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Positive Solutions for Third Order Three-Point Boundary Value Problems with p-Laplacian*

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Abstract In this paper, the existence of positive solutions of the following third-order three-point boundary value problem with p-Laplacian

$$\begin{cases} (\phi_p(u''(t)))' + f(t,u(t)) = 0,\ t \in (0,1), \\ u(0) = \alpha u(\eta), u(1) = \alpha u(\eta), u''(0) = 0, \end{cases}$$

is studied, where $\phi_p(s) = |s|^{p-2}s,\ p > 1$. By using the fixed point index method, we establish sufficient conditions for the existence of at least one or at least two positive solutions for the above boundary value problem. The main result is demonstrated by providing an example as an application.

Keywords Positive solution, three-point boundary value problem, fixed point index, p-Laplacian operator

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1. Introduction

The purpose of this paper is to study the existence of positive solutions for the following third-order three-point boundary value problem (BVP for short) with p-Laplacian

$$(\phi_p(u''(t)))' + f(t,u(t)) = 0,\ t \in (0,1), \tag{1.1}$$

$$u(0) = \alpha u(\eta), u(1) = \alpha u(\eta), u''(0) = 0, \tag{1.2}$$

where $\phi_p(s) = |s|^{p-2}s$, $p > 1$, $\phi_p^{-1} = \phi_q$, $\dfrac{1}{p} + \dfrac{1}{q} = 1$, $0 < \alpha,\ \eta < 1$.

There has been an extensive study on boundary value problems with diverse boundary conditions via many methods [1, 11, 21]. The equation with p-Laplacian operator arises in the modeling of different physical and natural phenomena, non-Newtonian mechanics [4, 10], combustion theory [18], population biology [16, 17] and nonlinear flow laws [5, 13]. Therefore, there exist a very lager number of

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papers devoted to the existence of solutions to the p-Laplacian boundary value problems with various boundary conditions, which have been studied by many authors [7, 14, 15, 19, 20] and references therein. Iyase [8] proved the existence of solutions for a third-order multipoint boundary value problem at resonance by using the coincidence degree arguments. Additionally, Iyase and Imaga [9] applied Leray-Schauder continuation principle to establish at least one solution to the third-order p-Laplacian boundary value problem.

Recently, Li [12] has studied the existence of positive solutions for the third-order boundary value problem with p-Laplacian operator.

$$\begin{aligned} &(\phi_p(u''(t)))' + f(t, u(t), u'(t), u''(t)) = 0,\ t \in (0, 1), \\ &au(0) - bu'(0) = 0,\ cu(1) + du'(1) = 0,\ u''(0) = 0, \end{aligned} \tag{1.3}$$

where $\phi_p(s) = |s|^{p-2}s$, $p > 1$. By using the fixed point theorem of Krasnosel'skii, the author established the existence results for positive solutions to (1.3).

Motivated by the above works, our purpose here is to give existence of positive solutions for a third-order three-point boundary value problem with p-Laplacian operator. In this paper, we construct a Green function and study its properties, and then transform BVP (1.1) and (1.2) into an equivalent integral equation. Next, applying the fixed point index theorem, we establish the existence of at least one or at least two positive solutions for the above boundary value problem. For convenience, we list the following assumptions:

(H_1) $f : [0, 1] \times [0, \infty) \to [0, \infty)$ is continuous;
(H_2) $0 < \alpha,\ \eta < 1$.

2. Preliminaries and several important lemmas

In this section, we provide some basic concepts and properties of fixed point index for compact maps.

Let $E = C[0, 1]$, $C^+[0, 1] = \{x \in C[0, 1] |\ x(t) \geq 0,\ t \in [0, 1]\}$, then E is a Banach space with norm $\|u\| = \max\limits_{t\in[0,1]} |u(t)|$.

Definition 2.1.([6]) Let E be a real Banach space. Let P be a nonempty, convex closed set in E. We say that P is a cone if it satisfies the following properties:
(i) $\lambda u \in P$ for $u \in P, \lambda \geq 0$;
(ii) $u, -u \in P$ implies $u = \theta$ (θ denotes the null element of E).

If $P \subset E$ is a cone, we denote the order induced by P on E by $\leq$. For $u, v \in P$, we write $u \leq v$ if and only if $v - u \in P$.

Lemma 2.1. *Assume that (H_1) holds and $\alpha \neq 1$. Then for any $x \in C^+[0, 1]$, the problem*

$$(\phi_p(u''(t)))' + f(t, x(t)) = 0,\ t \in (0, 1), \tag{2.1}$$

$$u(0) = \alpha u(\eta), u(1) = \alpha u(\eta), u''(0) = 0 \tag{2.2}$$

has the unique solution

$$u(t) = \int_0^1 G(t, s)\phi_q(\int_0^s f(\tau, x(\tau))\mathrm{d}\tau)\mathrm{d}s + \frac{\alpha}{1-\alpha}\int_0^1 G(\eta, s)\phi_q(\int_0^s f(\tau, x(\tau))\mathrm{d}\tau)\mathrm{d}s, \tag{2.3}$$

where

$$G(t,s)=\begin{cases} t(1-s), & t\le s,\\ s(1-t), & s\le t.\end{cases} \tag{2.4}$$

Proof. Integrating the equation (2.1) over the interval $[0,t]$ for $t\in[0,1]$, we obtain $\phi_p(u''(t))-\phi_p(u''(0))=-\int_0^t f(s,x(s))\mathrm{d}s$,
from the boundary condition (2.2) we get
$\phi_p(u''(t))=-\int_0^t f(s,x(s))\mathrm{d}s$,
which implies

$$u''(t)=-\phi_q(\int_0^t f(s,x(s))\mathrm{d}s). \tag{2.5}$$

By integration of (2.5), it follows that
$u'(t)=u'(0)-\int_0^t \phi_q\left(\int_0^s f(\tau,x(\tau))\mathrm{d}\tau\right)\mathrm{d}s$,

$$u(t)=u(0)+u'(0)t-\int_0^t (t-s)\phi_q\left(\int_0^s f(\tau,x(\tau))\mathrm{d}\tau\right)\mathrm{d}s. \tag{2.6}$$

