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Fractional Langevin Equation at Resonance*

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Abstract The study of fractional Langevin equation has obtained abundant results in recent years. However, there are few studies on resonant fractional Langevin equation. In this paper, we investigate boundary value problems for fractional Langevin equation at resonance. By virtue of Banach contraction mapping principle and Leray-Schauder fixed point theorem, we obtain the uniqueness and existence of solutions. In addition, we get different stability results, including Ulam-Hyres stability and generalized Ulam-Hyres stability. Finally, give relevant examples to demonstrate the main results.

Keywords Fractional order, Langevin equation, resonance, boundary value problems, fixed point theorem

MSC(2010) 26A33, 82C31, 34F15, 39A27, 47H10.

1. Introduction

Langevin equation is a motion equation proposed by Langevin in 1908 when he studied Brownian motion. It is well known that Langevin equation is widely used to describe the physical phenomena of waves in evolving environments [1,2]. However, in the complex evolution environment, Langevin equation of integer order cannot be described correctly. The fractional Langevin equation appears, that is, the integer derivative is replaced by the fractional derivative in the equation [3]. Compared with integer order, fractional calculus has non-local and memory properties and can better describe physical phenomena, such as heat conduction of soft matter, fractal phenomena and image processing [4,5].

The fractional Langevin equation is widely used in various fields [6–9]. In biochemistry, Langevin equation can be used to study protein folding dynamics [6]. In physics, Langevin equation can be used to study quantum noise [7]. For example, the stable two-state update model can be abstracted into the following fractional Langevin equation, which describes the underdiffusion process of potential fields and external signal forces [8]

$$ {}_0^cD_t^{\alpha}x(t) = ax - bx^3 + A\cos(2\pi ft) + \xi(t), \quad 0 < a < 1. $$

Based on the extensive application of the fractional Langevin equation, the solutions to its initial value problem and boundary value problem are concerned by

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scholars [10–15]. For example, Ahmad et al. [10] considered the existence and uniqueness of solutions for three-point boundary value problem of fractional Langevin equation

$$\begin{cases} {}^cD^{\beta}({}^cD^{\alpha}+\lambda)x(t)=f(t,x(t)), \quad 0<t<1, \\ x(0)=0, x(\eta)=0, x(1)=0, \quad 0<\alpha\le 1, 1<\beta\le 2. \end{cases}$$

Baleanu et al. [11] studied the existence of solutions for fractional Langevin equation involving Atangana-Baleanu operators

$$\begin{cases} {}^{ABD}D^{\beta}({}^{ABD}D^{\alpha}+\lambda)x(t)=f(t,x(t)), \quad 0<t<1, \\ x(0)=\gamma_1, x'(0)=\gamma_2, \quad 0<\alpha,\beta\le 1. \end{cases}$$

Wang et al. [12] investigated the fractional Langevin equation with integro-differential strip-multi-point boundary conditions

$$\begin{cases} {}_0^{\beta}D^{\alpha}({}_0^{\gamma}D^{\alpha}+\lambda)x(t)=f(t,x(t),({}_0^{\gamma}D^{\alpha}+\lambda)x(t)), \quad 0<t\le d, \\ t^{\alpha(1-\gamma)}x(t)|_{t=0}=\omega_1\int_0^{\eta}x(s)ds+\sum\limits_{i=1}^{m}\mu_i x(\xi_i), \quad 0<\alpha,\beta,\gamma<1, \\ t^{\alpha(1-\beta)}({}_0^{\gamma}D^{\alpha}+\lambda)x(t)|_{t=0}=\omega_2\int_0^{\eta}{}_0^{\gamma}D^{\alpha}x(s)ds+\sum\limits_{i=1}^{m}\nu_i{}_0^{\gamma}D^{\alpha}x(\xi_i). \end{cases}$$

For the study of fractional Langevin equation, as far as we know, it is considered in the case of non-resonance, and the resonance case has not been studied at present. Based on this, this paper establishes the fractional Langevin equation at resonance. Aiello [16] used the Langevin equation to describe the random movement of pollen under the action of water molecules. In the network public opinion environment, netizens' opinions are like the movement of pollen under the action of water molecules, which is very similar to the evolutionary structure of Langevin equation [17]. The resonance phenomenon between multiple events in network public opinions can be abstracted into the fractional Langevin equation at resonance, which makes the study of the resonant fractional Langevin equation not only have theoretical significance [18, 19], but also have certain practical value.

In addition to the existence and uniqueness of solutions, it is important to know the stability of them. As one of the qualitative theories of fractional differential equations, stability theory has gradually penetrated into various fields [20]. In the process of establishing the differential model, there are inevitably small disturbances that cannot be estimated, and these disturbances make the stability of differential equations fundamentally change. The stability of the system is an important basis to judge whether the system can run normally, so the stability of the system is worth further study. Antoniadou [21] proposed five different resonance orbitals, and found that the resonance orbitals with stability generally have larger eccentricity. Wang et al. [22] investigated the Ulam-Hyers stability of fractional Langevin equations

$$\begin{cases} {}^cD_t^{\beta}({}^cD_t^{\alpha}+\lambda)x(t)=f(t,x(t)), \quad t\in J', \\ \Delta x(t_k)=x(t_k^+)-x(t_k^-)=I_k, \quad I_k\in\mathbb{R}. \end{cases}$$

In this paper, we consider the following boundary value problem for fractional Langevin equation at resonance

$$\begin{cases} {}^cD_{0^+}^{\beta}({}^cD_{0^+}^{\alpha}-\lambda)x(t)=f(t,x(t)), \quad t\in(0,1), \\ x(0)=0, {}^cD_{0^+}^{\alpha}x(0)=0, x(1)=\eta x(\xi), \end{cases} \tag{1.1}$$

where $0 < \alpha \leq 1$, $1 < \beta \leq 2$, $0 < \beta - \alpha < 1$, $\lambda > 0$, $\eta > 0$, $0 < \xi < 1$, $\sum_{k=0}^{+\infty} \frac{\lambda^k(1-\eta\xi^{k\alpha+\alpha+1})}{\Gamma(k\alpha+\alpha+2)} = 0$, ${}^cD_{0^+}^{\alpha}$ is Caputo fractional derivative and $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous.

Boundary value problem (1.1) is resonant since the corresponding homogeneous boundary value problem

$$\begin{cases} {}^cD_{0^+}^{\beta}({}^cD_{0^+}^{\alpha} - \lambda)x(t) = 0, \quad t \in (0,1), \\ x(0) = 0, {}^c D_{0^+}^{\alpha}x(0) = 0, x(1) = \eta x(\xi) \end{cases} \tag{1.2}$$

has a nontrivial solution $\sum_{k=0}^{+\infty} \frac{\lambda^k t^{k\alpha+\alpha+1}}{\Gamma(k\alpha+\alpha+2)}$.

