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Interval Oscillation Criteria for Nonlinear Neutral Impulsive Differential Equations*

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Abstract In this paper, we study the interval oscillation for nonlinear neutral impulsive differential equations. Sufficient condition for the interval oscillation of the equations is obtained by using Riccati transformation and estimating the ratio of unknown functions $y(t-\sigma(t))$ and $y(t)$. Some known results are generalized and improved. An example is given to illustrate the results.

Keywords Interval oscillation theory, neutral differential equation, impulsive

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1. Introduction

The theory of impulsive differential equations has been extensively investigated and developed due to their potential applications in many fields such as bursting rhythm models in medicine and biology, optimal control models in economics, pharmacokinetics and frequency modulation systems, etc. The monographs [4, 6, 7, 10] are the applications of impulsive differential equations.

In 2007, Liu and Xu [8] gave interval oscillation criteria for equations of the form

$$\begin{cases} (r(t)x'(t))' + q(t)\Phi_\gamma(x(t)) = f(t), \quad t \neq \tau_i, \\ x(\tau_k^+) = a_k x(\tau_k), \quad x'(\tau_k^+) = b_k x'(\tau_k), \quad k = 1, 2, ... \end{cases}$$

They studied the interval oscillation of impulsive differential equations without delay parameters.

In 2010, Huang and Feng [3] considered the following second-order nonlinear impulsive differential equations with constant delay

$$\begin{cases} x''(t) + q(t)g(x(t-\sigma)) = f(t), \quad t \neq \tau_i, \\ x(\tau_k^+) = a_k x(\tau_k), \quad x'(\tau_k^+) = b_k x'(\tau_k), \quad k = 1, 2, ... \end{cases}$$

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In 2016, Zhou and Wang [16] studied the second-order nonlinear impulsive differential equations with variable delay of the form

$$\begin{cases} (r(t)x'(t))' + q(t)g(x(t-\sigma(t))) = f(t), & t \neq \tau_i, \\ x(\tau_k^+) = a_k x(\tau_k), \quad x'(\tau_k^+) = b_k x'(\tau_k), & k = 1, 2, ..., \end{cases}$$

and obtained some results which developed some known results of [3] and [13].

For the impulsive equations, almost all of the interval oscillation results in the existing literature were established for the case of "without delay" or "with constant delay" or "with variable delay" in a non-neutral case (see [2,5,12,15,17]). However, for the case of "with neutral", there are only oscillation results (see [1,9,11,14]), and the research on interval oscillation of impulsive differential equations is very scarce.

Motivated by the above papers, we consider the interval oscillatory behavior of solutions to the following neutral impulsive differential equation

$$\begin{gathered} (r(t)(x(t)+p(t)x(t-\sigma(t)))')' + q(t)g(x(t-\sigma(t))) = f(t), \quad t \geqslant t_0, \quad t \neq \tau_k, \\ x(\tau_k^+) = a_k x(\tau_k), \quad x'(\tau_k^+) = b_k x'(\tau_k), \quad k = 1, 2, ..., \end{gathered} \tag{1.1}$$

where $\{\tau_k\}$, $k = 1, 2, ...$ denotes the impulse moment sequence, $0 \leqslant t_0 = \tau_0 < \tau_1 < \tau_2 < ... < \tau_k < ...$ and $\lim\limits_{k\to\infty} \tau_k = \infty$, $x(\tau_k^+) = \lim\limits_{\varepsilon\to 0^+} x(\tau_k+\varepsilon)$ and $x(\tau_k^-) = \lim\limits_{\varepsilon\to 0^-} x(\tau_k+\varepsilon)$ represent the right and left limits of $x(t)$ at $t = \tau_k$. Denote $\pi(a,b) := \int_a^b \frac{1}{r(s)} ds$, and $y(t) = x(t) + p(t)x(t-\sigma(t))$.

By a solution to equation (1.1), we mean that a function $x(t)$ is piecewise continuous on the interval (t_0, ∞) with discontinuities of the first kind only at $t = \tau_k$, $k = 1, 2, ...$, i.e., at the moments of impulse the following relation is satisfied:

$$x(\tau_k^+) = a_k x(\tau_k), \quad x'(\tau_k^+) = b_k x'(\tau_k), \quad k = 1, 2, ...$$

As is customary, a nontrivial solution $x(t)$ of (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative. Otherwise, it is said to be nonoscillatory. In other words, a solution is said to be oscillatory if there exists an increasing divergent sequence $\{\xi_k\}_{k\in\mathbb{N}} \subset [t_0, \infty)$ such that $x(\xi_k^+)x(\xi_k^-) \leqslant 0$ for all $k \in \mathbb{N}$.

This paper is structured as follows. In Section 2, we present necessary notations, lemmas and definitions. In Section 3, we state and prove our main results. At last, one illustrative example is proposed.

2. Preliminaries

In this section, we will present some necessary knowledge and notations.

Let $I \subset \mathbb{R}$ be an interval. A function set $PLC(I, \mathbb{R})$ is defined as follows:

$PLC(I, \mathbb{R}) := \{z : I \to \mathbb{R} \mid z$ is continuous on $I/\{\tau_i\}$ and at each τ_i, $z(\tau_i^+)$ and $z(\tau_i^-)$ exist, and the left continuity of z is assumed, i.e., $z(\tau_i^-) = z(\tau_i)$, $i \in \mathbb{N}\}$.

