Journal of Nonlinear Modeling and Analysis http://jnma-online.com
Volume 6, Number 2, June 2024, 371–391 DOI:10.12150/jnma.2024.371

Bifurcation of Limit Cycles of a Perturbed Pendulum Equation*

Jihua Yang[1,†]

Abstract This paper investigates the limit cycle bifurcation problem of the pendulum equation on the cylinder of the form $\dot{x} = y, \dot{y} = -\sin x$ under perturbations of polynomials of $\sin x$, $\cos x$ and y of degree n with a switching line $y = 0$. We first prove that the corresponding first order Melnikov functions can be expressed as combinations of anti-trigonometric functions and the complete elliptic functions of first and second kind with polynomial coefficients in both the oscillatory and rotary regions for arbitrary n. We also verify the independence of coefficients of these polynomials. Then, the lower bounds for the number of limit cycles are obtained using their asymptotic expansions with $n = 1, 2, 3$.

Keywords Pendulum equation, complete elliptic function, Melnikov function, limit cycle

MSC(2010) 34C08, 34C07.

1. Introduction and main results

Consider the following non-smooth near-integrable differential equation

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{cases} \begin{pmatrix} p^+(x,y) + \varepsilon f^+(x,y) \\ q^+(x,y) + \varepsilon g^+(x,y) \end{pmatrix}, & y \geq 0, \\ \begin{pmatrix} p^-(x,y) + \varepsilon f^-(x,y) \\ q^-(x,y) + \varepsilon g^-(x,y) \end{pmatrix}, & y < 0, \end{cases} \tag{1.1}$$

where $\varepsilon > 0$ is a small parameter, and $p^\pm(x,y)$, $q^\pm(x,y)$, $f^\pm(x,y)$ and $g^\pm(x,y)$ are C^∞ smooth functions. When $\varepsilon = 0$, system (1.1) is a non-smooth integrable differential equation and has a family of piecewise smooth closed orbits to form a generalized annulus (for short, period annulus). In recent years, the limit cycle bifurcation problems of system (1.1) have received considerable attention from mathematical scholars, and some important results have been obtained, when $p^\pm(x,y)$, $q^\pm(x,y)$, $f^\pm(x,y)$ and $g^\pm(x,y)$ are polynomials of x and y. See [3, 4, 7, 7, 11–14, 18–21, 23, 24] and the references therein. For example, the authors in [7, 14] established a formula for the first order Melnikov function (called Melnikov function method), which

†the corresponding author.
Email address: jihua1113@163.com
[1]School of Mathematics and Computer Science, Ningxia Normal University, Xueyuan Road, 756000 Guyuan, China
*The author was supported by Construction of First-class Disciplines of Higher Education of Ningxia(Pedagogy)(NXYLXK2021B11).

plays a key role in studying the number of limit cycles of system (1.1). The authors in [4, 12, 13] developed the averaging method to non-smooth differential systems. Then the authors in [6, 10] showed the equivalence of these two methods. The authors in [20] studied the limit cycle problems of two kinds of quadratic reversible systems with non-smooth polynomial perturbation by using Picard-Fuchs equation.

But when $p^{\pm}(x,y)$, $q^{\pm}(x,y)$, $f^{\pm}(x,y)$ and $g^{\pm}(x,y)$ are not polynomials of x and y, such as trigonometric functions or trigonometric polynomials, there are a few results. For instance, the authors in [2] considered a pendulum-like equation of the form

$$\ddot{x} + \sin x = \varepsilon \sum_{s=0}^{m} Q_{n,s}(x)\dot{x}^s,$$

where $Q_{n,s}(x)$ are trigonometric polynomials of degree n, and got the upper bounds on the number of zeros of its associated first order Melnikov functions, in both the oscillatory and rotary regions. Another interesting perturbed whirling pendulum is the equation

$$\dot{x} = y,\ \ \dot{y} = \sin x(\cos x - r) + \varepsilon y(\cos x + a), \tag{1.2}$$

where a and $r \geq 0$ are real parameters, and $\varepsilon > 0$ is a small parameter, which was considered in [9]. The authors proved that, depending on the value of the parameter, the period function of system (1.2) is either monotone or has exactly one critical point using Picard-Fuchs equation method. By using the averaging method of first order, the authors in [1] obtained the exact number of limit cycles of the equation

$$\dot{x} = -y,\ \ \dot{y} = x + \varepsilon(1 + \cos^m(\theta))Q(x,y),$$

where $\varepsilon > 0$ is a small parameter, $\theta = \arctan(y/x)$ and $Q(x,y)$ is a polynomial of degree n. The non-smooth form of the above equation is

$$\dot{x} = -y,\ \ \dot{y} = x + \varepsilon(1 + \cos^m(\theta))\sum_{k=1}^{2} \chi_{S_k}(x,y)Q_k(x,y),$$

where χ_S is the characteristic function of a set S, $Q_k(x,y)$ is a polynomial of degree n, and $S_1 = \{(x,y) : y \geq 0\}$ and $S_2 = \{(x,y) : y \leq 0\}$, considered by [16]. The authors got the exact number of limit cycles of this differential equation by using the averaging method of first order. Recently, the authors in [17] established some general methods on the existence of limit cycles bifurcating from closed orbits of a near-Hamiltonian system on the cylinder by the Melnikov function method and derived the expansions of the first order Melnikov function, which were used to consider the bifurcation problem of limit cycles near a double homoclinic loop.

In the current work, we will give the lower bounds of the number of limit cycles of the single pendulum

$$\dot{x} = y, \quad \dot{y} = -\sin x \tag{1.3}$$

under perturbation of polynomials of $\sin x$, $\cos x$ and y of degree n with the switching line $y = 0$. The pendulum equation (1.3) is a Hamiltonian system with total energy

$$H(x,y) = \frac{1}{2}y^2 - \cos x + 1, \tag{1.4}$$

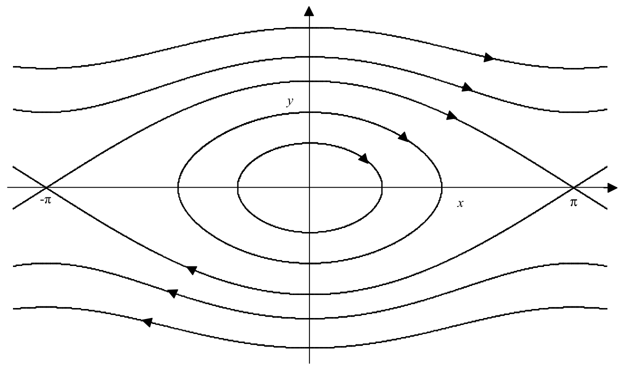

Fig. 1. The phase portrait of system (1.3).

which in fact can be considered on the cylinder $[-\pi, \pi] \times \mathbb{R}$. It is easy to get that system (1.3) has equilibria $(j\pi, 0)$. $(2j\pi, 0)$ are stable equilibria−centers, and $((2j+1)\pi, 0)$ are unstable equilibria−saddle points, $j = 0, \pm 1, \pm 2, \cdots$. See Fig. 1.

Since $\sin x$ is a periodic function with the period 2π, the physical state of system (1.3) described by points $(2k\pi + x, y)$ is the same for $k \in \mathbb{Z}$. If these points are considered as a single point, we obtain the phase cylinder as follows

$$P = \{(x, y) | x \in [-\pi, \pi], y \in \mathbb{R}\}.$$

In fact, it is formed by cutting the Euclidean plane along $x = \pm\pi$ and then bonding it at the cutting. There are two types of simple closed curves on the phase cylinder P. One type divides P into two regions, one of which is a bounded domain. These closed curves are homotopic to zero, i.e. the closed orbit can be deformed continuously to a point. The corresponding closed orbit is called a periodic orbit of type I. The other type divides P into two unbounded regions, and these closed curves cannot be homotopic to zero. The corresponding closed orbit is called a periodic orbit of type II.

Thus, for $h \in (0, 2)$, the levels $\{(x, y) : H(x, y) = h\}$ are of type I, while the corresponding levels are of type II for $h \in (2, +\infty)$, and they wind around the cylinder. See Fig. 2. The region corresponding the energies $h \in (0, 2)$ is usually called *oscillatory region*, and the regions with energies $h \in (2, +\infty)$ and $\pm y > 0$ are called *rotary region*.

It is well known that any real polynomial of $\sin x$, $\cos x$ and y of degree n can be written as

$$\sum_{i+j=0}^{n} a_{i,j} y^j \cos^i x + \sum_{i+j=0}^{n-1} b_{i,j} y^j \cos^i x \sin x, \;\; i, j \in \mathbb{N},$$

where $a_{i,j}, b_{i,j} \in \mathbb{R}$. Thus, the perturbed pendulum equation with a switching line

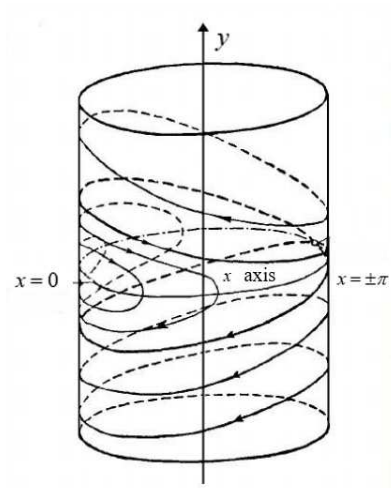

Fig. 2. The phase cylinder of system (1.3).