Using the boundary condition (2.2), we can easily get

$$\begin{aligned} u(0)=&\frac{\alpha}{1-\alpha}\left[\eta\int_0^1(1-s)\phi_q\left(\int_0^s f(\tau,x(\tau))\mathrm{d}\tau\right)\mathrm{d}s\right.\\ &\left.-\int_0^\eta(\eta-s)\phi_q\left(\int_0^s f(\tau,x(\tau))\mathrm{d}\tau\right)\mathrm{d}s\right],\\ u'(0)=&\int_0^1(1-s)\phi_q\left(\int_0^s f(\tau,x(\tau))\mathrm{d}\tau\right)\mathrm{d}s. \end{aligned} \tag{2.7}$$

By (2.6) and (2.7), we have

$$\begin{aligned} u(t)=&\frac{\alpha}{1-\alpha}\left[\eta\int_0^1(1-s)\phi_q(\int_0^s f(\tau,x(\tau))\mathrm{d}\tau)\mathrm{d}s\right.\\ &\left.-\int_0^\eta(\eta-s)\phi_q(\int_0^s f(\tau,x(\tau))\mathrm{d}\tau)\mathrm{d}s\right]\\ &+t\int_0^1(1-s)\phi_q(\int_0^s f(\tau,x(\tau))\mathrm{d}\tau)\mathrm{d}s-\int_0^t(t-s)\phi_q(\int_0^s f(\tau,x(\tau))\mathrm{d}\tau)\mathrm{d}s\\ =&\int_0^t s(1-t)\phi_q(\int_0^s f(\tau,x(\tau))\mathrm{d}\tau)\mathrm{d}s+\int_t^1 t(1-s)\phi_q(\int_0^s f(\tau,x(\tau))\mathrm{d}\tau)\mathrm{d}s\\ &+\frac{\alpha}{1-\alpha}\left[\int_0^\eta s(1-\eta)\phi_q(\int_0^s f(\tau,x(\tau))\mathrm{d}\tau)\mathrm{d}s\right.\\ &\left.+\int_\eta^1\eta(1-s)\phi_q(\int_0^s f(\tau,x(\tau))\mathrm{d}\tau)\mathrm{d}s\right]. \end{aligned}$$

Finally

$$u(t)=\int_0^1 G(t,s)\phi_q(\int_0^s f(\tau,x(\tau))\mathrm{d}\tau)\mathrm{d}s+\frac{\alpha}{1-\alpha}\int_0^1 G(\eta,s)\phi_q(\int_0^s f(\tau,x(\tau))\mathrm{d}\tau)\mathrm{d}s,$$

which achieves the proof of Lemma 2.1. □

In order to discuss the existence of positive solutions, we need some properties of functions $G(t,s)$.

Lemma 2.2. *For $G(t,s)$ given as in (2.4), we have the following results:*

(i) $0 \le G(t,s) \le G(s,s)$, *for* $t \in [0,1]$ *and* $s \in [0,1]$,

(ii) $G(t,s) \ge \frac{1}{4} G(s,s)$, *for* $t \in [\frac{1}{4}, \frac{3}{4}]$ *and* $s \in [0,1]$.

Proof. (i) According to the definition of $G(t,s)$, we can easily obtain

$0 \le G(t,s) \le G(s,s)$, for $t \in [0,1]$ and $s \in [0,1]$.

(ii) For all $t \in [\frac{1}{4}, \frac{3}{4}]$ and $s \in [0,1]$, from (2.4), we have

$$\frac{G(t,s)}{G(s,s)} = \begin{cases} \dfrac{t(1-s)}{s(1-s)} = \dfrac{t}{s} \ge t \ge \dfrac{1}{4}, & t \le s, \\ \dfrac{s(1-t)}{s(1-s)} = \dfrac{1-t}{1-s} \ge 1-t \ge \dfrac{1}{4}, & s \le t. \end{cases}$$

Therefore, $G(t,s) \ge \frac{1}{4} G(s,s)$, for $t \in [\frac{1}{4}, \frac{3}{4}]$ and $s \in [0,1]$. □

Let $P = \{x \in E |\ x(t) \ge 0, t \in [0,1]\}$. Then P is a cone in E. For $\forall\ u \in P$, we define an operator T by

$$\begin{aligned}(Tu)(t) = &\int_0^1 G(t,s)\phi_q\Big(\int_0^s f(\tau,u(\tau))\mathrm{d}\tau\Big)\mathrm{d}s \\ &+ \frac{\alpha}{1-\alpha}\int_0^1 G(\eta,s)\phi_q\Big(\int_0^s f(\tau,u(\tau))\mathrm{d}\tau\Big)\mathrm{d}s.\end{aligned}$$

Lemma 2.3. *Assume that* (H_1) *and* (H_2) *hold. Then the operator* $T : P \to P$ *is completely continuous.*

Proof. From (H_1), (H_2), Lemma 2.2 and the definition of T, it is easy to prove that $T : P \to P$. Now we show that $T : P \to P$ is completely continuous.

Let $\Omega \subset P$ be a bounded set. Then there exists $R > 0$ satisfying $\|u\| \le R$, for any $u \in \Omega$. Set $M = \max\{f(t,u) | t \in [0,1], u \in \overline{\Omega}\}$. For any $u \in \Omega$, we have

$$\begin{aligned} &(Tu)(t)| \\ =& \left|\int_0^1 G(t,s)\phi_q\left(\int_0^s f(\tau,u(\tau))\mathrm{d}\tau\right)\mathrm{d}s + \frac{\alpha}{1-\alpha}\int_0^1 G(\eta,s)\phi_q\left(\int_0^s f(\tau,u(\tau)\right)\mathrm{d}s\right| \\ \le& \left|\int_0^1 G(t,s)\phi_q\left(\int_0^1 f(\tau,u(\tau))\mathrm{d}\tau\right)\mathrm{d}s + \frac{\alpha}{1-\alpha}\int_0^1 G(\eta,s)\phi_q\left(\int_0^1 f(\tau,u(\tau)\right)\mathrm{d}s\right| \\ \le& \int_0^1 G(s,s)\phi_q(M)\mathrm{d}s + \frac{\alpha}{1-\alpha}\int_0^1 G(s,s)\phi_q(M)\mathrm{d}s = \frac{\phi_q(M)}{1-\alpha}\int_0^1 G(s,s)\mathrm{d}s, \end{aligned}$$

