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Asymptotic Expansion of Solutions to Singular Perturbation Problems in Critical Cases*

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Abstract This paper investigates the problem of singular perturbed integral initial values and Robin boundary values in the critical case. Based on the boundary layer function method, we not only construct the asymptotic approximation of the original equation, but also prove the uniform validity of the asymptotic solution by successive approximation. At the same time, we give an example to prove the validity of the theoretical results.

Keywords Singularly perturbed problem, critical case, boundary function method, approximate solution

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1. Introduction

Singularly perturbed differential equation boundary value problems are frequently utilized as mathematical models to represent biomechanical and physical phenomena [1, 2, 18], such as chemical kinetics [19] and semiconductor simulation [20] phenomena. They correspond to mathematical models in which the degradation equation of singularly perturbed differential equation boundary value problems does not have an isolated root, but has a series of roots depending on one or more parameters instead. This case will be called the critical case [6]. Compared with the singularly perturbed initial boundary value problem with isolated roots of other degradation equations, the critical case has the following three difficulties: first, the zero order regular approximation solution is an unknown arbitrary function, which needs to be obtained by subsequent conditions; secondly, the solution process of the zero-order boundary layer is also very complicated; thirdly, the coupling of equations related to $k\,(k \geqslant 1)$-order boundary layer functions can be reduced by a specific diagonalization transformation. Therefore, it is very meaningful and valuable to study the critical case in singularly perturbed problems.

The research methods of singular perturbation critical problems mainly include boundary layer function method [13]. Through the theory of boundary layer function method, Vasil'eva and Butuzov [6] were the first to study initial value problems for singularly perturbed systems in the critical case. Subsequently, in recent years, some related issues in critical case have been solved [3–5, 7, 15, 17]. To the

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best of our knowledge, the above problems only pertain to Dirichlet or Neumann boundary value conditions, while the Robin boundary value conditions and integral initial value conditions [8] are seldom investigated. In the past several decades, authors [9–11, 14, 16] have discussed the singularly perturbed integral initial boundary value and Robin initial boundary value problems in various non-critical cases. However, until now, the singularly perturbed problem in critical cases with initial integral value and Robin boundary value condition has not been studied. Inspired by this, we fill in the gap of this kind of problem, and give formal asymptotic solutions and numerical examples.

The structure of this paper is as follows. The boundary layer function method [13] is used in sections 2 and 3 to introduce corresponding singular perturbation critical problems and to obtain the zero-order approximate solution and the first-order approximate solution of the original equation; Section 4 uses the successive approximation method [4, 12] to demonstrate the existence of the solution; Section 5 provides an example to illustrate the second part of the theoretical results; the last section presents our conclusions.

2. Statement of the problem

Consider the following class of singularly perturbed initial boundary value problems

$$\begin{cases} \mu\frac{dz}{dt} = A\left(y,t\right)y + \mu B\left(z,t\right), \\ \mu\frac{dy}{dt} = C\left(t\right)y + \mu D\left(z,t\right), \end{cases} a \leqslant t \leqslant b, \tag{2.1}$$

the corresponding integral initial value and Robin boundary value condition

$$z(a) = z^0 + \int_a^b p_1\left(z\left(s,\mu\right)\right)ds, \qquad z(b) - z'\left(b\right) = z^1, \tag{2.2}$$

where z, y are scalar functions, and $0 < \mu \ll 1$ is a small parameter. Systems (2.1)-(2.2) must satisfy the following requirements:
(H_1) Assume that functions $A\left(y,t\right), B\left(z,t\right), C\left(t\right), D\left(z,t\right)$ and $p_1\left(z\right)$ are sufficiently smooth on $G = \{(z,y,t)\,||z| \le l, |y| \le l, a \le t \le b\}$, and l, z^0, z^1 are all real numbers. To be definite, assume that $C\left(t\right) < 0$.

The degradation equation of problem (2.1) is as follows.

$$\begin{cases} A\left(\bar{y},t\right)\bar{y} = 0, \\ C\left(t\right)\bar{y} = 0. \end{cases} \tag{2.3}$$

We can get from the degradation equation (2.3) that $\bar{y}\left(t\right) = 0$. However, since the degradation equation (2.3) does not include $\bar{z}\left(t\right)$, $\bar{z}\left(t\right)$ is an unknown function about the variable t. Assume that $\bar{z}\left(t\right) = \alpha\left(t\right)$ for the convenience of calculation. Then $\alpha\left(t\right)$ is also an arbitrary unknown function.

