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Painlevé Analysis and Auto-Bäcklund Transformation for a General Variable Coefficient Burgers Equation with Linear Damping Term*

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Abstract This paper investigates a general variable coefficient (gVC) Burgers equation with linear damping term. We derive the Painlevé property of the equation under certain constraint condition of the coefficients. Then we obtain an auto-Bäcklund transformation of this equation in terms of the Painlevé property. Finally, we find a large number of new explicit exact solutions of the equation. Especially, infinite explicit exact singular wave solutions are obtained for the first time. It is worth noting that these singular wave solutions will blow up on some lines or curves in the (x, t) plane. These facts reflect the complexity of the structure of the solution of the gVC Burgers equation with linear damping term. It also reflects the complexity of nonlinear wave propagation in fluid from one aspect.

Keywords Painlevé property, auto-Bäcklund transformation, a gVC Burgers equation with linear damping term, exact solutions

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1. Introduction

The well-known Burgers equation

$$u_t + uu_x - \nu u_{xx} = 0, \tag{1.1}$$

as the simplest nonlinear model in fluid dynamics balances the effect of nonlinear convection and the linear diffusion [5]. It was originally derived to describe the propagation of nonlinear waves in dissipative media, where $\nu(> 0)$ is the kinematic viscosity, and $u(x, t)$ represents the fluid velocity field. It plays an important role in explaining two fundamental effects characteristic of any turbulence: the nonlinear redistribution of energy over the spectrum and the action of viscosity in small scales. For the classical Burgers equation, a large number of literatures have discussed its Painlevé integrability, Bäcklund transformation and other mathematical physics properties [1, 4, 9, 10, 24].

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In the real processes, the influence of damping is inevitable. It even plays an important role in some problems. The Burgers equation with a linear damping which describes the plane motion of a continuous medium for which the constitutive relation for the stress contains a large linear term proportional to the strain, a small term which is quadratic in the strain, and a small dissipative term proportional to the strain-rate, has been studied for single hump conditions by singular perturbation approach by Lardner and Arya [11]. The N-wave solutions for this equation have been obtained by Sachdev and Joseph [18]. Vaganan and Kumaran [20] discussed similarity solutions of the Burgers equation with linear damping and obtained a trivial solution of the Burgers equation with linear damping by Lie's group analysis method. According to a relationship between the solutions of the damped Burgers equation and the cylindrical Burgers equation obtained by Sachdev and Vaganan [19], they also obtained a solution of the cylindrical Burgers equation. Peng and Chen [15] obtained another trivial solution of the Burgers equation with linear damping by using the direct method of Clarkson and Kruskal [3].

In the actual physical situations, the inhomogeneity of the medium, the roughness or non smoothness of the fluid bottom, and the non-uniformity of the boundary must also be considered. The variable coefficient partial differential equations often provide more powerful and realistic model than their constant coefficient counterparts in several physical situations. It seems more meaningful to consider the variable coefficient Burgers equation with damping.

In 1991, Oliveri [14] considered a generalized Burgers' equation containing an arbitrary function of time

$$u_t + uu_x - u_{xx} + f(t)u = 0. \tag{1.2}$$

He proved that equation (1.2) possesses the Painlevé property if and only if $f(t) = 0$, i.e. it reduces to the classical Burgers equation. He also determined some classes of functional forms for the function $f(t)$ compatible with the existence of similarity solutions of equation (1.2) by means of the Lie group techniques.

Qu Changzheng [17] derived a further generalized Eq.(1.1) with variable nonlinear and dissipative coefficients, i.e.

$$u_t + b(t)uu_x + a(t)u_{xx} = 0. \tag{1.3}$$

which can provide more useful models in many complicated physical situations, such as the propagation of a long shock wave in an inhomogeneous two-layer shallow liquid [8]. The allowed transformations, symmetry classes, Painlevé property, and Bäcklund transformation have been discussed in [8, 17] by the application of the truncated Painlevé expansion and symbolic computation method. Hong found kink-type solitonic solution under the conditions $a(t), b(t) \sim e^{-\alpha_1 t}$ with $\alpha_1 \ll 1$. In the past two decades, many authors have studied the exact linearization, Bäcklund transformation, and similarity reduction of Burgers equation with variable coefficients by using various methods. For more information, we suggest readers to read the literature [2, 6, 13, 16, 21, 22] and references therein.