Inspired by the above work, in this paper we study the existence, uniqueness and stability of solutions for fractional Langevin equation at resonance. The structure of this paper is as follows. In part 2, we review the basic definitions, theorems and lemmas. In part 3, the existence of solutions is obtained by Leray-Schauder fixed point theorem and the uniqueness of solution is studied by Banach contraction mapping principle. In part 4, we give different stability results, including Ulam-Hyers stability and generalised Ulam-Hyers stability. In part 5, examples are given to verify our main results.

2. Preliminaries

Definition 2.1. [23] The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $x : (0, +\infty) \to \mathbb{R}$ is given by

$$I_{0^+}^{\alpha}x(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}x(s)ds,$$

provided the right-hand side is point-wise defined on $(0, +\infty)$.

Definition 2.2. [23] The Caputo fractional derivative of order $\alpha > 0$ of a function $x : (0, +\infty) \to \mathbb{R}$ is given by

$${}^cD_{0^+}^{\alpha}x(t) = \int_0^t \frac{(t-s)^{n-\alpha-1}}{\Gamma(n-\alpha)}x^{(n)}(s)ds,$$

provided the right-hand side is point-wise defined on $(0, +\infty)$, where n is the smallest integer greater than or equal to α.

Lemma 2.1. *[23] Let $\alpha > 0$. If $x \in C[0,1]$, then the following equation holds*

$$I_{0^+}^{\alpha}({}^cD_{0^+}^{\alpha}x(t)) = x(t) - c_0 - c_1 t - c_2 t^2 - \cdots - c_{n-1}t^{n-1},$$

where $c_i \in \mathbb{R}, i = 0, 1, \cdots, n-1$, and n is the smallest integer greater than or equal to α.

Lemma 2.2. *[23] Let $\alpha > 0$, $\beta > 0$. If $x \in C[0,1]$, then the following equations hold*

$$I_{0^+}^{\alpha}(I_{0^+}^{\beta}x(t)) = I_{0^+}^{\alpha+\beta}x(t),$$
$${}^cD_{0^+}^{\alpha}(I_{0^+}^{\alpha}x(t)) = x(t).$$

Consider the following boundary value problem which is equivalent to (1.1)

$$\begin{cases} {}^cD^{\beta}_{0+}({}^cD^{\alpha}_{0+}x(t)) = f(t, x(t)) + \lambda {}^cD^{\beta}_{0+}x(t), & t \in (0,1), \\ x(0) = 0, {}^cD^{\alpha}_{0+}x(0) = 0, x(1) = \eta x(\xi). \end{cases} \tag{2.1}$$

Definition 2.3. Function $x \in AC^2[0,1]$ and $y \in AC[0,1]$ satisfying (2.3) is called the solution of boundary value problem (2.2).

Lemma 2.3. *If $x \in AC^2[0,1]$, $y \in AC[0,1]$, then the unique solution of problem*

$$\begin{cases} {}^cD^{\beta}_{0+}({}^cD^{\alpha}_{0+}x(t)) = y(t), & t \in (0,1), \\ x(0) = 0, {}^cD^{\alpha}_{0+}x(0) = 0, x(1) = \eta x(\xi) \end{cases} \tag{2.2}$$

is

$$\begin{aligned} x(t) = &\int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} y(s)ds + \frac{t^{\alpha+1}}{1-\eta\xi^{\alpha+1}} \\ &\Big[\eta \int_0^{\xi} \frac{(\xi-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} y(s)ds - \int_0^1 \frac{(1-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} y(s)ds\Big], \quad t \in [0,1]. \end{aligned} \tag{2.3}$$

Proof. (i) We present that if $x(t)$ is the solution of(2.2), it can be expressed as (2.3).

Applying I^{β}_{0+} to the both sides of (2.2)

$$I^{\beta}_{0+}({}^cD^{\beta}_{0+}({}^cD^{\alpha}_{0+}x(t))) = I^{\beta}_{0+}y(t).$$

From Lemma 2.1, we get

$${}^cD^{\alpha}_{0+}x(t) = I^{\beta}_{0+}y(t) + c_0 + ct.$$

By the boundary condition ${}^cD^{\alpha}_{0+}x(0) = 0$, we have $c_0 = 0$. Then

$${}^cD^{\alpha}_{0+}x(t) = I^{\beta}_{0+}y(t) + ct.$$

Applying I^{α}_{0+} to the both sides and by Lemmas 2.1 and 2.2, we obtain

$$x(t) - c_2 = I^{\alpha+\beta}_{0+}y(t) + I^{\alpha}_{0+}(ct).$$

From the boundary condition $x(0) = 0$, we get $c_2 = 0$. Then

$$x(t) = I^{\alpha+\beta}_{0+}y(t) + I^{\alpha}_{0+}(ct).$$

By Definition 2.1, we have

$$x(t) = \int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} y(s)ds + \frac{ct^{\alpha+1}}{\Gamma(\alpha+2)}.$$

Then

$$x(1) = \int_0^1 \frac{(1-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} y(s)ds + \frac{c}{\Gamma(\alpha+2)},$$

$$x(\xi) = \int_0^{\xi} \frac{(\xi-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} y(s)ds + \frac{c\xi^{\alpha+1}}{\Gamma(\alpha+2)}.$$

Notice that $\sum\limits_{k=0}^{+\infty} \frac{\lambda^k(1-\eta\xi^{k\alpha+\alpha+1})}{\Gamma(k\alpha+\alpha+2)} = 0$ and $0 < \xi < 1$, we get $\eta\xi^{\alpha+1} \neq 1$. Otherwise, if $\eta\xi^{\alpha+1} = 1$, we have

$$\sum_{k=0}^{+\infty} \frac{\lambda^k(1-\eta\xi^{k\alpha+\alpha+1})}{\Gamma(k\alpha+\alpha+2)} = \sum_{k=0}^{+\infty} \frac{\lambda^k(1-\xi^{k\alpha})}{\Gamma(k\alpha+\alpha+2)} > 0,$$

which is a contradiction. By the boundary condition $x(1) = \eta x(\xi)$, we obtain

$$c = \frac{\Gamma(\alpha+2)}{1-\eta\xi^{\alpha+1}}\Big[\eta\int_0^\xi \frac{(\xi-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}y(s)ds - \int_0^1 \frac{(1-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}y(s)ds\Big].$$

Then $x(t)$ can be expressed as follows

$$\begin{aligned} x(t) = &\int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}y(s)ds + \frac{t^{\alpha+1}}{1-\eta\xi^{\alpha+1}} \\ &\Big[\eta\int_0^\xi \frac{(\xi-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}y(s)ds - \int_0^1 \frac{(1-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}y(s)ds\Big]. \end{aligned}$$