$y = 0$ can be written as

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{cases} \begin{pmatrix} y + \varepsilon f^{+}(x,y) \\ -\sin x + \varepsilon g^{+}(x,y) \end{pmatrix}, & y \geq 0, \\ \begin{pmatrix} y + \varepsilon f^{-}(x,y) \\ -\sin x + \varepsilon g^{-}(x,y) \end{pmatrix}, & y < 0, \end{cases} \tag{1.5}$$

where $\varepsilon > 0$ is a small parameter,

$$f^{\pm}(x,y) = \sum_{i+j=0}^{n} a_{i,j}^{\pm} y^j \cos^i x + \sum_{i+j=0}^{n-1} b_{i,j}^{\pm} y^j \cos^i x \sin x,$$
$$g^{\pm}(x,y) = \sum_{i+j=0}^{n} c_{i,j}^{\pm} y^j \cos^i x + \sum_{i+j=0}^{n-1} d_{i,j}^{\pm} y^j \cos^i x \sin x, \; i,j \in \mathbb{N},$$

and $a_{i,j}^{\pm}, b_{i,j}^{\pm}, c_{i,j}^{\pm}, d_{i,j}^{\pm} \in \mathbb{R}$. By the main results in [7, 14], one knows that the first order Melnikov functions corresponding to three families of periodic orbits of system (1.5) are as follows

$$\begin{aligned} M_0(h) = & \int_{\Gamma_{h,0}^{+}} g^{+}(x,y)dx - f^{+}(x,y)dy \\ & + \int_{\Gamma_{h,0}^{-}} g^{-}(x,y)dx - f^{-}(x,y)dy, \; h \in (0,2), \end{aligned} \tag{1.6}$$

$$M_{+}(h) = \int_{\Gamma_h^{+}} g^{+}(x,y)dx - f^{+}(x,y)dy, \; h \in (2,+\infty), \tag{1.7}$$

$$M_{-}(h) = \int_{\Gamma_h^{-}} g^{-}(x,y)dx - f^{-}(x,y)dy, \; h \in (2,+\infty), \tag{1.8}$$

where

$$\begin{aligned}\Gamma_{h,0}^{+} &= \{H(x,y)=h, h\in(0,2), y>0\},\\ \Gamma_{h,0}^{-} &= \{H(x,y)=h, h\in(0,2), y<0\},\\ \Gamma_{h}^{+} &= \{H(x,y)=h, h\in(2,+\infty), y>0\},\\ \Gamma_{h}^{-} &= \{H(x,y)=h, h\in(2,+\infty), y<0\}.\end{aligned}$$

There is a beautiful relationship between the limit cycles and the zeros of the first order Melnikov function by the mains results in [7, 8, 14, 17]: the total number of zeros of the first non-vanishing Melnikov functions (1.6)-(1.8) can control the number of limit cycles bifurcating from the period annuli of system (1.5). In [22], the authors got the upper bounds of the number of limit cycles of system (1.5) using Picard-Fuchs equation. In this article, we first obtain the algebraic structure of the first order Melnikov functions (1.6)-(1.8). Then, the lower bounds for the number of limit cycles of system (1.5) are obtained, by applying their asymptotic expansions with $n=1,2,3$. Our main results are the following two theorems.

Theorem 1.1. *For $n\in\mathbb{N}$ and $n\geq 4$, the following statements hold:*

(i) If $h\in(0,2)$, then

$$\begin{aligned}M_0(h) =&\Big(\sum_{i=0}^{[\frac{n}{2}]} a_i h^i\Big)\arccos(1-h)+\Big(\sum_{i=0}^{n-1} b_i h^i\Big)\sqrt{2h-h^2}\\ &+\Big(\sum_{i=0}^{n} c_i h^i\Big)K(\sqrt{h/2})+\Big(\sum_{i=0}^{n} d_i h^i\Big)E(\sqrt{h/2}),\end{aligned}\tag{1.9}$$

where a_i $(i=0,1,\cdots,[\frac{n}{2}])$, b_i $(i=0,1,\cdots,n-1)$, c_i $(i=1,2,\cdots,n-2)$ and d_i $(i=0,1,\cdots,n-1)$ can be chosen arbitrarily, and $K(\cdot)$ and $E(\cdot)$ are the complete elliptic integrals of first and second kind and defined by (2.11).

(ii) If $h\in(2,+\infty)$, then

$$M_{\pm}(h)=\sum_{i=0}^{[\frac{n}{2}]} a_i^{\pm} h^i+\Big(\sqrt{h}\sum_{i=0}^{n} b_i^{\pm} h^i\Big)K(\sqrt{2/h})+\Big(\sqrt{h}\sum_{i=0}^{n} c_i^{\pm} h^i\Big)E(\sqrt{2/h}),\tag{1.10}$$

where $a_i^{\pm}$ $(i=0,1,\cdots,[\frac{n}{2}])$, $b_i^{\pm}$ $(i=1,\cdots,n-2)$ and $c_i^{\pm}$ $(i=0,1,\cdots,n-1)$ can be chosen arbitrarily.

Theorem 1.2. *By properly choosing the perturbation coefficients $a_{ij}^{\pm}$, $b_{ij}^{\pm}$, $c_{ij}^{\pm}$ and $d_{ij}^{\pm}$ in the perturbed pendulum equation (1.5), $M_0(h)$ can have at least 9 zeros if $n=3$; 6 zeros if $n=2$; 3 zeros if $n=1$, when $h\in(0,2)$; and $M_+(h)$ can have at least 6 zeros if $n=3$; 4 zeros if $n=2$; 2 zero if $n=1$, when $h\in(2,+\infty)$. The same result holds for $M_-(h)$.*

Remark 1.1. By Theorem 2.1 in [17], the total number of zeros of $M_0(h)$ and $M_{\pm}(h)$ provide an upper bound for the number of limit cycles of system (1.5) bifurcating from the corresponding period annulus, and the existence of multiple simple zeros provides a lower bound for the number of limit cycles.

This paper is organized as follows. In Section 2, we obtain the detailed expressions of the first order Melnikov functions and verify the independence of coefficients of the coefficient polynomials of several generated integrals by using mathematical induction. Section 3 is devoted to studying the lower bounds for the number of limit cycles of system (1.5) for $n = 1, 2, 3$.

2. The algebraic structure of the first order Melnikov function

In order to estimate the number of zeros of the first order Melnikov functions $M_0(h)$ and $M_\pm(h)$, one should study the algebraic structure of $M(h)$. To this end, we denote

$$I_{i,j}(h) = \int_{\Gamma^+_{h,0}} y^j \cos^i x dx,\ h \in (0,2),$$
$$J_{i,j}(h) = \int_{\Gamma^+_{h}} y^j \cos^i x dx,\ h \in (2,+\infty).$$

The next lemma shows that $M_0(h)$ and $M_\pm(h)$ can be expressed as a combination of several curvilinear integrals with polynomial coefficients and some coefficients of these polynomials can be taken as free parameters.

Lemma 2.1. *For $n \in \mathbb{N}$ and $n \geq 4$. Then the following statements hold:*

(i) Let $h \in (0,2)$. Then

$$\begin{aligned} M_0(h) =& \Big(\sum_{i=0}^{[\frac{n}{2}]} \alpha_i h^i\Big) I_{0,0}(h) + \Big(\sum_{i=0}^{n-1} \beta_i h^i\Big) I_{1,0}(h) \\ &+ \Big(\sum_{i=0}^{n-2} \gamma_i h^i\Big) I_{0,1}(h) + \Big(\sum_{i=0}^{n-1} \delta_i h^i\Big) I_{1,1}(h), \end{aligned} \tag{2.1}$$

where α_i, β_i, γ_i and δ_i are constants and α_i $(i = 0, 1, \cdots, [\frac{n}{2}])$, β_i $(i = 0, 1, \cdots, n-1)$, γ_i $(i = 0, 1, \cdots, n-3)$ and δ_i $(i = 0, 1, \cdots, n-1)$ can be chosen arbitrarily.

(ii) Let $h \in (2,+\infty)$. Then

$$\begin{aligned} M_\pm(h) =& \Big(\sum_{i=0}^{[\frac{n}{2}]} \alpha_i^\pm h^i\Big) J_{0,0}(h) + \Big(\sum_{i=0}^{n-1} \beta_i^\pm h^i\Big) J_{1,0}(h) \\ &+ \Big(\sum_{i=0}^{n-2} \gamma_i^\pm h^i\Big) J_{0,1}(h) + \Big(\sum_{i=0}^{n-1} \delta_i^\pm h^i\Big) J_{1,1}(h), \end{aligned} \tag{2.2}$$

where $\alpha_i^\pm$, $\beta_i^\pm$, $\gamma_i^\pm$ and $\delta_i^\pm$ are constants and $\alpha_i^\pm$ $(i = 0, 1, \cdots, [\frac{n}{2}])$, $\beta_i^\pm$ $(i = 0, 1, \cdots, n-1)$, $\gamma_i^\pm$ $(i = 0, 1, \cdots, n-3)$ and $\delta_i^\pm$ $(i = 0, 1, \cdots, n-1)$ can be chosen arbitrarily.

Proof. We only prove item (i) for $M_0(h)$. The proofs of $M_\pm(h)$ in (ii) follow in the same way. An easy calculation gives that

$$\begin{aligned}
&\int_{\Gamma^{\pm}_{h,0}} y^j \cos^i x \sin x dx = \int_{\Gamma^{\pm}_{h,0}} y^j \cos^i x dy = 0,\\
&\int_{\Gamma^{-}_{h,0}} y^j \cos^i x dx = (-1)^{j+1} I_{i,j}(h),\\
&\int_{\Gamma^{-}_{h,0}} y^j \cos^i x \sin x dy = (-1)^j \int_{\Gamma^{+}_{h,0}} y^j \cos^i x \sin x dy.
\end{aligned} \tag{2.3}$$

Using the Green's Formula, one has that

$$\int_{\Gamma^{+}_{h,0}} y^j \cos^i x \sin x dy = \begin{cases} \frac{i}{j+1} I_{i-1,j+1}(h) - \frac{i+1}{j+1} I_{i+1,j+1}(h), & i \geq 1, j \geq 0,\\ -\frac{i+1}{j+1} I_{i+1,j+1}(h), & i = 0, j \geq 0, \end{cases} \tag{2.4}$$

and

$$\int_{\Gamma^{-}_{h,0}} y^j \cos^i x \sin x dy = \begin{cases} (-1)^j \Big(\frac{i}{j+1} I_{i-1,j+1}(h) - \frac{i+1}{j+1} I_{i+1,j+1}(h)\Big), & i \geq 1, j \geq 0,\\ (-1)^{j+1} \frac{i+1}{j+1} I_{i+1,j+1}(h), & i = 0, j \geq 0, \end{cases} \tag{2.5}$$

and in (2.5) we have used that the second equality of (2.3). From (2.3), (2.4) and (2.5) it follows that

$$\begin{aligned}
M_0(h) =& \sum_{i+j=0}^{n} [c^{+}_{i,j} - (-1)^j c^{-}_{i,j}] I_{i,j}(h)\\
&- \sum_{i+j=0}^{n-1} [b^{+}_{i,j} + (-1)^j b^{-}_{i,j}] \Big[\frac{i}{j+1} I_{i-1,j+1}(h) - \frac{i+1}{j+1} I_{i+1,j+1}(h)\Big]\\
=& \sum_{i+j=0}^{n} [c^{+}_{i,j} - (-1)^j c^{-}_{i,j}] I_{i,j}(h) + \sum_{\substack{i+j=2,\\ i\geq 1, j\geq 1}}^{n+1} \frac{i}{j} [b^{+}_{i-1,j-1} - (-1)^j b^{-}_{i-1,j-1}] I_{i,j}(h)\\
&- \sum_{\substack{i+j=1,\\ i\geq 0, j\geq 1}}^{n-1} \frac{i+1}{j} [b^{+}_{i+1,j-1} - (-1)^j b^{-}_{i+1,j-1}] I_{i,j}(h)\\
\triangleq& \sum_{i+j=0}^{n} \xi_{i,j} I_{i,j}(h) + \sum_{\substack{i+j=n+1,\\ i\geq 1, j\geq 1}} \xi_{i,j} I_{i,j}(h),
\end{aligned} \tag{2.6}$$

where

$$\xi_{i,j} = \begin{cases} c^{+}_{i,j} - c^{-}_{i,j}, & 0 \leq i \leq n,\ j = 0,\\ c^{+}_{i,j} - (-1)^j c^{-}_{i,j} - \frac{i+1}{j} [b^{+}_{i+1,j-1} - (-1)^j b^{-}_{i+1,j-1}] & \\ \qquad + \frac{i}{j} [b^{+}_{i-1,j-1} - (-1)^j b^{-}_{i-1,j-1}], & 0 \leq i+j \leq n-1,\ j \geq 1,\\ c^{+}_{i,j} - (-1)^j c^{-}_{i,j} + \frac{i}{j} [b^{+}_{i-1,j-1} - (-1)^j b^{-}_{i-1,j-1}], & i+j = n,\ j \geq 1,\\ \frac{i}{j} [b^{+}_{i-1,j-1} - (-1)^j b^{-}_{i-1,j-1}], & i+j = n+1,\ j \geq 1. \end{cases}$$

If the subscript i in $b^{\pm}_{i,j}$ is less than zero in the above equalities, then $b^{\pm}_{i,j} = 0$. It is easy to check that $\xi_{i,j}$ in (2.6) can be taken as free parameters.