Set

$$F = \begin{pmatrix} A\left(y,t\right)y + \mu B\left(z,t\right) \\ C\left(t\right)y + \mu D\left(z,t\right) \end{pmatrix}, \qquad x = \left(z,y\right)^T.$$

Then the matrix F_x at the equilibrium point $(\alpha(t), 0)$ has the following form

$$\begin{pmatrix} 0 & A(0,t) \\ 0 & C(t) \end{pmatrix}.$$

Therefore, it is easy to get the eigenvalue $\lambda \equiv 0$ of the matrix F_x and the other eigenvalue $\lambda_1 = C(t) < 0$. Thus, we get the case where the critical condition is stable. Under the assumption of (H_1), there may exist a solution $x(t,\mu)$ with only one boundary layer that occurs at $t = a$.

Due to the introduction of the integral initial value condition, the discussion of problems (2.1)-(2.2) is much more complicated than that of the conventional fixed initial value. Therefore, problems (2.1)-(2.2) are transformed into the following equivalent singularly perturbed Robin boundary value problem

$$\begin{cases} \mu \frac{dz}{dt} = A(y,t)\, y + \mu B(z,t), \\ \mu \frac{dy}{dt} = C(t)\, y + \mu D(z,t), \qquad a \leqslant t \leqslant b, \\ \frac{dk_1}{dt} = p_1(z), \end{cases} \tag{2.4}$$

$$\begin{aligned} & z(a) = z^0 - k_1(a), \qquad k_1(b) = 0, \\ & z(b) = z^1 + z'(b). \end{aligned} \tag{2.5}$$

3. Construction of the asymptotic expansion

We use the boundary layer function method [13] to construct an asymptotic solution to problems (2.4)-(2.5) of the form $u = (x, k_1)^T$

$$u(t,\mu) = \begin{cases} x = \sum\limits_{l=0}^{\infty} \mu^l \left[\bar{x}_l(t) + L_l x(\tau_0)\right], \\ k_1 = \sum\limits_{l=0}^{\infty} \mu^l \left[\bar{k}_{1l}(t) + L_l k_1(\tau_0)\right], \end{cases} \quad a \leqslant t \leqslant b, \tag{3.1}$$

where $\tau_0 = \frac{t-a}{\mu}$, and $\bar{x}_l(t), \bar{k}_{1l}(t)$ are coefficients of regular terms; $L_l x(\tau_0), L_l k_1(\tau_0)$ are coefficients of boundary layer terms at $t = a$; through the initial boundary value condition (2.5), we can get

$$\begin{aligned} & \bar{z}_0(a) + L_0 z(0) = z^0 - \bar{k}_{10}(a) - L_0 k_1(0), \qquad \bar{k}_{10}(b) = 0; \\ & \bar{z}_0(b) = z^1 + \bar{z}_0'(b); \\ & \bar{z}_l(a) + L_l z(0) = -\bar{k}_{1l}(a) - L_l k_1(0), \qquad \bar{k}_{1l}(b) = 0; \\ & \bar{z}_l(b) = \bar{z}_l'(b). \end{aligned}$$

First, we study the zero-order regularization part of the asymptotic solution of problems (2.4)-(2.5), and find that the first two equations of the zero-order regularization part are the same as the degradation equation (2.3)

$$\begin{cases} A(\bar{y}_0, t)\, \bar{y}_0 = 0, \\ C(t)\, \bar{y}_0 = 0, \\ \frac{d\bar{k}_{10}}{dt} = \bar{p}_{10}(\bar{z}_0(t)), \end{cases} \tag{3.2}$$