which implies that $T(\Omega)$ is uniformly bounded. Further for any $u \in \Omega$ and $t \in [0,1]$, we have

$$|(Tu)'(t)| = \left|-\int_0^t s\phi_q\left(\int_0^s f(\tau,u(\tau))\mathrm{d}\tau\right)\mathrm{d}s + \int_t^1 (1-s)\phi_q\left(\int_0^s f(\tau,u(\tau)\right)\mathrm{d}s\right|$$
$$\le \int_0^t s\phi_q\left(\int_0^1 f(\tau,u(\tau)\right)\mathrm{d}\tau)\mathrm{d}s + \int_t^1 (1-s)\phi_q\left(\int_0^1 f(\tau,u(\tau))\mathrm{d}\tau\right)\mathrm{d}s$$
$$\le \int_0^1 s\phi_q(M)\mathrm{d}s + \int_0^1 (1-s)\phi_q(M)\mathrm{d}s = \phi_q(M).$$

Hence $\|(Tu)'\| \le \phi_q(M)$. For any $0 \le t_1 \le t_2 \le 1$ and $u \in \Omega$,

$$|(Tu)(t_1) - (Tu)(t_2)| = \left|\int_{t_1}^{t_2} (Tu)'(t)\mathrm{d}t\right| \le \int_{t_1}^{t_2} |(Tu)'(t)|\mathrm{d}t \le \phi_q(M)|t_2 - t_1|.$$

Therefore, we can easily prove that $T(\Omega)$ is equi-continuous, that is, $T(\Omega)$ is a relatively compact set according to the Ascoli-Arzela theorem. In view of the continuity of f and the Lebesgue dominated convergence theorem, we know that T is continuous on Ω. Thus $T : P \to P$ is completely continuous. □

To obtain positive solutions of BVPs (1.1) and (1.2), we require some knowledge of the classical fixed point index for compact maps, and the index has the following properties.

Theorem 2.1 ([6]). *Let K be a closed convex set in a Banach space E and let D be a bounded open set such that $D_k := D \cap K \neq \emptyset$. Let $T : \overline{D_k} \to K$ be a compact map. Suppose that $x \neq Tx$ for all $x \in \partial D_k$.*
(A_1) (Existence) If $i(T, D_k, K) \neq 0$, then T has a fixed point in D_k.
(A_2) (Normalization) If $u \in D_k$, then $i(\widehat{u}, D_k, K) = 1$, where $\widehat{u}(x) = u$ for $x \in D_k$.
(A_3) (Homotopy) Let $\mu : [0,1] \times \overline{D_k} \to K$ be a compact map such that $x \neq \mu(t,x)$ for $x \in \partial D_k$ and $t \in [0,1]$. Then
$i(\mu(0,\cdot), D_k, K) = i(\mu(1,\cdot), D_k, K)$.
(A_4) (Additivity) If U_1 and U_2 are disjoint relatively open subsets of D_k such that $x \neq Tx$ for $x \in \overline{D_k}\backslash(U_1 \cup U_2)$, then
$i(T, D_k, K) = i(T, U_1, K) + i(T, U_2, K)$,
where $i(T, U_j, K) = i(T|_{U_j}, U_j, K)$.

Theorem 2.2 ([2], [3]). *Let P be a cone in a Banach space E. For $q > 0$, define $\Omega_q = \{x \in P \mid \|x\| < q\}$. Assume that $T : \overline{\Omega_q} \to P$ is a compact map such that $x \neq Tx$ for $x \in \partial\Omega_q$.*
(i) If $\|x\| \le \|Tx\|$ for $x \in \partial\Omega_q$, then $i(T, \Omega_q, P) = 0$;
(ii) If $\|x\| \ge \|Tx\|$ for $x \in \partial\Omega_q$, then $i(T, \Omega_q, P) = 1$.

Now, for the sake of convenience, we use the following notations. Let

$$f^0 = \lim_{u\to 0^+}\sup\max_{t\in[0,1]}\frac{f(t,u)}{\phi_p(u)}, \qquad f^\infty = \lim_{u\to\infty}\sup\max_{t\in[0,1]}\frac{f(t,u)}{\phi_p(u)},$$
$$f_0 = \lim_{u\to 0^+}\inf\min_{t\in[\frac{1}{4},\frac{3}{4}]}\frac{f(t,u)}{\phi_p(u)}, \qquad f_\infty = \lim_{u\to\infty}\inf\min_{t\in[\frac{1}{4},\frac{3}{4}]}\frac{f(t,u)}{\phi_p(u)},$$
$$l = 4\rho\left(\int_{\frac{1}{4}}^{\frac{3}{4}} G(\frac{1}{2},s)(s-\frac{1}{4})^{q-1}\mathrm{d}s\right)^{-1} \quad (\rho > 1),$$
$$m = (1-\alpha)\left(\int_0^1 s(1-s)\mathrm{d}s\right)^{-1} = 6(1-\alpha).$$

3. The main results and proofs

Now, we give our results for the existence of positive solutions of BVPs (1.1) and (1.2).

Theorem 3.1. *Suppose that* (H_1) *and* (H_2) *hold and there exist different constants* $b, c > 0$ *with* $b < \min\left\{\frac{m}{l}, \frac{1}{4}\right\} c$. *If the following conditions hold,*

(H_3) $f(t,u) \geq \phi_p(lb)$, *for* $\frac{1}{4} \leq t \leq \frac{3}{4}$, $\quad (1-\alpha)b \leq u \leq 4b$;

(H_4) $f(t,u) \leq \phi_p(mc)$, *for* $0 \leq t \leq 1, \quad 0 \leq u \leq c$.