(ii) We show that if $x(t)$ can be expressed as (2.3), it is the solution of (2.2). In fact,

$$\begin{aligned} x(t) &= \int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}y(s)ds + \frac{ct^{\alpha+1}}{\Gamma(\alpha+2)} \\ &= I_{0+}^{\alpha+\beta}y(t) + I_{0+}^{\alpha}(ct), \end{aligned}$$

where

$$c = \frac{\Gamma(\alpha+2)}{1-\eta\xi^{\alpha+1}}\Big[\eta\int_0^\xi \frac{(\xi-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}y(s)ds - \int_0^1 \frac{(1-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}y(s)ds\Big].$$

Then, we have

$${}^cD_{0+}^{\alpha}x(t) = {}^cD_{0+}^{\alpha}(I_{0+}^{\alpha+\beta}y(t)) + {}^cD_{0+}^{\alpha}(I_{0+}^{\alpha}(ct)).$$

By Lemma 2.2, we get

$${}^cD_{0+}^{\alpha}x(t) = I_{0+}^{\beta}y(t) + ct.$$

Applying ${}^cD_{0+}^{\beta}$ to the both sides

$${}^cD_{0+}^{\beta}({}^cD_{0+}^{\alpha}x(t)) = {}^cD_{0+}^{\beta}(I_{0+}^{\beta}y(t)) + {}^cD_{0+}^{\beta}(ct).$$

From Lemma 2.2, we have

$${}^cD_{0+}^{\beta}({}^cD_{0+}^{\alpha}x(t)) = y(t).$$

This completes the proof. □

Lemma 2.4. *[24] Let E be a Banach space and there exists a completely continuous operator $T : E \to E$, such that*

$$V = \{x \in E : x = \mu Tx, 0 < \mu < 1\}$$

is a bounded set. Then the operator T has fixed points on E.

Lemma 2.5. *[25] Let E be a Banach space and $D \subset E$ be a non-empty closed subset, the mapping $T : D \to D$ is a contraction, that is,*

$$\forall x, y \in D, \quad \|Tx - Ty\| \le k\|x - y\| \quad (0 < k < 1).$$

Then there is a unique point $x^* \in D$, such that $Tx^* = x^*$, that is, T exists a unique fixed point on D.

3. The solvability of fractional Langevin equation

Let $E = \{x : x, {}^c D^\beta_{0^+} x \in C[0,1]\}$ be a Banach space with the norm

$$\|x\| = \max_{0\le t\le 1} |x(t)| + \max_{0\le t\le 1} |{}^c D^\beta_{0^+} x(t)|. \tag{3.1}$$

Define operator $T : E \to E$ as follows

$$\begin{aligned} Tx(t) = &\int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}[f(s,x(s)) + \lambda^c D^\beta_{0^+} x(s)]ds + \frac{t^{\alpha+1}}{1-\eta\xi^{\alpha+1}} \\ &\Big[\eta \int_0^\xi \frac{(\xi-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}[f(s,x(s)) + \lambda^c D^\beta_{0^+} x(s)]ds \\ &- \int_0^1 \frac{(1-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}[f(s,x(s)) + \lambda^c D^\beta_{0^+} x(s)]ds\Big]. \end{aligned} \tag{3.2}$$

For convenience, we make the following notations

$$A_1 = \frac{1}{\Gamma(\alpha+\beta+1)}, \quad A_2 = \frac{\eta+1}{|1-\eta\xi^{\alpha+1}|\Gamma(\alpha+\beta+1)},$$

$$A_3 = \frac{1}{\Gamma(\alpha+1)}, \quad A_4 = \frac{(\eta+1)\Gamma(\alpha+2)}{|1-\eta\xi^{\alpha+1}|\Gamma(2-\beta+\alpha)\Gamma(\alpha+\beta+1)}.$$

Theorem 3.1. *If the following assumptions hold*

(H_1) $f : [0,1]\times\mathbb{R} \to \mathbb{R}$ is continuous;

(H_2) there exists constants $m, n > 0$ such that

$$|f(t,x)| \le m + n|x|, \quad \forall (t,x) \in [0,1]\times\mathbb{R}.$$

Then boundary value problem (1.1) has at least a solution provided that

$$0 < (n+\lambda)(A_1 + A_2 + A_3 + A_4) < 1.$$

Proof. Take $D = \{x \in E : \|x\| \le r\}$ where

$$\frac{m(A_1 + A_2 + A_3 + A_4)}{1-(n+\lambda)(A_1 + A_2 + A_3 + A_4)} \le r. \tag{3.3}$$

(i) First, we prove that $T(D) \subset D$. For $\forall x \in D$, $\forall t \in [0,1]$, from (3.2) and (H_2), we get

$$
\begin{aligned}
|Tx(t)| = \Big| \int_0^t & \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}[f(s,x(s)) + \lambda^c D_{0^+}^{\beta} x(s)]ds + \frac{t^{\alpha+1}}{1-\eta\xi^{\alpha+1}} \\
& \Big[\eta \int_0^{\xi} \frac{(\xi-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}[f(s,x(s)) + \lambda^c D_{0^+}^{\beta} x(s)]ds \\
& - \int_0^1 \frac{(1-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}[f(s,x(s)) + \lambda^c D_{0^+}^{\beta} x(s)]ds\Big]\Big| \\
\leq & \int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}(|f(s,x(s))| + |\lambda^c D_{0^+}^{\beta} x(s)|)ds + \frac{t^{\alpha+1}}{|1-\eta\xi^{\alpha+1}|} \\
& \Big[\eta \int_0^{\xi} \frac{(\xi-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}(|f(s,x(s))| + |\lambda^c D_{0^+}^{\beta} x(s)|)ds \\
& + \int_0^1 \frac{(1-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}(|f(s,x(s))| + |\lambda^c D_{0^+}^{\beta} x(s)|)ds\Big] \\
\leq & \int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}(m + n|x(s)| + \lambda|^c D_{0^+}^{\beta} x(s)|)ds + \frac{t^{\alpha+1}}{|1-\eta\xi^{\alpha+1}|} \\
& \Big[\eta \int_0^{\xi} \frac{(\xi-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}(m + n|x(s)| + \lambda|^c D_{0^+}^{\beta} x(s)|)ds \\
& + \int_0^1 \frac{(1-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}(m + n|x(s)| + \lambda|^c D_{0^+}^{\beta} x(s)|)ds\Big] \\
\leq & \frac{(m+nr+\lambda r)t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \frac{t^{\alpha+1}}{|1-\eta\xi^{\alpha+1}|}\frac{(m+nr+\lambda r)(\eta\xi^{\alpha+\beta}+1)}{\Gamma(\alpha+\beta+1)} \\
\leq & (m+nr+\lambda r)\Big[\frac{1}{\Gamma(\alpha+\beta+1)} + \frac{\eta+1}{|1-\eta\xi^{\alpha+1}|\Gamma(\alpha+\beta+1)}\Big] \\
= & (m+nr+\lambda r)(A_1 + A_2).
\end{aligned}
$$