Now we claim that (2.1) holds and α_i $(i=0,1,\cdots,[\frac{n}{2}])$, β_i $(i=0,1,\cdots,n-1)$, γ_i $(i=0,1,\cdots,n-3)$ and δ_i $(i=0,1,\cdots,n-1)$ can be chosen arbitrarily. In fact, similar to (2.17) and (2.19) in [22], one has that

$$I_{i,j}(h)=2(h-1)I_{i,j-2}(h)+2I_{i+1,j-2}(h), \tag{2.7}$$

and

$$\begin{aligned}I_{i,j}(h)=&\frac{1}{2i+j}\big[2(i-2)(h-1)I_{i-3,j}(h)-2(i-1)(h-1)I_{i-1,j}(h)\\&+(2i+j-2)I_{i-2,j}(h)\big].\end{aligned} \tag{2.8}$$

Next we will prove the claim by induction on n. Indeed, a direct computation using the above two equalities (2.7) and (2.8) gives that

$$\begin{cases}I_{0,2}(h)=2(h-1)I_{0,0}(h)+2I_{1,0}(h),\\ I_{2,0}(h)=\frac{1}{2}I_{0,0}(h)-\frac{1}{2}(h-1)I_{1,0}(h),\\ I_{0,3}(h)=2(h-1)I_{0,1}(h)+2I_{1,1}(h),\\ I_{1,2}(h)=(h-1)I_{1,0}(h)+I_{0,0}(h),\\ I_{2,1}(h)=\frac{3}{5}I_{0,1}(h)-\frac{2}{5}(h-1)I_{1,1}(h),\\ I_{3,0}(h)=(\frac{1}{3}h^2-\frac{2}{3}h+1)I_{1,0}(h),\\ I_{0,4}(h)=2(2h^2-4h+3)I_{0,0}(h)+6(h-1)I_{1,0}(h),\\ I_{1,3}(h)=\frac{6}{5}I_{0,1}(h)+\frac{6}{5}(h-1)I_{1,1}(h),\\ I_{2,2}(h)=(h-1)I_{0,0}(h)-\frac{1}{3}(h^2-2h-3)I_{1,0}(h),\\ I_{3,1}(h)=-\frac{2}{35}(h-1)I_{0,1}(h)+\frac{1}{35}(8h^2-16h+33)I_{1,1}(h),\\ I_{4,0}(h)=\frac{3}{8}I_{0,0}(h)-\frac{1}{8}(2h^3-6h^2+9h-5)I_{1,0}(h),\end{cases}$$

and

$$\begin{cases}I_{0,5}(h)=\frac{4}{5}(5h^2-10h+8)I_{0,1}(h)+\frac{32}{5}(h-1)I_{1,1}(h),\\ I_{1,4}(h)=4(h-1)I_{0,0}(h)+\frac{4}{3}(h^2-2h+3)I_{1,0}(h),\\ I_{2,3}(h)=\frac{38}{35}(h-1)I_{0,1}(h)-\frac{2}{35}(6h^2-12h-19)I_{1,1}(h),\\ I_{3,2}(h)=\frac{3}{4}I_{0,0}(h)+\frac{1}{12}(2h^3-6h^2+13h-9)I_{1,0}(h),\\ I_{4,1}(h)=\frac{1}{105}(4h^2-8h+53)I_{0,1}(h)-\frac{4}{105}(4h^3-12h^2+21h-13)I_{1,1}(h),\\ I_{5,0}(h)=\frac{1}{15}(3h^4-12h^3+22h^2-20h+15)I_{1,0}(h),\\ I_{1,5}(h)=\frac{32}{7}(h-1)I_{0,1}(h)+\frac{4}{7}(3h^2-6h+8)I_{1,1}(h),\\ I_{2,4}(h)=\frac{1}{2}(4h^2-8h+7)I_{0,0}(h)-\frac{1}{6}(2h^3-6h^2-17h+21)I_{1,0}(h),\\ I_{3,3}(h)=-\frac{2}{105}(2h^2-4h-47)I_{0,1}(h)+\frac{2}{105}(8h^3-24h^2+63h-47)I_{1,1}(h),\\ I_{4,2}(h)=\frac{3}{4}(h-1)I_{0,0}(h)-\frac{1}{60}(6h^4-24h^3+49h^2-50h-45)I_{1,0}(h),\\ I_{5,1}(h)=-\frac{4}{1155}(8h^3-24h^2+41h-25)I_{0,1}(h)\\ \qquad\qquad+\frac{1}{1155}(128h^4-512h^3+1020h^2-1016h1055)I_{1,1}(h).\end{cases}$$

Hence, one has that

$$\begin{aligned}\sum_{i+j=0}^{5}\xi_{i,j}I_{i,j}(h)=&\sum_{i+j=0}^{4}\xi_{i,j}I_{i,j}(h)+\sum_{\substack{i+j=5,\\ i\geq1,j\geq1}}\xi_{i,j}I_{i,j}(h)\\ =&\big(\check{\alpha}_2h^2+\check{\alpha}_1h+\check{\alpha}_0\big)I_{0,0}(h)+\big(\check{\beta}_3h^3+\check{\beta}_2h^2+\check{\beta}_1h+\check{\beta}_0\big)I_{1,0}(h)\\&+\big(\check{\gamma}_2h^2+\check{\gamma}_1h+\check{\gamma}_0\big)I_{0,1}(h)+\big(\check{\delta}_3h^3+\check{\delta}_2h^2+\check{\delta}_1h+\check{\delta}_0\big)I_{1,1}(h),\end{aligned}$$

and

$$\begin{aligned}\sum_{i+j=0}^{6}\xi_{i,j}I_{i,j}(h)=&\sum_{i+j=0}^{5}\xi_{i,j}I_{i,j}(h)+\sum_{\substack{i+j=6,\\ i\geq 1,j\geq 1}}\xi_{i,j}I_{i,j}(h)\\ =&\big(\hat\alpha_2h^2+\hat\alpha_1h+\hat\alpha_0\big)I_{0,0}(h)\\ &+\big(\hat\beta_4h^4+\hat\beta_3h^3+\hat\beta_2h^2+\hat\beta_1h+\hat\beta_0\big)I_{1,0}(h)\\ &+\big(\hat\gamma_3h^3+\hat\gamma_2h^2+\hat\gamma_1h+\hat\gamma_0\big)I_{0,1}(h)\\ &+\big(\hat\delta_4h^4+\hat\delta_3h^3+\hat\delta_2h^2+\hat\delta_1h+\hat\delta_0\big)I_{1,1}(h),\end{aligned}$$

where

$$\begin{cases}\check\alpha_2=4\xi_{0,4},\ \check\alpha_1=2\xi_{0,2}-8\xi_{0,4}+\xi_{2,2}+4\xi_{1,4},\\ \check\alpha_0=\xi_{0,0}-2\xi_{0,2}+\frac12\xi_{2,0}+\xi_{1,2}-\xi_{2,2}+\frac38\xi_{4,0}+6\xi_{0,4}-4\xi_{1,4}+\frac34\xi_{2,2},\\ \check\beta_3=-\frac14\xi_{4,0}+\frac16\xi_{3,2},\check\beta_2=\frac13\xi_{3,0}-\frac13\xi_{2,2}+\frac34\xi_{4,0}+\frac43\xi_{1,4}-\frac12\xi_{3,2},\\ \check\beta_1=\xi_{1,2}-\frac23\xi_{3,0}-\frac12\xi_{2,0}+6\xi_{0,4}+\frac23\xi_{2,2}-\frac98\xi_{4,0}-\frac83\xi_{1,4}+\frac{13}{12}\xi_{3,2},\\ \check\beta_0=\xi_{1,0}+2\xi_{0,2}+\frac12\xi_{2,0}-\xi_{1,2}+\xi_{3,0}-6\xi_{0,4}+\xi_{2,2}+\frac58\xi_{4,0}+4\xi_{1,4}-\frac34\xi_{3,2},\\ \check\gamma_2=\frac{4}{105}\xi_{4,1},\ \check\gamma_1=2\xi_{0,3}-\frac{2}{35}\xi_{3,1}+\frac{38}{35}\xi_{2,3}-\frac{8}{105}\xi_{4,1},\\ \check\gamma_0=\xi_{0,1}-2\xi_{0,3}+\frac35\xi_{2,1}+\frac65\xi_{1,3}+\frac{2}{35}\xi_{3,1}-\frac{38}{35}\xi_{2,3}+\frac{53}{105}\xi_{4,1},\\ \check\delta_3=-\frac{16}{105}\xi_{4,1},\ \check\delta_2=\frac{8}{35}\xi_{3,1}-\frac{12}{35}\xi_{2,3}+\frac{16}{35}\xi_{4,1},\\ \check\delta_1=-\frac25\xi_{2,1}+\frac65\xi_{1,3}-\frac{16}{35}\xi_{3,1}+\frac{24}{35}\xi_{2,3}-\frac45\xi_{4,1},\\ \check\delta_0=\xi_{1,1}+\frac25\xi_{2,1}+2\xi_{0,3}-\frac65\xi_{1,3}+\frac{33}{35}\xi_{3,1}+\frac{38}{35}\xi_{2,3}+\frac{52}{105}\xi_{4,1},\end{cases}$$