Then the BVPs (1.1) *and* (1.2) *have at least one positive solution* u^* *with* $\|u^*\| \leq c$ *and* $\min\limits_{t\in[\frac{1}{4},\frac{3}{4}]} u^*(t) > (1-\alpha)b$.

Proof. For any $u \in P$ and $t \in [0,1]$, from Lemma 2.2, we obtain

$$\begin{aligned}
&(Tu)(t)\\
=&\int_0^1 G(t,s)\phi_q\left(\int_0^s f(\tau,u(\tau))\mathrm{d}\tau\right)\mathrm{d}s + \frac{\alpha}{1-\alpha}\int_0^1 G(\eta,s)\phi_q\left(\int_0^s f(\tau,u(\tau))\mathrm{d}\tau\right)\mathrm{d}s\\
\leq&\int_0^1 G(s,s)\phi_q\left(\int_0^s f(\tau,u(\tau))\mathrm{d}\tau\right)\mathrm{d}s + \frac{\alpha}{1-\alpha}\int_0^1 G(s,s)\phi_q\left(\int_0^s f(\tau,u(\tau))\mathrm{d}\tau\right)\mathrm{d}s\\
=&\frac{1}{1-\alpha}\int_0^1 G(s,s)\phi_q\left(\int_0^s f(\tau,u(\tau))\mathrm{d}\tau\right)\mathrm{d}s.
\end{aligned}$$

Thus,

$$\|Tu\| \leq \frac{1}{1-\alpha}\int_0^1 G(s,s)\phi_q\left(\int_0^s f(\tau,u(\tau))\mathrm{d}\tau\right)\mathrm{d}s.$$

Further, for $u \in P$ and $t \in [\frac{1}{4}, \frac{3}{4}]$, from lemma 2.2 we get

$$\begin{aligned}
\min_{t\in[\frac{1}{4},\frac{3}{4}]}(Tu)(t) =& \min_{t\in[\frac{1}{4},\frac{3}{4}]}\left[\int_0^1 G(t,s)\phi_q\left(\int_0^s f(\tau,u(\tau))\mathrm{d}\tau\right)\mathrm{d}s\right.\\
&\left.+\frac{\alpha}{1-\alpha}\int_0^1 G(\eta,s)\phi_q\left(\int_0^s f(\tau,u(\tau))\mathrm{d}\tau\right)\mathrm{d}s\right]\\
\geq& \min_{t\in[\frac{1}{4},\frac{3}{4}]}\int_0^1 G(t,s)\phi_q\left(\int_0^s f(\tau,u(\tau))\mathrm{d}\tau\right)\mathrm{d}s\\
\geq&\frac{1}{4}\int_0^1 G(s,s)\phi_q\left(\int_0^s f(\tau,u(\tau))\mathrm{d}\tau\right)\mathrm{d}s \geq \frac{1-\alpha}{4}\|Tu\|.
\end{aligned}$$

If $u \in \overline{P}_c = \{u \in P| \ \|u\| \le c\}$, then $0 \le u(t) \le c$ for $t \in [0,1]$. From (H_4), we have

$$\begin{aligned}
\|Tu\| &= \max_{t\in[0,1]} |(Tu)(t)| \\
&= \max_{t\in[0,1]} \left[\int_0^1 G(t,s)\phi_q\left(\int_0^s f(\tau,u(\tau))\mathrm{d}\tau\right)\mathrm{d}s \right.\\
&\quad \left. + \frac{\alpha}{1-\alpha}\int_0^1 G(\eta,s)\phi_q\left(\int_0^s f(\tau,u(\tau))\mathrm{d}\tau\right)\mathrm{d}s\right] \\
&\le \int_0^1 G(s,s)\phi_q\left(\int_0^1 f(\tau,u(\tau))\mathrm{d}\tau\right)\mathrm{d}s \\
&\quad + \frac{\alpha}{1-\alpha}\int_0^1 G(s,s)\phi_q\left(\int_0^1 f(\tau,u(\tau))\mathrm{d}\tau\right)\mathrm{d}s \\
&= \frac{1}{1-\alpha}\int_0^1 G(s,s)\phi_q\left(\int_0^1 f(\tau,u(\tau))\mathrm{d}\tau\right)\mathrm{d}s \\
&\le \frac{1}{1-\alpha}\cdot mc \cdot \int_0^1 G(s,s)\mathrm{d}s = c.
\end{aligned}$$

Hence, $T : \overline{P}_c \to \overline{P}_c$ and

$$\min_{t\in[\frac14,\frac34]} (Tu)(t) \ge \frac{1-\alpha}{4}\|Tu\|, \quad \forall u \in \overline{P}_c. \tag{3.1}$$

Next, for $(1-\alpha)b \le u(t) \le 4b,\ t \in [\frac{1}{4}, \frac{3}{4}]$, by (H_3) we know that $f(t,u) \ge \phi_p(lb)$ for $t \in [\frac{1}{4}, \frac{3}{4}]$ and then we have

$$\begin{aligned}
(Tu)\left(\frac12\right) &= \int_0^1 G\left(\frac12, s\right)\phi_q\left(\int_0^s f(\tau,u(\tau))\mathrm{d}\tau\right)\mathrm{d}s \\
&\quad + \frac{\alpha}{1-\alpha}\int_0^1 G(\eta,s)\phi_q\left(\int_0^s f(\tau,u(\tau))\mathrm{d}\tau\right)\mathrm{d}s \\
&\ge \int_0^1 G\left(\frac12, s\right)\phi_q\left(\int_0^s f(\tau,u(\tau))\mathrm{d}\tau\right)\mathrm{d}s \\
&\ge \int_{\frac14}^{\frac34} G\left(\frac12, s\right)\phi_q\left(\int_{\frac14}^s f(\tau,u(\tau))\mathrm{d}\tau\right)\mathrm{d}s \\
&\ge lb\int_{\frac14}^{\frac34} G\left(\frac12, s\right)\left(s-\frac14\right)^{q-1}\mathrm{d}s = 4\rho b > 4b,
\end{aligned}$$

which implies $\|Tu\| > 4b$.