By the definition of the operator T and (H_2), we have

$$
\begin{aligned}
& |^c D_{0^+}^{\beta} Tx(t)| \\
= & \Big| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}[f(s,x(s)) + \lambda^c D_{0^+}^{\beta} x(s)]ds + \frac{\Gamma(\alpha+2)t^{1-\beta+\alpha}}{(1-\eta\xi^{\alpha+1})\Gamma(2-\beta+\alpha)} \\
& \Big[\eta \int_0^{\xi} \frac{(\xi-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}[f(s,x(s)) + \lambda^c D_{0^+}^{\beta} x(s)]ds \\
& - \int_0^1 \frac{(1-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}[f(s,x(s)) + \lambda^c D_{0^+}^{\beta} x(s)]ds\Big]\Big| \\
\leq & \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}(|f(s,x(s))| + |\lambda^c D_{0^+}^{\beta} x(s)|)ds + \frac{\Gamma(\alpha+2)t^{1-\beta+\alpha}}{|1-\eta\xi^{\alpha+1}|\Gamma(2-\beta+\alpha)} \\
& \Big[\eta \int_0^{\xi} \frac{(\xi-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}(|f(s,x(s))| + |\lambda^c D_{0^+}^{\beta} x(s)|)ds \\
& + \int_0^1 \frac{(1-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}(|f(s,x(s))| + |\lambda^c D_{0^+}^{\beta} x(s)|)ds\Big] \\
\leq & \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}(m + n|x(s)| + \lambda|^c D_{0^+}^{\beta} x(s)|)ds + \frac{\Gamma(\alpha+2)t^{1-\beta+\alpha}}{|1-\eta\xi^{\alpha+1}|\Gamma(2-\beta+\alpha)}
\end{aligned}
$$

$$
\begin{aligned}
&\Big[\eta\int_0^\xi \frac{(\xi-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}(m+n|x(s)|+\lambda|{}^cD_{0^+}^\beta x(s)|)ds\\
&+\int_0^1 \frac{(1-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}(m+n|x(s)|+\lambda|{}^cD_{0^+}^\beta x(s)|)ds\Big]\\
\le&\frac{(m+nr+\lambda r)t^\alpha}{\Gamma(\alpha+1)}+\frac{\Gamma(\alpha+2)t^{1-\beta+\alpha}}{|1-\eta\xi^{\alpha+1}|\Gamma(2-\beta+\alpha)}\frac{(m+nr+\lambda r)(\eta\xi^{\alpha+\beta}+1)}{\Gamma(\alpha+\beta+1)}\\
\le&(m+nr+\lambda r)\Big[\frac{1}{\Gamma(\alpha+1)}+\frac{(\eta+1)\Gamma(\alpha+2)}{|1-\eta\xi^{\alpha+1}|\Gamma(2-\beta+\alpha)\Gamma(\alpha+\beta+1)}\Big]\\
=&(m+nr+\lambda r)(A_3+A_4).
\end{aligned}
$$

Therefore, from (3.1), we obtain

$$
\begin{aligned}
\|Tx\|=&\max_{0\le t\le 1}|Tx(t)|+\max_{0\le t\le 1}|{}^cD_{0^+}^\beta Tx(t)|\\
\le&(m+nr+\lambda r)(A_1+A_2+A_3+A_4)\\
\le&r.
\end{aligned}
$$

Then $T(D)\subset D$. It's clear that operator T is uniformly bounded on D and from the continuity of f we know that operator T is continuous.

Next, we show that operator T is equicontinuous. For $\forall x\in D$ and $0\le t_1<t_2\le 1$, by (3.2) and (H_2), we have

$$
\begin{aligned}
&|Tx(t_2)-Tx(t_1)|\\
=&\Big|\int_0^{t_2}\frac{(t_2-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}[f(s,x(s))+\lambda^cD_{0^+}^\beta x(s)]ds\\
&-\int_0^{t_1}\frac{(t_1-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}[f(s,x(s))+\lambda^cD_{0^+}^\beta x(s)]ds\\
&+\frac{(t_2^{\alpha+1}-t_1^{\alpha+1})}{1-\eta\xi^{\alpha+1}}\Big[\eta\int_0^\xi\frac{(\xi-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}[f(s,x(s))+\lambda^cD_{0^+}^\beta x(s)]ds\\
&-\int_0^1\frac{(1-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}[f(s,x(s))+\lambda^cD_{0^+}^\beta x(s)]ds\Big]\Big|\\
\le&\Big|\int_0^{t_2}\frac{(t_2-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}[f(s,x(s))+\lambda^cD_{0^+}^\beta x(s)]ds\\
&-\int_0^{t_1}\frac{(t_1-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}[f(s,x(s))+\lambda^cD_{0^+}^\beta x(s)]ds\Big|\\
&+\frac{(t_2^{\alpha+1}-t_1^{\alpha+1})}{|1-\eta\xi^{\alpha+1}|}\Big[\eta\int_0^\xi\frac{(\xi-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\Big|f(s,x(s))+\lambda^cD_{0^+}^\beta x(s)\Big|ds\\
&+\int_0^1\frac{(1-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\Big|f(s,x(s))+\lambda^cD_{0^+}^\beta x(s)\Big|ds\Big]\\
\le&\frac{(m+nr+\lambda r)(t_2^{\alpha+\beta}-t_1^{\alpha+\beta})}{\Gamma(\alpha+\beta+1)}+\frac{(t_2^{\alpha+1}-t_1^{\alpha+1})}{|1-\eta\xi^{\alpha+1}|}\frac{(m+nr+\lambda r)(\eta\xi^{\alpha+\beta}+1)}{\Gamma(\alpha+\beta+1)}\\
\le&(m+nr+\lambda r)\Big[\frac{(t_2^{\alpha+\beta}-t_1^{\alpha+\beta})}{\Gamma(\alpha+\beta+1)}+\frac{(\eta+1)(t_2^{\alpha+1}-t_1^{\alpha+1})}{|1-\eta\xi^{\alpha+1}|\Gamma(\alpha+\beta+1)}\Big]\\
=&(m+nr+\lambda r)[A_1(t_2^{\alpha+\beta}-t_1^{\alpha+\beta})+A_2(t_2^{\alpha+1}-t_1^{\alpha+1})].
\end{aligned}
$$