$\bar{y}_0(t) = 0$ may be found in (3.2), where $\bar{z}_0(t) = \alpha(t)$ is an unknown arbitrary function. In this case, $\alpha(t)$ needs to be determined in the equation for the first-order regular part. Write the differential equations and initial boundary value conditions that can determine the zero-order boundary layer coefficients in (3.1)

$$\begin{cases} \frac{dL_0 z(\tau_0)}{d\tau_0} = A\left(L_0 y\left(\tau_0\right), a\right) L_0 y\left(\tau_0\right), \\ \frac{dL_0 y(\tau_0)}{d\tau_0} = C\left(a\right) L_0 y\left(\tau_0\right), \\ \frac{dL_0 k_1(\tau_0)}{d\tau_0} = 0. \end{cases} \tag{3.3}$$

$$\begin{aligned} &\alpha(a) + L_0 z(0) = z^0 - \bar{k}_{10}(a) - L_0 k_1(0), \\ &\alpha(b) = z^1 + \alpha'(b), \qquad \bar{k}_{10}(b) = 0, \\ &L_0 y(+\infty) = L_0 z(+\infty) = L_0 k_1(+\infty) = 0. \end{aligned} \tag{3.4}$$

$L_0 k_1(\tau_0) = 0$ can be obtained by applying the constant-change method to the third differential equation in (3.3) and the initial boundary value conditions in (3.4). Next, the following is the differential equation concerning the first-order regular part of (3.1)

$$\begin{cases} \frac{d\bar{z}_0}{dt} = \bar{A}\bar{y}_1 + \bar{\bar{B}}, \\ \frac{d\bar{y}_0}{dt} = C(t)\,\bar{y}_1 + \bar{\bar{D}}, \\ \frac{d\bar{k}_{11}(t)}{dt} = q_{1z}\left(\alpha(t)\right)\bar{z}_1. \end{cases} \tag{3.5}$$

The values at the points $(\alpha(t), t)$ and $(0, t)$ are denoted by the symbols "=" and "-", respectively. In order to obtain $\alpha(t)$, it is necessary to use the first-order regular partial equation of (3.5). As a result, by computing the first two equations in equation (3.5), we may obtain

$$\bar{y}_1(t) = -\frac{\bar{\bar{D}}(\alpha, t)}{C(t)}, \qquad \frac{d\alpha(t)}{dt} = -\bar{A}(0, t)\,\frac{\bar{\bar{D}}(\alpha, t)}{C(t)} + \bar{\bar{B}}(\alpha, t). \tag{3.6}$$

Substituting the second equation of equation (3.6) into the boundary value condition (3.4), we can find $\alpha(b) = \alpha^0$. According to the existence of solutions to the boundary value problem, $\alpha(t)$ can be determined by the first-order differential equation for $\alpha(t)$ in (3.6) and the boundary value condition $\alpha(b) = \alpha^0$. Then, $\bar{k}_{10}(t)$ is determined by the third equation of (3.2) and the boundary value condition $\bar{k}_{10}(b) = 0$. Finally, $\bar{u}_0(t)$ is fully determined. Moreover, $\bar{y}_1(t)$ is also worked out.

We may determine the relationship between $L_0 z(\tau_0)$ and $L_0 y(\tau_0)$ as $L_0 z(\tau_0) = \int_0^{L_0 y} \frac{A(s,a)}{C(a)} ds$ by using the first two equations of system (3.3) and the boundary value condition (3.4). By substituting the initial boundary value condition (3.4), $L_0 y(0) = \beta^0$ can be obtained, so that $L_0 y(\tau_0) = \beta^0 e^{C(a)\tau_0}$ can be determined by the constant variation method, and then $L_0 z(\tau_0)$ can be determined accordingly. So far, $L_0 u(\tau_0)$ has been completely determined. At the same time, the zeroth-order principal terms of the asymptotic solutions to the form (2.4)-(2.5) have all been found. For the first-order regular part of issue (2.4)-(2.5), only $\bar{y}_1(t)$ is currently determined, while $\bar{z}_1(t)$ and $\bar{k}_{11}(t)$ need to be determined by the regular part $\bar{u}_2(t)$

of the system. Next, we need to determine the first-order boundary layer term $L_1u(\tau_0)$ of the asymptotic solution, and solve it with the following form

$$\begin{cases} \frac{dL_1z(\tau_0)}{d\tau_0} = \left(\tilde{A} + \tilde{A}_y L_0 y(\tau_0)\right) L_1 y(\tau_0) + h_1, \\ \frac{dL_1y(\tau_0)}{d\tau_0} = C(a) L_1 y(\tau_0) + \hat{D} - \widehat{D}, \\ \frac{dL_1k_1(\tau_0)}{d\tau_0} = q_1(\tilde{z}) - q_1(\alpha(a)), \end{cases} \tag{3.7}$$