and

$$\begin{cases}\hat\alpha_2=4\xi_{0,4}+2\xi_{2,4},\ \hat\alpha_1=2\xi_{0,2}-8\xi_{0,4}+\xi_{2,2}+4\xi_{1,4}+\frac34\xi_{4,2}-4\xi_{2,4},\\ \hat\alpha_0=\xi_{0,0}-2\xi_{0,2}+\frac12\xi_{2,0}+\xi_{1,2}-\xi_{2,2}+\frac38\xi_{4,0}\\ \qquad+6\xi_{0,4}-4\xi_{1,4}+\frac34\xi_{3,2}-\frac34\xi_{4,2}+\frac72\xi_{2,4},\\ \hat\beta_4=\frac15\xi_{5,0}-\frac{1}{10}\xi_{4,2},\ \hat\beta_3=-\frac14\xi_{4,0}+\frac16\xi_{3,2}-\frac45\xi_{5,0}+\frac25\xi_{4,2}-\frac13\xi_{2,4},\\ \hat\beta_2=\frac13\xi_{3,0}-\frac13\xi_{2,2}+\frac34\xi_{4,0}+\frac43\xi_{1,4}-\frac12\xi_{3,2}+\frac{22}{15}\xi_{5,0}-\frac{49}{60}\xi_{4,2}+\xi_{2,4},\\ \hat\beta_1=\xi_{1,2}-\frac23\xi_{3,0}-\frac12\xi_{2,0}+6\xi_{0,4}+\frac23\xi_{2,2}-\frac98\xi_{4,0}\\ \qquad-\frac83\xi_{1,4}+\frac{13}{12}\xi_{3,2}-\frac43\xi_{5,0}+\frac56\xi_{4,2}+\frac{17}{6}\xi_{2,4},\\ \hat\beta_0=\xi_{1,0}+2\xi_{0,2}+\frac12\xi_{2,0}-\xi_{1,2}+\xi_{3,0}-6\xi_{0,4}+\xi_{2,2}\\ \qquad+\frac58\xi_{4,0}+4\xi_{1,4}-\frac34\xi_{3,2}+\xi_{5,0}+\frac34\xi_{4,2}-\frac72\xi_{2,4},\\ \hat\gamma_3=-\frac{32}{1155}\xi_{5,1},\ \hat\gamma_2=4\xi_{0,5}+\frac{4}{105}\xi_{4,1}+\frac{32}{385}\xi_{5,1}-\frac{4}{105}\xi_{3,3},\\ \hat\gamma_1=2\xi_{0,3}-\frac{2}{35}\xi_{3,1}-8\xi_{0,5}+\frac{38}{35}\xi_{2,3}-\frac{8}{105}\xi_{4,1}-\frac{164}{1155}\xi_{5,1}+\frac{8}{105}\xi_{3,3}+\frac{32}{7}\xi_{1,5},\\ \hat\gamma_0=\xi_{0,1}-2\xi_{0,3}+\frac35\xi_{2,1}+\frac65\xi_{1,3}+\frac{2}{35}\xi_{3,1}+\frac{32}{5}\xi_{0,5}-\frac{38}{35}\xi_{2,3}\\ \qquad+\frac{53}{105}\xi_{4,1}+\frac{20}{231}\xi_{5,1}+\frac{94}{105}\xi_{3,3}-\frac{32}{7}\xi_{1,5},\\ \hat\delta_4=\frac{128}{1155}\xi_{5,1},\ \hat\delta_3=-\frac{16}{105}\xi_{4,1}-\frac{512}{1155}\xi_{5,1}+\frac{16}{105}\xi_{3,3},\\ \hat\delta_2=\frac{8}{35}\xi_{3,1}-\frac{12}{35}\xi_{2,3}+\frac{16}{35}\xi_{4,1}+\frac{68}{77}\xi_{5,1}-\frac{16}{35}\xi_{3,3}+\frac{12}{7}\xi_{1,5},\\ \hat\delta_1=-\frac25\xi_{2,1}+\frac65\xi_{1,3}-\frac{16}{35}\xi_{3,1}+\frac{32}{5}\xi_{0,5}+\frac{24}{35}\xi_{2,3}-\frac45\xi_{4,1}\\ \qquad-\frac{1016}{1155}\xi_{5,1}+\frac65\xi_{3,3}-\frac{24}{7}\xi_{1,5},\\ \hat\delta_0=\xi_{1,1}+\frac25\xi_{2,1}+2\xi_{0,3}-\frac65\xi_{1,3}+\frac{33}{35}\xi_{3,1}-\frac{32}{5}\xi_{0,5}+\frac{38}{35}\xi_{2,3}\\ \qquad+\frac{52}{105}\xi_{4,1}+\frac{211}{231}\xi_{5,1}-\frac{94}{105}\xi_{3,3}+\frac{32}{7}\xi_{1,5}.\end{cases}$$

A simple computation shows that

$$\det \mathbf{A}_4 = \det\left(\frac{\partial(\hat\alpha_2,\hat\alpha_1,\hat\alpha_0,\hat\beta_3,\hat\beta_2,\hat\beta_1,\hat\beta_0,\hat\delta_3,\hat\delta_2,\hat\delta_1,\hat\delta_0,\hat\gamma_1,\hat\gamma_0)}{\partial(\xi_{0,4},\xi_{0,2},\xi_{0,0},\xi_{4,0},\xi_{3,0},\xi_{2,0},\xi_{1,0},\xi_{4,1},\xi_{3,1},\xi_{2,1},\xi_{1,1},\xi_{2,3},\xi_{1,3})}\right) = \frac{17664}{214375},$$

and

$$\mathbf{A}_5 = \frac{\partial(\hat\alpha_2,\hat\alpha_1,\hat\alpha_0,\hat\beta_3,\hat\beta_2,\hat\beta_1,\hat\beta_0,\hat\delta_3,\hat\delta_2,\hat\delta_1,\hat\delta_0,\hat\gamma_1,\hat\gamma_0,\hat\beta_{\mathbf{4}},\hat\delta_{\mathbf{4}},\hat\gamma_{\mathbf{2}})}{\partial(\xi_{0,4},\xi_{0,2},\xi_{0,0},\xi_{4,0},\xi_{3,0},\xi_{2,0},\xi_{1,0},\xi_{4,1},\xi_{3,1},\xi_{2,1},\xi_{1,1},\xi_{2,3},\xi_{1,3},\xi_{\mathbf{5},\mathbf{0}},\xi_{\mathbf{5},\mathbf{1}},\xi_{\mathbf{3},\mathbf{3}})} = \begin{pmatrix} \mathbf{A}_4 & \bar{\mathbf{A}}_1 & \bar{\mathbf{A}}_2 & \bar{\mathbf{A}}_3 \\ \mathbf{0} & \frac{1}{5} & 0 & 0 \\ \mathbf{0} & 0 & \frac{128}{1155} & 0 \\ \mathbf{0} & 0 & \frac{32}{385} & -\frac{4}{105} \end{pmatrix},$$

where $\mathbf{0}$ is a row vector, and $\bar{\mathbf{A}}_i, i = 1,2,3$ are column vectors. A straightforward calculation gives that

$$\det \mathbf{A}_5 = -\det \mathbf{A}_4 \times \frac{1}{5} \times \frac{128}{1155} \times \frac{4}{105} = \frac{3014656}{43330546875}.$$

The above discussions yield that the claim holds for $n = 4, 5$.

Now assume that the claim holds for $\sum_{i+j=0}^{k+1} \xi_{i,j} I_{i,j}(h), k \le n-1 (n \ge 4)$. In view of (2.7) and (2.8), one gets that

$$\begin{pmatrix} I_{0,n} \\ I_{1,n} \\ I_{2,n-1} \\ \vdots \\ I_{n-1,2} \\ I_{n,1} \\ I_{n,0} \end{pmatrix} = \begin{pmatrix} 2(h-1)I_{0,n-2} + 2I_{1,n-2} \\ 2(h-1)I_{1,n-2} + 2I_{2,n-2} \\ 2(h-1)I_{2,n-3} + 2I_{3,n-3} \\ \vdots \\ 2(h-1)I_{n-1,0} + 2I_{n,0} \\ \frac{1}{2n+1}\left[(2n-1)I_{n-2,1} - (2n-2)(h-1)I_{n-1,1} + (2n-4)(h-1)I_{n-3,1}\right] \\ \frac{1}{n}\left[(n-1)I_{n-2,0} - (n-1)(h-1)I_{n-1,0} + (n-2)(h-1)I_{n-3,0}\right] \end{pmatrix}.$$

Therefore, by the induction hypothesis and noticing that $I_{0,n}(h)$ and $I_{n,0}(h)$ do not appear in $\sum\limits_{i+j=0}^{k+1}\xi_{i,j}I_{i,j}(h), k\le n-1$, one finds that, for $k=n$

$$\begin{aligned}
\sum_{i+j=0}^{n+1}\xi_{i,j}I_{i,j}(h)=&\sum_{i+j=0}^{n-1}\xi_{i,j}I_{i,j}(h)+\sum_{\substack{i+j=1\\ i\ge1,j\ge1}}^{n}\xi_{i,j}I_{i,j}(h)\\
&+\xi_{0,n}I_{0,n}(h)+\xi_{n,0}I_{n,0}(h)+\sum_{\substack{i+j=n+1\\ i\ge1,j\ge1}}\xi_{i,j}I_{i,j}(h)\\
=&\Big(\sum_{i=0}^{[\frac{n-1}{2}]}\tilde{\alpha}_ih^i\Big)I_{0,0}+\Big(\sum_{i=0}^{n-2}\tilde{\beta}_ih^i\Big)I_{1,0}+\Big(\sum_{i=0}^{n-3}\tilde{\gamma}_ih^i\Big)I_{0,1}+\Big(\sum_{i=0}^{n-2}\tilde{\delta}_ih^i\Big)I_{1,1}\\
&+2\xi_{0,n}[(h-1)I_{0,n-2}(h)+I_{1,n-2}(h)]+\frac{\xi_{n,0}}{n}[(n-1)I_{n-2,0}(h)\\
&-(n-1)(h-1)I_{n-1,0}(h)+(n-2)(h-1)I_{n-3,0}(h)]\\
&+2\xi_{1,n}[2(h-1)I_{1,n-2}(h)+I_{2,n-2}(h)]\\
&+2\xi_{2,n-1}[2(h-1)I_{2,n-3}(h)+I_{3,n-3}(h)]+\cdots\\
&+\frac{\xi_{n,1}}{2n+1}[(2n-1)I_{n-2,1}(h)-2(n-1)(h-1)I_{n-1,1}(h)\\
&+2(n-2)(h-1)I_{n-3,1}(h)]\\
\triangleq&\Big(\sum_{i=0}^{[\frac{n}{2}]}\bar{\alpha}_ih^i\Big)I_{0,0}(h)+\Big(\sum_{i=0}^{n-1}\bar{\beta}_ih^i\Big)I_{1,0}(h)\\
&+\Big(\sum_{i=0}^{n-2}\bar{\gamma}_ih^i\Big)I_{0,1}(h)+\Big(\sum_{i=0}^{n-1}\bar{\delta}_ih^i\Big)I_{1,1}(h),
\end{aligned}\tag{2.9}$$

where $\tilde{\alpha}_i$, $\tilde{\beta}_i$, $\tilde{\gamma}_i$, $\tilde{\delta}_i$, $\bar{\alpha}_i$, $\bar{\beta}_i$, $\bar{\gamma}_i$ and $\bar{\delta}_i$ are constants.

