,t,\lambda)dy. \tag{2.21}$$

From the transformation (2.6), it can be known that $\mu_1(x,t,\lambda)e^{-i(\lambda\sigma_3 x+\alpha\lambda^3\sigma_3 t)}$ and $\mu_2(x,t,\lambda)e^{-i(\lambda\sigma_3 x+\alpha\lambda^3\sigma_3 t)}$ are the two linear correlation matrix solutions of Lax pair (2.7) and (2.8), so we have

$$\mu_1(x,t,\lambda)=\mu_2(x,t,\lambda)e^{-i\theta(\lambda)\hat{\sigma}_3}S(\lambda), \tag{2.22}$$

where

$$\theta(\lambda)=\lambda\frac{x}{t}+\alpha\lambda^3,\quad S(\lambda)=\begin{pmatrix} s_{11}(\lambda) & s_{12}(\lambda)\\ s_{21}(\lambda) & s_{22}(\lambda)\end{pmatrix},$$

$S(\lambda)$ is irrelevant to x and t, and it is called the spectral matrix function. Nextly, we study the analyticity of $\mu_1(x,t,\lambda)$, $\mu_2(x,t,\lambda)$ and $S(\lambda)$. For the integral equation (2.20), a direct calculation shows that

$$e^{i\lambda(y-x)\hat{\sigma}_3}M(\lambda;\xi,t)=\begin{pmatrix} 0 & ue^{2i\lambda(y-x)}\\ \bar{u}e^{2i\lambda(y-x)} & 0\end{pmatrix},\tag{2.23}$$

and

$$2i\lambda(y-x)=2i(Re\lambda+iIm\lambda)(y-x)=2iRe\lambda(y-x)-2Im\lambda(y-x),$$

so that the first column of $\mu_1(x,t,\lambda)$ is analytical in the upper half plane $\mathbb{C}_+$, the second column of $\mu_1(x,t,\lambda)$ is analytical in the lower half plane $\mathbb{C}_-$, and $\mu_1(x,t,\lambda)$ can be written as

$$\mu_1=\begin{pmatrix} \mu_1^{(11)} & \mu_1^{(12)}\\ \mu_1^{(21)} & \mu_1^{(22)}\end{pmatrix}=(\mu_1^+,\mu_1^-).\tag{2.24}$$

Similarly, the first column of $\mu_2(x,t,\lambda)$ is analytical in the lower half plane $\mathbb{C}_-$, the second column of $\mu_2(x,t,\lambda)$ is analytical in the upper half plane $\mathbb{C}_+$, and $\mu_2(x,t,\lambda)$ can be written as

$$\mu_2=\begin{pmatrix} \mu_2^{(11)} & \mu_2^{(12)}\\ \mu_2^{(21)} & \mu_2^{(22)}\end{pmatrix}=(\mu_2^-,\mu_2^+).\tag{2.25}$$

Theorem 2.1. *The eigenfunctions $\mu_1(x,t,\lambda)$, $\mu_2(x,t,\lambda)$ and spectral matrix $S(\lambda)$ have the following symmetry properties*

$$\mu_j(x,t,\lambda)=\sigma_1\overline{\mu_j(x,t,\overline{\lambda})}\sigma_1,\ \ (j=1,2),\tag{2.26}$$

$$S(\lambda)=\sigma_1\overline{S(\overline{\lambda})}\sigma_1,\tag{2.27}$$

where $\sigma_1=\begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}$.

Proof. Through Lax pair (2.7), we have

$$\mu_{j,x}(x,t,\lambda)+i\lambda[\sigma_3,\mu_j(x,t,\lambda)]=M\mu_j(x,t,\lambda).\tag{2.28}$$

Substituting $\overline{\lambda}$ for λ and taking the conjugate of the left and right sides of equation (2.28) gives

$$\overline{\mu_{j,x}(x,t,\overline{\lambda})}-i\lambda[\sigma_3,\overline{\mu_j(x,t,\overline{\lambda})}]=\overline{M}\overline{\mu_j(x,t,\overline{\lambda})}.\tag{2.29}$$

Multiplying the left and right sides of equation (2.29) by σ_1 leads to

$$[\sigma_1\overline{\mu_j(x,t,\overline{\lambda})}\sigma_1]_x-i\lambda\sigma_1\sigma_3\overline{\mu_j(x,t,\overline{\lambda})}\sigma_1+i\lambda\sigma_1\overline{\mu_j(x,t,\overline{\lambda})}\sigma_3\sigma_1=\sigma_1\overline{M}\overline{\mu_j(x,t,\overline{\lambda})}\sigma_1.\tag{2.30}$$

By calculation, we get

$$[\sigma_1\overline{\mu_j(x,t,\overline{\lambda})}\sigma_1]_x+i\lambda[\sigma_3,\sigma_1\overline{\mu_j(x,t,\overline{\lambda})}\sigma_1]=M\sigma_1\overline{\mu_j(x,t,\overline{\lambda})}\sigma_1.\tag{2.31}$$

Comparing equation (2.7) with equation (2.31), we know that $\mu_j(x,t,\lambda)$ and $\sigma_1\overline{\mu_j(x,t,\overline{\lambda})}\sigma_1$ satisfy the same differential equation and have the same asymptotic property

$$\mu_j(x,t,\lambda), \quad \sigma_1\overline{\mu_j(x,t,\overline{\lambda})}\sigma_1 \to I, \ x \to \infty. \tag{2.32}$$

Therefore,

$$\mu_j(x,t,\lambda) = \sigma_1\overline{\mu_j(x,t,\overline{\lambda})}\sigma_1, \tag{2.33}$$

that is

$$\begin{pmatrix} \mu_{11}(\lambda) & \mu_{12}(\lambda) \\ \mu_{21}(\lambda) & \mu_{22}(\lambda) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \overline{\mu_{11}(\overline{\lambda})} & \overline{\mu_{12}(\overline{\lambda})} \\ \overline{\mu_{21}(\overline{\lambda})} & \overline{\mu_{22}(\overline{\lambda})} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \overline{\mu_{22}(\overline{\lambda})} & \overline{\mu_{21}(\overline{\lambda})} \\ \overline{\mu_{12}(\overline{\lambda})} & \overline{\mu_{11}(\overline{\lambda})} \end{pmatrix}. \tag{2.34}$$

By comparing the two sides of the above equation, we get $\mu_{11}(\lambda) = \overline{\mu_{22}(\overline{\lambda})}$, and $\mu_{12}(\lambda) = \overline{\mu_{21}(\overline{\lambda})}$. From equation (2.33), we get the symmetry property of spectral matrix $S(\lambda)$:

$$\sigma_1\overline{S(\overline{\lambda})}\sigma_1 = S(\lambda), \tag{2.35}$$

that is $\overline{s_{11}(\overline{\lambda})} = s_{22}(\lambda)$ and $\overline{s_{12}(\overline{\lambda})} = s_{21}(\lambda)$. □

Theorem 2.2. *The eigenfunctions $\mu_1(x,t,\lambda)$, $\mu_2(x,t,\lambda)$ and spectral matrix function $S(\lambda)$ also have the following symmetry properties*

$$\sigma_3\mu_j^H(x,t,\overline{\lambda})\sigma_3 = \mu_j^{-1}(x,t,\lambda), \tag{2.36}$$

$$\sigma_3 S^H(\overline{\lambda})\sigma_3 = S^{-1}(\lambda), \tag{2.37}$$

where superscript H denotes conjugate transpose.