Wang, Zhang, Li et.al [23] considered the following generalized variable coefficient Burgers equation with linear damping term

$$u_t + \alpha(t)uu_x - \beta(t)u_{xx} + \gamma(t)u = 0, \tag{1.4}$$

which can describe the propagation of nonlinear waves in a liquid subject not only to thermal conductivity but also to convective diffusion effects associated with the viscosity (damping effect), where $u = u(x,t)$ represents temperature (or concentration). $\alpha(t)$, $\beta(t)$ and $\gamma(t)$ are all arbitrary time-dependent functions. When $\alpha(t) = 1$, $\beta(t) = \sigma$,$\gamma(t) = 0$, Eq. (1.4) is known as the Burgers equation [24]. When $\alpha(t) = \beta(t) = 1$, Eq. (1.4) becomes the form of [14]. When $\gamma(t) = 0$, Eq. (1.4) gives the generalized Burgers equation [8]. When $\alpha(t) = \beta(t) = 1$, $\gamma(t) = \sigma$, Eq. (1.4) becomes the equation considered in [11,15,20]. The authors have got the generalized Cole-Hopf transformation by using the simplified homogeneous balance method and obtained some exact solutions by means of the given transformation.

As far as we know, the integrability of Eq. (1.4) is rarely studied at present. Therefore, we hope to apply Painlevé analysis technique in this paper to investigate whether Eq. (1.4) has the Painlevé property. We derive the constraint conditions that variable coefficient functions $\alpha(t), \beta(t)$ and $\gamma(t)$ satisfy when equation (1.4) has the Painlevé property and an auto-Bäcklund transformation. By means of the auto-Bäcklund transformation, many new explicit exact solutions of the gVC Burgers equation with linear damping term (1.4) were immediately obtained.

This paper is organized as follows. In Section 2, Painlevé analysis and auto-Bäcklund transformation of the Eq. (1.4) are derived. In Section 3, some new explicit exact solutions were obtained in terms of different seed solution. In Section 4, several particular equations are discussed.

2. Painlevé analysis and auto-Bäcklund transformation

Painlevé property is one of the important properties of integrable nonlinear models. A partial differential equation is said to possess the Painlevé if its solutions are single valued in a neighbourhood of a non-characteristic movable singularity manifold. A partial differential equation called Painlevé integrable if it possess the Painlevé test. The Painlevé test proposed by Weiss, Tabor and Carnevale to prove the integrability of partial differential equations [24].

According to standard WTC(Weiss-Tabor-Carnevale) method [24], for a nonlinear partial differential equation(NLPDE), we always assume that the solution of NLPDE has the following form

$$u(x,t) = \phi^p \sum_{j=0}^{\infty} u_j \phi^j, \tag{2.1}$$

where $\phi = \phi(x,t)$,$u_j = u_j(x,t)$ are analytic functions of the independent variables in a neighborhood of the singular manifold $\phi(x,t) = 0$. Here p is a negative integer and $u_0 \neq 0$.

Firstly, employ the leading order analysis. Let

$$u(x,t) \sim u_0(x,t)\phi^p.$$

According to the balance between the highest order derivative term $(\beta(t)u_{xx})$ and the highest order nonlinear terms $(\alpha(t)uu_x)$, we get

$$p = -1, \quad u_0 = \frac{-2\beta(t)}{\alpha(t)}\phi_x,$$

so that (2.1) becomes

$$u(x,t) = \sum_{j=0}^{\infty} u_j(x,t)\phi^{j-1}(x,t). \tag{2.2}$$

Substituting (2.2) into Eq. (1.4) with the help of Maple, and taking the coefficient of lowest power term $\phi^{j-3}(x,t)$ to be zero, we can conclude the recursion relation as follow

$$(j+1)(j-2)\beta(t)\phi_x^2 u_j = F(u_{j-1}, u_{j-2}, ..., u_0, \phi_x, \phi_t, ...), \tag{2.3}$$

where

$$\begin{aligned} F(u_{j-1}, ..., u_0, \phi_x, \phi_t, ...) =& \frac{\alpha(t)}{2}\Big\{\sum_{k=1}^{j-2}(u_k u_{j-k-1})_x + (j-2)\sum_{k=1}^{j-1} u_k u_{j-k}\phi_x\Big\} \\ & - \beta(t)\Big\{2(j-1)u_{j-1,x}\phi_x + j u_{j-1}\phi_{xx} + u_{j-2,xx}\Big\} \\ & + u_{j-2,t} + (j-2)u_{j-1}\phi_t + \gamma(t)u_{j-2}. \end{aligned}$$

From (2.3), it is easily noticed that the resonances occur at $j=-1,2$. $j=-1$ corresponds to the arbitrariness of $\phi(x,t)$. When $j=2$, the left-hand side of (2.3) is zero. If the right-hand side is also equal to zero, Eq. (1.4) is said to be compatible, corresponding to the arbitrariness of $u_2(x,t)$.