Next, we will prove that $\bar{\alpha}_i$, $i=0,1,2,\cdots,[\frac{n}{2}]$, $\bar{\beta}_i$, $i=0,1,2,\cdots,n-1$, $\bar{\gamma}_i$, $i=0,1,2,\cdots,n-3$ and $\bar{\delta}_i$, $i=0,1,2,\cdots,n-1$ can be taken as free parameters. In fact, by the induction hypothesis, one obtains that $\tilde{\alpha}_i$, $i=0,1,2,\cdots,[\frac{n-1}{2}]$, $\tilde{\beta}_i$, $i=0,1,2,\cdots,n-2$, $\tilde{\gamma}_i$, $i=0,1,2,\cdots,n-4$ and $\tilde{\delta}_i$, $i=0,1,2,\cdots,n-2$ are independent of each other. That is, the determinant of the following Jacobian matrix

$$\mathbf{A}_0=\frac{\partial\Big(\tilde{\alpha}_{[\frac{n-1}{2}]},\cdots,\tilde{\alpha}_0,\tilde{\beta}_{n-2},\cdots,\tilde{\beta}_0,\tilde{\delta}_{n-2},\cdots,\tilde{\delta}_0,\tilde{\gamma}_{n-4},\cdots,\tilde{\gamma}_0\Big)}{\partial\Big(\xi_{i_0,j_{[\frac{n-1}{2}]}},\cdots,\xi_{i_{[\frac{n-1}{2}]},j_0},\xi_{k_0,l_{n-2}},\cdots,\xi_{k_{n-2},l_0},\xi_{p_0,q_{n-2}},\cdots,\xi_{p_{n-2},q_0},\xi_{s_0,t_{n-4}},\cdots,\xi_{s_{n-4},t_0}\Big)}$$

is not equal to zero. Here $\xi_{0,n}$ and $\xi_{n,0}$ do not appear in the above Jacobian matrix $\mathbf{A}_0$, and the sum of subscripts of $\xi_{i,j}$ in the Jacobian matrix is less than or equal to n . When $n\ge4$ is an even number, some explicit computations give the following

Jacobian matrix

$$\mathbf{A} = \frac{\partial\Big(\bar{\alpha}_{[\frac{n-1}{2}]},\cdots,\bar{\alpha}_0,\bar{\beta}_{n-2},\cdots,\bar{\beta}_0,\bar{\delta}_{n-2},\cdots,\bar{\delta}_0, \bar{\gamma}_{n-4},\cdots,\bar{\gamma}_0,\bar{\alpha}_{[\frac{\mathbf{n}}{\mathbf{2}}]},\bar{\beta}_{\mathbf{n-1}},\bar{\delta}_{\mathbf{n-1}},\bar{\gamma}_{\mathbf{n-3}}\Big)}{\partial\Big(\xi_{i_0,j_{[\frac{n-1}{2}]}},\cdots,\xi_{i_{[\frac{n-1}{2}]},j_0},\xi_{k_0,l_{n-2}},\cdots,\xi_{k_{n-2},l_0},\xi_{p_0,q_{n-2}},\cdots,\xi_{p_{n-2},q_0}, \xi_{s_0,t_{n-4}},\cdots,\xi_{s_{n-4},t_0},\xi_{\mathbf{0},\mathbf{n}},\xi_{\mathbf{n},\mathbf{0}},\xi_{\mathbf{n},\mathbf{1}},\xi_{\mathbf{n-2},\mathbf{3}}\Big)}$$

$$= \begin{pmatrix} \mathbf{A}_0 & \mathbf{A}_1 & \mathbf{A}_2 & \mathbf{A}_3 & \mathbf{A}_4 \\ \mathbf{0} & 2^{[\frac{n}{2}]} & 0 & 0 & 0 \\ \mathbf{0} & 0 & -\frac{1}{n} & 0 & 0 \\ \mathbf{0} & 0 & 0 & \prod\limits_{i=2}^{n}\frac{2(1-i)}{2i+1} & 0 \\ \mathbf{0} & 0 & 0 & * & \nu \end{pmatrix},$$

in view of (2.7) and (2.8), where $\mathbf{0}$ is a row vector, $\mathbf{A}_i, i = 1,2,3,4$ are column vectors and

$$\nu = \begin{cases} \frac{38}{35}, & n = 4, \\ (-1)^n \frac{9}{2}\frac{(2n-6)!!}{(2n-1)!!}, & n \geq 5. \end{cases} \tag{2.10}$$

It is easy to get that

$$|\mathbf{A}| = -\frac{2^{[\frac{n}{2}]}\upsilon}{n}\prod_{i=2}^{n}\frac{2(1-i)}{2i+1}|\mathbf{A}_0| \neq 0,$$

which implies that $\bar{\alpha}_i,\ i = 0,1,2,\cdots,[\frac{n}{2}],\ \bar{\beta}_j,\ j = 0,1,2,\cdots,n-1,\ \bar{\gamma}_i,\ i = 0,1,2,\cdots,n-3$ and $\bar{\delta}_i,\ i = 0,1,2,\cdots,n-1$ can be taken as free parameters. When $n \geq 4$ is an odd number, a similar calculation gives that

$$\mathbf{A} = \frac{\partial\Big(\bar{\alpha}_{[\frac{n-1}{2}]},\cdots,\bar{\alpha}_0,\bar{\beta}_{n-2},\cdots,\bar{\beta}_0,\bar{\delta}_{n-3},\cdots,\bar{\delta}_0, \bar{\gamma}_{n-5},\cdots,\bar{\gamma}_0,\bar{\beta}_{\mathbf{n-1}},\bar{\delta}_{\mathbf{n-1}},\bar{\gamma}_{\mathbf{n-3}}\Big)}{\partial\Big(\xi_{i_0,j_{[\frac{n-1}{2}]}},\cdots,\xi_{i_{[\frac{n-1}{2}]},j_0},\xi_{k_0,l_{n-2}},\cdots,\xi_{k_{n-2},l_0},\xi_{p_0,q_{n-3}},\cdots,\xi_{p_{n-3},q_0}, \xi_{s_0,t_{n-5}},\cdots,\xi_{s_{n-5},t_0},\xi_{\mathbf{n},\mathbf{0}},\xi_{\mathbf{n},\mathbf{1}},\xi_{\mathbf{n-2},\mathbf{3}}\Big)}$$

$$= \begin{pmatrix} \mathbf{A}_0 & \mathbf{A}_5 & \mathbf{A}_6 & \mathbf{A}_7 \\ \mathbf{0} & \frac{1}{n} & 0 & 0 \\ \mathbf{0} & 0 & \prod\limits_{i=2}^{n-1}\frac{2(1-i)}{2i+1} & 0 \\ \mathbf{0} & 0 & * & \nu \end{pmatrix},$$

where $\mathbf{A}_i, i = 5,6,7$ are column vectors. And

$$|\mathbf{A}| = \frac{\upsilon}{n}\prod_{i=2}^{n-1}\frac{2(1-i)}{2i+1}|\mathbf{A}_0| \neq 0.$$

This ends the proof. □

Remark 2.1. When $n = 5$, in view of (2.7) and (2.8), one can obtain that the coefficient polynomials of $I_{0,1}(h)$ and $I_{1,1}(h)$ are $-\frac{32}{1155}\xi_{5,1}h^3 + \cdots$ and $\frac{128}{1155}\xi_{5,1}h^4 + \cdots$, respectively. That is, $\gamma_3 = -\frac{32}{1155}\xi_{5,1}$ and $\delta_4 = \frac{128}{1155}\xi_{5,1}$, which shows that γ_3 and δ_4 are interdependent. Hence, all the coefficients of the coefficient polynomials of $I_{0,0}(h)$, $I_{1,0}(h)$, $I_{0,1}(h)$ and $I_{1,1}(h)$ in (2.1) can not be chosen arbitrarily unless one of γ_{n-2} and δ_{n-1} is removed.

Next we want to express the first order Melnikov functions in terms of the complete elliptic integrals of first and second kind K and E. We need a result derived in [2], which we state in Lemma 2.2 below.

Lemma 2.2. *[2]. Let the complete elliptic integrals of first and second kind be*

$$K(k) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 - k^2 \sin^2\theta}}, \quad E(k) = \int_0^{\frac{\pi}{2}} \sqrt{1 - k^2 \sin^2\theta}\, d\theta, \; k \in (-1, 1). \tag{2.11}$$

Then, the following statements hold:

(1) Let $h \in (0, 2)$. Then

$$\begin{aligned} I_{0,1}(h) &= 2\sqrt{2}\big[(h-2)K(\sqrt{h/2}) + 2E(\sqrt{h/2})\big], \\ I_{1,1}(h) &= \frac{2\sqrt{2}}{3}\big[(2-h)K(\sqrt{h/2}) + 2(h-1)E(\sqrt{h/2})\big]; \end{aligned} \tag{2.12}$$

(2) Let $h \in (2, +\infty)$. Then

$$\begin{aligned} J_{0,1}(h) &= 2\sqrt{2h}E(\sqrt{2/h}), \\ J_{1,1}(h) &= \frac{2}{3}\sqrt{2h}\big[(2-h)K(\sqrt{2/h}) + (h-1)E(\sqrt{2/h})\big]. \end{aligned} \tag{2.13}$$

Proof of Theorem 1.1. (i) A straightforward calculation yields that

$$I_{0,0}(h) = 2\arccos(1-h), \; I_{1,0}(h) = 2\sqrt{2h - h^2}. \tag{2.14}$$

Inserting (2.12) and (2.14) into (2.1) gives that (1.9). Moreover,

$$\begin{aligned} & a_i = 2\alpha_i, \; 0 \le i \le [\frac{n}{2}], \; b_i = 2\beta_i, \; 0 \le i \le n-1, \\ & c_0 = -4\sqrt{2}\gamma_0 + \frac{4\sqrt{2}}{3}\delta_0, \\ & c_i = 2\sqrt{2}\gamma_{i-1} - 4\sqrt{2}\gamma_i - \frac{2\sqrt{2}}{3}\delta_{i-1} + \frac{4\sqrt{2}}{3}\delta_i, \; 1 \le i \le n-2, \\ & c_{n-1} = 2\sqrt{2}\gamma_{n-2} - \frac{2\sqrt{2}}{3}\delta_{n-2} + \frac{4\sqrt{2}}{3}\delta_{n-1}, \; c_n = -\frac{2\sqrt{2}}{3}\delta_{n-1}, \\ & d_0 = 4\sqrt{2}\gamma_0 - \frac{4\sqrt{2}}{3}\delta_0, \\ & d_i = 4\sqrt{2}\gamma_i + \frac{4\sqrt{2}}{3}\delta_{i-1} - \frac{4\sqrt{2}}{3}\delta_i, \; 1 \le i \le n-2, \\ & d_{n-1} = \frac{4\sqrt{2}}{3}\delta_{n-2} - \frac{4\sqrt{2}}{3}\delta_{n-1}, \; d_n = \frac{4\sqrt{2}}{3}\delta_{n-1}. \end{aligned} \tag{2.15}$$