where $h_1(\tau_0) = (\tilde{A}_y \bar{y}_1(a) + \tilde{A}_t \tau_0) L_0 y(\tau_0) + (\tilde{A} - \breve{A})\bar{y}_1(a) + \hat{B} - \widehat{B}$. Here $\tilde{A}$, $\tilde{A}_y$, and $\tilde{A}_t$ take values at the point $(L_0y(\tau_0), a)$, $\breve{A}$ takes values at the point $(0, a)$, $\hat{B}, \hat{D}$ take values at the point $(\alpha(a) + L_0z(\tau_0), a)$, and $\widehat{B}, \widehat{D}$ take values at the point $(\alpha(a), a)$. The corresponding boundary value conditions for these equations are

$$\begin{aligned} &\bar{z}_1(a) + L_1z(0) = -\bar{k}_{11}(a) - L_1k_1(0), \qquad && \bar{k}_{11}(b) = 0; \\ &\bar{z}_1(b) = \bar{z}_1'(b), && L_1u(+\infty) = 0. \end{aligned} \tag{3.8}$$

Moreover, the equation for solving the regular part $\bar{u}_2(t)$ is

$$\begin{cases} \frac{d\bar{z}_1(t)}{dt} = \bar{A}\bar{y}_2 + \bar{A}_y \bar{y}_1^2 + \bar{\bar{B}}_z \bar{z}_1, \\ \frac{d\bar{y}_1(t)}{dt} = C(t)\bar{y}_2, \\ \frac{d\bar{k}_{12}(t)}{dt} = q_{1z}(\alpha(t))\bar{z}_2 + S_{12}, \end{cases} \tag{3.9}$$

where $S_{12} = q_{1z}(\alpha(t))\bar{z}_1^2$ is an unknown composite function. The following first-order differential equation for $\bar{z}_1(t)$ may be obtained by computing the first two equations of (3.9)

$$\begin{cases} \frac{d\bar{z}_1}{dt} = \bar{\bar{B}}_z \bar{z}_1 + \gamma(t), \\ \bar{z}_1(b) = \frac{\gamma(b)}{1 - \bar{\bar{B}}_z(b)}, \end{cases} \tag{3.10}$$

where $\gamma(t) = \bar{A}\frac{\bar{y}_1'(t)}{C(t)} + \bar{A}_y \bar{y}_1^2$. Then we can obtain

$$\bar{z}_1(t) = e^{\int_b^t \bar{\bar{B}}_z(\kappa)d\kappa} \left(\int_b^t \gamma(s) e^{\int_b^s -\bar{\bar{B}}_z(p)dp} ds + \frac{\gamma(b)}{1 - \bar{\bar{B}}_z(b)} \right). \tag{3.11}$$

At this point, $\bar{z}_1(t)$ has been determined. Meanwhile, $\bar{k}_{11}(t)$ can be determined by the third equation in the system (3.5) and the corresponding boundary value condition $\bar{k}_{11}(b) = 0$, i.e. $\bar{k}_{11}(t) = \int_b^t q_{1z}(\alpha(s))\bar{z}_1(s)\,ds$. Therefore, the first regularization part $\bar{u}_1(t)$ has been obtained, and then the boundary layer part $L_1u(\tau_0)$ is determined at once. In $L_1u(\tau_0)$, the boundary layer function $L_1k(\tau_0)$ is first determined, that is, $L_1k_1(\tau_0) = \int_{+\infty}^{\tau_0} q_1(\alpha(a) + L_0z(s)) - q_1(\alpha(a))ds$. And then, by using the constant-change method, we obtain a particular solution

$$L_1y(\tau_0) = e^{C(a)\tau_0} \left(\int_0^{\tau_0} \left(\hat{D} - \widehat{D}\right) e^{-C(a)s} ds + L_1y(0) \right) \tag{3.12}$$

of $\frac{dL_1 y(\tau_0)}{d\tau_0} = C\,(a)\,L_1 y\,(\tau_0) + \hat{D} - \widehat{D}$. Finally, we have

$$\begin{aligned} L_1 z\,(\tau_0) = & \int_{+\infty}^{\tau_0} \left[\left(\tilde{A} + \tilde{A}_y L_0 y\,(p) \right) e^{C(a)p} \int_0^p \left(\hat{D}\,(s) - \widehat{D}\,(a) \right) e^{-C(a)s} ds \right] dp \\ & + \int_{+\infty}^{\tau_0} \left[\left(\tilde{A} + \tilde{A}_y L_0 y\,(p) \right) e^{C(a)p} L_1 y\,(0) + h_1\,(p) \right] dp. \end{aligned} \tag{3.13}$$