Making the coefficients of like powers of $\phi(x,t)$ to be zero, we can find that

$$j=0 : -\alpha(t)u_0^2\phi_x - 2\beta(t)u_0\phi_x^2 = 0, \tag{2.4}$$

$$j=1 : \alpha(t)(u_0u_{0x} - u_1u_0\phi_x) + \beta(t)(2u_{0x}\phi_x + u_0\phi_{xx}) - u_0\phi_t = 0, \tag{2.5}$$

$$j=2 : u_{0t} + \alpha(t)(u_0u_{1x} + u_1u_{0x}) - \beta(t)u_{0xx} + \gamma(t)u_0 = 0. \tag{2.6}$$

Generally, we suppose that $\phi_x \neq 0$ and $u_0 \neq 0$. From (2.4) above, we get

$$u_0 = \frac{-2\beta(t)}{\alpha(t)}\phi_x, \tag{2.7}$$

plugging (2.7) into (2.5) yields

$$\alpha(t)u_1\phi_x - \beta(t)\phi_{xx} + \phi_t = 0, \tag{2.8}$$

c ombining with $\phi_x \neq 0$, so we have

$$u_1 = \frac{\beta(t)\phi_{xx} - \phi_t}{\alpha(t)\phi_x}. \tag{2.9}$$

Through (2.4) and (2.5) , we get expression of u_0 and u_1 given as (2.7) and (2.9). But from (2.6), we can not get u_2 uniquely. In other words, Eq. (2.3) is always satisfied for arbitrary u_2. As (2.6) involves determined quantities, we substitute (2.7) and (2.9) into (2.6) in the case of $\phi_x \neq 0$, which yields

$$\phi_x[-\frac{2\gamma(t)\beta(t)}{\alpha(t)} - \frac{2\beta^{'}(t)}{\alpha(t)} + \frac{2\beta(t)\alpha^{'}(t)}{\alpha(t)^2}] = 0,$$

where $\beta^{'}(t) = \frac{\mathrm{d}\beta(t)}{\mathrm{d}t}$ and $\alpha^{'}(t) = \frac{\mathrm{d}\alpha(t)}{\mathrm{d}t}$. Therefore, for the resonance term $j = 2$, we conclude a constraint condition as follows

$$\gamma(t) = \frac{\mathrm{d}}{\mathrm{d}t}[\ln(\frac{\alpha(t)}{\beta(t)})], \tag{2.10}$$

which is the same relationship as that Wang, Zhang, Li et. al [23] got by using the simplified homogeneous balance method.

In addition, our constraint condition (2.10) obtained here includes the compatibility conditions obtained in other literatures(such as [24], [**?**], and [8]) as special cases. From the above analysis, we obtain the following result:

Theorem 2.1. *If the coefficients $\alpha(t)$, $\beta(t)$, and $\gamma(t)$ in Eq. (1.4) satisfy the constraint condition (2.10) and $\phi(x,t)$ satisfies (2.8), then Eq. (1.4) possesses the Painlevé property.*

Next, we consider the case of $j = 3$.

$$\begin{aligned} j = 3 : u_{1t} + u_2\phi_t + \alpha(t)(u_0u_3\phi_x + u_1u_2\phi_x + u_2u_{0x} + u_0u_{2x} + u_1u_{1x}) \\ -\beta(t)(2u_3\phi_x^2 + u_2\phi_{xx} + 2u_{2x}\phi_x + u_{1xx}) + \gamma(t)u_1 = 0. \end{aligned} \tag{2.11}$$