Proof. Through direct calculation, we get

$$\begin{aligned}
&\sigma_3\mu_j^H(x,t,\overline{\lambda})\sigma_3 \\
&= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \mu_{j,11}(x,t,\overline{\lambda}) & \mu_{j,12}(x,t,\overline{\lambda}) \\ \mu_{j,21}(x,t,\overline{\lambda}) & \mu_{j,22}(x,t,\overline{\lambda}) \end{pmatrix}^H \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \overline{\mu_{j,11}}(x,t,\overline{\lambda}) & \overline{\mu_{j,21}}(x,t,\overline{\lambda}) \\ \overline{\mu_{j,12}}(x,t,\overline{\lambda}) & \overline{\mu_{j,22}}(x,t,\overline{\lambda}) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \mu_{j,22}(\lambda) & \mu_{j,12}(\lambda) \\ \mu_{j,21}(\lambda) & \mu_{j,11}(\lambda) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\
&= \begin{pmatrix} \mu_{j,22}(\lambda) & -\mu_{j,12}(\lambda) \\ -\mu_{j,21}(\lambda) & \mu_{j,11}(\lambda) \end{pmatrix} \\
&= \mu_j^{-1}(x,t,\lambda).
\end{aligned} \tag{2.38}$$

Similarly, we can get

$$\sigma_3 S^H(\overline{\lambda})\sigma_3 = S^{-1}(\lambda). \tag{2.39}$$

□

From equation (2.22), we obtain that

$$s_{11}(\lambda) = 1 + \int_{-\infty}^{+\infty} u\mu_{1,21} d\xi, s_{21}(\lambda) = \int_{-\infty}^{+\infty} \overline{u} e^{-2i\lambda\xi} \mu_{1,11} d\xi. \tag{2.40}$$

3. The construction of an RH problem

Define the reflection coefficient as $r(\lambda) = \frac{s_{21}(\lambda)}{s_{11}(\lambda)}$, and a piecewise analytical function $m(x,t,\lambda)$ that

$$m(x,t,\lambda) = \begin{cases} \left(\frac{\mu_1^+}{s_{11}}, \mu_2^+\right), & Im\lambda > 0, \\ \left(\mu_2^-, \frac{\mu_1^-}{s_{22}}\right), & Im\lambda < 0. \end{cases} \tag{3.1}$$

By applying the analyticities and symmetries of the eigenfunctions and the spectral matrix, the RH problem corresponding to the initial value problem of the complex mKdV equation can be obtained.
RH problem 1 :

$$\begin{aligned} &\bullet \quad m_\pm(x,t,\lambda) \ \ is \ \ analytical \ \ in \ \ \mathbb{C}_\pm, \\ &\bullet \quad m_+(x,t,\lambda) = m_-(x,t,\lambda) v(x,t,\lambda), \\ &\bullet \quad m_\pm(x,t,\lambda) \to I, as \ \ \lambda \to \infty, \end{aligned} \tag{3.2}$$

where the jump matrix is

$$v(x,t,\lambda) = \begin{pmatrix} 1 - |r(\lambda)|^2 & -e^{-2it\theta}\overline{r(\overline{\lambda})} \\ e^{2it\theta} r(\lambda) & 1 \end{pmatrix}. \tag{3.3}$$

This is an RH problem defined on the real axis, as shown in figure 1 and the solution $u(x,t)$ to the initial value problem of the complex mKdV equation can be expressed as the RH problem above

$$u(x,t) = 2i \lim_{\lambda\to\infty} (\lambda\mu(x,t,\lambda))_{12} = 2i \lim_{\lambda\to\infty} (\lambda m(x,t,\lambda))_{12} = 2i \lim_{\lambda\to\infty} (m_1(x,t,\lambda))_{12}. \tag{3.4}$$

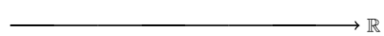

Figure 1. The oriented contour of $m(\lambda)$

4. Triangular decomposition of jump matrix

We write the oscillating term of the jump matrix as

$$e^{it\theta(\lambda)} = e^{t\varphi(\lambda)}, \varphi(\lambda) = i\theta(\lambda),$$

to obtain two steady state phase points $\pm\lambda_0 = \pm\sqrt{-\frac{x}{3\alpha t}}$. Since

$$\theta(\lambda) = \alpha\left((\lambda+\lambda_0)^3 - 3\lambda_0(\lambda+\lambda_0)^2 + 2\lambda_0^3\right), \tag{4.1}$$

and

$$\theta(\lambda)=\alpha\left((\lambda-\lambda_0)^3+3\lambda_0(\lambda-\lambda_0)^2-2\lambda_0^3\right), \tag{4.2}$$

the complex plane can be divided into two types of regions on the basis of the exponential decay of $e^{it\theta}$, see figure 2. The jump matrix $v(x,t,\lambda)$ has the lower/upper triangular decomposition

$$v(x,t,\lambda)=\begin{pmatrix}1 & -\overline{r(\overline{\lambda})}e^{-2it\theta(\lambda)}\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ r(\lambda)e^{2it\theta(\lambda)} & 1\end{pmatrix},\quad (\lambda\to\infty) \tag{4.3}$$

and the upper/diagonal/lower decomposition

$$v(x,t,\lambda) = \begin{pmatrix}1 & 0\\ e^{2it\theta(\lambda)}\frac{r(\lambda)}{1-r(\lambda)\overline{r(\overline{\lambda})}} & 1\end{pmatrix}\begin{pmatrix}1-r(\lambda)\overline{r(\overline{\lambda})} & 0\\ 0 & \frac{1}{1-r(z)\overline{r(\overline{\lambda})}}\end{pmatrix}\begin{pmatrix}1 & -e^{-2it\theta(\lambda)}\frac{\overline{r(\overline{\lambda})}}{1-r(\lambda)\overline{r(\overline{\lambda})}}\\ 0 & 1\end{pmatrix}, \tag{4.4}$$

$\lambda\in(-\lambda_0,\lambda_0)$.

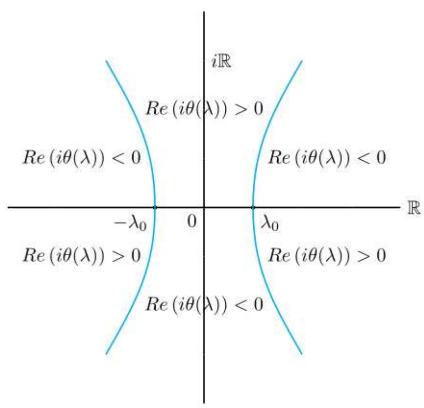

Figure 2. Symbol distribution map of $Re(i\theta(\lambda))$.