At this time, $L_1 y\,(0) = \beta^1$ can be obtained through (3.13) and the initial value condition (3.4). So $L_1 y\,(\tau_0)$ and $L_1 z\,(\tau_0)$ can be determined. Finally, all the boundary layer coefficients of the first-order asymptotic solution are obtained. The process of solving the term coefficients $k\,(k \geqslant 2)$ of the asymptotic solution is completely similar to the case of $k = 1$, and will not be repeated here.

4. The existence of the solution and the remainder estimate

Theorem 4.1. *If the condition (H_1) is satisfied, there must be positive constants μ_0, δ and C, so that when $0 < \mu \leq \mu_0$, the boundary value problem (2.4)-(2.5) has a unique solution $u\,(t, \mu)$, and satisfies the inequality*

$$|u\,(t, \mu) - U\,(t, \mu)| \leq C\mu^{n+1}, \quad a \leq t \leq b, \tag{4.1}$$

Proof. Let $\xi = z - Z_{n+1}, \lambda = y - Y_{n+1}$, and $\varsigma = k_1 - K_{1,n+1}$, where (z, y, k_1) is the solution of (2.4)-(2.5) and $(Z_{n+1}, Y_{n+1}, K_{1,n+1})$ is determined by (4.2)

$$U_n(t, \mu) = \sum_{k=0}^{n} \mu^k [\bar{u}_k(t) + L_k u(\tau_0)]. \tag{4.2}$$

Therefore, we get the following system of equations

$$\begin{cases} \mu \frac{d\xi}{dt} = A(0, t)\lambda + G_1(\xi, \lambda, t, \mu), \\ \mu \frac{d\lambda}{dt} = C\,(t)\,\lambda + G_2(\xi, t, \mu), \\ \frac{d\varsigma}{dt} = q_{1z}(\alpha(t))\xi + G_3(\xi, t, \mu), \\ \xi\,(0, \mu) = 0, \lambda\,(0, \mu) = 0, \varsigma\,(0, \mu) = 0. \end{cases} \tag{4.3}$$

$$\begin{aligned} G_1(\xi, \lambda, t, \mu) &= A(\lambda + Y_{n+1}, t)(\lambda + Y_{n+1}) + \mu B(\xi + Z_{n+1}, t) \\ &\quad - \mu \tfrac{dZ_{n+1}}{dt} - A(0, t)\lambda, \\ G_2(\xi, t, \mu) &= C\,(t)\,Y_{n+1} + \mu D(\xi + Z_{n+1}, t) - \mu \tfrac{dY_{n+1}}{dt}, \\ G_3(\xi, t, \mu) &= q_1(\xi + Z_{n+1}) - \tfrac{dK_{1,n+1}}{dt} - q_{1z}(\alpha(t))\xi. \end{aligned}$$

$G_1\,(\xi, \lambda, t, \mu)\,, G_2\,(\xi, t, \mu)\,,$ and $G_3\,(\xi, t, \mu)$ have the following two important properties.

(1) $G_1\,(0, 0, t, \mu) = G_2\,(0, t, \mu) = G_3\,(0, t, \mu) = O\left(\mu^{n+1}\right)$.

(2) For all $\varepsilon = O\,(\mu) > 0$, there are constants $\delta = \delta\,(\varepsilon)$ and $\mu_0 = \mu_0(\varepsilon)$ so that as

long as $|\xi_i| < \delta, |\lambda_i| < \delta, |\varsigma_i| < \delta, (i = 1, 2)$, $0 < \mu \le \mu_0$, then

$|G_1(\xi_1, \lambda_1, t, \mu) - G_1(\xi_2, \lambda_2, t, \mu)| \le \mu |\xi_1 - \xi_2| + \mu |\lambda_1 - \lambda_2|$,

$|G_2(\xi_1, t, \mu) - G_2(\xi_2, t, \mu)| \le \mu |\xi_1 - \xi_2|$,

$|G_3(\xi_1, t, \mu) - G_3(\xi_2, t, \mu)| \le \mu |\xi_1 - \xi_2|$.