Clearly, (2.11) is equivalent to the following form

$$\begin{aligned} 4\beta(t)u_3\phi_x^2 =& u_{1t} + u_2\phi_t + \alpha(t)(u_1u_2\phi_x + u_2u_{0x} + u_0u_{2x} + u_1u_{1x}) \\ &- \beta(t)(u_2\phi_{xx} + 2u_{2x}\phi_x + u_{1xx}) + \gamma(t)u_1, \end{aligned}$$

which shows that u_3 is also an arbitrary function when u_2 is an arbitrary function. If $u_2 = 0$, then $u_3 = 0$, and we similarly prove that $u_j = 0$ ($j \geq 2$). By simplifying (2.11), we get

$$u_{1t} + \alpha(t)u_1u_{1x} - \beta(t)u_{1xx} + \gamma(t)u_1 = 0, \tag{2.12}$$

so $u_1 = u_1(x,t)$ is a solution of Eq. (1.4). Thus we have the following conclusion:

Theorem 2.2. *If the coefficient functions $\alpha(t)$, $\beta(t)$, and $\gamma(t)$ in Eq. (1.4) satisfy the constraint condition (2.10), then there exists an auto-Bäcklund transformation*

$$u(x,t) = \frac{-2\beta(t)}{\alpha(t)}\frac{\phi_x}{\phi} + u_1(x,t), \tag{2.13}$$

where $u_1(x,t)$ is a solution of Eq. (1.4) and $\phi(x,t)$ satisfies a linear parabolic equation (2.8) .

Remark 2.1. Obviously, $u_1 = 0$ is a trivial solution of Eq.(1.4). When taking seed solution as trivial solution $u_1 = 0$, the auto-Bäcklund transformation (2.13) degenerates the generalized Cole-Hopf transformation (15) in [23]. Our result is an extension of the conclusion in [23].

3. Explicit exact solutions

Seeking the exact solutions of nonlinear partial differential equations is not only of great theoretical significance in mathematics, but also of great application value in physics. For nonlinear partial differential equations with constant coefficients, there are many methods for seeking explicit exact solutions, such as the plane dynamic system method [?, ?], Hirota bilinear method [7, 25], etc. In [7], Gaillard construct rational solutions to the KdV equation by particular polynomials. In this section, we will construct infinite explicit exact singular wave solutions to the gVC Burgers equation with the linear damping term.

According to auto-Bäcklund transformation (2.13), we can find out various kinds of solutions for Eq. (1.4) by choosing different $\phi(x,t)$ and $u_1(x,t)$. In particular, choosing $u_1 = 0$ as a "seed" solution, from (2.8) in case of $\phi_x \neq 0$, we get

$$\phi_t - \beta(t)\phi_{xx} = 0. \tag{3.1}$$

Letting $\tau = \int^t \beta(\xi)d\xi$, Eq. (3.1) becomes

$$\phi_\tau - \phi_{xx} = 0. \tag{3.2}$$

According to the solutions of the linear heat conduction equation (3.2), we get the following explicit exact solutions of equation (1.4).

Case 1. Taking $\phi(x,\tau) = 1 + exp(kx + k^2\tau + c)$, we obtain a solitary wave-like solution

$$u(x,t) = \frac{-k\beta(t)}{\alpha(t)}[1 + tanh\frac{k}{2}(x + k\int^t \beta(\tau)d\tau + C)]. \tag{3.3}$$

Remark 3.1. The solution (3.3) is a smooth global solution. For specific $\alpha(t), \beta(t)$, this solution has the waveform profile of kink-typed solitary wave.

Case 2. Taking $\phi(x,\tau) = A + B\exp(-\lambda^2\tau)cos(\lambda x)$, we obtain explicit exact solutions of Eq. (1.4) as follows

$$u(x,t) = \frac{2\beta(t)}{\alpha(t)}\frac{\lambda B\exp(-\lambda^2\int^t \beta(\tau)d\tau)sin(\lambda x)}{A + B\exp(-\lambda^2\int^t \beta(\tau)d\tau)cos(\lambda x)}. \tag{3.4}$$

Case 3. Taking $\phi(x,\tau) = A+B\exp(-\lambda^2\tau)sin(\lambda x)$, we obtain the explicit exact solutions of Eq. (1.4) given as

$$u(x,t) = \frac{-2\beta(t)}{\alpha(t)}\frac{\lambda B\exp(-\lambda^2\int^t \beta(\tau)d\tau)cos(\lambda x)}{A + B\exp(-\lambda^2\int^t \beta(\tau)d\tau)sin(\lambda x)}. \tag{3.5}$$

Remark 3.2. It is worth pointing out that solutions (3.4) and (3.5) are all periodic functions with respect to the independent variable x. And these solutions asymptotically decay to zero when t tend to zero $+\infty$ (when $\beta(t) > 0$). It must also be pointed out that these solutions are all blow-up solutions, i.e. they will blow up on some curves in the (x,t) plane.