We introduce a function set $\Omega(c, d)$ as follows:

$$\Omega(c, d) := \{w \in C^1[c, d] : w(t) \not\equiv 0, w(c) = w(d) = 0\}.$$

Throughout this paper, we assume that the following hypotheses are satisfied:

(H_1) $r(t),\ p(t) \in C([t_0,\infty),(0,\infty))$, and $r(t)$ is nondecreasing; $q(t),\ f(t) \in PLC([t_0,\ \infty),\mathbb{R})$; $g(x) \in C(\mathbb{R},\mathbb{R})$, and there exists a positive constant α such that $\frac{g(x)}{x} \geqslant \alpha$ for all $x \in \mathbb{R}/\{0\}$; $\sigma(t) \in C^1([t_0,\infty),[0,\infty))$ and there exists a nonnegative constant σ_0 such that $0 \leqslant \sigma(t) \leqslant \sigma_0$ for all $t \geqslant t_0$ and $\tau_{k+1}-\tau_k \geqslant 2\sigma_0$ for all $k = 1,2,...$;

(H_2) For any $T \geqslant t_0$, and there exist $c_j,\ d_j \notin \{\tau_k\},\ j=1,2$ such that $T < c_1 - 2\sigma_0 < c_1 < d_1 < c_2 - 2\sigma_0 < c_2 < d_2$ and $(-1)^j f(t) \geqslant 0$ for $t \in [c_j,d_j]/\{\tau_k\}$; $p(t) \in C^1([t_0,\infty),[0,\infty))$, $1-p(t)\frac{\pi(t-\sigma(t)-\sigma(t-\sigma(t)),d_j)}{\pi(t-\sigma(t),d_j)} > 0$ for $t \in [c_j,d_j]$, $j=1,2$;

(H_3) $\{a_k\}$ and $\{b_k\}$ are real–valued sequences satisfying $b_k \geqslant a_k > 0,\ k=1,2,....$

We define two functions (called " interval delay function", see [16])

$$D_k(t) = t-\tau_k-\sigma(t),\quad t \in [\tau_k,\tau_{k+1}],\ k=1,2,...,$$

$$\tilde{D}_k(t) = t-\tau_k-\sigma(t)-\sigma(t-\sigma(t)),\quad t \in [\tau_k,\tau_{k+1}],\ k=1,2,...,$$

and suppose that the following condition always holds.

(H_4) There is a zero point $t_k \in (\tau_k,\tau_{k+1}]$ such that $D_k(t_k)=0$, $D_k(t)<0$ for $t \in (\tau_k,t_k)$ and $D_k(t)>0$ for $t \in (t_k,\tau_{k+1}]$, and a zero point $\tilde{t}_k \in (\tau_k,\tau_{k+1}]$ such that $\tilde{D}_k(\tilde{t}_k)=0$, $\tilde{D}_k(t)<0$ for $t \in (\tau_k,\tilde{t}_k)$ and $\tilde{D}_k(t)>0$ for $t \in (\tilde{t}_k,\tau_{k+1}]$.

In the remainder of this article, for two constants $c,\ d \notin \{\tau_k\}$ with $c<d$ and $k(c)<k(d)$, where $k(s) := \max\{i : t_0 < \tau_i < s\}$, let

$$\int_{[c,d]} := \int_c^{\tau_{k(c)+1}} + \sum_{i=k(c)+1}^{k(d)-1}\left(\int_{\tau_i}^{t_i} + \int_{t_i}^{\tau_{i+1}}\right) + \int_{\tau_{k(d)}}^{t_{k(d)}} + \int_{t_{k(d)}}^{d}.$$

For $\phi \in PLC([c,d],\mathbb{R})$, we define a functional $J : PLC([c,d],\mathbb{R}) \to \mathbb{R}$ by

$$\begin{aligned}
J_c^d[\phi] := &\int_c^{\tau_{k(c)+1}} \phi(t)\frac{t-\tau_{k(c)}-\sigma(t)}{t-\tau_{k(c)}}\left(1-p(t-\sigma(t))\frac{t-\tau_{k(c)}-2\sigma(t)}{t-\tau_{k(c)}-\sigma(t)}\right)dt\\
&+\sum_{i=k(c)+1}^{k(d)-1}\left(\int_{\tau_i}^{t_i}\phi(t)\frac{t-\tau_i}{b_i(t-\tau_i+\sigma(t))}\left(1-p(t-\sigma(t))\frac{t-\tau_i-\sigma(t)}{b_i(t-\tau_i)}\right)dt\right)\\
&+\sum_{i=k(c)+1}^{k(d)-1}\left(\int_{t_i}^{\tilde{t}_i}\phi(t)\frac{t-\tau_i-\sigma(t)}{t-\tau_i}\left(1-p(t-\sigma(t))\frac{t-\tau_i-\sigma(t)}{b_i(t-\tau_i)}\right)dt\right)\\
&+\sum_{i=k(c)+1}^{k(d)-1}\left(\int_{\tilde{t}_i}^{\tau_{i+1}}\phi(t)\frac{t-\tau_i-\sigma(t)}{t-\tau_i}\left(1-p(t-\sigma(t))\frac{t-\tau_i-2\sigma(t)}{t-\tau_i-\sigma(t)}\right)dt\right)\\
&+\int_{\tau_{k(d)}}^{t_{k(d)}}\phi(t)\frac{t-\tau_{k(d)}}{b_{k(d)}(t-\tau_{k(d)}+\sigma(t))}\left(1-p(t-\sigma(t))\frac{t-\tau_{k(d)}-\sigma(t)}{b_{k(d)}(t-\tau_{k(d)})}\right)dt\\
&+\int_{t_{k(d)}}^{\tilde{t}_{k(d)}}\phi(t)\frac{t-\tau_{k(d)}-\sigma(t)}{t-\tau_{k(d)}}\left(1-p(t-\sigma(t))\frac{t-\tau_{k(d)}-\sigma(t)}{b_{k(d)}(t-\tau_{k(d)})}\right)dt\\
&+\int_{\tilde{t}_{k(d)}}^{d}\phi(t)\frac{t-\sigma(t)-\tau_{k(d)}}{t-\tau_{k(d)}}\,(1-p(t-\sigma(t))\,\frac{t-2\sigma(t)-\tau_{k(d)}}{t-\tau_{k(d)}-\sigma(t)})dt,
\end{aligned} \tag{2.1}$$

and for $\varphi \in C([c,d],\mathbb{R})$, we define a functional $Q : C([c,d],\mathbb{R}) \to \mathbb{R}$ for some constant $\bar{r}$ by

$$Q_c^d[\varphi] := \varphi(\tau_{k(c)+1})\frac{(b_{k(c)+1}-a_{k(c)+1})\bar{r}}{a_{k(c)+1}(\tau_{k(c)+1}-c)} + \sum_{i=k(c)+2}^{k(d)} \varphi(\tau_i)\frac{(b_i-a_i)\bar{r}}{a_i(\tau_i-\tau_{i-1})}. \tag{2.2}$$

3. Main results

Now, we are in a position to state and prove our main results. In the sequel, we consider the following case for $j = 1, 2$:

(S_1) $\tau_{k(c_j)} + 2\sigma_0 < c_j$ and $\tau_{k(d_j)} + 2\sigma_0 < d_j$, for $k(c_j) < k(d_j)$,

$(\bar{S}_1)$ $\tau_{k(c_j)} + 2\sigma_0 < c_j$, for $k(c_j) = k(d_j)$.