$\lambda(t, \mu) = \mu^{-1} e^{\mu^{-1} \int_0^t C(s) ds} \int_0^t G_2(\xi, p, \mu) e^{\mu^{-1} \int_0^p C(\kappa) d\kappa} dp$ can be calculated using the second equation in (4.3), and then be substituted into the first equation in (4.3) to obtain

$$\mu \frac{d\xi}{dt} = A(0, t) \mu^{-1} e^{\mu^{-1} \int_0^t C(s) ds} \int_0^t G_2(\xi, p, \mu) e^{\mu^{-1} \int_0^p C(\kappa) d\kappa} dp + G_1(\xi, \lambda, t, \mu). \quad (4.4)$$

Then, the equivalent integral equation of (4.4) can be written as

$$\begin{aligned} \xi(t, \mu) = & \int_0^t \mu^{-2} e^{\mu^{-1} \int_0^q C(s) ds} \int_0^q G_2(\xi, p, \mu) e^{\mu^{-1} \int_0^p C(\kappa) d\kappa} dp dq \\ & + \int_0^t \mu^{-1} G_1(\xi, \lambda, q, \mu) dq \equiv J(\xi, \lambda, t, \mu). \end{aligned} \quad (4.5)$$

It is not difficult to see that J and G_i $(i = 1, 2)$ have the same properties. Since the right-hand side of the unknown ς in the last equation in (4.3) is only related to ξ and is continuous on $a \leqslant t \leqslant b$, the first two equations in (4.3) have nothing to do with ς. Therefore, as long as we show that ξ, λ exist and are unique, then ς exists and is unique. Finally, using the successive approximation method for (4.5), we can prove the existence of the solution to equations (4.3) and estimate that $|\xi(t, \mu)| \leqslant C\mu^{n+1}, |\lambda(t, \mu)| \leqslant C\mu^{n+1}, |\varsigma(t, \mu)| \leqslant C\mu^{n+1}$, which not only deduces the existence of solutions y, z, but also provides the remainder estimation formula (4.1). □

5. Numerical example

Consider the following singularly perturbed boundary value problem

$$\begin{aligned} & \mu \frac{dz}{dt} = (y+1) y + \frac{3}{2} \mu z, \qquad \mu \frac{dy}{dt} = -2y + \mu z, \qquad 0 \leqslant t \leqslant 1, \\ & z(0, \mu) = \frac{1}{2} e^{-2} - \frac{5}{4} + \int_0^1 z(s, \mu) ds, \qquad y(1, \mu) - y'(1, \mu) = -1. \end{aligned} \quad (5.1)$$

Obviously, conditions $(H_1), (H_2)$ hold. In order to reduce the computational complexity, system (5.1) can be changed into the following form:

$$\begin{aligned} & \mu \frac{dz}{dt} = (y+1) y + \frac{3}{2} \mu z, \quad \mu \frac{dy}{dt} = -2y + \mu z, \quad \frac{dk_1}{dt} = z, \quad 0 \leqslant t \leqslant 1, \\ & z(0, \mu) = \frac{1}{2} e^{-2} - \frac{5}{4} - k_1(0), \; k_1(1) = 0, \; z(1, \mu) - z'(1, \mu) = -1. \end{aligned} \quad (5.2)$$

$\bar{z}_0(t), \bar{y}_0(t), \bar{k}_{10}(t), L_0 z(\tau_0), L_0 y(\tau_0)$ and $L_0 k_1(\tau_0)$ are given by the following systems respectively.

$$\begin{aligned}
&(\bar{y}_0+1)\,\bar{y}_0=0, \qquad -2\bar{y}_0=0, \qquad \frac{d\bar{k}_{10}}{dt}=\bar{z}_0,\\
&\frac{dL_0z}{d\tau_0}=L_0^2y+L_0y, \quad \frac{dL_0y}{d\tau_0}=-2L_0y, \quad \frac{dL_0k_1}{d\tau_0}=0,\\
&\frac{d\bar{z}_0}{dt}=(\bar{y}_0+1)\,\bar{y}_1+\frac{3}{2}\bar{z}_0, \qquad \frac{d\bar{y}_0}{dt}=-2\bar{y}_1+\bar{z}_0,\\
&\frac{d\bar{k}_{11}}{dt}=\bar{z}_1, \quad \bar{z}_0(1)=\bar{z}_0'(1)-1, \qquad \bar{k}_{10}(1)=0,\\
&\bar{z}_0(0)+L_0z(0)=\frac{1}{2}e^{-2}-\frac{5}{4}-\bar{k}_{10}(0)+L_0k_1(0).
\end{aligned} \tag{5.3}$$