In order to remove the intermediate diagonal matrix in the decomposition for $\lambda\in(-\lambda_0,\lambda_0)$, we introduce a scalar RH problem:

$$\begin{aligned}&\bullet\, \delta(\lambda)\ \ is\ \ analytical\ \ in\ \ \mathbb{C}\backslash\mathbb{R},\\ &\bullet\, \delta_+(\lambda)=\delta_-(\lambda)\left(1-r(\lambda)\overline{r(\overline{\lambda})}\right),\quad \lambda\in(-\lambda_0,\lambda_0),\\ &\bullet\, \delta_+(\lambda)=\delta_-(\lambda),\quad \lambda\to\infty,\\ &\bullet\, \delta(\lambda)\to 1,\quad \lambda\to\infty.\end{aligned} \tag{4.5}$$

According to Plemelj formula, we get the unique solution of the above RH problem:

$$\delta(\lambda)=\exp\left(\frac{1}{2\pi i}\int_{-\lambda_0}^{\lambda_0}\frac{log(1-|r(\xi)|^2)}{\xi-\lambda}d\xi\right)=\exp\left(i\int_{-\lambda_0}^{\lambda_0}\frac{\nu(\xi)}{\xi-\lambda}d\xi\right), \tag{4.6}$$

where

$$\nu(\xi) = -\tfrac{1}{2\pi}\log\left(1-|r(\xi)|^2\right).$$

Assuming that $r(\lambda) \in L^\infty \cap L^2$ and $\|r(\lambda)\|_{L^\infty} \le \rho < 1$, $\delta(\lambda)$ has the following properties:
(1) $\delta(\lambda)$ is analytic in $C\backslash(-\lambda_0, \lambda_0)$;
(2)$\delta(\lambda)\overline{\delta(\overline{\lambda})} = 1$, $\|\delta_\pm - 1\|_{L^2} \le \frac{c\|r\|_{L^2}}{1-\rho}$;
(3)$(1-\rho^2)^{\frac{1}{2}} \le |\delta(\lambda)| \le (1-\rho^2)^{-\frac{1}{2}}$.
Further, we rewrite $\delta(\lambda)$ as

$$\begin{aligned}
&\delta(\lambda)\\
=&\exp\left(i\int_{-\lambda_0}^{0}\frac{\nu(\xi)}{\xi-\lambda}d\xi + i\int_0^{\lambda_0}\frac{\nu(\xi)}{\xi-\lambda}d\xi\right)\\
=&\exp\left(i\int_{-\lambda_0}^{0}\frac{\nu(\xi)-\chi_2(\xi)\nu(-\lambda_0)(\xi+\lambda_0-1)}{\xi-\lambda}d\xi + i\nu(-\lambda_0)\int_{-\lambda_0}^{-\lambda_0+1}\frac{\xi+\lambda_0-1}{\xi-\lambda}d\xi\right)\\
&\cdot\exp\left(i\int_0^{\lambda_0}\frac{\nu(\xi)-\chi_1(\xi)\nu(\lambda_0)(\xi-\lambda_0+1)}{\xi-\lambda}d\xi + i\nu(\lambda_0)\int_{\lambda_0-1}^{\lambda_0}\frac{\xi-\lambda_0+1}{\xi-\lambda}d\xi\right)\\
=&\exp(i\beta_2(\lambda,-\lambda_0) + i\nu(-\lambda_0) + i\nu(-\lambda_0)[(\lambda+\lambda_0-1)\log(\lambda+\lambda_0-1) - (\lambda+\lambda_0)\\
&\log(\lambda+\lambda_0)]) \cdot \exp(i\nu(-\lambda_0)\log(\lambda+\lambda_0) + +i\nu(\lambda_0)\log(\lambda-\lambda_0)) \cdot \exp\\
&(i\beta_1(\lambda,\lambda_0) + i\nu(\lambda_0) + i\nu(\lambda_0)[(\lambda-\lambda_0)\log(\lambda-\lambda_0) - (\lambda-\lambda_0+1)\log(\lambda-\lambda_0)+1])\\
=&\exp\left(i\nu(-\lambda_0) + i\beta_2(\lambda,-\lambda_0)\right)(\lambda+\lambda_0)^{i\nu(-\lambda_0)}\exp\left(i\nu(\lambda_0)+i\beta_1(\lambda,\lambda_0)\right)(\lambda-\lambda_0)^{i\nu(\lambda_0)}\\
&\cdot\exp\left(i\nu(-\lambda_0)[(\lambda+\lambda_0-1)\log(\lambda+\lambda_0-1) - (\lambda+\lambda_0)\log(\lambda+\lambda_0)]\right)\\
&\cdot\exp\left(i\nu(\lambda_0)[(\lambda-\lambda_0)\log(\lambda-\lambda_0) - (\lambda-\lambda_0+1)\log(\lambda-\lambda_0+1)]\right),
\end{aligned} \tag{4.7}$$

where $\chi_1(\xi)$ is an eigenfunction defined on the $(-\lambda_0, -\lambda_0+1)$, $\chi_2(\xi)$ is an eigenfunction defined on the (λ_0-1, λ_0), and

$$\begin{aligned}
\beta_1(\lambda,\lambda_0) &= \textstyle\int_0^{\lambda_0}\frac{\nu(\xi)-\chi_1(\xi)\nu(\lambda_0)(\xi-\lambda_0+1)}{\xi-\lambda},\\
\beta_2(\lambda,-\lambda_0) &= \textstyle\int_{-\lambda_0}^{0}\frac{\nu(\xi)-\chi_2(\xi)\nu(-\lambda_0)(\xi+\lambda_0+1)}{\xi-\lambda}.
\end{aligned}$$

Letting $\Sigma^{(1)} = \mathbb{R}$, we make a transformation

$$m^{(1)}(\lambda) = m(\lambda)\delta(\lambda)^{-\sigma_3}, \tag{4.8}$$

and $m^{(1)}(\lambda)$ satisfies the following RH problem.
RH problem 2 :

$$\begin{aligned}
&\bullet\, m^{(1)}(x,t,\lambda)\ \ \textit{is analytical in}\ \ \mathbb{C}\backslash\Sigma^{(1)},\\
&\bullet\, m_+^{(1)}(x,t,\lambda) = m_-^{(1)}(x,t,\lambda)v^{(1)}(x,t,\lambda), \quad \lambda\in\Sigma^{(1)},\\
&\bullet\, m^{(1)}(x,t,\lambda)\to I, \quad \textit{as}\ \lambda\to\infty;
\end{aligned} \tag{4.9}$$

for $\lambda \in (-\lambda_0, \lambda_0)$,

$$v^{(1)}(x,t,\lambda) = \begin{pmatrix} 1 & 0 \\ \frac{r(\lambda)}{1-r(\lambda)\overline{r(\bar{\lambda})}}\delta_-^{-2}e^{2it\theta(\lambda)} & 1 \end{pmatrix} \begin{pmatrix} 1 & -\frac{\overline{r(\bar{\lambda})}}{1-r(\lambda)\overline{r(\bar{\lambda})}}\delta_+^2 e^{-2it\theta(\lambda)} \\ 0 & 1 \end{pmatrix}; \tag{4.10}$$

for $\lambda \to \infty$,

$$v^{(1)}(x,t,\lambda) = \begin{pmatrix} 1 & -\overline{r(\bar{\lambda})}\delta_-^2 e^{-2it\theta(\lambda)} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ r(\lambda)\delta_-^{-2}e^{2it\theta(\lambda)} & 1 \end{pmatrix}. \tag{4.11}$$