Consequently for $(1-\alpha)b \le u(t) \le 4b,\ t \in [\frac{1}{4}, \frac{3}{4}]$, we have

$$\min_{t\in[\frac14,\frac34]} (Tu)(t) \ge \frac{1-\alpha}{4}\|Tu\| > \frac{1-\alpha}{4}\times 4b = (1-\alpha)b. \tag{3.2}$$

Now we can conclude from Theorem 2.1, (3.1) and (3.2) that T has at least one nonzero fixed point. In fact, let $U = \{x \in \overline{P}_c | \min_{t\in[\frac{1}{4},\frac{3}{4}]} x(t) > (1-\alpha)b\}$. Evidently, U is a nonempty bounded, convex open set (for $4b \in U$) in $\overline{P}_c$.

Firstly we prove $Tx \neq x$ for $x \in \partial U$. Suppose that there is $x_0 \in \partial U$ such that $Tx_0 = x_0$, then we have $\min_{t\in[\frac{1}{4},\frac{3}{4}]} x_0(t) = (1-\alpha)b$ and either (i) $x_0 \in \{x \in \overline{P}_c | \ \|x\| \leq 4b, \ \min_{t\in[\frac{1}{4},\frac{3}{4}]} x(t) \geq (1-\alpha)b\}$ or (ii) $x_0 \in \{x \in \overline{P}_c | \ \|x\| > 4b, \ \min_{t\in[\frac{1}{4},\frac{3}{4}]} x(t) \geq (1-\alpha)b\}$.

For case (i), we know that $(1-\alpha)b \leq x_0(t) \leq 4b$ for $t \in [\dfrac{1}{4}, \dfrac{3}{4}]$. From (3.2), we have

$$(1-\alpha)b = \min_{t\in[\frac{1}{4},\frac{3}{4}]} x_0(t) = \min_{t\in[\frac{1}{4},\frac{3}{4}]} (Tx_0)(t) > (1-\alpha)b.$$

This is a contradiction.

For case (ii), by (3.1), we get

$$(1-\alpha)b = \min_{t\in[\frac{1}{4},\frac{3}{4}]} x_0(t) = \min_{t\in[\frac{1}{4},\frac{3}{4}]} (Tx_0)(t) \geq \frac{1-\alpha}{4}\|Tx_0\| = \frac{1-\alpha}{4}\|x_0\| > (1-\alpha)b.$$

This is a contradiction. Hence, $Tx \neq x$ for $x \in \partial U$. Therefore, $i(T, U, \overline{P}_c)$ is meaningful.

Secondly, we take $u_0 \in P$ such that $\min_{t\in[\frac{1}{4},\frac{3}{4}]} u_0(t) > (1-\alpha)b$, $\|u_0\| \leq 4b$. Set $\mu(t,x) = tu_0 + (1-t)Tx$, and then $\mu : \ [0,1] \times \overline{U} \to \overline{P}_c$ is completely continuous. Suppose that there is $(t_0, x_0) \in [0,1] \times \partial U$ such that $\mu(t_0, x_0) = x_0$. Then $\min_{t\in[\frac{1}{4},\frac{3}{4}]} x_0(t) = (1-\alpha)b$.

We distinguish two cases: (i) If $\|Tx_0\| > 4b$, then by (3.1), we get $\min_{t\in[\frac{1}{4},\frac{3}{4}]} (Tx_0)(t) \geq \dfrac{1-\alpha}{4}\|Tx_0\| > (1-\alpha)b$. Hence,

$$\begin{aligned}
(1-\alpha)b &= \min_{t\in[\frac{1}{4},\frac{3}{4}]} x_0(t) = \min_{t\in[\frac{1}{4},\frac{3}{4}]} [t_0u_0(t) + (1-t_0)(Tx_0)(t)] \\
&\geq \min_{t\in[\frac{1}{4},\frac{3}{4}]} t_0u_0(t) + \min_{t\in[\frac{1}{4},\frac{3}{4}]} (1-t_0)(Tx_0)(t) \\
&> t_0(1-\alpha)b + (1-t_0)(1-\alpha)b = (1-\alpha)b.
\end{aligned}$$

This is a contradiction.

(ii) If $\|Tx_0\| \leq 4b$, then we obtain

$$\|x_0\| = \|t_0u_0 + (1-t_0)Tx_0\| \leq t_0\|u_0\| + (1-t_0)\|Tx_0\| \leq t_0 \cdot 4b + (1-t_0) \cdot 4b = 4b.$$

That is, $(1-\alpha)b \leq x_0(t) \leq 4b$ for $t \in [\dfrac{1}{4}, \dfrac{3}{4}]$, thus, we have by (3.2)

$$\begin{aligned}
(1-\alpha)b &= \min_{t\in[\frac{1}{4},\frac{3}{4}]} x_0(t) = \min_{t\in[\frac{1}{4},\frac{3}{4}]} [t_0u_0(t) + (1-t_0)(Tx_0)(t)] \\
&\geq \min_{t\in[\frac{1}{4},\frac{3}{4}]} t_0u_0(t) + \min_{t\in[\frac{1}{4},\frac{3}{4}]} (1-t_0)(Tx_0)(t) \\
&> t_0(1-\alpha)b + (1-t_0)(1-\alpha)b = (1-\alpha)b.
\end{aligned}$$

This is a contradiction. So we have $\mu(t,x) \neq x$ for $(t,x) \in [0,1] \times \partial U$. Finally, applying (A_2) and (A_3) in Theorem 2.1, we get

$$i(T, U, \overline{P}_c) = i(u_0, U, \overline{P}_c) = 1.$$

It follows from (A_1) in Theorem 2.1 that T has a fixed point $u^* \in U$. Further, $\min\limits_{t\in[\frac{1}{4},\frac{3}{4}]} u^*(t) > (1-\alpha)b$, $\|u^*\| \leq c$. That is, u^* is a nonzero fixed point of T in $\overline{P}_c$. That is to say, BVPs (1.1) and (1.2) have at least one positive solution. □

Theorem 3.2. *Suppose that (H_1) and (H_2) hold. If the following conditions hold,*

(H_5) $f_0 = f_\infty = \infty$;
(H_6) *there exists a constant $\rho_1 > 0$ such that*

$$f(t,u) < \phi_p(m\rho_1), for t \in [0,1], \quad u \in [0,\rho_1].$$

Then, BVPs (1.1) and (1.2) have at least two positive solutions u_1 and u_2 such that $0 < \|u_1\| < \rho_1 < \|u_2\|$.