Lemma 3.1. *If $x(t)$ is a positive solution to* (1.1)*, and $y(t) = x(t)+p(t)x(t-\sigma(t))$, then*

$$\left(\frac{y(t)}{\pi(t,\tau_{i+1})}\right)' \geqslant 0, \quad t \in [c_1,d_1]/\{\tau_k\}. \tag{3.1}$$

Proof. If $x(t)$ is a positive solution, we know that $x(t) > 0$, $x(t-\sigma(t)) > 0$ and $x(t-2\sigma(t)) > 0$ for $t \geqslant T$. From (1.1), we have

$$(r(t)y'(t))' = -q(t)g(x(t-\sigma(t))) + f(t) \leqslant 0, \quad t \in [c_1,d_1]/\{\tau_k\}.$$

Therefore, $r(t)y'(t)$ is nonincreasing on $(\tau_i,\tau_{i+1}]$, $i = k_{c_1}+1, ..., k_{d_1}-1$. Following the monotonicity of $r(t)y'(t)$, we have

$$\begin{aligned} y(t) &\geqslant -\int_t^{\tau_{i+1}} r^{-1}(s)r(s)y'(s)ds \\ &\geqslant -r(t)y'(t)\int_t^{\tau_{i+1}} r^{-1}(s)ds, \ t \in (\tau_i,\tau_{i+1}]. \end{aligned}$$

Therefore,

$$y(t) \geqslant -r(t)y'(t)\pi(t,\tau_{i+1}), \quad t \in (\tau_i,\tau_{i+1}]. \tag{3.2}$$

From (3.2), we conclude that $\frac{y(t)}{\pi(t)}$ is nondecreasing, as

$$\begin{aligned} \left(\frac{y(t)}{\pi(t,\tau_{i+1})}\right)' &= \frac{y'(t)\pi(t,\tau_{i+1}) - y(t)\pi'(t,\tau_{i+1})}{\pi^2(t,\tau_{i+1})} \\ &= \frac{r(t)y'(t)\pi(t,\tau_{i+1}) + y(t)}{r(t)\pi^2(t,\tau_{i+1})} \geqslant 0, \quad t \in [c_1,d_1]/\{\tau_k\}. \end{aligned}$$

The proof is complete. □

Lemma 3.2. *If $x(t)$ is a positive solution to* (1.1) *on $[T,\infty)$, then there exist the following estimations of $\frac{y(t-\sigma(t))}{y(t)}$.*

(I) When $k(c_1) < k(d_1)$, and $t_i \in (\tau_i,\tau_{i+1}]$ for $i = k(c_1)+1, ...k(d_1)-1$:

(a) $\frac{y(t-\sigma(t))}{y(t)} > \frac{t-\tau_i-\sigma(t)}{t-\tau_i}$, $t \in (t_i,\tau_{i+1}]$;

(b) $\frac{y(t-\sigma(t))}{y(t)} > \frac{t-\tau_i}{b_i(t+\sigma(t)-\tau_i)}$, $t \in (\tau_i,t_i]$;

(c) $\frac{y(t-\sigma(t))}{y(t)} > \frac{t-\tau_{k(c_1)}-\sigma(t)}{t-\tau_{k(c_1)}}$, $t \in (c_1,\tau_{k(c_1)+1}]$;

(d) $\frac{y(t-\sigma(t))}{y(t)} > \frac{t-\tau_{k(d_1)}-\sigma(t)}{t-\tau_{k(d_1)}}$, $t \in (t_{k(d_1)}, d_1]$;

(e) $\frac{y(t-\sigma(t))}{y(t)} > \frac{t-\tau_{k(d_1)}}{b_{k(d_1)}(t+\sigma(t)-\tau_{k(d_1)})}$, $t \in (\tau_{k(d_1)}, t_{k(d_1)}]$.

(II) When $k(c_1) = k(d_1)$*:*

(f) $\frac{y(t-\sigma(t))}{y(t)} > \frac{t-\tau_{k(c_1)}-\sigma(t)}{t-\tau_{k(c_1)}}$, $t \in (c_1, d_1]$.

Proof. If $x(t)$ is a positive solution, from (1.1), (H_1) and (H_2), we obtain that for $t \in [c_1, d_1]/\{\tau_k\}$,

$$(r(t)y'(t))' \leqslant f(t) - \alpha q(t)x(t-\sigma(t)) \leqslant 0.$$

Hence, $r(t)y'(t)$ is nonincreasing on $[c_1, d_1]/\{\tau_k\}$. Next, we give the proof of case (a) and (b) only. For other cases, the proof is similar and will be omitted.