Similarly, we get

$$
\begin{aligned}
&|{}^cD_{0^+}^{\beta}Tx(t_2) -{}^c D_{0^+}^{\beta}Tx(t_1)|\\
=&\Big|\int_0^{t_2}\frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}[f(s,x(s))+\lambda^c D_{0^+}^{\beta}x(s)]ds\\
&-\int_0^{t_1}\frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}[f(s,x(s))+\lambda^c D_{0^+}^{\beta}x(s)]ds\\
&+\frac{\Gamma(\alpha+2)(t_2^{1-\beta+\alpha}-t_1^{1-\beta+\alpha})}{(1-\eta\xi^{\alpha+1})\Gamma(2-\beta+\alpha)}\Big[\eta\int_0^{\xi}\frac{(\xi-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}[f(s,x(s))+\lambda^c D_{0^+}^{\beta}x(s)]ds\\
&-\int_0^1\frac{(1-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}[f(s,x(s))+\lambda^c D_{0^+}^{\beta}x(s)]ds\Big]\Big|\\
\le&\Big|\int_0^{t_2}\frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}[f(s,x(s))+\lambda^c D_{0^+}^{\beta}x(s)]ds\\
&-\int_0^{t_1}\frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}[f(s,x(s))+\lambda^c D_{0^+}^{\beta}x(s)]ds\Big|\\
&+\frac{\Gamma(\alpha+2)(t_2^{1-\beta+\alpha}-t_1^{1-\beta+\alpha})}{|1-\eta\xi^{\alpha+1}|\Gamma(2-\beta+\alpha)}\Big[\eta\int_0^{\xi}\frac{(\xi-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\Big|f(s,x(s))+\lambda^c D_{0^+}^{\beta}x(s)\Big|ds\\
&+\int_0^1\frac{(1-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\Big|f(s,x(s))+\lambda^c D_{0^+}^{\beta}x(s)\Big|ds\Big]\\
\le&\frac{(m+nr+\lambda r)(t_2^{\alpha}-t_1^{\alpha})}{\Gamma(\alpha+1)}+\frac{\Gamma(\alpha+2)(t_2^{1-\beta+\alpha}-t_1^{1-\beta+\alpha})}{|1-\eta\xi^{\alpha+1}|\Gamma(2-\beta+\alpha)}\frac{(m+nr+\lambda r)(\eta\xi^{\alpha+\beta}+1)}{\Gamma(\alpha+\beta+1)}\\
\le&(m+nr+\lambda r)\Big[\frac{(t_2^{\alpha}-t_1^{\alpha})}{\Gamma(\alpha+1)}+\frac{(\eta+1)\Gamma(\alpha+2)(t_2^{1-\beta+\alpha}-t_1^{1-\beta+\alpha})}{|1-\eta\xi^{\alpha+1}|\Gamma(2-\beta+\alpha)\Gamma(\alpha+\beta+1)}\Big]\\
=&(m+nr+\lambda r)[A_3(t_2^{\alpha}-t_1^{\alpha})+A_4(t_2^{1-\beta+\alpha}-t_1^{1-\beta+\alpha})].
\end{aligned}
$$

Therefore, by (3.1), we have

$$
\begin{aligned}
\|Tx_2-Tx_1\|=&\max_{0\le t\le 1}|Tx(t_2)-Tx(t_1)|+\max_{0\le t\le 1}|{}^cD_{0^+}^{\beta}Tx(t_2)-{}^c D_{0^+}^{\beta}Tx(t_1)|\\
\le&[A_1(t_2^{\alpha+\beta}-t_1^{\alpha+\beta})+A_2(t_2^{\alpha+1}-t_1^{\alpha+1})+A_3(t_2^{\alpha}-t_1^{\alpha})\\
&+A_4(t_2^{1-\beta+\alpha}-t_1^{1-\beta+\alpha})](m+nr+\lambda r)\to 0,\quad t_1\to t_2.
\end{aligned}
$$

Then operator T is equicontinuous. From Arzela-Ascoli theorem, we obtain that operator $T:D\to D$ is completely continuous.

(ii) Finally, we present that the following set is bounded,

$$V=\{x\in E: x=\mu Tx, 0<\mu<1\}.$$

For $\forall x\in V$, $\forall t\in[0,1]$, we obtain

$$
\begin{aligned}
|x(t)|=&|\mu Tx(t)|\le|Tx(t)|\\
\le&\int_0^t\frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}(|f(s,x(s))|+|\lambda^c D_{0^+}^{\beta}x(s)|)ds+\frac{t^{\alpha+1}}{|1-\eta\xi^{\alpha+1}|}\\
&\Big[\eta\int_0^{\xi}\frac{(\xi-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}(|f(s,x(s))|+|\lambda^c D_{0^+}^{\beta}x(s)|)ds\\
&+\int_0^1\frac{(1-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}(|f(s,x(s))|+|\lambda^c D_{0^+}^{\beta}x(s)|)ds\Big]
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{(m+n\|x\|+\lambda\|x\|)t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}+\frac{t^{\alpha+1}}{|1-\eta\xi^{\alpha+1}|}\frac{(m+n\|x\|+\lambda\|x\|)(\eta\xi^{\alpha+\beta}+1)}{\Gamma(\alpha+\beta+1)}\\
&\leq (m+n\|x\|+\lambda\|x\|)\Big[\frac{1}{\Gamma(\alpha+\beta+1)}+\frac{\eta+1}{|1-\eta\xi^{\alpha+1}|\Gamma(\alpha+\beta+1)}\Big]\\
&= (m+n\|x\|+\lambda\|x\|)(A_1+A_2),
\end{aligned}
$$

and

$$
\begin{aligned}
&|{}^cD^{\beta}_{0^+}x(t)| = |\mu{}^cD^{\beta}_{0^+}Tx(t)| \leq |{}^cD^{\beta}_{0^+}Tx(t)|\\
&\leq \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}(|f(s,x(s))|+|\lambda{}^cD^{\beta}_{0^+}x(s)|)ds+\frac{\Gamma(\alpha+2)t^{1-\beta+\alpha}}{|1-\eta\xi^{\alpha+1}|\Gamma(2-\beta+\alpha)}\\
&\Big[\eta\int_0^{\xi}\frac{(\xi-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}(|f(s,x(s))|+|\lambda{}^cD^{\beta}_{0^+}x(s)|)ds\\
&+\int_0^1\frac{(1-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}(|f(s,x(s))|+|\lambda{}^cD^{\beta}_{0^+}x(s)|)ds\Big]\\
&\leq \frac{(m+n\|x\|+\lambda\|x\|)t^{\alpha}}{\Gamma(\alpha+1)}+\frac{\Gamma(\alpha+2)t^{1-\beta+\alpha}}{|1-\eta\xi^{\alpha+1}|\Gamma(2-\beta+\alpha)}\frac{(m+n\|x\|+\lambda\|x\|)(\eta\xi^{\alpha+\beta}+1)}{\Gamma(\alpha+\beta+1)}\\
&\leq (m+n\|x\|+\lambda\|x\|)\Big[\frac{1}{\Gamma(\alpha+1)}+\frac{(\eta+1)\Gamma(\alpha+2)}{|1-\eta\xi^{\alpha+1}|\Gamma(2-\beta+\alpha)\Gamma(\alpha+\beta+1)}\Big]\\
&= (m+n\|x\|+\lambda\|x\|)(A_3+A_4).
\end{aligned}
$$