Proof. Define $P_1 = \{u | u \in P, \min\limits_{t\in[\frac{1}{4},\frac{3}{4}]} u(t) \geq \dfrac{1-\alpha}{4}\|u\|\}$. It is clear that P_1 is also a cone. By the same proof of Lemma 2.3, we can get that $T : P_1 \to P_1$ is completely continuous .

To begin with, in view of $f_0 = \infty$, there exists $r_1 \in (0,\rho_1)$ such that $f(t,u) \geq \phi_p(\lambda_1 u)$ for $t \in [\dfrac{1}{4}, \dfrac{3}{4}]$ and $0 < u \leq r_1$, where $\lambda_1 \in \left(\dfrac{l}{\rho(1-\alpha)}, +\infty\right)$.

Let $\Omega_{r_1} = \{u \in P_1 | \; \|u\| < r_1\}$. Then for any $u \in \partial\Omega_{r_1}$, by using the same calculation in the proof of Theorem 3.1, we have

$$\begin{aligned}
(Tu)\left(\frac{1}{2}\right) = & \int_0^1 G\left(\frac{1}{2}, s\right) \phi_q\left(\int_0^s f(\tau, u(\tau))\mathrm{d}\tau\right) \mathrm{d}s \\
& + \frac{\alpha}{1-\alpha}\int_0^1 G(\eta, s)\phi_q\left(\int_0^s f(\tau, u(\tau))\mathrm{d}\tau\right) \mathrm{d}s \\
\geq & \int_0^1 G\left(\frac{1}{2}, s\right) \phi_q\left(\int_0^s f(\tau, u(\tau))\mathrm{d}\tau\right) \mathrm{d}s \\
\geq & \int_{\frac{1}{4}}^{\frac{3}{4}} G\left(\frac{1}{2}, s\right) \phi_q\left(\int_{\frac{1}{4}}^s f(\tau, u(\tau))\mathrm{d}\tau\right) \mathrm{d}s \\
\geq & \int_{\frac{1}{4}}^{\frac{3}{4}} G\left(\frac{1}{2}, s\right) \phi_q\left(\int_{\frac{1}{4}}^s (\lambda_1 u(\tau))^{p-1}\mathrm{d}\tau\right) \mathrm{d}s \\
\geq & \int_{\frac{1}{4}}^{\frac{3}{4}} G\left(\frac{1}{2}, s\right) \phi_q\left(\int_{\frac{1}{4}}^s \left(\lambda_1 \cdot \frac{1-\alpha}{4}\|u\|\right)^{p-1}\mathrm{d}\tau\right) \mathrm{d}s \\
\geq & \lambda_1 \cdot \frac{1-\alpha}{4}\|u\| \int_{\frac{1}{4}}^{\frac{3}{4}} G\left(\frac{1}{2}, s\right) \left(s - \frac{1}{4}\right)^{q-1} \mathrm{d}s = \frac{\lambda_1\rho(1-\alpha)}{l}\|u\| > \|u\|,
\end{aligned}$$

which implies $\|Tu\| > \|u\|$ for $u \in \partial\Omega_{r_1}$. Thus, from theorem 2.2, it follows that

$$i(T, \Omega_{r_1}, P_1) = 0. \tag{3.3}$$

Next, since $f_\infty = \infty$, there exists $R_1 > \rho_1$ such that $f(t,u) \geq \phi_p(\lambda_2 u)$ for $t \in [\frac{1}{4}, \frac{3}{4}]$ and $u \geq R_1$, where $\lambda_2 \in \left(\frac{l}{\rho(1-\alpha)}, +\infty \right)$.

Let $r_2 > \frac{4}{1-\alpha} R_1 > \rho_1$ and set $\Omega_{r_2} = \{u \in P_1 |\ \|u\| < r_2\}$. If $u \in \partial\Omega_{r_2}$, then $\min\limits_{t\in[\frac{1}{4},\frac{3}{4}]} u(t) \geq \frac{1-\alpha}{4}\|u\| > R_1$. Therefore, for any $u \in \partial\Omega_{r_2}$, by using the method to get (3.3), we get

$$(Tu)\left(\frac{1}{2}\right) > \frac{\lambda_2\rho(1-\alpha)}{l}\|u\| > \|u\|,$$

which implies $\|Tu\| > \|u\|$ for $u \in \partial\Omega_{r_2}$. Hence, by theorem 2.2, it follows that

$$i(T,\ \Omega_{r_2}, P_1) = 0. \tag{3.4}$$

Finally, let $\Omega_{\rho_1} = \{u \in P_1 |\ \|u\| < \rho_1\}$. Then for any $u \in \partial\Omega_{\rho_1}$, from (H_6), we have

$$\begin{aligned}
(Tu)(t) = &\int_0^1 G(t,s)\phi_q\left(\int_0^s f(\tau,u(\tau))\mathrm{d}\tau\right)\mathrm{d}s \\
&+ \frac{\alpha}{1-\alpha}\int_0^1 G(\eta,s)\phi_q\left(\int_0^s f(\tau,u(\tau))\mathrm{d}\tau\right)\mathrm{d}s \\
\leq &\frac{1}{1-\alpha}\int_0^1 G(s,s)\phi_q\left(\int_0^1 f(\tau,u(\tau))\mathrm{d}\tau\right)\mathrm{d}s \\
< &\frac{1}{1-\alpha} m\rho_1 \int_0^1 G(s,s)\mathrm{d}s = \rho_1 = \|u\|,
\end{aligned}$$

which implies $\|Tu\| < \|u\|$ for $u \in \partial\Omega_{\rho_1}$. Hence, by theorem 2.2, we obtain

$$i(T,\ \Omega_{\rho_1}, P_1) = 1. \tag{3.5}$$

Note that $r_1 < \rho_1 < r_2$; it follows from the additivity of fixed point index and (3.3)-(3.5) that

$$i(T,\ \Omega_{\rho_1}\backslash\overline{\Omega}_{r_1}, P_1) = i(T,\ \Omega_{\rho_1}, P_1) - i(T,\ \Omega_{r_1}, P_1) = 1,$$

and

$$i(T,\ \Omega_{r_2}\backslash\overline{\Omega}_{\rho_1}, P_1) = i(T,\ \Omega_{r_2}, P_1) - i(T,\ \Omega_{\rho_1}, P_1) = -1.$$