Through the above calculation, we obtain

$$\begin{aligned}
&\bar{z}_0(t)=e^{2(t-1)}, \quad \bar{y}_0(t)=0, \quad \bar{k}_{10}(t)=\frac{1}{2}e^{2(t-1)}-\frac{1}{2},\\
&L_0y(\tau_0)=e^{-2\tau_0}, L_0k_1(\tau_0)=0, L_0z(\tau_0)=-\frac{1}{2}e^{-2\tau_0}-\frac{1}{4}e^{-4\tau_0}.
\end{aligned} \tag{5.4}$$

$\bar{z}_1(t), \bar{y}_1(t), \bar{k}_{11}(t), L_1z(\tau_0), L_1y(\tau_0)$ and $L_1k_1(\tau_0)$ are given by the following systems respectively.

$$\begin{aligned}
&\frac{d\bar{k}_{12}}{dt}=\bar{z}_2, \ \frac{d\bar{y}_1}{dt}=-2\bar{y}_2+\bar{z}_1, \ \frac{dL_1k_1}{d\tau_0}=L_0z, \frac{d\bar{z}_1}{dt}=\bar{y}_1^2+\bar{y}_2+\frac{3}{2}\bar{z}_1,\\
&\frac{dL_1z}{d\tau_0}=(2L_0y+1)\,L_1y+2\bar{y}_1L_0y+\frac{3}{2}L_0z, \ \frac{dL_1y}{d\tau_0}=-2L_1y+L_0z,\\
&\bar{z}_1(0)+L_1z(0)=-\bar{k}_{11}(0)-L_1k_1(0), \qquad \bar{z}_1(1)=\bar{z}_1'(1),\\
&L_1z(+\infty)=L_1y(+\infty)=L_1k_1(+\infty)=0, \qquad \bar{k}_{11}(1)=0.
\end{aligned} \tag{5.5}$$

Using the above theoretical part of the method, we can obtain

$$\begin{aligned}
&\bar{k}_{11}(t)=\frac{1}{32}e^{4(t-1)}-\frac{1}{4}te^{2(t-1)}+\frac{7}{16}e^{2(t-1)}-\frac{7}{32}, \qquad \bar{y}_1(t)=\frac{1}{2}e^{2(t-1)},\\
&\bar{z}_1(t)=\frac{1}{8}e^{4(t-1)}-\left(\frac{1}{2}t-\frac{5}{8}\right)e^{2(t-1)}, \quad L_1k_1(\tau_0)=\frac{1}{4}e^{-2\tau_0}+\frac{1}{16}e^{-4\tau_0},\\
&L_1y(\tau_0)=\left(-\frac{5}{32e^4}-\frac{1}{16e^2}-\frac{89}{96}\right)e^{-2\tau_0}-\frac{1}{2}\tau_0e^{-2\tau_0}+\frac{1}{8}e^{-4\tau_0},\\
&L_1z(\tau_0)=\left(\frac{1}{64e^4}-\frac{31}{32e^2}+\frac{185}{192}\right)e^{-2\tau_0}+\left(\frac{1}{64e^4}+\frac{1}{32e^2}+\frac{113}{192}\right)e^{-4\tau_0}\\
&\qquad +\frac{1}{4}\tau_0e^{-2\tau_0}+\frac{1}{4}\tau_0e^{-4\tau_0}-\frac{1}{24}e^{-6\tau_0}.
\end{aligned} \tag{5.6}$$