Therefore, from (3.1), we have

$$\|x\| \leq (m+n\|x\|+\lambda\|x\|)(A_1+A_2+A_3+A_4),$$

and

$$\|x\| \leq \frac{m(A_1+A_2+A_3+A_4)}{1-(n+\lambda)(A_1+A_2+A_3+A_4)}.$$

Then V is bounded. From Leray-Schauder fixed point theorem, we get boundary value problem (1.1) has at least a solution on D. This completes the proof. □

Theorem 3.2. *If the following assumptions hold*
(H_3) $f:[0,1]\times\mathbb{R}\to\mathbb{R}$ *is continuous;*
(H_4) *there exists a constant* $k>0$ *such that*

$$|f(t,x_1)-f(t,x_2)| \leq k|x_1-x_2|, \quad \forall (t,x_i)\in[0,1]\times\mathbb{R}, i=1,2,$$

then boundary value problem (1.1) *has a unique solution provided that*

$$0<(k+\lambda)(A_1+A_2+A_3+A_4)<1.$$

Proof. For $\forall x,y\in E$, $\forall t\in[0,1]$, by (3.2) and (H_4), we obtain

$$
\begin{aligned}
&|Tx(t)-Ty(t)|\\
&\leq \int_0^t\frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\big(\big|f(s,x(s))-f(s,y(s))\big|+\big|\lambda{}^cD^{\beta}_{0^+}x(s)-\lambda{}^cD^{\beta}_{0^+}y(s)\big|\big)ds\\
&+\frac{t^{\alpha+1}}{|1-\eta\xi^{\alpha+1}|}\Big[\eta\int_0^{\xi}\frac{(\xi-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\big(\big|f(s,x(s))-f(s,y(s))\big|\\
&+\big|\lambda{}^cD^{\beta}_{0^+}x(s)-\lambda{}^cD^{\beta}_{0^+}y(s)\big|\big)ds+\int_0^1\frac{(1-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\big(\big|f(s,x(s))-f(s,y(s))\big|\\
&+\big|\lambda{}^cD^{\beta}_{0^+}x(s)-\lambda{}^cD^{\beta}_{0^+}y(s)\big|\big)ds\Big]
\end{aligned}
$$

$$
\begin{aligned}
\leq & \int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\big(k\big|x(s)-y(s)\big|+\lambda\big|{}^cD_{0^+}^{\beta}x(s)-{}^c D_{0^+}^{\beta}y(s)\big|\big)ds \\
& + \frac{t^{\alpha+1}}{|1-\eta\xi^{\alpha+1}|}\Big[\eta\int_0^{\xi} \frac{(\xi-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\big(k\big|x(s)-y(s)\big|+\lambda\big|{}^cD_{0^+}^{\beta}x(s)-{}^c D_{0^+}^{\beta}y(s)\big|\big)ds \\
& + \int_0^1 \frac{(1-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\big(k\big|x(s)-y(s)\big|+\lambda\big|{}^cD_{0^+}^{\beta}x(s)-{}^c D_{0^+}^{\beta}y(s)\big|\big)ds\Big] \\
\leq & \frac{(k+\lambda)\|x-y\|t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}+\frac{t^{\alpha+1}}{|1-\eta\xi^{\alpha+1}|}\frac{(k+\lambda)\|x-y\|(\eta\xi^{\alpha+\beta}+1)}{\Gamma(\alpha+\beta+1)} \\
\leq & (k+\lambda)\Big[\frac{1}{\Gamma(\alpha+\beta+1)}+\frac{\eta+1}{|1-\eta\xi^{\alpha+1}|\Gamma(\alpha+\beta+1)}\Big]\|x-y\| \\
= & (k+\lambda)(A_1+A_2)\|x-y\|,
\end{aligned}
$$

and

$$
\begin{aligned}
& |{}^cD_{0^+}^{\beta}Tx(t)-{}^c D_{0^+}^{\beta}Ty(t)| \\
\leq & \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\big(\big|f(s,x(s))-f(s,y(s))\big|+\big|\lambda{}^cD_{0^+}^{\beta}x(s)-\lambda{}^cD_{0^+}^{\beta}y(s)\big|\big)ds \\
& + \frac{\Gamma(\alpha+2)t^{1-\beta+\alpha}}{|1-\eta\xi^{\alpha+1}|\Gamma(2-\beta+\alpha)} \\
& \Big[\eta\int_0^{\xi} \frac{(\xi-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\big(\big|f(s,x(s))-f(s,y(s))\big|+\big|\lambda{}^cD_{0^+}^{\beta}x(s)-\lambda{}^cD_{0^+}^{\beta}y(s)\big|\big)ds \\
& + \int_0^1 \frac{(1-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\big(\big|f(s,x(s))-f(s,y(s))\big|+\big|\lambda{}^cD_{0^+}^{\beta}x(s)-\lambda{}^cD_{0^+}^{\beta}y(s)\big|\big)ds\Big] \\
\leq & \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\big(k\big|x(s)-y(s)\big|+\lambda\big|{}^cD_{0^+}^{\beta}x(s)-{}^c D_{0^+}^{\beta}y(s)\big|\big)ds \\
& + \frac{\Gamma(\alpha+2)t^{1-\beta+\alpha}}{|1-\eta\xi^{\alpha+1}|\Gamma(2-\beta+\alpha)} \\
& \Big[\eta\int_0^{\xi} \frac{(\xi-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\big(k\big|x(s)-y(s)\big|+\lambda\big|{}^cD_{0^+}^{\beta}x(s)-{}^c D_{0^+}^{\beta}y(s)\big|\big)ds \\
& + \int_0^1 \frac{(1-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\big(k\big|x(s)-y(s)\big|+\lambda\big|{}^cD_{0^+}^{\beta}x(s)-{}^c D_{0^+}^{\beta}y(s)\big|\big)ds\Big] \\
\leq & \frac{(k+\lambda)\|x-y\|t^{\alpha}}{\Gamma(\alpha+1)}+\frac{\Gamma(\alpha+2)t^{1-\beta+\alpha}}{|1-\eta\xi^{\alpha+1}|\Gamma(2-\beta+\alpha)}\frac{(k+\lambda)\|x-y\|(\eta\xi^{\alpha+\beta}+1)}{\Gamma(\alpha+\beta+1)} \\
\leq & (k+\lambda)\Big[\frac{1}{\Gamma(\alpha+1)}+\frac{(\eta+1)\Gamma(\alpha+2)}{|1-\eta\xi^{\alpha+1}|\Gamma(2-\beta+\alpha)\Gamma(\alpha+\beta+1)}\Big]\|x-y\| \\
= & (k+\lambda)(A_3+A_4)\|x-y\|.
\end{aligned}
$$