Therefore, T has a fixed point u_1 in $\Omega_{\rho_1}\backslash\overline{\Omega}_{r_1}$, and has a fixed point u_2 in $\Omega_{r_2}\backslash\overline{\Omega}_{\rho_1}$. Both are positive solutions of the BVP (1.1), (1.2) and $0 < \|u_1\| < \rho_1 < \|u_2\|$. The proof is completed. □

□

Theorem 3.3. *Assume that* (H_1) *and* (H_2) *hold. If the following conditions hold,*

(H_7) $f^0 = f^\infty = 0$;
(H_8) *there exists a constant* $\rho_2 > 0$ *such that*
$f(t,u) > \phi_p(\frac{l\rho_2}{4})$, *for* $t \in [\frac{1}{4}, \frac{3}{4}]$, $\quad u \in [\frac{1-\alpha}{4}\rho_2, \rho_2]$.
Then the BVPs (1.1) *and* (1.2) *have at least two positive solutions* u_1 *and* u_2 *such that* $0 < \|u_1\| < \rho_2 < \|u_2\|$.

Proof. Firstly, since $f^0 = 0$, there exists $r_1 \in (0, \rho_2)$ such that $f(t,u) \le \phi_p(\varepsilon_1 u)$ for $t \in [0,1]$ and $0 < u \le r_1$, where $\varepsilon_1 \in (0, m)$.

Let $\Omega_{r_1} = \{u \in P_1 | \ \|u\| < r_1\}$. Then for any $u \in \partial\Omega_{r_1}$, by using the same calculation in the proof of Theorem 3.2, we have

$$\begin{aligned}(Tu)(t) =& \int_0^1 G(t,s)\phi_q\left(\int_0^s f(\tau,u(\tau))\mathrm{d}\tau\right)\mathrm{d}s \\ &+ \frac{\alpha}{1-\alpha}\int_0^1 G(\eta,s)\phi_q\left(\int_0^s f(\tau,u(\tau))\mathrm{d}\tau\right)\mathrm{d}s \\ \le& \frac{1}{1-\alpha}\int_0^1 G(s,s)\phi_q\left(\int_0^1 f(\tau,u(\tau))\mathrm{d}\tau\right)\mathrm{d}s \\ \le& \frac{1}{1-\alpha}\varepsilon_1\|u\|\int_0^1 G(s,s)\mathrm{d}s = \frac{\varepsilon_1}{m}\|u\| < \|u\|,\end{aligned}$$

which implies $\|Tu\| < \|u\|$ for $u \in \partial\Omega_{r_1}$. Therefore, by theorem 2.2,

$$i(T, \Omega_{r_1}, P_1) = 1. \tag{3.6}$$

Secondly, in view of $f^\infty = 0$, there exists $R_2 > \rho_2$ such that $f(t,u) \le \phi_p(\varepsilon_2 u)$ for $t \in [0,1]$ and $u \ge R_2$, where $\varepsilon_2 \in (0,m)$.

We divide the proof into two cases: f is bounded and f is unbounded.

Case (i). Suppose that f is bounded, which implies that there exists $\overline{M} > 0$ such that $f(t,u) \le \phi_p(\overline{M})$ for all $t \in [0,1]$ and $u \in [0,+\infty)$.

Now, choosing $r_2 > \max\left\{\dfrac{\overline{M}}{m}, R_2\right\}$ for $u \in P_1$, then with $\|u\| = r_2$, we get

$$\begin{aligned}(Tu)(t) =& \int_0^1 G(t,s)\phi_q\left(\int_0^s f(\tau,u(\tau))\mathrm{d}\tau\right)\mathrm{d}s \\ &+ \frac{\alpha}{1-\alpha}\int_0^1 G(\eta,s)\phi_q\left(\int_0^s f(\tau,u(\tau))\mathrm{d}\tau\right)\mathrm{d}s \\ \le& \frac{1}{1-\alpha}\int_0^1 G(s,s)\phi_q\left(\int_0^1 f(\tau,u(\tau))\mathrm{d}\tau\right)\mathrm{d}s \\ \le& \frac{1}{1-\alpha}\overline{M}\int_0^1 G(s,s)\mathrm{d}s = \frac{\overline{M}}{m} < r_2 = \|u\|.\end{aligned}$$

Case (ii). Suppose that f is unbounded. Then, for $f : [0,1]\times[0,+\infty)$ is continuous, we know that there exist $t_0 \in [0,1]$ and $r_2 > R_2 > \rho_2$ such that $f(t,u) \le f(t_0, r_2)$

for $t \in [0,1]$ and $0 < u \leq r_2$. Then for $u \in P_1$, with $\|u\| = r_2$, we have

$$\begin{aligned}(Tu)(t) =& \int_0^1 G(t,s)\phi_q\left(\int_0^s f(\tau,u(\tau))\mathrm{d}\tau\right)\mathrm{d}s\\ &+ \frac{\alpha}{1-\alpha}\int_0^1 G(\eta,s)\phi_q\left(\int_0^s f(\tau,u(\tau))\mathrm{d}\tau\right)\mathrm{d}s\\ \leq& \frac{1}{1-\alpha}\int_0^1 G(s,s)\phi_q\left(\int_0^1 f(\tau,u(\tau))\mathrm{d}\tau\right)\mathrm{d}s\\ \leq& \frac{1}{1-\alpha}\int_0^1 G(s,s)\phi_q\left(\int_0^1 f(t_0,r_2)\mathrm{d}\tau\right)\mathrm{d}s\\ \leq& \frac{1}{1-\alpha}\varepsilon_2 r_2\int_0^1 G(s,s)\mathrm{d}s = \frac{\varepsilon_2}{m}r_2 < r_2 = \|u\|.\end{aligned}$$