Due to $\delta(\lambda) \to I$, $\lambda \to \infty$, the relation between the solution of the complex mKdV equation and the solution of the corresponding RH problem is

$$u(x,t) = 2i \lim_{\lambda\to\infty} (\lambda m^{(1)}(\lambda)\delta^{\sigma_3})_{12} = 2i \lim_{\lambda\to\infty} (\lambda m^{(1)}(\lambda))_{12}. \tag{4.12}$$

5. A hybrid $\bar{\partial}$-problem

We extended the scattering data by the $\bar{\partial}$-steepest descent method:
(1)$r(\lambda)$ extends to the regions Ω_{11} and Ω_{21};
(2)$\overline{r(\bar{\lambda})}$ extends to the regions Ω_{16} and Ω_{26};
(3)$\frac{\overline{r(\bar{\lambda})}}{1-r(\lambda)\overline{r(\bar{\lambda})}}$ extends to the regions Ω_{13} and Ω_{23};
(4)$\frac{r(\bar{\lambda})}{1-r(\lambda)\overline{r(\bar{\lambda})}}$ extends to the regions Ω_{14} and Ω_{24}.

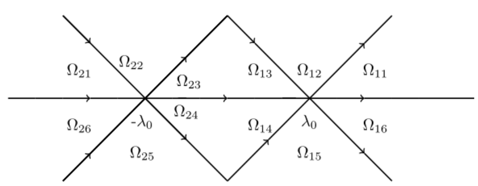

Figure 3. Regions Ω_{ij}.

Proposition 5.1. *There exist functions $R_{nj}(\lambda) \to C$, $n = 1,2$, $j = 1,3,4,6$ satisfy the following boundary conditions*

$$R_{11}(\lambda) = \begin{cases} r(\lambda), & \lambda \in (\lambda_0, \infty), \\ f_{11} = \hat{r}_{10}(\lambda-\lambda_0)^{-2i\nu(\lambda_0)}\delta^2, & \lambda \in \Sigma_{11}, \end{cases} \tag{5.1}$$

$$R_{13}(\lambda) = \begin{cases} -\frac{\overline{r(\bar{\lambda})}}{1-|r(\lambda)|^2}, & \lambda \in (0, \lambda_0), \\ f_{13} = -\frac{\hat{\bar{r}}_{10}}{1-|\hat{r}_{10}|^2}(\lambda-\lambda_0)^{2i\nu(\lambda_0)}\delta^{-2}, & \lambda \in \Sigma_{12}, \end{cases} \tag{5.2}$$

$$R_{14}(\lambda) = \begin{cases} \frac{r(\lambda)}{1-|r(\lambda)|^2}, & \lambda \in (0, \lambda_0), \\ f_{14} = \frac{\hat{r}_{10}}{1-|\hat{r}_{10}|^2}(\lambda-\lambda_0)^{-2i\nu(\lambda_0)}\delta^2, & \lambda \in \Sigma_{13}, \end{cases} \tag{5.3}$$

$$R_{16}(\lambda)=\begin{cases}-\overline{r(\bar{\lambda})}, & \lambda\in(\lambda_0,\infty),\\ f_{16}=-\hat{\bar{r}}_{10}(\lambda-\lambda_0)^{2i\nu(\lambda_0)}\delta^{-2}, & \lambda\in\Sigma_{14},\end{cases} \tag{5.4}$$

$$R_{21}(\lambda)=\begin{cases}r(\lambda), & \lambda\in(-\infty,-\lambda_0),\\ f_{21}=\hat{r}_{20}(\lambda+\lambda_0)^{-2i\nu(-\lambda_0)}\delta^{2}, & \lambda\in\Sigma_{21},\end{cases} \tag{5.5}$$

$$R_{23}(\lambda)=\begin{cases}-\frac{\overline{r(\bar{\lambda})}}{1-|r(\lambda)|^2}, & \lambda\in(-\lambda_0,0),\\ f_{23}=-\frac{\hat{\bar{r}}_{20}}{1-|\hat{r}_{20}|^2}(\lambda+\lambda_0)^{2i\nu(-\lambda_0)}\delta^{-2}, & \lambda\in\Sigma_{22},\end{cases} \tag{5.6}$$

$$R_{24}(\lambda)=\begin{cases}\frac{r(\lambda)}{1-|r(\lambda)|^2}, & \lambda\in(-\lambda_0,0),\\ f_{24}=\frac{\hat{r}_{20}}{1-|\hat{r}_{20}|^2}(\lambda+\lambda_0)^{-2i\nu(-\lambda_0)}\delta^{2}, & \lambda\in\Sigma_{23},\end{cases} \tag{5.7}$$

$$R_{26}(\lambda)=\begin{cases}-\overline{r(\bar{\lambda})}, & \lambda\in(-\infty,\lambda_0),\\ f_{26}=-\hat{\bar{r}}_{20}(\lambda+\lambda_0)^{2i\nu(-\lambda_0)}\delta^{-2}, & \lambda\in\Sigma_{24},\end{cases} \tag{5.8}$$

where $\hat{r}_{10}=r(\lambda_0)e^{-2i\nu(\lambda_0)-2\beta_1(\lambda_0,\lambda_0)}$, $\hat{r}_{20}=r(-\lambda_0)e^{-2i\nu(-\lambda_0)-2\beta_2(-\lambda_0,-\lambda_0)}$ *and* $R_{nj}(\lambda)$ *have the following estimations*

$$|\bar{\partial}R_{nj}(\lambda)|\le c_1|\lambda-\lambda_0|^{-\frac{1}{2}}+c_2|r^{'}(Re(\lambda))|, \tag{5.9}$$

$$|R_{nj}(\lambda)|\le c_1\sin^2(arg\lambda)+c_1(Re\lambda)^{-\frac{1}{2}}. \tag{5.10}$$