Therefore, by (3.1), we get

$$
\begin{aligned}
\|Tx-Ty\| = & \max_{0\leq t\leq 1}|Tx(t)-Ty(t)|+\max_{0\leq t\leq 1}|{}^cD_{0^+}^{\beta}Tx(t)-{}^c D_{0^+}^{\beta}Ty(t)| \\
\leq & (k+\lambda)(A_1+A_2+A_3+A_4)\|x-y\|.
\end{aligned}
$$

Then T is a contraction. By contraction mapping principle, we obtain that boundary value problem (1.1) has a unique solution on E. This completes the proof. □

4. The stability of fractional Langevin equation

Definition 4.1. The solution is Ulam-Hyers (UH) stable, if there exist constants $M_1 \geq 0$ and $\varepsilon > 0$, for each solution $x \in C([0,1],\mathbb{R})$ of

$$\left|{}^cD^{\beta}_{0^+}({}^cD^{\alpha}_{0^+} - \lambda)x(t) - f(t,x(t))\right| \leq \varepsilon, \quad t \in [0,1], \tag{4.1}$$

and a solution $x^* \in C([0,1],\mathbb{R})$ of (1.1), such that $|x - x^*| \leq M_1\varepsilon$. The solution is generalised Ulam-Hyers (GUH) stable, if there exists $\psi \in C(\mathbb{R}^+,\mathbb{R}^+)$ with $\psi(0) = 0$, such that $|x - x^*| \leq M_1\psi(\varepsilon)$.

Lemma 4.1. *Function $x \in C([0,1],\mathbb{R})$ is the solution of (4.1), if there exists a function $\varpi \in C([0,1],\mathbb{R})$ depending on x, such that*
(i) $\left|\varpi(t)\right| \leq \varepsilon$, $t \in [0,1]$;
(ii) ${}^cD^{\beta}_{0^+}({}^cD^{\alpha}_{0^+} - \lambda)x(t) - f(t,x(t)) - \varpi(t) = 0$.

Proof. If (i) and (ii) hold, we obtain

$$\left|{}^cD^{\beta}_{0^+}({}^cD^{\alpha}_{0^+} - \lambda)x(t) - f(t,x(t))\right| = \left|\varpi(t)\right| \leq \varepsilon.$$

If $x \in C([0,1],\mathbb{R})$ is the solution of (4.1), we get

$$-\varepsilon \leq^c D^{\beta}_{0^+}({}^cD^{\alpha}_{0^+} - \lambda)x(t) - f(t,x(t)) \leq \varepsilon.$$

Hence, there exists $\varpi(t) \in [-\varepsilon,\varepsilon]$, such that

$${}^cD^{\beta}_{0^+}({}^cD^{\alpha}_{0^+} - \lambda)x(t) - f(t,x(t)) = \varpi(t).$$

This completes the proof. $\square$

Lemma 4.2. *If $u \in C([0,1],\mathbb{R})$ is the solution of*

$$\begin{cases} {}^cD^{\beta}_{0^+}({}^cD^{\alpha}_{0^+} - \lambda)x(t) = f(t,x(t)) + \varpi(t), \quad t \in (0,1), \\ x(0) = 0, {}^c D^{\alpha}_{0^+}x(0) = 0, x(1) = \eta x(\xi), \end{cases} \tag{4.2}$$

then, u satisfies the following inequality

$$\left|x(t) - Tx(t)\right| \leq m_1\varepsilon,$$

where

$$m_1 = \Big[\frac{1}{\Gamma(\alpha+\beta+1)} + \frac{1}{|1-\eta\xi^{\alpha+1}|}\Big(\eta\frac{1}{\Gamma(\alpha+\beta+1)} - \frac{1}{\Gamma(\alpha+\beta+1)}\Big)\Big].$$

Proof. If $x \in C([0,1],\mathbb{R})$ is the solution of (4.2), then

$$\begin{aligned} x(t) =& Tx(t) + \int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\varpi(s)ds + \frac{t^{\alpha+1}}{1-\eta\xi^{\alpha+1}} \\ & \Big[\eta\int_0^{\xi}\frac{(\xi-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\varpi(s)ds - \int_0^1\frac{(1-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\varpi(s)ds\Big]. \end{aligned}$$

In view of Lemma 2.3, we obtain

$$\begin{aligned}\big|x(t)-Tx(t)\big| \leq & \int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\big|\varpi(s)\big|ds+\frac{t^{\alpha+1}}{|1-\eta\xi^{\alpha+1}|}\\ & \Big[\eta\int_0^\xi \frac{(\xi-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\big|\varpi(s)\big|ds-\int_0^1 \frac{(1-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\big|\varpi(s)\big|ds\Big]\\ \leq & \varepsilon\Big[\int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}ds+\frac{t^{\alpha+1}}{|1-\eta\xi^{\alpha+1}|}\\ & \Big(\eta\int_0^\xi \frac{(\xi-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}ds-\int_0^1 \frac{(1-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}ds\Big)\Big]\\ \leq & \varepsilon\Big[\frac{1}{\Gamma(\alpha+\beta+1)}+\frac{1}{|1-\eta\xi^{\alpha+1}|}\Big(\eta\frac{1}{\Gamma(\alpha+\beta+1)}-\frac{1}{\Gamma(\alpha+\beta+1)}\Big)\Big]\\ = & m_1\varepsilon.\end{aligned}$$

This completes the proof. □

Theorem 4.1. *If* (H_3) *and* (H_4) *hold and*

$$(k+\lambda)(A_1+A_2+A_3+A_4)<1,$$

then boundary value problem (1.1) *is* UH *stable and* GUH *stable.*

Proof. For each solution $x\in C([0,1],\mathbb{R})$ of (4.2) and the unique solution x^* of (1.1), by Theorem 3.2 and Lemma 4.2, we get

$$\begin{aligned}\big\|x-x^*\big\| &= \big\|x-Tx^*\big\| = \big\|x-Tx+Tx-Tx^*\big\|\\ &\leq \big\|x-Tx\big\|+\big\|Tx-Tx^*\big\|\\ &\leq m_1\varepsilon+k\big\|x-x^*\big\|.\end{aligned}$$

Therefore,

$$\big\|x-x^*\big\| \leq \frac{m_1}{1-k}\varepsilon := M_1\varepsilon.$$

Then boundary value problem (1.1) is UH stable. For $\psi(\varepsilon)=\varepsilon$, boundary value problem (1.1) is GUH stable, which can be seen in Figure 1. □

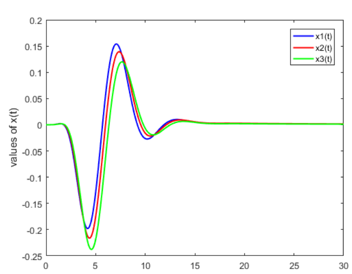

Figure 1. Illustration of proof.