Hence, in either case, we may always set $\Omega_{r_2} = \{u \in P_1 |\ \|u\| < r_2\}$ such that

$$\|Tu\| < \|u\| \ \text{ for } \ u \in \partial\Omega_{r_2}.$$

Thus, by Theorem 2.2, it follows that

$$i(T,\ \Omega_{r_2}, P_1) = 1. \tag{3.7}$$

Finally, let $\Omega_{\rho_2} = \{u \in P_1 |\ \|u\| < \rho_2\}$. Since $u \in \partial\Omega_{\rho_2} \subset P_1$, $\min\limits_{t\in[\frac{1}{4},\frac{3}{4}]} u(t) \geq \frac{1-\alpha}{4}\|u\| = \frac{1-\alpha}{4}\rho_2$. Hence for any $u \in \partial\Omega_{\rho_2}$, from (H_8), we have

$$\begin{aligned}(Tu)\left(\frac{1}{2}\right) =& \int_0^1 G\left(\frac{1}{2},s\right)\phi_q\left(\int_0^s f(\tau,u(\tau))\mathrm{d}\tau\right)\mathrm{d}s\\ &+ \frac{\alpha}{1-\alpha}\int_0^1 G(\eta,s)\phi_q\left(\int_0^s f(\tau,u(\tau))\mathrm{d}\tau\right)\mathrm{d}s\\ \geq& \int_0^1 G\left(\frac{1}{2},s\right)\phi_q\left(\int_0^s f(\tau,u(\tau))\mathrm{d}\tau\right)\mathrm{d}s\\ \geq& \int_{\frac{1}{4}}^{\frac{3}{4}} G\left(\frac{1}{2},s\right)\phi_q\left(\int_{\frac{1}{4}}^s f(\tau,u(\tau))\mathrm{d}\tau\right)\mathrm{d}s\\ \geq& \frac{l\rho_2}{4}\int_{\frac{1}{4}}^{\frac{3}{4}} G\left(\frac{1}{2},s\right)\left(s-\frac{1}{4}\right)^{q-1}\mathrm{d}s = \rho\cdot\rho_2 > \rho_2 = \|u\|,\end{aligned}$$

which yields $\|Tu\| > \|u\|$ for $u \in \partial\Omega_{\rho_2}$. Hence, by Theorem 2.2, we obtain

$$i(T,\ \Omega_{\rho_2}, P_1) = 0. \tag{3.8}$$

Note that $r_1 < \rho_2 < r_2$. As before, from (3.6)-(3.8), we get

$$i(T,\ \Omega_{\rho_2}\backslash\overline{\Omega}_{r_1}, P_1) = -1,\ i(T,\ \Omega_{r_2}\backslash\overline{\Omega}_{\rho_2}, P_1) = 1,$$

which shows that T has two fixed points, and consequently, BVPs (1.1) and (1.2) have two positive solutions. This completes the proof. □ □

3.1. Example

Now, we present an example to illustrate Theorem 3.1, the main result. Consider the following third-order three-point boundary value problem with p-Laplacian.

Example 4.1. Consider the following BVP

$$(\phi_3(u''(t)))' + [\varphi(t)h(u(t))]^2 = 0,\ t \in (0,1), \tag{3.9}$$

$$u(0) = u(1) = \frac{1}{2}u(\frac{1}{2}), \quad u''(0) = 0, \tag{3.10}$$

where $\varphi(t) = 4t,\ t \in [0,1]$, and

$$h(u) = \begin{cases} 480u, & 0 \le u \le \dfrac{1}{480}, \\ 1, & \dfrac{1}{480} < u \le \dfrac{1}{60}, \\ \dfrac{30}{119}u + \dfrac{237}{238}, & \dfrac{1}{60} < u \le 2, \\ \dfrac{357}{476}u, & u > 2. \end{cases}$$

In this example, we note that $p = 3$ and $\alpha = \eta = \frac{1}{2}$. Letting $\rho = \frac{9\sqrt{2}-2}{4}$, by a simple calculation, we get $q = \frac{3}{2}$, $G(s,s) = s(1-s)$ and $l = 4\rho\left(\int_{\frac{1}{4}}^{\frac{3}{4}} G(\frac{1}{2},s)(s-\frac{1}{4})^{q-1}\mathrm{d}s\right)^{-1}$ $= 240$, $m = (1-\alpha)\left(\int_0^1 (1-s)\mathrm{d}s\right)^{-1} = 6(1-\alpha) = 3$. We choose $b = \frac{1}{240}$ and $c = 2$. Evidently, $b < \min\{\frac{m}{l}, \frac{1}{4}\}c$ and

(i) for $t \in [\frac{1}{4}, \frac{3}{4}]$, $\frac{1}{480} \le u(t) \le \frac{1}{60}$, we have

$$f(t,u) = [\varphi(t)h(u(t))]^2 \ge \left[4 \times \frac{1}{4} \times 1\right]^2 = (lb)^2.$$

(ii) for $t \in [0,1], 0 \le u(t) \le 2$, we have

$$f(t,u) = [\varphi(t)h(u(t))]^2 \le \left[4 \times 1 \times \left(\frac{30}{119} \times 2 + \frac{237}{238}\right)\right]^2 = (mc)^2.$$

Hence, all the conditions of Theorem 3.1 are satisfied, then BVPs (3.9) and (3.10) have at least one positive solution u^* with $\|u^*\| \le 2$, $\min\limits_{t\in[\frac{1}{4},\frac{3}{4}]} u^*(t) > \frac{1}{480}$.

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