Case 4. Taking $\phi(x,\tau) = ax + b$, a, and b are two arbitrary constants that are not all zero, we obtain the explicit exact solution of rational type

$$u(x,t) = \frac{-2\beta(t)}{\alpha(t)}\frac{a}{ax + b}. \tag{3.6}$$

Remark 3.3. The solution of rational type (3.6) is a product form variable separation solution. This solution will blow up on the line $ax + b = 0$ in the (x,t) plane.

Case 5. Taking $\phi(x,\tau) = a(x^2+2\tau)+bx+c$, a, b, and c are three arbitrary constants that are not all zero, we obtain the exact rational fraction solutions

$$u(x,t) = \frac{-2\beta(t)}{\alpha(t)} \frac{2ax+b}{a(x^2+2\int^t \beta(\tau)d\tau)+bx+c}. \tag{3.7}$$

Remark 3.4. The solution of rational type (3.7) is also a blow-up solution. It will blow up on the curve $a(x^2+2\int^t \beta(\tau)d\tau)+bx+c=0$ in the (x,t) plane.

Case 6. Taking $\phi(x,\tau) = ax^3+bx^2+cx+d+(6ax+2b)\tau$, a, b, c, and d are four arbitrary constants that are not all zero,we obtain a solution

$$u(x,t) = \frac{-2\beta(t)}{\alpha(t)} \frac{3ax^2+2bx+c+6a\int^t \beta(\tau)d\tau}{ax^3+bx^2+cx+d+(6ax+2b)\int^t \beta(\tau)d\tau}. \tag{3.8}$$

Remark 3.5. The solution of rational type (3.9) is also a blow-up solution. It will blow up on the curve $ax^3+bx^2+cx+d+(6ax+2b)\int^t \beta(\tau)d\tau = 0$ in the (x,t) plane.

Case 7. In general, we can suppose that Eq. (3.2) has a solution in the following form

$$\phi(x,\tau) = \Sigma_{i=0}^{k} P_i(x)\tau^i, \quad k \geq 1, \tag{3.9}$$

where $P_i(x)$ is the polynomial of its variable x. By solving the following ordinary differential equations

$$\begin{cases} P_0''(x) + P_1(x) = 0, \\ P_1''(x) + 2P_2(x) = 0, \\ . \\ . \\ . \\ P_{k-1}''(x) + kP_k(x) = 0, \\ P_k''(x) = 0, \end{cases} \tag{3.10}$$

$P_i(x)$ can be determined. By using the auto-Bäcklund transformation (2.13), we can obtain infinite explicit exact solutions of rational type for Eq. (1.4).

Since Eq. (3.2) is a homogeneous linear equation, any linear combination of $\phi(x,\tau)$ functions obtained above is also its solution. By using the Bäcklund transformation , we can obtain more abundant new explicit exact solutions of Eq. (1.4). These results above greatly enrich those solutions obtained in [23]. Except for the solution (3.3), all other solutions are singular solutions that will blow up on a curve in the (x, t) plane.

When taking the "seed" solution $u_1 \neq 0$ in the auto-Bäcklund transformation (2.13), we can also find more new explicit exact solutions. A detailed discussion will be given in another article.

4. Conclusion

In this paper, a generalized variable coefficients Burgers equation with linear damping term has been discussed. The WTC method is used to investigate whether the equation has the Painlevé property. It turns out that this equation has the Painlevé property as long as the coefficient functions satisfy a constraint condition (2.10). By applying Painlevé truncation expansion, the auto-Bäcklund transformation has been obtained. According to the transformation obtained, a large number of new solutions have been got through choosing different $\phi(x,t)$. Especially, The infinite explicit exact singular wave solutions of the gVC Burgers equation with linear damping term are obtained for the first time. It is worth noting that these singular wave solutions of the gVC Burgers equation with linear damping term will blow up on some lines or curves in the (x,t) plane. These facts reflect the complexity of the structure of the solution of the gVC Burgers equation with linear damping term. It also reflects the complexity of fluid from one aspect.

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