5. Examples

Example 5.1. Consider the following boundary value problem

$$\begin{cases} {}^cD_{0^+}^{\frac{5}{4}}({}^cD_{0^+}^{\frac{1}{2}} - 0.01)x(t) = f(t, x(t)), \quad t \in (0, 1), \\ x(0) = 0, {}^c D_{0^+}^{\frac{1}{2}} x(0) = 0, x(1) = \eta x(\frac{1}{4}), \end{cases} \tag{5.1}$$

where $\alpha = \frac{1}{2}$, $\beta = \frac{5}{4}$, $\lambda = 0.01$, $\eta = \frac{A}{B}$, $A = \sum_{k=0}^{+\infty} \frac{\lambda^k}{\Gamma(k\alpha+\alpha+2)} \approx 0.7573$, $B = \sum_{k=0}^{+\infty} \frac{\lambda^k \xi^{k\alpha+\alpha+1}}{\Gamma(k\alpha+\alpha+2)} \approx 0.0943$, $\sum_{k=0}^{+\infty} \frac{\lambda^k(1-\eta\xi^{k\alpha+\alpha+1})}{\Gamma(k\alpha+\alpha+2)} = 0$, $\xi = \frac{1}{4}$, $r = 100$, ${}^cD_{0^+}^{\alpha}$ is Caputo fractional derivative and $f : [0, 1]\times\mathbb{R} \to \mathbb{R}$ is continuous and $f(t, x(t)) = t+\frac{x(t)}{10(t^2+10)}$. Then

$$\begin{aligned}
A_1 &= \frac{1}{\Gamma(\alpha + \beta + 1)} = \frac{1}{\Gamma(2.75)} \approx 0.6217, \\
A_2 &= \frac{\eta + 1}{|1 - \eta\xi^{\alpha+1}|\Gamma(\alpha + \beta + 1)} = \frac{9.03}{0.00375\Gamma(2.75)} \approx 1496, \\
A_3 &= \frac{1}{\Gamma(\alpha + 1)} = \frac{1}{\Gamma(1.5)} \approx 1.1284, \\
A_4 &= \frac{(\eta + 1)\Gamma(\alpha + 2)}{|1 - \eta\xi^{\alpha+1}|\Gamma(2 - \beta + \alpha)\Gamma(\alpha + \beta + 1)} = \frac{9.03\Gamma(2.5)}{0.00375\Gamma(1.25)\Gamma(2.75)} \approx 2195.
\end{aligned}$$

By (5.1), we get

$$|f(t, x(t))| \le 2 + \frac{1}{90}|x(t)|.$$

Then $m = 2$, $n = \frac{1}{90}$ and

$$\frac{m(A_1 + A_2 + A_3 + A_4)}{1 - (n + \lambda)(A_1 + A_2 + A_3 + A_4)} \approx 96 < 100.$$

Then from Theorem 3.1, boundary value problem (5.1) has at least a solution.

Example 5.2. Consider the following boundary value problem

$$\begin{cases} {}^cD_{0^+}^{\frac{5}{4}}({}^cD_{0^+}^{\frac{1}{2}} - 0.01)x(t) = f(t, x(t)), \quad t \in (0, 1), \\ x(0) = 0, {}^c D_{0^+}^{\frac{1}{2}} x(0) = 0, x(1) = \eta x(\frac{1}{4}), \end{cases} \tag{5.2}$$

where $\alpha = \frac{1}{2}$, $\beta = \frac{5}{4}$, $\lambda = 0.01$, $\eta = \frac{A}{B}$, $A = \sum_{k=0}^{+\infty} \frac{\lambda^k}{\Gamma(k\alpha+\alpha+2)} \approx 0.7573$, $B = \sum_{k=0}^{+\infty} \frac{\lambda^k \xi^{k\alpha+\alpha+1}}{\Gamma(k\alpha+\alpha+2)} \approx 0.0943$, $\sum_{k=0}^{+\infty} \frac{\lambda^k(1-\eta\xi^{k\alpha+\alpha+1})}{\Gamma(k\alpha+\alpha+2)} = 0$, $\xi = \frac{1}{4}$, $r = 100$, ${}^cD_{0^+}^{\alpha}$ is Caputo fractional derivative and $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is continuous and $f(t, x(t)) = \frac{x(t)}{10(t^2+10)}$.

Then

$$
\begin{aligned}
A_1 &= \frac{1}{\Gamma(\alpha+\beta+1)} = \frac{1}{\Gamma(2.75)} \approx 0.6217,\\
A_2 &= \frac{\eta+1}{|1-\eta\xi^{\alpha+1}|\Gamma(\alpha+\beta+1)} = \frac{9.03}{0.0375\Gamma(2.75)} \approx 1496,\\
A_3 &= \frac{1}{\Gamma(\alpha+1)} = \frac{1}{\Gamma(1.5)} \approx 1.1284,\\
A_4 &= \frac{(\eta+1)\Gamma(\alpha+2)}{|1-\eta\xi^{\alpha+1}|\Gamma(2-\beta+\alpha)\Gamma(\alpha+\beta+1)} = \frac{9.03\Gamma(2.5)}{0.0375\Gamma(1.25)\Gamma(2.75)} \approx 2195.
\end{aligned}
$$

From (5.2), we have

$$
\begin{aligned}
\left|f(t,x_1)-f(t,x_2)\right| &\le \frac{1}{10(t^2+10)}\left|x_1-x_2\right|\\
&\le \frac{1}{100}\left|x_1-x_2\right|.
\end{aligned}
$$

Then $k=\frac{1}{100}$ and

$$(k+\lambda)(A_1+A_2+A_3+A_4) \approx 0.3417 < 1.$$

Therefore by Theorem 3.2, boundary value problem (5.2) has a unique solution. From Theorem 4.1, we obtain (5.2) is UH stable and GUH stable.

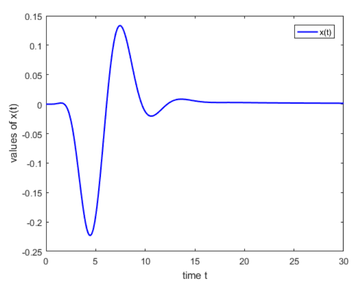

Figure 2. Illustration of proof.

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