Now the independence of the constant coefficients in (1.9) follows by induction. Indeed, using (2.15), one finds that

$$\mathbf{B}_3=\frac{\partial\Big(d_0,d_1,d_2,c_1\Big)}{\partial\Big(\delta_0,\delta_1,\delta_2,\gamma_0\Big)}=\begin{pmatrix}-\frac{4\sqrt{2}}{3} & 0 & 0 & 4\sqrt{2}\\ \frac{4\sqrt{2}}{3} & -\frac{4\sqrt{2}}{3} & 0 & 0\\ 0 & \frac{4\sqrt{2}}{3} & -\frac{4\sqrt{2}}{3} & 0\\ -\frac{2\sqrt{2}}{3} & \frac{4\sqrt{2}}{3} & 0 & 2\sqrt{2}\end{pmatrix},$$

$$\mathbf{B}_4=\frac{\partial\Big(d_0,d_1,d_2,c_1,\mathbf{d_3},\mathbf{c_2}\Big)}{\partial\Big(\delta_0,\delta_1,\delta_2,\gamma_0,\delta_{\mathbf{3}},\gamma_{\mathbf{1}}\Big)}=\left(\begin{array}{cccc|cc}-\frac{4\sqrt{2}}{3} & 0 & 0 & 4\sqrt{2} & 0 & 0\\ \frac{4\sqrt{2}}{3} & -\frac{4\sqrt{2}}{3} & 0 & 0 & 0 & 4\sqrt{2}\\ 0 & \frac{4\sqrt{2}}{3} & -\frac{4\sqrt{2}}{3} & 0 & 0 & 0\\ -\frac{2\sqrt{2}}{3} & \frac{4\sqrt{2}}{3} & 0 & 2\sqrt{2} & 0 & -4\sqrt{2}\\ \hline 0 & 0 & \frac{4\sqrt{2}}{3} & 0 & -\frac{4\sqrt{2}}{3} & 0\\ 0 & -\frac{2\sqrt{2}}{3} & \frac{4\sqrt{2}}{3} & 0 & 0 & 2\sqrt{2}\end{array}\right).$$

A direct calculation shows that $|\mathbf{B}_3|=-\frac{1024}{27}$. In the matrix $\mathbf{B}_4$, multiplying the second and third columns by three, respectively, and then adding them to the sixth column, one gets that

$$\bar{\mathbf{B}}_4=\left(\begin{array}{cccc|cc}-\frac{4\sqrt{2}}{3} & 0 & 0 & 4\sqrt{2} & 0 & 0\\ \frac{4\sqrt{2}}{3} & -\frac{4\sqrt{2}}{3} & 0 & 0 & 0 & 0\\ 0 & \frac{4\sqrt{2}}{3} & -\frac{4\sqrt{2}}{3} & 0 & 0 & 0\\ -\frac{2\sqrt{2}}{3} & \frac{4\sqrt{2}}{3} & 0 & 2\sqrt{2} & 0 & 0\\ \hline 0 & 0 & \frac{4\sqrt{2}}{3} & 0 & -\frac{4\sqrt{2}}{3} & 4\sqrt{2}\\ 0 & -\frac{2\sqrt{2}}{3} & \frac{4\sqrt{2}}{3} & 0 & 0 & 4\sqrt{2}\end{array}\right).$$

By the property of determinant, one obtains that

$$|\mathbf{B}_4|=|\bar{\mathbf{B}}_4|=\frac{32768}{81}.$$

Thus, d_0,d_1,d_2,d_3,c_1 and c_2 are independent. Now assume that the statement holds for all $i\le n-1$ $(n\ge 4)$. That is, the determinant of the following matrix

$$\mathbf{B}_0=\frac{\partial\Big(d_0,d_1,\cdots,d_{n-3},d_{n-2},c_1,c_2,\cdots,c_{n-3}\Big)}{\partial\Big(\delta_0,\delta_1,\cdots,\delta_{n-3},\delta_{n-2},\gamma_0,\gamma_1,\cdots,\gamma_{n-4}\Big)}$$

is different from zero. In view of (2.15), one has that

$$\mathbf{B} = \frac{\partial\Big(d_0, d_1, \cdots, d_{n-3}, d_{n-2}, c_1, c_2, \cdots, c_{n-3}, \mathbf{d_{n-1}}, \mathbf{c_{n-2}}\Big)}{\partial\Big(\delta_0, \delta_1, \cdots, \delta_{n-3}, \delta_{n-2}, \gamma_0, \gamma_1, \cdots, \gamma_{n-4}, \boldsymbol{\delta_{n-1}}, \boldsymbol{\gamma_{n-3}}\Big)}$$

$$= \left(\begin{array}{ccccccccc|cc}
-\frac{4\sqrt{2}}{3} & 0 & \cdots & 0 & 0 & 4\sqrt{2} & 0 & \cdots & 0 & 0 & 0 \\
\frac{4\sqrt{2}}{3} & -\frac{4\sqrt{2}}{3} & \cdots & 0 & 0 & 0 & 4\sqrt{2} & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & -\frac{4\sqrt{2}}{3} & 0 & 0 & 0 & \cdots & 0 & 0 & 4\sqrt{2} \\
0 & 0 & \cdots & \frac{4\sqrt{2}}{3} & -\frac{4\sqrt{2}}{3} & 0 & 0 & \cdots & 0 & 0 & 0 \\
-\frac{2\sqrt{2}}{3} & \frac{4\sqrt{2}}{3} & \cdots & 0 & 0 & 2\sqrt{2} & -4\sqrt{2} & \cdots & 0 & 0 & 0 \\
0 & -\frac{2\sqrt{2}}{3} & \cdots & 0 & 0 & 0 & 2\sqrt{2} & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & \frac{4\sqrt{2}}{3} & 0 & 0 & 0 & \cdots & 2\sqrt{2} & 0 & -4\sqrt{2} \\
\hline
0 & 0 & \cdots & 0 & \frac{4\sqrt{2}}{3} & 0 & 0 & \cdots & 0 & -\frac{4\sqrt{2}}{3} & 0 \\
0 & 0 & \cdots & -\frac{2\sqrt{2}}{3} & \frac{4\sqrt{2}}{3} & 0 & 0 & \cdots & 0 & 0 & 2\sqrt{2}
\end{array}\right).$$

Similarly, in the matrix $\mathbf{B}$, multiplying the $(n-2)$th and $(n-1)$th columns by three, respectively, and then adding them to the last column, one gets that

$$\bar{\mathbf{B}} = \left(\begin{array}{ccccccccc|cc}
-\frac{4\sqrt{2}}{3} & 0 & \cdots & 0 & 0 & 4\sqrt{2} & 0 & \cdots & 0 & 0 & 0 \\
\frac{4\sqrt{2}}{3} & -\frac{4\sqrt{2}}{3} & \cdots & 0 & 0 & 0 & 4\sqrt{2} & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & -\frac{4\sqrt{2}}{3} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & \cdots & \frac{4\sqrt{2}}{3} & -\frac{4\sqrt{2}}{3} & 0 & 0 & \cdots & 0 & 0 & 0 \\
-\frac{2\sqrt{2}}{3} & \frac{4\sqrt{2}}{3} & \cdots & 0 & 0 & 2\sqrt{2} & -4\sqrt{2} & \cdots & 0 & 0 & 0 \\
0 & -\frac{2\sqrt{2}}{3} & \cdots & 0 & 0 & 0 & 2\sqrt{2} & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & \frac{4\sqrt{2}}{3} & 0 & 0 & 0 & \cdots & 2\sqrt{2} & 0 & 0 \\
\hline
0 & 0 & \cdots & 0 & \frac{4\sqrt{2}}{3} & 0 & 0 & \cdots & 0 & -\frac{4\sqrt{2}}{3} & 4\sqrt{2} \\
0 & 0 & \cdots & -\frac{2\sqrt{2}}{3} & \frac{4\sqrt{2}}{3} & 0 & 0 & \cdots & 0 & 0 & 4\sqrt{2}
\end{array}\right).$$

Hence,

$$|\mathbf{B}| = |\bar{\mathbf{B}}| = -\frac{32}{3}|\mathbf{B}_0| \neq 0.$$

Observe that a_i is expressed by α_i and b_i is expressed by β_i. One has that a_i $(i = 0, 1, \cdots, [\frac{n}{2}])$, b_i $(i = 0, 1, \cdots, n-1)$, c_i $(i = 1, 2, \cdots, n-2)$ and d_i $(i = 0, 1, \cdots, n-1)$ can be chosen arbitrarily.

The proof of the conclusion (ii) follows by using the same arguments which we omit for the sake of brevity. This ends the proof of Theorem 1.1.

3. The lower bounds for the number of limit cycles

Once we are able to get the desired expansions for $M_0(h)$ and $M_\pm(h)$ with $n = 1, 2, 3$, we may use some known results [5, 7, 14, 15, 17] to obtain the lower bounds of the number of limit cycles of system (1.5).

Lemma 3.1. *For $n = 3$, the following statements hold.*

(i) If $0 < h \ll 1$, then

$$\begin{aligned} M_0(h) =&\sigma_1 h^{\frac{1}{2}} + \sigma_2 h + \sigma_3 h^{\frac{3}{2}} + \sigma_4 h^2 + \sigma_5 h^{\frac{5}{2}} + \sigma_6 h^3 \\ &+ \sigma_7 h^{\frac{7}{2}} + \sigma_8 h^4 + \sigma_9 h^{\frac{9}{2}} + \sigma_{10} h^{10} + o(h^{10}), \end{aligned} \tag{3.1}$$

where σ_i, $i = 1, 2, \cdots, 10$ are constants and can be chosen arbitrarily.

(ii) If $0 < h - 2 \ll 1$, then

$$\begin{aligned} M_\pm(h) =&\rho_0^\pm + \rho_1(h-2) + \rho_2^\pm(h-2)\ln(h-2) + \rho_3^\pm(h-2)^2 \\ &+ \rho_4^\pm(h-2)^2\ln(h-2) + \rho_5^\pm(h-2)^3 + o((h-2)^3), \end{aligned} \tag{3.2}$$

where $\rho_i^\pm$, $i = 0, 1, \cdots, 5$ are constants and can be chosen arbitrarily.