Case (a). If $t_i < t \leqslant \tau_{i+1}$, then $(t-\sigma(t), t) \subset (\tau_i, \tau_{i+1}]$. Thus, there is no impulsive moment in $(t-\sigma(t), t)$. For any $s \in (t-\sigma(t), t)$, there exists a $\xi_1 \in (\tau_i, s)$ such that

$$y(s) - y(\tau_i^+) = y'(\xi_1)(s-\tau_i). \tag{3.3}$$

Since $y(\tau_i^+) > 0$, $r(s)$ is nondecreasing and $r(t)y'(t)$ is nonincreasing on (τ_i, τ_{i+1}), we have

$$y(s) \geqslant \frac{r(\xi_1)}{r(s)}y(s) > \frac{r(\xi_1)}{r(s)}(y'(\xi_1)(s-\tau_i)) \geqslant \frac{r(s)y'(s)}{r(s)}(s-\tau_i) = y'(s)(s-\tau_i). \tag{3.4}$$

Therefore,

$$\frac{y'(s)}{y(s)} < \frac{1}{s-\tau_i}. \tag{3.5}$$

Integrating both sides of (3.5) from $t-\sigma(t)$ to t, we obtain

$$\ln(y(t)) - \ln(y(t-\sigma(t))) < \ln(t-\tau_i) - \ln(t-\sigma(t)-\tau_i),$$

i.e.,

$$\frac{y(t-\sigma(t))}{y(t)} > \frac{t-\sigma(t)-\tau_i}{t-\tau_i}. \tag{3.6}$$

Case (b). If $\tau_i < t < t_i$, then $\tau_i - \sigma_0 < t-\sigma(t) < \tau_i < t$. There is an impulsive moment τ_i in $(t-\sigma(t), t)$. For any $t \in (\tau_i, t_i)$, we have

$$y(t) - y(\tau_i^+) = y'(\xi_2)(t-\tau_i), \quad \xi_2 \in (\tau_i, t). \tag{3.7}$$

For the impulsive condition of (1.1), we can conclude

$$y(\tau_i^+) = x(\tau_i^+) + p(\tau_i^+)x(\tau_i^+ - \sigma(\tau_i^+)) = a_i x(\tau_i) + p(\tau_i)a_i x(\tau_i - \sigma(\tau_i)) = a_i y(\tau_i). \tag{3.8}$$

$$\begin{aligned} y'(\tau_i^+) &= x'(\tau_i^+) + p'(\tau_i^+)x(\tau_i^+ - \sigma(\tau_i^+)) + p(\tau_i^+)x'(\tau_i^+ - \sigma(\tau_i^+))(1+\sigma'(\tau_i^+)) \\ &= b_i x'(\tau_i) + p'(\tau_i)a_i x(\tau_i - \sigma(\tau_i)) + p(\tau_i)b_i x'(\tau_i - \sigma(\tau_i))(1+\sigma'(\tau_i)) \\ &\leqslant b_i x'(\tau_i) + p'(\tau_i)b_i x(\tau_i - \sigma(\tau_i)) + p(\tau_i)b_i x'(\tau_i - \sigma(\tau_i))(1+\sigma'(\tau_i)) \\ &= b_i y'(\tau_i). \end{aligned} \tag{3.9}$$

Using (3.8), (3.9) and the monotone properties of $r(t)$, $r(t)y'(t)$ in (3.7), we get

$$y(t)-a_iy(\tau_i)=\frac{r(\xi_2)}{r(\xi_2)}y'(\xi_2)(t-\tau_i) \leqslant \frac{r(\tau_i^+)y'(\tau_i^+)}{r(\tau_i^+)}(t-\tau_i) \leqslant b_iy'(\tau_i)(t-\tau_i),\ \ \xi_2 \in (\theta_i, t).$$

Since $y(\tau_i) > 0$, we have

$$\frac{y(t)}{y(\tau_i)} - a_i \leqslant b_i\frac{y'(\tau_i)}{y(\tau_i)}(t-\tau_i). \tag{3.10}$$

In addition,

$$y(\tau_i) > y(\tau_i) - y(\tau_i - \sigma(t)) = y'(\xi_3)\sigma(t), \quad \xi_3 \in (\tau_i - \sigma(t), \tau_i). \tag{3.11}$$

Similar to the analysis of (3.3), (3.4) and (3.5), we have

$$\frac{y'(\tau_i)}{y(\tau_i)} < \frac{1}{\sigma(t)}. \tag{3.12}$$

From (3.10) and (3.12), we get

$$\frac{y(t)}{y(\tau_i)} \leqslant a_i + \frac{b_i}{\sigma(t)}(t-\tau_i).$$

In view of $a_k \leqslant b_k$, we have

$$\frac{y(\tau_i)}{y(t)} \geqslant \frac{\sigma(t)}{a_i\sigma(t) + b_i(t-\tau_i)} \geqslant \frac{\sigma(t)}{b_i(t+\sigma(t)-\tau_i)}. \tag{3.13}$$

On the other hand, using the similar analysis of (3.3), (3.4) and (3.5), we get

$$\frac{y'(s)}{y(s)} < \frac{1}{s-\tau_i+\sigma(t)}, \quad s \in (\tau_i - \sigma(t), \tau_i). \tag{3.14}$$

Integrating (3.14) from $t-\sigma(t)(> \tau_i - \sigma(t))$ to τ_i, $t \in (\tau_i, t_i)$, we have

$$\frac{y(t-\sigma(t))}{y(\tau_i)} > \frac{t-\tau_i}{\sigma(t)} \geqslant 0. \tag{3.15}$$

From (3.13) and (3.15), we obtain

$$\frac{y(t-\sigma(t))}{y(t)} > \frac{t-\tau_i}{b_i(t+\sigma(t)-\tau_i)}.$$

The proof is complete. □

Lemma 3.3. *Let $x(t)$ be a positive solution to (1.1) and $u(t)$ be defined by*

$$u(t) := \frac{r(t)y'(t)}{y(t)}, \quad t \in [c_1, d_1]. \tag{3.16}$$

If $k(c_1) < k(d_1)$ and τ_i, $i = k(c_1)+1,...k(d_1)$ are impulse moments in $[c_1, d_1]$, then there exist the following estimations of $u(t)$:

(g) There exists a constant $\bar{r}$ such that $u(\tau_i) \leqslant \frac{\bar{r}}{\tau_i - \tau_{i-1}}$ for $\tau_i \in [c_1, d_1]$, $i = k(c_1)+2, ..., k(d_1)$;

(h) There exists a constant $\bar{r}$ such that $u(\tau_{k(c_1)+1}) \leqslant \frac{\bar{r}}{\tau_{k(c_1)+1} - c_1}$ for $\tau_{k(c_1)+1} \in [c_1, d_1]$.