Define the contour

$$\Sigma^{(2)}=\Sigma_{11}\cup\Sigma_{12}\cup\Sigma_{13}\cup\Sigma_{14}\cup\Sigma_{21}\cup\Sigma_{22}\cup\Sigma_{23}\cup\Sigma_{24},$$

$$R^{(2)}(\lambda)=\begin{cases}\begin{pmatrix}1 & 0\\ R_{11}e^{2it\theta(\lambda)}\delta^{-2} & 1\end{pmatrix}^{-1}, & \lambda\in\Omega_{11},\\ \begin{pmatrix}1 & R_{13}e^{-2it\theta(\lambda)}\delta^{2}\\ 0 & 1\end{pmatrix}^{-1}, & \lambda\in\Omega_{13},\\ \begin{pmatrix}1 & 0\\ R_{14}e^{2it\theta(\lambda)}\delta^{-2} & 1\end{pmatrix}, & \lambda\in\Omega_{14},\\ \begin{pmatrix}1 & R_{16}e^{-2it\theta(\lambda)}\delta^{2}\\ 0 & 1\end{pmatrix}, & \lambda\in\Omega_{16},\\ \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, & \lambda\in\Omega_{12}\cup\Omega_{15};\end{cases} \tag{5.11}$$

and

$$R^{(2)}(\lambda)=\begin{cases}\begin{pmatrix}1 & 0\\ R_{21}e^{2it\theta(\lambda)}\delta^{-2} & 1\end{pmatrix}^{-1}, & \lambda\in\Omega_{21},\\ \begin{pmatrix}1 & R_{23}e^{-2it\theta(\lambda)}\delta^{2}\\ 0 & 1\end{pmatrix}^{-1}, & \lambda\in\Omega_{23},\\ \begin{pmatrix}1 & 0\\ R_{24}e^{2it\theta(\lambda)}\delta^{-2} & 1\end{pmatrix}, & \lambda\in\Omega_{24},\\ \begin{pmatrix}1 & R_{26}e^{-2it\theta(\lambda)}\delta^{2}\\ 0 & 1\end{pmatrix}, & \lambda\in\Omega_{26},\\ \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, & \lambda\in\Omega_{22}\cup\Omega_{25}.\end{cases}\tag{5.12}$$

Due to the boundedness of $\delta(\lambda)$ and $R_{nj}(\lambda)$, and the exponential decay of $e^{\pm 2it\theta}$, we have

$$R^{(2)}(\lambda)\sim I,\quad t\to\infty.$$

We make a transformation

$$m^{(2)}(\lambda)=m^{(1)}(\lambda)R^{(2)}(\lambda),\tag{5.13}$$

then the RH problem on $\Sigma^{(1)}$ becomes the RH problem on $\Sigma^{(2)}$.

RH problem 3 :

$$\begin{aligned}&\bullet\, m^{(2)}(x,t,\lambda)\ \textit{is continuousin}\ \mathbb{C}\backslash\Sigma^{(2)},\\ &\bullet\, m_{+}^{(2)}(x,t,\lambda)=m_{-}^{(2)}(x,t,\lambda)v^{(2)}(x,t,\lambda),\quad \lambda\in\Sigma^{(2)},\\ &\bullet\, m_{+}^{(2)}(x,t,\lambda)\longrightarrow I,\quad \textit{as}\quad \lambda\to\infty,\end{aligned}\tag{5.14}$$

the relation between the solution of the complex mKdV equation and the solution of corresponding RH problem is

$$u(x,t)=2i\lim_{\lambda\to\infty}(\lambda m^{(2)}(x,t,\lambda))_{12}.\tag{5.15}$$

The exact expression for the jump matrix $v^{(2)}(\lambda)$ is

$$v^{(2)}(\lambda)=(R_{-}^{2}(\lambda))^{-1}v^{(1)}(\lambda)R_{+}^{2}(\lambda),\tag{5.16}$$

that is

$$v^{(2)}(\lambda)=\begin{cases}\begin{pmatrix}1 & 0\\ R_{n1}e^{2it\theta(\lambda)}\delta^{-2} & 1\end{pmatrix}, & \lambda\in\Sigma_{n1},\\ \begin{pmatrix}1 & R_{n3}e^{-2it\theta(\lambda)}\delta^{2}\\ 0 & 1\end{pmatrix}, & \lambda\in\Sigma_{n2},\\ \begin{pmatrix}1 & 0\\ R_{n4}e^{2it\theta(\lambda)}\delta^{-2} & 1\end{pmatrix}, & \lambda\in\Sigma_{n3},\\ \begin{pmatrix}1 & R_{n6}e^{-2it\theta(\lambda)}\delta^{2}\\ 0 & 1\end{pmatrix}, & \lambda\in\Sigma_{n4}.\end{cases}\tag{5.17}$$

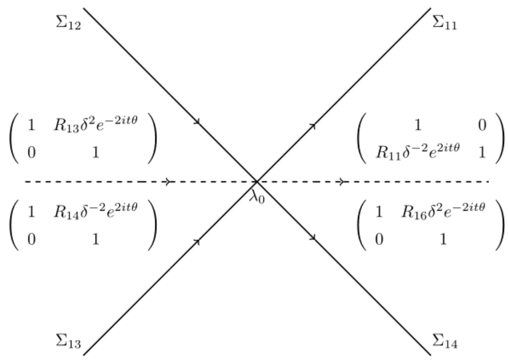

Figure 4. The jump matrix $v^{(2)}(\lambda)$.

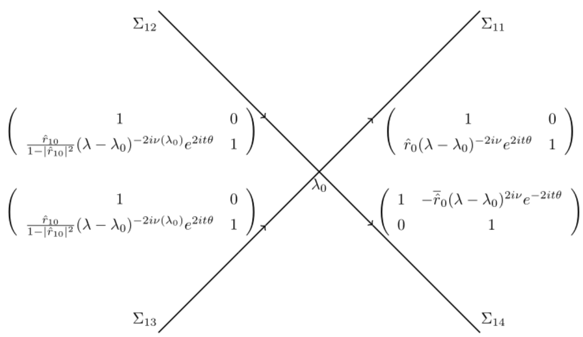

Figure 5. The jump matrix $v^{(2)}(\lambda)$

In order for $v^{(2)}(\lambda)$ to match the jump matrix $v^{(pc)}(\lambda)$ of the parabolic cylindrical RH problem, we make the scale transformation

$$r_{10} = \hat{r}_{10} e^{i\nu \log(12\alpha t\lambda_0) - 4i\alpha t\lambda_0^3}, \tag{5.18}$$

then the jump matrix $v^{(pc)}[\sqrt{12\alpha t\lambda_0}(\lambda-\lambda_0)]$ corresponding to $m^{(pc)}[\sqrt{12\alpha t\lambda_0}(\lambda-\lambda_0)]$ is consistent with $v^{(2)}(\lambda)$. We make the transformation

$$r_{20} = \hat{r}_{20} e^{i\nu \log(-12\alpha t\lambda_0) + 4i\alpha t\lambda_0^3}, \tag{5.19}$$

the jump matrix $v^{(pc)}[\sqrt{-12\alpha t\lambda_0}(\lambda+\lambda_0)]$ corresponding to $m^{(pc)}[\sqrt{12\alpha t\lambda_0}(\lambda+\lambda_0)]$ is consistent with $v^{(2)}(\lambda)$. Therefore, we infer that

$$v^{(2)}(\lambda) = (R_-^{(2)})^{-1} v^{(1)}(\lambda) R_+^{(2)} = v^{(pc)}[\sqrt{12\alpha t\lambda_0}(\lambda-\lambda_0)]. \tag{5.20}$$

On the boundary (λ_0, ∞), we have

$$v^{(pc)}[\sqrt{12\alpha t\lambda_0}(\lambda-\lambda_0)] = I,\ v^{(1)}(\lambda) = \begin{pmatrix} 1 & -\overline{r(\overline{\lambda})}\delta_-^2 e^{-2it\theta(\lambda)} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ r(\lambda)\delta_-^{-2} e^{2it\theta(\lambda)} & 1 \end{pmatrix}.$$