Proof. (i) By (2.9), one gets that

$$\begin{aligned} M_0(h) =& \sum_{i+j=0}^{3} \xi_{i,j} I_{i,j}(h) + \sum_{\substack{i+j=4,\\ i\geq 1, j\geq 1}} \xi_{i,j} I_{i,j}(h) \\ =&\big[(2\xi_{0,2} + \xi_{2,2})h + \xi_{0,0} - 2\xi_{0,2} + \frac{1}{2}\xi_{2,0} + \xi_{1,2} - \xi_{2,2}\big] I_{0,0}(h) \\ &+ \big[\frac{1}{3}(\xi_{3,0} - \xi_{2,2})h^2 + \big(\xi_{1,2} - \frac{2}{3}\xi_{3,0} - \frac{1}{2}\xi_{2,0} + \frac{2}{3}\xi_{2,2}\big)h \\ &+ \xi_{1,0} + 2\xi_{0,2} + \frac{1}{2}\xi_{2,0} - \xi_{1,2} + \xi_{3,0} + \xi_{2,2}\big] I_{1,0}(h) \\ &+ \big[(2\xi_{0,3} - \frac{2}{35}\xi_{3,1})h + \xi_{0,1} - 2\xi_{0,3} + \frac{3}{5}\xi_{2,1} + \frac{6}{5}\xi_{1,3} + \frac{2}{35}\xi_{3,1}\big] I_{0,1}(h) \\ &+ \big[\frac{8}{35}\xi_{3,1}h^2 + \big(\frac{6}{5}\xi_{1,3} - \frac{16}{35}\xi_{3,1} - \frac{2}{5}\xi_{2,1}\big)h \\ &+ \xi_{1,1} + \frac{2}{5}\xi_{2,1} + 2\xi_{0,3} - \frac{6}{5}\xi_{1,3} + \frac{33}{35}\xi_{3,1}\big] I_{1,1}(h) \\ =&\big(\alpha_1 h + \alpha_0\big) I_{0,0}(h) + \big(\beta_2 h^2 + \beta_1 h + \beta_0\big) I_{1,0}(h) \\ &+ \big(\gamma_1 h + \gamma_0\big) I_{0,1}(h) + \big(\delta_2 h^2 + \delta_1 h + \delta_0\big) I_{1,1}(h). \end{aligned} \tag{3.3}$$

It is easy to check that α_i $(i = 0, 1)$, β_j $(j = 0, 1, 2)$, γ_k $(k = 0, 1)$ and δ_l $(l = 0, 1, 2)$ are independent of each other.

With the help of the command "series" in Maple 18, one has the following generalized series expansions of $I_{0,0}(h)$ and $I_{1,0}(h)$ for $0 < h \ll 1$

$$\begin{aligned} I_{0,0}(h) &= 2\sqrt{2}h^{\frac{1}{2}} + \frac{\sqrt{2}}{6}h^{\frac{3}{2}} + \frac{3\sqrt{2}}{80}h^{\frac{5}{2}} + \frac{5\sqrt{2}}{448}h^{\frac{7}{2}} + \frac{35\sqrt{2}}{9216}x^{\frac{9}{2}} + o(h^{\frac{9}{2}}), \\ I_{1,0}(h) &= 2\sqrt{2}h^{\frac{1}{2}} - \frac{\sqrt{2}}{2}h^{\frac{3}{2}} - \frac{\sqrt{2}}{16}h^{\frac{5}{2}} - \frac{\sqrt{2}}{64}h^{\frac{7}{2}} - \frac{5\sqrt{2}}{1024}h^{\frac{9}{2}} + o(h^{\frac{9}{2}}). \end{aligned} \tag{3.4}$$

To prove the desired conclusion, it suffices to obtain the expansions of $K(k)$ and $E(k)$ for $k \in (-1, 1)$. Taking $x = k^2 \sin^2\theta$ in the following series

$$(1-x)^{-\frac{1}{2}} = 1 + \frac{1}{2}x + \frac{3!!}{4!!}x^2 + \cdots + \frac{(2n-1)!!}{(2n)!!}x^n + \cdots,\ x \in [-1, 1),$$

one gets that, for $k \in (-1, 1)$,

$$\begin{aligned}(1-k^2\sin^2\theta)^{-\frac{1}{2}} =& 1 + \frac{1}{2}k^2\sin^2\theta + \frac{3!!}{4!!}k^4\sin^4\theta + \cdots \\ &+ \frac{(2n-1)!!}{(2n)!!}k^{2n}\sin^{2n}\theta + \cdots.\end{aligned} \tag{3.5}$$

Since the series (3.5) is uniformly convergent, integrating it item by item from 0 to $\frac{\pi}{2}$ gives that

$$K(k) = \frac{\pi}{2} + \frac{\pi}{2}\sum_{n=1}^{+\infty}\Big(\frac{(2n-1)!!}{(2n)!!}\Big)^2 k^{2n},\ k \in (-1, 1). \tag{3.6}$$

Similarly, one obtains that

$$E(k) = \frac{\pi}{2} - \frac{\pi}{2}\sum_{n=1}^{+\infty}\Big(\frac{(2n-1)!!}{(2n)!!}\Big)^2 \frac{k^{2n}}{2n-1},\ k \in (-1, 1), \tag{3.7}$$

in view of

$$(1-x)^{\frac{1}{2}} = 1 - \frac{1}{2}x - \frac{1}{4!!}x^2 - \frac{3!!}{6!!}x^3 - \cdots - \frac{(2n-3)!!}{(2n)!!}x^n - \cdots,\ x \in [-1, 1].$$

A direct calculation using (2.12), (3.3), (3.4), (3.6) and (3.7) yields that

$$\begin{aligned}M_0(h) =& \sigma_1 h^{\frac{1}{2}} + \sigma_2 h + \sigma_3 h^{\frac{3}{2}} + \sigma_4 h^2 + \sigma_5 h^{\frac{5}{2}} \\ &+ \sigma_6 h^3 + \sigma_7 h^{\frac{7}{2}} + \sigma_8 h^4 + \sigma_9 h^{\frac{9}{2}} + \sigma_{10} h^5 + o(h^5),\end{aligned}$$

where

$$\begin{aligned}
\sigma_1 &= 2\sqrt{2}(\alpha_0 + \beta_0),\ \sigma_2 = \frac{\sqrt{2}\pi}{2}(\gamma_0 + \delta_0),\\
\sigma_3 &= \frac{\sqrt{2}}{6}\alpha_0 + 2\sqrt{2}\alpha_1 - \frac{\sqrt{2}}{2}\beta_0 + 2\sqrt{2}\beta_1,\\
\sigma_4 &= \frac{\sqrt{2}\pi}{32}\gamma_0 + \frac{\sqrt{2}\pi}{2}\gamma_1 - \frac{3\sqrt{2}\pi}{32}\delta_0 + \frac{\sqrt{2}\pi}{2}\delta_1,\\
\sigma_5 &= \frac{3\sqrt{2}}{80}\alpha_0 + \frac{\sqrt{2}}{6}\alpha_1 - \frac{\sqrt{2}}{16}\beta_0 - \frac{\sqrt{2}}{2}\beta_1 + 2\sqrt{2}\beta_2,\\
\sigma_6 &= \frac{3\sqrt{2}\pi}{512}\gamma_0 + \frac{\sqrt{2}\pi}{32}\gamma_1 - \frac{5\sqrt{2}\pi}{512}\delta_0 - \frac{3\sqrt{2}\pi}{32}\delta_1 + \frac{\sqrt{2}\pi}{2}\delta_2,\\
\sigma_7 &= \frac{5\sqrt{2}}{448}\alpha_0 + \frac{3\sqrt{2}}{80}\alpha_1 - \frac{\sqrt{2}}{64}\beta_0 - \frac{\sqrt{2}}{16}\beta_1 - \frac{\sqrt{2}}{2}\beta_2,\\
\sigma_8 &= \frac{25\sqrt{2}\pi}{16384}\gamma_0 + \frac{3\sqrt{2}\pi}{512}\gamma_1 - \frac{35\sqrt{2}\pi}{16384}\delta_0 - \frac{5\sqrt{2}\pi}{512}\delta_1 - \frac{3\sqrt{2}\pi}{32}\delta_2,\\
\sigma_9 &= \frac{35\sqrt{2}}{9216}\alpha_0 + \frac{5\sqrt{2}}{448}\alpha_1 - \frac{5\sqrt{2}}{1024}\beta_0 - \frac{\sqrt{2}}{64}\beta_1 - \frac{\sqrt{2}}{16}\beta_2,\\
\sigma_{10} &= \frac{245\sqrt{2}\pi}{524288}\gamma_0 + \frac{25\sqrt{2}\pi}{16384}\gamma_1 - \frac{315\sqrt{2}\pi}{524288}\delta_0 - \frac{35\sqrt{2}\pi}{16384}\delta_1 - \frac{5\sqrt{2}\pi}{512}\delta_2.
\end{aligned}$$

Then, using the above equalities, one finds that the determinant of the following Jacobian matrix

$$\frac{\partial(\sigma_1,\sigma_3,\sigma_5,\sigma_7,\sigma_9,\sigma_2,\sigma_4,\sigma_6,\sigma_8,\sigma_{10})}{\partial(\alpha_0,\alpha_1,\beta_0,\beta_1,\beta_2,\gamma_0,\gamma_1,\delta_0,\delta_1,\delta_2)}$$

is $\frac{1}{6193152}\pi^5$, which means that σ_i, $i=1,2,\cdots,10$ can be chosen arbitrarily.