Thus, we obtain the first-order asymptotic solution of (5.1)

$$\begin{cases} y(t,\mu) = e^{-2\tau_0} + \mu\frac{1}{2}e^{2(t-1)} + \mu\left[\left(-\frac{5}{32e^4} - \frac{1}{16e^2} - \frac{89}{96}\right)e^{-2\tau_0}\right. \\ \qquad\qquad \left. -\frac{1}{2}\tau_0 e^{-2\tau_0} + \frac{1}{8}e^{-4\tau_0}\right] + O\left(\mu^2\right), \\ z(t,\mu) = e^{2(t-1)} - \frac{1}{2}e^{-2\tau_0} - \frac{1}{4}e^{-4\tau_0} + \mu\left[\frac{1}{8}e^{4(t-1)} - \left(\frac{1}{2}t - \frac{5}{8}\right)e^{2(t-1)}\right. \\ \qquad\qquad + \left(\frac{1}{64e^4} - \frac{31}{32e^2} + \frac{185}{192}\right)e^{-2\tau_0} + \left(\frac{1}{64e^4} + \frac{1}{32e^2} + \frac{113}{192}\right)e^{-4\tau_0} \\ \qquad\qquad \left. + \frac{1}{4}\tau_0 e^{-2\tau_0} + \frac{1}{4}\tau_0 e^{-4\tau_0} - \frac{1}{24}e^{-6\tau_0}\right] + O\left(\mu^2\right), \end{cases} \tag{5.7}$$

where $0 \leqslant t \leqslant 1, 0 < \mu \ll 1, \tau_0 = \frac{t-0}{\mu}$.

Therefore, these results were obtained using the boundary layer function method in [13]. Figure 1-4 displays the Matlab-produced result. The exact and approximate solutions corresponding to different line types have been marked in the figure. These graphs show that if we use a smaller value of μ, then it will make the asymptotic solution almost coincide with the exact solution. At the same time, the following image of the exact solution versus the approximate solution also illustrates the nature of the exponential decay of the boundary layer functions.

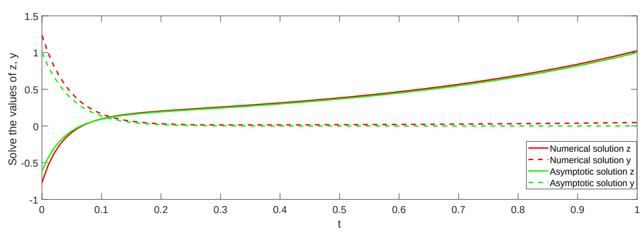

Figure 1. Zero-order asymptotic solution and numerical solution to problem (5.1) ($\mu = 0.1$);

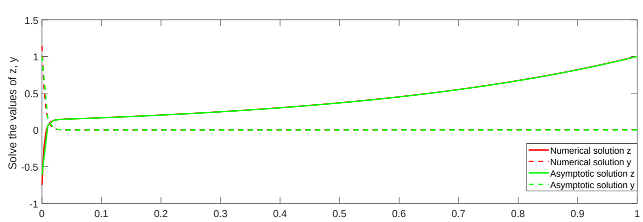

Figure 2. Zero-order asymptotic solution and numerical solution to problem (5.1) ($\mu = 0.01$);

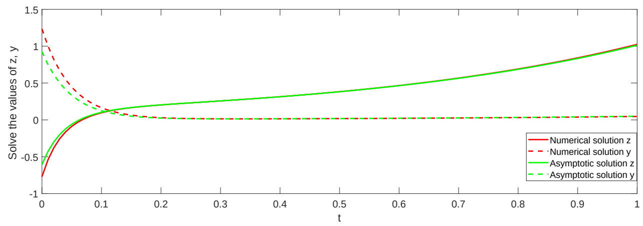

Figure 3. First-order asymptotic solution and numerical solution to problem (5.1) ($\mu = 0.1$);

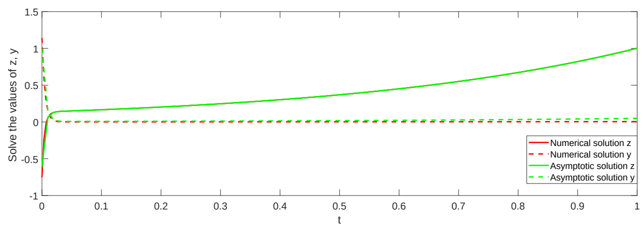

Figure 4. First-order asymptotic solution and numerical solution to problem (5.1)($\mu = 0.01$).

6. Conclusive remarks

In the above sections, we study a class of singular perturbations with initial integral values and Robin boundary value conditions in a critical case. By using boundary layer function method and successive approximation principle, we not only construct the formal asymptotic solution of the original problem, but also prove the existence of the solution. In addition, an example is given to verify the theoretical results by the numerical method. Furthermore, these theoretical results can be applied to some chemical kinetics, enzyme kinetics and chemical bonding problems, etc.

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