Therefore, equation (5.20) derives

$$v^{(2)}(\lambda)$$

$$= (R^{(2)}(\lambda)|_{\Omega_{16}})^{-1} \begin{pmatrix} 1 & -\overline{r(\overline{\lambda})}\delta_-^2 e^{-2it\theta(\lambda)} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ r(\lambda)\delta_-^{-2} e^{2it\theta(\lambda)} & 1 \end{pmatrix} R^{(2)}(\lambda)|_{\Omega_{11}}$$

$$= I. \tag{5.21}$$

We define

$$R^{(2)}(\lambda) = \begin{pmatrix} 1 & 0 \\ r\delta^{-2}e^{2it\theta} & 1 \end{pmatrix}^{-1}, \quad \lambda \in \Omega_{11}; \qquad R^{(2)}(\lambda) = \begin{pmatrix} 1 & -\overline{r}\delta^2 e^{-2it\theta} \\ 0 & 1 \end{pmatrix}, \quad \lambda \in \Omega_{16}, \tag{5.22}$$

where $R_{11}(\lambda)$ and $R_{16}(\lambda)$ on the line (λ_0, ∞) have the boundary values

$$R_{11}(\lambda) = r(\lambda), \quad R_{16}(\lambda) = \overline{r}(\lambda), \quad \lambda \in (\lambda_0, \infty). \tag{5.23}$$

On Σ_{11}, we have

$$v^1(\lambda) = I, \; v^{(pc)}[\sqrt{12\alpha t\lambda_0}(\lambda - \lambda_0)] = e^{-it\theta\hat{\sigma}_3}(\lambda - \lambda_0)^{i\nu\hat{\sigma}_3} \begin{pmatrix} 1 & 0 \\ \hat{r}_0 & 1 \end{pmatrix}.$$

By using (5.23), (5.16) can be written as

$$v^{(2)}(\lambda) = \begin{pmatrix} 1 & 0 \\ R_1\delta^{-2}e^{2it\theta} & 1 \end{pmatrix} \cdot I \cdot R^{(2)}(\lambda)|_{\Omega_2} = \begin{pmatrix} 1 & 0 \\ \hat{r}_{10}e^{2it\theta}(\lambda - \lambda_0)^{-2i\nu} & 1 \end{pmatrix}. \tag{5.24}$$

We take

$$R^{(2)}(\lambda) = I, \; \lambda \in \Omega_2,$$

Comparing the elements at position 21 of the jump matrix (5.24), we get

$$R_1(\lambda)\delta^{-2}e^{2it\theta} = f_1\delta^{-2}e^{2it\theta} = \hat{r}_{10}e^{2it\theta}(\lambda - \lambda_0)^{-2i\nu}. \tag{5.25}$$

Therefore, the boundary value of $R_{11}(\lambda)$ on Σ_{11} is

$$R_{11}(\lambda) = \hat{r}_{10}(\lambda - \lambda_0)^{-2i\nu}\delta^2, \quad \lambda \in \Sigma_{11}. \tag{5.26}$$

The proofs for the other regions are similar.
From the literature [39], we get

$$M_{\lambda_0}^{PC}(\xi) = I + \frac{M_1^{PC}(\lambda_0)}{i\xi} + O(\xi^{-2}), \tag{5.27}$$

where

$$M_1^{PC}(\lambda_0) = \begin{pmatrix} 0 & \beta_{12}(\hat{r}_{10}) \\ \beta_{21}(\hat{r}_{10}) & 0 \end{pmatrix} \tag{5.28}$$

with

$$\beta_{12}(\hat{r}_{10}) = \frac{-\sqrt{2\pi}e^{\frac{i\pi}{4}}e^{-\frac{\pi\nu(\lambda_0)}{2}}}{\hat{r}_{10}\Gamma(-i\nu(\lambda_0))}. \tag{5.29}$$

Similarly, through the literature [39], we obtain

$$M_{-\lambda_0}^{PC}(\xi) = I + \frac{M_1^{PC}(-\lambda_0)}{i\xi} + O(\xi^{-2}), \tag{5.30}$$

where

$$M_1^{PC}(-\lambda_0) = \begin{pmatrix} 0 & \beta_{12}(\hat{r}_{20}) \\ \beta_{21}(\hat{r}_{20}) & 0 \end{pmatrix} \tag{5.31}$$

with

$$\beta_{12}(\hat{r}_{10}) = \frac{-\sqrt{2\pi}e^{\frac{i\pi}{4}}e^{-\frac{\pi\nu(-\lambda_0)}{2}}}{\hat{r}_{20}\Gamma(-i\nu(-\lambda_0))}. \tag{5.32}$$

Since $m^{(1)}(\lambda)$ is analytical in the regions $\Omega_{nj}, n=1,2, j=1,3,4,6$, and $(R^{(2)}(\lambda))^{-1}\overline{\partial}R^{(2)}(\lambda) = \overline{\partial}R^{(2)}(\lambda)$, it follows that

$$\overline{\partial}m^{(2)}(\lambda) = m^{(1)}(\lambda)\overline{\partial}R^{(2)}(\lambda) = m^{(2)}(\lambda)(R^{(2)}(\lambda))^{-1}\overline{\partial}R^{(2)}(\lambda) = m^{(2)}(\lambda)\overline{\partial}R^{(2)}(\lambda), \tag{5.33}$$

where

$$\overline{\partial}R^{(2)}(\lambda) = \begin{cases} \begin{pmatrix} 0 & 0 \\ -\overline{\partial}R_{n1}e^{2it\theta}\delta^{-2} & 0 \end{pmatrix}, & \lambda \in \Omega_{n1}, \\ \begin{pmatrix} 0 & -\overline{\partial}R_{n3}e^{-2it\theta}\delta^{2} \\ 0 & 1 \end{pmatrix}, & \lambda \in \Omega_{n3}, \\ \begin{pmatrix} 0 & 0 \\ \overline{\partial}R_{n4}e^{2it\theta}\delta^{-2} & 0 \end{pmatrix}, & \lambda \in \Omega_{n4}, \\ \begin{pmatrix} 0 & \overline{\partial}R_{n6}e^{-2it\theta}\delta^{2} \\ 0 & 0 \end{pmatrix}, & \lambda \in \Omega_{n6}, \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, & \lambda \in \Omega_{n2}\cup\Omega_{n5}. \end{cases} \tag{5.34}$$

$n = 1, 2.$

6. Pure $\overline{\partial}$-problem and asymptotics of its solutions

We define

$$E(\lambda) = m^{(2)}(\lambda)[m^{(pc)}(\sqrt{12\alpha t\lambda_0}(\lambda - \lambda_0))], \tag{6.1}$$