(ii) We only prove the conclusion for $M_+(h)$. The proof of $M_-(h)$ follows in the same way. Similar to (3.3), one can obtain that

$$\begin{aligned} M_+(h) =& \big(\alpha_1^+ h+\alpha_0^+\big)J_{0,0}(h)+\big(\beta_2^+h^2+\beta_1^+h+\beta_0^+\big)J_{1,0}(h) \\ &+\big(\gamma_1^+h+\gamma_0^+\big)J_{0,1}(h)+\big(\delta_2^+h^2+\delta_1^+h+\delta_0^+\big)J_{1,1}(h), \end{aligned} \tag{3.8}$$

and α_i^+ $(i=0,1)$, β_j^+ $(j=0,1,2)$, γ_k^+ $(k=0,1)$ and δ_l^+ $(l=0,1,2)$ are independent of each other. Note that $J_{0,0}(h)=2\pi$, $J_{1,0}(h)=0$. Using the expansions for $0<h-2\ll 1$ which are obtained by the command "series" in Maple 18,

$$\begin{aligned} \sqrt{h}(2-h)K(\sqrt{2/h}) =& -\frac{5\sqrt{2}\ln 2}{2}(h-2)+\frac{\sqrt{2}}{2}(h-2)\ln(h-2)-\frac{\sqrt{2}}{16}(15\ln 2+2)(h-2)^2 \\ &+\frac{3\sqrt{2}}{16}(h-2)^2\ln(h-2)+\frac{\sqrt{2}}{512}(35\ln 2-11)(h-2)^3 \\ &-\frac{7\sqrt{2}}{512}(h-2)^3\ln(h-2)+o((h-2)^4), \\ \sqrt{h}E(\sqrt{2/h}) =& \sqrt{2}+\frac{\sqrt{2}}{8}(1+5\ln 2)(h-2)-\frac{\sqrt{2}}{8}(h-2)\ln(h-2) \\ &+\frac{\sqrt{2}}{256}(3-10\ln 2)(h-2)^2+\frac{\sqrt{2}}{128}(h-2)^2\ln(h-2) \\ &+\frac{3\sqrt{2}}{2048}(5\ln 2-2)(h-2)^3-\frac{3\sqrt{2}}{2048}(h-2)^3\ln(h-2)+o((h-2)^4), \\ \sqrt{h}(h-1)E(\sqrt{2/h}) =& \sqrt{2}+\frac{\sqrt{2}}{8}(9+5\ln 2)(h-2)-\frac{\sqrt{2}}{8}(h-2)\ln(h-2) \\ &+\frac{5\sqrt{2}}{256}(7+30\ln 2)(h-2)^2-\frac{15\sqrt{2}}{128}(h-2)^2\ln(h-2) \\ &-\frac{\sqrt{2}}{2048}(65\ln 2-18)(h-2)^3+\frac{13\sqrt{2}}{2048}(h-2)^3\ln(h-2) \\ &+o((h-2)^4), \end{aligned}$$

one obtains that

$$\begin{aligned} M_+(h) =& \rho_0^+ +\rho_1(h-2)+\rho_2^+(h-2)\ln(h-2)+\rho_3^+(h-2)^2+\rho_4^+(h-2)^2\ln(h-2) \\ &+\rho_5^+(h-2)^3+\rho_6^+(h-2)^3\ln(h-2)+o((h-2)^4), 0<h-2\ll 1, \end{aligned} \tag{3.9}$$

where

$$
\begin{aligned}
\rho_0^+ =& 2\pi\alpha_0^+ + 4\pi\alpha_1^+ + 4\gamma_0^+ + 8\gamma_1^+ + \frac{4}{3}\delta_0^+ + \frac{8}{3}\delta_1^+ + \frac{16}{3}\delta_2^+,\\
\rho_1^+ =& 2\pi\alpha_1^+ + \frac{1}{2}(5\ln 2 + 1)\gamma_0^+ + 5(\ln 2 + 1)\gamma_1^+\\
&+ \frac{1}{2}(3 - 5\ln 2)\delta_0^+ + \frac{1}{3}(13 - 15\ln 2)\delta_1^+ + \frac{2}{3}(17 - 15\ln 2)\delta_2^+,\\
\rho_2^+ =& -\frac{1}{2}\gamma_0^+ - \gamma_1^+ + \frac{1}{2}\delta_0^+ + \delta_1^+ + 2\delta_2^+,\\
\rho_3^+ =& \frac{1}{64}(3 - 10\ln 2)\gamma_0^+ + \frac{1}{32}(70\ln 2 + 19)\gamma_1^+ + \frac{1}{64}(1 - 30\ln 2)\delta_0^+\\
&+ \frac{1}{32}(49 - 110\ln 2)\delta_1^+ + \frac{5}{48}(71 - 114\ln 2)\delta_2^+,\\
\rho_4^+ =& \frac{1}{32}\gamma_0^+ - \frac{7}{16}\gamma_1^+ + \frac{3}{32}\delta_0^+ + \frac{11}{16}\delta_1^+ + \frac{19}{8}\delta_2^+,\\
\rho_5^+ =& \frac{3}{512}(5\ln 2 - 2)\gamma_0^+ + \frac{1}{256}(6 - 25\ln 2)\gamma_1^+ + \frac{1}{1536}(75\ln 2 - 26)\delta_0^+\\
&- \frac{1}{768}(285\ln 2 + 14)\delta_1^+ + \frac{1}{384}(574 - 1605\ln 2)\delta_2^+,\\
\rho_6^+ =& -\frac{3}{512}\gamma_0^+ + \frac{5}{256}\gamma_1^+ - \frac{5}{512}\delta_0^+ + \frac{19}{256}\delta_1^+ + \frac{107}{128}\delta_2^+.
\end{aligned}
$$

The independence of ρ_i^+ $(i = 0, 1, \cdots, 6)$ follows from

$$
\det\Big(\frac{\partial(\rho_0^+, \rho_1^+, \rho_2^+, \rho_3^+, \rho_4^+, \rho_5^+, \rho_6^+)}{\partial(\alpha_0^+, \alpha_1^+, \gamma_0^+, \gamma_1^+, \delta_0^+, \delta_1^+, \delta_2^+)}\Big) = \frac{175}{4608}\pi^2.
$$

This ends the proof of the Lemma. □

Proof of Theorem 1.2. From Lemma 3.1, one knows that σ_i, $i = 1, 2, \cdots, 10$ can be chosen arbitrarily. Then, one can choose

$$
0 < -\sigma_1 \ll \sigma_2 \ll -\sigma_3 \ll \sigma_4 \ll -\sigma_5 \ll \sigma_6 \ll -\sigma_7 \ll \sigma_8 \ll -\sigma_9 \ll \sigma_{10}
$$

such that the sign of $M_0(h)$ has been changed 9 times. In other words, $M_0(h)$ can have 9 simple zeros near $h = 0$. The other cases for $M_\pm(h)$ can be proved similarly. This ends the proof of Theorem 1.2 by Remark 1.1.

4. Conclusion

In this paper, we get the detailed algebraic structure of the first order Melnikov functions of the perturbed pendulum equation (1.5) by using two iterative formulas. In order to obtain the lower bounds of the number of limit cycles of system (1.5), we also verify the independence of coefficients of the coefficient polynomials of several curvilinear integrals by induction on n. Then, for $n = 1, 2, 3$, we get the lower bounds of the number of limit cycles of system (1.5) in both the oscillatory and rotary regions. But for the general natural number n, it is difficult to get the lower bounds, because the Melnikov functions involve the complete elliptic functions of first and second kind. These results may be further improved in the future by developing more powerful computational tools or methods.

Acknowledgements

I would like to express my gratitude to Professor Maoan Han and Doctor Meilan Cai for their invaluable suggestions, which greatly enhanced the quality of this paper. Additionally, I would like to extend my thanks to the referee and the editor for their valuable input, which greatly contributed to the improvement of the paper's presentation.

References

[1] T. Chen, J. Llibre, *Limit cycles of a second-order differential equation*, Appl. Math. Letters, 2019, 88, 111–117.

[2] A. Gasull, A. Geyer, F. Mañosas, *On the number of limit cycles for perturbed pendulum equations*, J. Differential Equations, 2016, 261, 2141–2167.

[3] S. Gong, M. Han, *An estimate of the number of limit cycles bifurcating from a planar integrable system*, Bull. Sci. math., 2022, 176, 103118.

[4] M. Han, *On the maximum number of periodic solutions of piecewise smooth periodic equations by average method*, J. Appl. Anal. Comput. 7, 788–794.

[5] M. Han, *Bifurcation Theory of limit cycles,* Science Press, Beijing, 2013.

[6] M. Han, V. Romanovski, X. Zhang, *Equivalence of the Melnikov function method and the averaging method*, Qualitative Theory of Dynamical Systems, 2016, 15, 471–479.

[7] M. Han, L. Sheng, *Bifurcation of limit cycles in piecewise smooth systems via Melnikov function*, J. Appl. Anal. Comput., 2015, 5, 809–815.

[8] M. Han, J. Yang, *The maximum number of zeros of functions with parameters and application to differential equations*, Journal of Nonlinear Modeling and Analysis, 2021, 3(1), 13–34.

[9] H. Lichardova, *The period of a whirling pendulum*, Mathematica Bohemica, 2001, 3, 593–606.

[10] S. Liu, M. Han, J. Li, *Bifurcation methods of periodic orbits for piecewise smooth systems*, J. Differential Equations, 2021, 275, 204–233.

[11] F. Liang, M. Han, V. Romanovski, *Bifurcation of limit cycles by perturbing a piecewise linear Hamiltonian system with a homoclinic loop*, Nonlinear Anal., 2012, 75, 4355–4374.

[12] J. Llibre, A. Mereu, D. Novaes, *Averaging theory for discontinuous piecewise differential systems*, J. Differential Equations, 2015, 258, 4007–4032.

[13] J. Llibre, D. Novaes, M. Teixeira, *On the birth of limit cycles for non-smooth dynamical systems*, Bull. Sci. Math., 2015, 139, 229–244.

[14] X. Liu, M. Han, *Bifurcation of limit cycles by perturbing piecewise Hamiltonian systems*, Internat. J. Bifur. Chaos Appl. Sci. Engrg, 2010, 20, 1379–1390.

[15] L. Pontryagin, *On dynamical systems close to Hamiltonian ones*, Zh. Exp. Theor. Phys., 1934, 4, 234–238.

[16] H. Shi, Y. Bai, M. Han, *On the maximum number of limit cycles for a piecewise smooth differential system*, Bull. Sci. math., 2020, 163, 102887.

[17] Y. Shi, M. Han, L. Zhang, *Homoclinic bifurcation of limit cycles in near-Hamiltonian systems on the cylinder*, Journal of Differential Equations, 2021, 304, 1–28.

[18] Y. Xiong, M. Han, *Limit cycles appearing from a generalized heteroclinic loop with a cusp and a nilpotent saddle*, J. Differential Equations, 2021, 303, 575–607.

[19] Y. Xiong, J. Hu, *A class of reversible quadratic systems with piecewise polynomial perturbations*, Applied Mathematics and Computation, 2019, 362, 124527.

[20] J. Yang, L. Zhao, *Bounding the number of limit cycles of discontinuous differential systems by using Picard-Fuchs equations*, J. Differential Equations, 2018, 264, 5734–5757.

[21] J. Yang, *Bifurcation of limit cycles of the nongeneric quadratic reversible system with discontinuous perturbations*, Science China Mathematics, 2020, 63(5), 873–886.

[22] J. Yang, *On the number of limit cycles of a pendulum-like equation with two switching lines*, Chaos, Solitons and Fractals, 2021, 150, 111092.

[23] J. Yang, L. Zhao, *Limit cycle bifurcations for piecewise smooth Hamiltonian systems with a generalized eye-figure loop*, International J. Bifurcation and Chaos, 2016, 26, 1650204.

[24] J. Yang, L. Zhao, *Bifurcation of limit cycles of a piecewise smooth Hamiltonian system*, Qualitative Theory of Dynamical Systems, 2022, 21, 142.