Proof. If $x(t)$ is a positive solution, for $t \in (\tau_{i-1}, \tau_i] \subset [c_1, d_1]$, and $i = k(c_1) + 2, ..., k(d_1)$, we have

$$y(t) - y(\tau_{i-1}) = y'(\xi)(t - \tau_{i-1}), \quad \xi \in (\tau_{i-1}, t).$$

In view of $y(\tau_{i-1}^+) > 0$, and the monotone properties of $r(t)y'(t)$, we obtain

$$y(t) > y'(\xi)(t - \tau_{i-1}) = \frac{r(\xi)y'(\xi)}{r(\xi)}(t - \tau_{i-1}) \geqslant \frac{r(t)y'(t)}{r(\xi)}(t - \tau_{i-1}).$$

That is,

$$\frac{r(t)y'(t)}{y(t)} \leqslant \frac{r(\xi)}{t - \tau_{i-1}}.$$

Let $t \to \tau_i^-$. It follows

$$u(\tau_i) = \frac{r(\tau_i)y'(\tau_i)}{y(\tau_i)} \leqslant \frac{r(\xi)}{\tau_i - \tau_{i-1}}.$$

Using the similar analysis on $(c_1, \tau_{k(c_1)+1}]$, we can get

$$u(\tau_{k(c_1)+1}) \leqslant \frac{r(\xi_1)}{\tau_{k(c_1)+1} - c_1}.$$

The proof is complete. □

Remark 3.1. Let $x(t)$ be a negative solution of (1.1). Then, the results in Lemmas 3.1–3.3 are correct with the replacement of $[c_1, d_1]$ by $[c_2, d_2]$.

Theorem 3.1. *If there exist $w_j(t) \in \Omega(c_j, d_j)$ such that, for $k(c_j) < k(d_j),\ j = 1, 2,$*

$$\alpha J_{c_j}^{d_j}[q(t)\omega_j^2(t)] - \int_{c_j}^{d_j} r(t)\omega_j'^2(t)dt \geqslant Q_{c_j}^{d_j}[\omega_j^2], \tag{3.17}$$

and for $k(c_j) = k(d_j),\ j = 1, 2,$

$$\int_{c_j}^{d_j} r(t)\omega_j'^2(t)dt - \alpha \int_{c_j}^{d_j} q(t)P(t - \sigma(t))\omega_j^2(t)\frac{t - \tau_{k(c_j)} - \sigma(t)}{t - \tau_{k(c_j)}}dt \leqslant 0, \tag{3.18}$$

where $P(t - \sigma(t)) := 1 - p(t - \sigma(t))\frac{\pi(t - \sigma(t) - \sigma(t - \sigma(t)), d_j)}{\pi(t - \sigma(t), d_j)}$. Then, (1.1) is oscillatory.

Proof. Assume, to the contrary, that $x(t)$ is a non–oscillatory solution to (1.1). If $x(t)$ is a positive solution, we choose the interval $[c_1, d_1]$ for consideration. Differentiating $u(t)$ defined by (3.16), in view of (1.1) and condition (H_1), we obtain for $t \neq \tau_k$,

$$\begin{aligned} u'(t) &= \frac{(r(t)y'(t))'y(t) - r(t)y'(t)y'(t)}{y^2(t)} = \frac{(r(t)y'(t))'}{y(t)} - \frac{1}{r(t)}\left(\frac{r(t)y'(t)}{y(t)}\right)^2 \\ &\leqslant \frac{f(t) - \alpha q(t)x(t - \sigma(t))}{y(t)} - \frac{1}{r(t)}u^2(t) \\ &\leqslant -\alpha q(t)\frac{x(t - \sigma(t))}{y(t)} - \frac{1}{r(t)}u^2(t). \end{aligned} \tag{3.19}$$

If $k(c_1) < k(d_1)$, choosing a $\omega_1(t) \in \Omega(c_1, d_1)$, multiplying both sides of (3.19) by $\omega_1^2(t)$ and integrating it from c_1 to d_1, we obtain

$$\int_{[c_1,d_1]} u'(t)\omega_1^2(t)dt \leqslant -\int_{[c_1,d_1]} \alpha q(t)\omega_1^2(t)\frac{x(t-\sigma(t))}{y(t)}dt - \int_{[c_1,d_1]} \frac{1}{r(t)}u^2(t)\omega_1^2(t)dt. \tag{3.20}$$

Using the integration by parts formula on the left side of (3.20) and noting the condition $\omega_1(c_1) = \omega_1(d_1) = 0$, we obtain

$$\begin{aligned}\sum_{i=k(c_1)+1}^{k(d_1)} \omega_1^2(\tau_i)[u(\tau_i) - u(\tau_i^+)] \leqslant & -\int_{[c_1,d_1]} \alpha q(t)\omega_1^2(t)\frac{x(t-\sigma(t))}{y(t)}dt \\ & + \int_{[c_1,d_1]} V(\omega_1^2(t), u(t))dt,\end{aligned} \tag{3.21}$$

where

$$\begin{aligned}V(\omega_1^2(t), u(t)) &= 2\omega_1'(t)\omega_1(t)u(t) - \frac{\omega_1^2(t)}{r(t)}u^2(t) \\ &= r(t)\omega_1'^2(t) - \left(\omega_1(t)u(t)\frac{1}{\sqrt{r(t)}} - \omega_1'(t)\sqrt{r(t)}\right)^2 \leqslant r(t)\omega_1'^2(t).\end{aligned} \tag{3.22}$$

Meanwhile, due to (3.8) and (3.9), for $t = \tau_k,\ k = 1, 2, ...,$ we have

$$u(\tau_k^+) = \frac{r(\tau_k^+)y'(\tau_k^+)}{y(\tau_k^+)} \leqslant \frac{b_k r(\tau_k)y'(\tau_k)}{a_k y(\tau_k)} = \frac{b_k}{a_k}u(\tau_k).$$

Hence,

$$\sum_{i=k(c_1)+1}^{k(d_1)} \omega_1^2(\tau_i)[u(\tau_i) - u(\tau_i^+)] = \sum_{i=k(c_1)+1}^{k(d_1)} \frac{a_i - b_i}{a_i}\omega_1^2(\tau_i)u(\tau_i). \tag{3.23}$$