$E(\lambda)$ is continuous in $\mathbb{C}$ without jumping. Through (5.14) and (6.1), on $\Sigma_{1j}, j = 1,2,3,4$, have

$$\begin{aligned} E_-^{-1}(\lambda)E_+(\lambda) &= m_-^{(pc)}(m_-^{(2)})^{-1}m_+^{(2)}(m_+^{(pc)})^{-1} \\ &= m_-^{(pc)}v^{(2)}(m_-^{(pc)}v^{(pc)})^{-1} \\ &= m_-^{(pc)}v^{(2)}(m_-^{(pc)}v^{(2)})^{-1} \\ &= I. \end{aligned} \tag{6.2}$$

Therefore, we get a pure $\overline{\partial}$-problem

$$\begin{aligned} &\bullet E(\lambda) \ \ \textit{is continuous in} \ \ \mathbb{C}, \\ &\bullet \overline{\partial}E(\lambda) = E(\lambda)W(\lambda), \quad \lambda \in \mathbb{C}, \\ &\bullet E(\lambda) \sim I, \quad \lambda \to \infty, \end{aligned} \tag{6.3}$$

where

$$W(\lambda)=\begin{cases} m^{(pc)}(\lambda)\begin{pmatrix}0 & 0\\ \overline{\partial}R_{11}e^{2it\theta}\delta^{-2} & 0\end{pmatrix}(m^{(pc)}(\lambda))^{-1},\ \lambda\in\Omega_{11},\\ m^{(pc)}(\lambda)\begin{pmatrix}0 & -\overline{\partial}R_{13}e^{-2it\theta}\delta^{2}\\ 0 & 1\end{pmatrix}(m^{(pc)}(\lambda))^{-1},\ \lambda\in\Omega_{13},\\ m^{(pc)}(\lambda)\begin{pmatrix}0 & 0\\ \overline{\partial}R_{14}e^{2it\theta}\delta^{-2} & 0\end{pmatrix}(m^{(pc)}(\lambda))^{-1},\ \lambda\in\Omega_{14},\\ m^{(pc)}(\lambda)\begin{pmatrix}0 & -\overline{\partial}R_{16}e^{-2it\theta}\delta^{2}\\ 0 & 0\end{pmatrix}(m^{(pc)}(\lambda))^{-1},\ \lambda\in\Omega_{16},\\ \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix},\ \lambda\in\Omega_{12}\cup\Omega_{15}.\end{cases} \tag{6.4}$$

$\overline{\partial}$-problem (6.3) is equivalent to the following integral equation

$$E(\lambda)=I-\frac{1}{\pi}\iint_{\mathbb{C}}\frac{E(s)W(s)}{s-\lambda}\ \mathrm{d}A(s), \tag{6.5}$$

where $dA(s)$ is the Lebesgue measure on the real plane. Equation (6.5) can also be expressed as an operator

$$(1-S)E(\lambda)=I, \tag{6.6}$$

where S is a Cauchy operator

$$S[f](\lambda)=-\frac{1}{\pi}\iint_{\mathbb{C}}\frac{f(s)W(s)}{s-\lambda}\ \mathrm{d}A(s). \tag{6.7}$$

Proposition 6.1. *For sufficiently large t, the operator S is a small norm, $(1-S)^{-1}$ exists, and*

$$\|S\|_{L^\infty\to L^\infty}\le ct^{-\frac{1}{4}}. \tag{6.8}$$

From equation (6.7), further expand $E(\lambda)$ as

$$E(\lambda)=I+\frac{E_1(\lambda)}{\lambda}+O(\lambda^{-2}), \tag{6.9}$$

where

$$E_1(\lambda)=\tfrac{1}{\pi}\iint_{\Omega_{11}}E(s)W(s)\ \mathrm{d}A(s),$$

and satisfies the following estimation

$$|E_1(\lambda)|\le ct^{-\frac{3}{4}}. \tag{6.10}$$

In addition, we define

$$T(\lambda)=m^{(2)}(\lambda)[m^{(pc)}(\sqrt{-12\alpha t\lambda_0}(\lambda+\lambda_0))], \tag{6.11}$$

and $T(\lambda)$ is continuous in $\mathbb{C}$ without jumping. Through (5.14) and (6.11), on $\Sigma_{2j}, j=1,2,3,4$, we have

$$T_-^{-1}(\lambda)T_+(\lambda)=m_-^{(pc)}(m_-^{(2)})^{-1}m_+^{(2)}(m_+^{(pc)})^{-1}$$

$$
\begin{aligned}
&= m_-^{(pc)} v^{(2)} (m_-^{(pc)} v^{(pc)})^{-1} \\
&= m_-^{(pc)} v^{(2)} (m_-^{(pc)} v^{(2)})^{-1} \\
&= I.
\end{aligned} \tag{6.12}
$$

Therefore, we get a pure $\overline{\partial}$-problem

$$
\begin{aligned}
&\bullet\, T(\lambda) \ \ \textit{is} \ \ \textit{continuous} \ \ \textit{in} \ \ \mathbb{C}, \\
&\bullet\, \overline{\partial} T(\lambda) = T(\lambda) M(\lambda), \quad \lambda \in \mathbb{C}, \\
&\bullet\, T(\lambda) \sim I, \quad \lambda \to \infty,
\end{aligned} \tag{6.13}
$$

where

$$
M(\lambda) = \begin{cases}
m^{(pc)}(\lambda) \begin{pmatrix} 0 & 0 \\ \overline{\partial} R_{21} e^{2it\theta} \delta^{-2} & 0 \end{pmatrix} (m^{(pc)}(\lambda))^{-1}, \ \lambda \in \Omega_{21}, \\
m^{(pc)}(\lambda) \begin{pmatrix} 0 & -\overline{\partial} R_{23} e^{-2it\theta} \delta^{2} \\ 0 & 1 \end{pmatrix} (m^{(pc)}(\lambda))^{-1}, \ \lambda \in \Omega_{23}, \\
m^{(pc)}(\lambda) \begin{pmatrix} 0 & 0 \\ \overline{\partial} R_{24} e^{2it\theta} \delta^{-2} & 0 \end{pmatrix} (m^{(pc)}(\lambda))^{-1}, \ \lambda \in \Omega_{24}, \\
m^{(pc)}(\lambda) \begin{pmatrix} 0 & -\overline{\partial} R_{26} e^{-2it\theta} \delta^{2} \\ 0 & 0 \end{pmatrix} (m^{(pc)}(\lambda))^{-1}, \ \lambda \in \Omega_{26}, \\
\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \ \lambda \in \Omega_{22} \cup \Omega_{25}.
\end{cases} \tag{6.14}
$$

$\overline{\partial}$-problem (6.13) is equivalent to the following integral equation

$$
T(\lambda) = I - \frac{1}{\pi} \iint_{\mathbb{C}} \frac{EW}{s - \lambda} \ \mathrm{d}A(s), \tag{6.15}
$$

where $dA(s)$ is the Lebesgue measure on the real plane. Equation (6.11) can also be represented as an operator

$$
(1 - S) T(\lambda) = I, \tag{6.16}
$$

where S is a Cauchy operator

$$
S[f](\lambda) = -\frac{1}{\pi} \iint_{\mathbb{C}} \frac{f(s) W(s)}{s - \lambda} \ \mathrm{d}A(s). \tag{6.17}
$$