Therefore, from (3.21)–(3.23), we can get

$$\begin{aligned}\sum_{i=k(c_1)+1}^{k(d_1)} \frac{a_i - b_i}{a_i}\omega_1^2(\tau_i)u(\tau_i) \leqslant & -\alpha\int_{[c_1,d_1]} q(t)\omega_1^2(t)\frac{x(t-\sigma(t))}{y(t)}dt \\ & + \int_{[c_1,d_1]} r(t)\omega_1'^2(t)dt.\end{aligned} \tag{3.24}$$

From $y(t) = x(t) + p(t)x(\sigma(t))$, we can conclude

$$\begin{aligned}\frac{x(t-\sigma(t))}{y(t)} &= \frac{y(t-\sigma(t))}{y(t)} - \frac{p(t-\sigma(t))x(t-\sigma(t)-\sigma(t-\sigma(t)))}{y(t)} \\ &\geqslant \frac{y(t-\sigma(t))}{y(t)} - \frac{p(t-\sigma(t))y(t-\sigma(t)-\sigma(t-\sigma(t)))}{y(t)} \\ &= \frac{y(t-\sigma(t))}{y(t)} - p(t-\sigma(t))\frac{y(t-\sigma(t)-\sigma(t-\sigma(t)))}{y(t-\sigma(t))}\frac{y(t-\sigma(t))}{y(t)}.\end{aligned} \tag{3.25}$$

We also conclude from (3.24) and (3.25) that

$$\begin{aligned}&\sum_{i=k(c_1)+1}^{k(d_1)}\frac{a_i-b_i}{a_i}\omega_1^2(\tau_i)u(\tau_i)\\ &\leqslant -\alpha\int_{[c_1,d_1]}q(t)\omega_1^2(t)\frac{y(t-\sigma(t))}{y(t)}\left(1-p(t-\sigma(t))\frac{y(t-\sigma(t)-\sigma(t-\sigma(t)))}{y(t-\sigma(t))}\right)dt\\ &\quad+\int_{[c_1,d_1]}r(t)\omega_1'^2(t)dt.\end{aligned}\tag{3.26}$$

From (3.24), Lemma 3.2 and Lemma 3.3, we easily obtain

$$\alpha J_{c_1}^{d_1}[q(t)\omega_1^2(t)]-\int_{c_1}^{d_1}r(t)\omega_1'^2(t)dt\leqslant Q_{c_1}^{d_1}[\omega_1^2],$$

where functions J and Q are defined by (2.1) and (2.2), which contradicts condition (3.17) for $j=1$.

If $k(c_1)=k(d_1)$, there is no impulsive moment in $[c_1,d_1]$. Because of the definition of $y(t)$, Lemma 3.1 and condition $(\bar{S}_1)$, we have for $t\in[c_1,d_1]$

$$\begin{aligned}x(t-\sigma(t))&=y(t-\sigma(t))-p(t)x(t-\sigma(t)-\sigma(t-\sigma(t)))\\ &\geqslant y(t-\sigma(t))-p(t)y(t-\sigma(t)-\sigma(t-\sigma(t)))\\ &\geqslant y(t-\sigma(t))-p(t-\sigma(t))\frac{\pi(t-\sigma(t)-\sigma(t-\sigma(t)),d_1)}{\pi(t-\sigma(t),d_1)}y(t-\sigma(t))\\ &=y(t-\sigma(t))\left(1-p(t-\sigma(t))\frac{\pi(t-\sigma(t)-\sigma(t-\sigma(t)),d_1)}{\pi(t-\sigma(t),d_1)}\right).\end{aligned}\tag{3.27}$$

We can conclude from (3.19) and (3.27) that

$$u'(t)\leqslant-\alpha q(t)P(t-\sigma(t))\frac{y(t-\sigma(t))}{y(t)}-\frac{1}{r(t)}u^2(t),\tag{3.28}$$

where $P(t-\sigma(t)):=1-p(t-\sigma(t))\frac{\pi(t-\sigma(t)-\sigma(t-\sigma(t)),d_1)}{\pi(t-\sigma(t),d_1)}$. Multiplying both sides of (3.28) by $\omega_1^2(t)$ and integrating it from c_1 to d_1, we obtain

$$\begin{aligned}\int_{c_1}^{d_1}u'(t)\omega_1^2(t)dt\leqslant&-\int_{c_1}^{d_1}\alpha q(t)P(t-\sigma(t))\omega_1^2(t)\frac{y(t-\sigma(t))}{y(t)}dt\\ &-\int_{c_1}^{d_1}\frac{1}{r(t)}u^2(t)\omega_1^2(t)dt.\end{aligned}\tag{3.29}$$

Using the integration by parts formula on the left side of (3.29) and noting the condition $\omega_1(c_1)=\omega_1(d_1)=0$, we obtain

$$\int_{c_1}^{d_1}V(\omega_1^2(t),u(t))dt-\alpha\int_{c_1}^{d_1}q(t)P(t-\sigma(t))\omega_1^2(t)\frac{y(t-\sigma(t))}{y(t)}dt\geqslant 0,$$

where $V(\omega_1^2(t),u(t))$ is defined by (3.22). We can conclude from Lemma 3.2 and (3.22) that

$$\int_{c_1}^{d_1}r(t)\omega_1^2(t)dt-\alpha\int_{c_1}^{d_1}q(t)P(t-\sigma(t))\omega_1'^2(t)\frac{t-\tau_{k(c_1)}-\sigma(t)}{t-\tau_{k(c_1)}}dt\geqslant 0,$$

which contradicts condition (3.18) for $j=1$. If $x(t)$ is a negative solution, we choose the interval $[c_2,d_2]$ to consider. The proof is similar to $x(t)$ is a positive solution, and will be omitted. $\square$

Remark 3.2. For the discussion of impulse moments of $y(t)$, $y(t-\sigma(t))$ and $y(t-2\sigma(t))$ on two intervals $[c_j,d_j](j=1,2)$, we need to consider the following possible cases for $k(c_j)<k(d_j)$:

(S_1) $\tau_{k(c_j)}+2\sigma_0<c_j$ and $\tau_{k(d_j)}+2\sigma_0<d_j$;
(S_2) $\tau_{k(c_j)}+2\sigma_0<c_j$, $\tau_{k(d_j)}+2\sigma_0>d_j$ and $\tau_{k(d_j)}+\sigma_0<d_j$;
(S_3) $\tau_{k(c_j)}+2\sigma_0<c_j$ and $\tau_{k(d_j)}+\sigma_0>d_j$;
(S_4) $\tau_{k(c_j)}+2\sigma_0>c_j$, $\tau_{k(c_j)}+\sigma_0<c_j$ and $\tau_{k(d_j)}+2\sigma_0<d_j$;
(S_5) $\tau_{k(c_j)}+2\sigma_0>c_j$, $\tau_{k(c_j)}+\sigma_0<c_j$, $\tau_{k(d_j)}+2\sigma_0>d_j$ and $\tau_{k(d_j)}+\sigma_0<d_j$;
(S_6) $\tau_{k(c_j)}+2\sigma_0>c_j$, $\tau_{k(c_j)}+\sigma_0<c_j$ and $\tau_{k(d_j)}+\sigma_0>d_j$;
(S_7) $\tau_{k(c_j)}+\sigma_0>c_j$ and $\tau_{k(d_j)}+2\sigma_0<d_j$;
(S_8) $\tau_{k(c_j)}+\sigma_0>c_j$, $\tau_{k(d_j)}+2\sigma_0>d_j$ and $\tau_{k(d_j)}+\sigma_0<d_j$;
(S_9) $\tau_{k(c_j)}+\sigma_0>c_j$ and $\tau_{k(d_j)}+\sigma_0>d_j$;

and six cases for $k(c_j)=k(d_j)$:

$(\bar{S}_1)$ $\tau_{k(c_j)}+2\sigma_0<c_j$;
$(\bar{S}_2)$ $\tau_{k(c_j)}+\sigma_0<c_j,\ \tau_{k(c_j)}+2\sigma_0>d_j$;
$(\bar{S}_3)$ $\tau_{k(c_j)}+\sigma_0<c_j<\tau_{k(c_j)}+2\sigma_0,\ \tau_{k(c_j)}+2\sigma_0<d_j$;
$(\bar{S}_4)$ $\tau_{k(c_j)}+\sigma_0>d_j$;
$(\bar{S}_5)$ $\tau_{k(c_j)}+\sigma_0>c_j,\ \tau_{k(c_j)}+\sigma_0<d_j<\tau_{k(c_j)}+2\sigma_0$;
$(\bar{S}_6)$ $\tau_{k(c_j)}+\sigma_0>c_j,\ \tau_{k(c_j)}+2\sigma_0<d_j$.

In order to save space, we only study equation (1.1) under the cases of S_1 and $(\bar{S}_1)$ throughout the paper.

4. Example

In this section, we will present an example to illustrate our main results.

Example 4.1. Consider the impulsive neutral differential equation

$$\begin{aligned}&(r(t)(x(t)+p(t)x(t-\sigma(t)))')'+q(t)g(x(t-\sigma(t)))=f(t),\quad t\geqslant t_0,\quad t\neq\tau_k,\\&x(\tau_k^+)=a_kx(\tau_k),\quad x'(\tau_k^+)=b_kx'(\tau_k),\quad k=1,2,...,\end{aligned}\tag{4.1}$$

where $r(t)=1$, $p(t)=\frac{1}{3}$, $\sigma(t)=\frac{1}{2}$, and $q(t)=1$; $g(x)=\alpha x$, where $\alpha=\frac{3(\pi+2)\pi}{2(\pi+1)(\pi-1)}$; $f(t)=(-1)^jt$, $j=1,2$; $t_0=0$; $\tau_k=(k-1)\pi,\ k=1,2,...$; $a_k=b_k=2$. Obviously, conditions (H_1) and (H_3) are satisfied. Then, letting $c_1=\frac{6}{4}\pi$, $d_1=\frac{7}{4}\pi$, $c_2=\frac{10}{4}\pi$ and $d_2=\frac{11}{4}\pi$, we get $k(\frac{6}{4}\pi)=k(\frac{7}{4}\pi)=2$, $k(\frac{10}{4}\pi)=k(\frac{11}{4}\pi)=3$, i.e., $\tau_{c_1}=\tau_{d_1}=\pi$, and $\tau_{c_2}=\tau_{d_2}=2\pi$. Next, we verify conditions (H_2) and (H_4). For $T=t_0=0$, we have

$$0<\frac{6}{4}\pi-1<\frac{6}{4}\pi<\frac{7}{4}\pi<\frac{10}{4}\pi-1<\frac{10}{4}\pi<\frac{11}{4}\pi\ \text{ and }\ (-1)^j(-1)^jt\geqslant 0,\ j=1,2,$$

$$1-p(t)\frac{\pi(t-\sigma(t)-\sigma(t-\sigma(t)),d_1)}{\pi(t-\sigma(t),d_1)}=1-\frac{1}{3}\frac{\int_{t-\frac{1}{2}-\frac{1}{2}}^{\frac{7}{4}\pi}1dt}{\int_{t-\frac{1}{2}}^{\frac{7}{4}\pi}1dt}=1-\frac{1}{3}\frac{\frac{7}{4}\pi-t+\frac{1}{2}+\frac{1}{2}}{\frac{7}{4}\pi-t+\frac{1}{2}}$$

$$=\frac{2}{3}-\frac{\frac{1}{6}}{\frac{7}{4}\pi-t+\frac{1}{2}}>\frac{2}{3}-\frac{\frac{1}{6}}{\frac{7}{4}\pi-\frac{6}{4}\pi+\frac{1}{2}}=\frac{2(\pi+1)}{3(\pi+2)}>0\ \text{ for }\ t\in[\frac{6}{4}\pi,\frac{7}{4}\pi],\tag{4.2}$$

$$1-p(t)\frac{\pi(t-\sigma(t)-\sigma(t-\sigma(t)),d_2)}{\pi(t-\sigma(t),d_2)}=1-\frac{1}{3}\frac{\int_{t-\frac{1}{2}-\frac{1}{2}}^{\frac{11}{4}\pi}1dt}{\int_{t-\frac{1}{2}}^{\frac{11}{4}\pi}1dt}=1-\frac{1}{3}\frac{\frac{11}{4}\pi-t+\frac{1}{2}+\frac{1}{2}}{\frac{11}{4}\pi-t+\frac{1}{2}}$$

$$=\frac{2}{3}-\frac{\frac{1}{6}}{\frac{11}{4}\pi-t+\frac{1}{2}}>\frac{2}{3}-\frac{\frac{1}{6}}{\frac{11}{4}\pi-\frac{6}{4}\pi+\frac{1}{2}}=\frac{2(5\pi+1)}{3(5\pi+2)}>0 \quad \text{for} \quad t\in[\frac{10}{4}\pi,\frac{11}{4}\pi]. \tag{4.3}$$

$$D_k(t)=t-(k-1)\pi-\frac{1}{2},\quad t\in[\tau_k,\tau_{k+1}]$$

and

$$\tilde{D}_k(t)=t-(k-1)\pi-1,\quad t\in[\tau_k,\tau_{k+1}].$$

It can be easily seen that both $D_k(t)$ and $\tilde{D}_k(t)$ have a zero point in $[\tau_k,\tau_{k+1}]$. Therefore, (H_2) and (H_4) are satisfied. Let $\omega_1(t)=-t^2+\frac{13}{4}\pi t-\frac{21}{8}\pi^2$. From (4.2), we get

$$\begin{aligned}
&-\alpha\int_{c_1}^{d_1}q(t)P(t-\sigma(t))\omega_1^2(t)\frac{t-\tau_{k(c_1)}-\sigma(t)}{t-\tau_{k(c_1)}}dt\\
=&-\frac{3(\pi+2)\pi}{2(\pi+1)(\pi-1)}\int_{\frac{6}{4}\pi}^{\frac{7}{4}\pi}\frac{2(\pi+1)}{3(\pi+2)}(-t^2+\frac{13}{4}\pi t-\frac{21}{8}\pi^2)^2\frac{t-\pi-\frac{1}{2}}{t-\pi}dt\\
\leqslant&-\frac{\pi}{\pi-1}\int_{\frac{6}{4}\pi}^{\frac{7}{4}\pi}(-t^2+\frac{13}{4}\pi t-\frac{21}{8}\pi^2)^2\frac{\pi-1}{\pi}dt\\
=&-\int_{\frac{6}{4}\pi}^{\frac{7}{4}\pi}(-t^2+\frac{13}{4}\pi t-\frac{21}{8}\pi^2)^2dt\\
\approx&-3.2552,
\end{aligned}$$

$$\int_{c_1}^{d_1}r(t)\omega_1'^2(t)dt=\int_{\frac{6}{4}\pi}^{\frac{7}{4}\pi}(-2t+\frac{13}{4}\pi)^2dt\approx 0.1615.$$

Thus,

$$\int_{c_1}^{d_1}r(t)\omega_1'^2(t)dt-\alpha\int_{c_1}^{d_1}q(t)P(t-\sigma(t))\omega_1^2(t)\frac{t-\tau_{k(c_1)}-\sigma(t)}{t-\tau_{k(c_1)}}dt\leqslant 0.$$

Let $\omega_2(t)=-t^2+\frac{21}{4}\pi t-\frac{55}{8}\pi^2$. From (4.3), we get

$$\begin{aligned}
&-\alpha\int_{c_2}^{d_2}q(t)P(t-\sigma(t))\omega_2^2(t)\frac{t-\tau_{k(c_2)}-\sigma(t)}{t-\tau_{k(c_2)}}dt\\
=&-\frac{3(\pi+2)\pi}{2(\pi+1)(\pi-1)}\int_{\frac{10}{4}\pi}^{\frac{11}{4}\pi}\frac{2(5\pi+1)}{3(5\pi+2)}(-t^2+\frac{21}{4}\pi t-\frac{55}{8}\pi^2)^2\frac{t-2\pi-\frac{1}{2}}{t-2\pi}dt\\
\leqslant&-\frac{(\pi+2)(5\pi+1)\pi}{(\pi+1)(5\pi+2)(\pi-1)}\int_{\frac{10}{4}\pi}^{\frac{11}{4}\pi}(-t^2+\frac{21}{4}\pi t-\frac{55}{8}\pi^2)^2\frac{\pi-1}{\pi}dt\\
=&-\frac{(\pi+2)(5\pi+1)}{(\pi+1)(5\pi+2)}\int_{\frac{10}{4}\pi}^{\frac{11}{4}\pi}(-t^2+\frac{21}{4}\pi t-\frac{55}{8}\pi^2)^2dt\\
\approx&-4045.1090,
\end{aligned}$$

$$\int_{c_2}^{d_2} r(t)\omega_2'^2(t)dt = \int_{\frac{10}{4}\pi}^{\frac{11}{4}\pi} (-2t + \frac{21}{4}\pi)^2 dt \approx -0.1245.$$

Thus,

$$\int_{c_2}^{d_2} r(t)\omega_2'^2(t)dt - \alpha \int_{c_2}^{d_2} q(t)P(t-\sigma(t))\omega_2^2(t)\frac{t-\tau_{k(c_2)}-\sigma(t)}{t-\tau_{k(c_2)}}dt < 0.$$

Therefore,

$$\int_{c_j}^{d_j} r(t)\omega_j'^2(t)dt - \alpha \int_{c_j}^{d_j} q(t)P(t-\sigma(t))\omega_j^2(t)\frac{t-\tau_{k(c_j)}-\sigma(t)}{t-\tau_{k(c_j)}}dt \leqslant 0.$$

Thus, (4.1) satisfies (3.18). By Theorem 3.1, all the solutions to (4.1) are oscillatory.

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