From (6.17), we further expand $T(\lambda)$ as

$$
T(\lambda) = I + \frac{T_1(\lambda)}{\lambda} + O(\lambda^{-2}), \tag{6.18}
$$

where

$$
T_1(\lambda) = \tfrac{1}{\pi} \textstyle\iint_{\Omega_{11}} TW \ \mathrm{d}A(s),
$$

and satisfies the following estimation

$$
|T_1(\lambda)| \le c t^{-\frac{3}{4}}. \tag{6.19}
$$

Recalling the series of transformations we have made (4.8), (5.13), (6.1) and (6.11), we can reverse these transformations

$$m(\lambda) \leftrightarrow m^{(1)}(\lambda) \leftrightarrow m^{(2)}(\lambda) \leftrightarrow (E(\lambda) + T(\lambda)), \tag{6.20}$$

and have

$$m(\lambda) = E(\lambda) M_{\lambda_0}^{PC} (R^{(2)}(\lambda))^{-1} \delta(\lambda)^{\sigma_3} + T(\lambda) M_{-\lambda_0}^{PC} (R^{(2)}(\lambda))^{-1} \delta(\lambda)^{\sigma_3}. \tag{6.21}$$

Particularly, considering $\lambda \to \infty$ in $\lambda \in \Omega_{12}$, Ω_{15}, Ω_{22}, Ω_{25}, $R^{(2)} = I$, then we have

$$\begin{aligned} m(\lambda) &= \left(I + \frac{E_1(\lambda)}{\lambda} + \dots\right)\left(I + \frac{({M_{\lambda_0}}^{PC})}{\sqrt{12\alpha t \lambda_0} + \dots}\right)\left(I + \frac{\triangle_1}{\lambda} + \dots\right) \\ &= \left(I + \frac{T_1(\lambda)}{\lambda} + \dots\right)\left(I + \frac{({M_{-\lambda_0}}^{PC})}{\sqrt{-12\alpha t \lambda_0} + \dots}\right)\left(I + \frac{\triangle_1}{\lambda} + \dots\right). \end{aligned} \tag{6.22}$$

There is also

$$m_1(\lambda) = E_1(\lambda) + T_1(\lambda) + \frac{({M_{\lambda_0}}^{PC})_{12}}{\sqrt{12\alpha t \lambda_0}} + 2i \frac{({M_{-\lambda_0}}^{PC})_{12}}{\sqrt{-12\alpha t \lambda_0}} + \triangle_1, \tag{6.23}$$

where $\triangle_1 = \begin{pmatrix} \delta_1 & 0 \\ 0 & -\delta_1 \end{pmatrix}$.

Therefore, we get

$$u(x,t) = 2i \frac{({M_{\lambda_0}}^{PC})_{12}}{\sqrt{12\alpha t \lambda_0}} + 2i \frac{({M_{-\lambda_0}}^{PC})_{12}}{\sqrt{-12\alpha t \lambda_0}} + O(t^{\frac{3}{4}}), \tag{6.24}$$

where $M_{\lambda_0}^{PC} = -i\beta_{12}(\hat{r}_{10})$, $M_{-\lambda_0}^{PC} = -i\beta_{12}(\hat{r}_{20})$.

Appendix A

Here we describe the solution to the parabolic cylindrical model problem introduced by [39], which has been widely used to study the long-time asymptotics of integrable systems in the literature [21, 40]. Define the contour

$$\Sigma^{pc} = \cup_{j=1}^{4} \Sigma_j, \Sigma_j = \left\{ \zeta = \mathbb{R}^+ e^{\frac{i(2j-1)\pi}{4}}, j = 1, 2, 3, 4 \right\}, \tag{6.25}$$

we have the following parabolic cylinder model problem.

RH Problem A.1 :

$$\bullet\ M^{(pc)}(\xi) \ \ is \ \ analytic \ \ in \ \ \mathbb{C} \backslash \Sigma^{pc}, \tag{6.26}$$

$$\bullet\ M_+^{(pc)}(\xi) = M_-^{(pc)}(\xi) V^{(pc)}(\xi), \quad \xi \in \Sigma^{pc}, \tag{6.27}$$

$$\bullet\ M^{(pc)}(\xi) = I + \frac{M_1}{\xi} + O(\xi^2), \quad \xi \to \infty. \tag{6.28}$$

where

$$v^{(pc)}(\xi) = \begin{cases} \xi^{i\nu\hat{\sigma}_3} e^{-\frac{i\xi^2}{4}\hat{\sigma}_3} \begin{pmatrix} 1 & 0 \\ r_0 & 1 \end{pmatrix}, & \xi \in \Sigma_1, \\ \xi^{i\nu\hat{\sigma}_3} e^{-\frac{i\xi^2}{4}\hat{\sigma}_3} \begin{pmatrix} 1 & \frac{-\bar{r}_0}{1-|r_0|^2} \\ 0 & 1 \end{pmatrix}, & \xi \in \Sigma_2, \\ \xi^{i\nu\hat{\sigma}_3} e^{-\frac{i\xi^2}{4}\hat{\sigma}_3} \begin{pmatrix} 1 & 0 \\ \frac{r_0}{1-|r_0|^2} & 1 \end{pmatrix}, & \xi \in \Sigma_3, \\ \xi^{i\nu\hat{\sigma}_3} e^{-\frac{i\xi^2}{4}\hat{\sigma}_3} \begin{pmatrix} 1 & -\bar{r}_0 \\ 0 & 1 \end{pmatrix}, & \xi \in \Sigma_4. \end{cases} \tag{6.29}$$

(See figure 6)

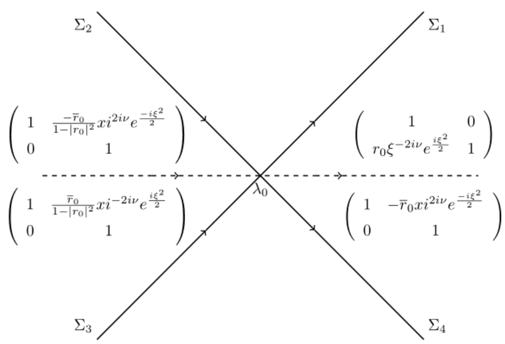

Figure 6. The jump matrix $v^{(pc)}(\xi)$

We know that the RH Problem A.1 admits the solution

$$M^{pc}(\zeta, r_0) = I + \frac{M_1^{pc}(r_0)}{i\zeta} + \mathcal{O}\left(\zeta^{-2}\right) \tag{6.30}$$

where

$$M_1^{pc}(r_0) = \begin{pmatrix} 0 & \beta_{12} \\ -\beta_{21} & 0 \end{pmatrix}$$

with β_{12} and β_{21} which are two complex constants

$$\beta_{12} = \frac{\sqrt{2\pi} e^{i\pi/4} e^{-\pi\nu/2}}{r_0 \Gamma(-iv)}, \quad \beta_{21} = -\frac{\sqrt{2\pi} e^{-i\pi/4} e^{-\pi v/2}}{\varepsilon_n \bar{r}_0 \Gamma(iv)}.$$

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