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Positive Solutions for Hilfer Fractional Differential Equation Boundary Value Problems at Resonance*

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Abstract In this paper, we investigate the positive solutions for Hilfer fractional differential equation boundary value problems at resonance. First, we give the expression of the solution with Mittag-Leffler function. Next, we obtain the existence of the positive solutions by using fixed point index theorem. Finally, we give relevant examples to prove our main results.

Keywords Fractional differential equation, resonance, Hilfer fractional derivative, positive solution, fixed point index

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1. Introduction

In this paper, we consider the following Hilfer fractional differential equation boundary value problem (FBVP) at resonance

$$\begin{cases} D_{0^+}^{\alpha,\beta}x(t)+f(t,x(t))=0, \quad t\in(0,1), \\ x(0)=0, x(1)=\eta x(\xi), \end{cases} \tag{1.1}$$

where $1<\alpha<2$, $0\le\beta\le1$, $1<\gamma=\alpha+\beta(2-\alpha)<2$, $0<\xi<1$, and $\eta\xi^{\gamma-1}=1$. $D_{0^+}^{\alpha,\beta}$ is Hilfer fractional derivative and $f:[0,1]\times[0,+\infty)\to\mathbb{R}$ is continuous.

Boundary value problem (BVP) (1.1) is resonant as the corresponding homogeneous BVP

$$\begin{cases} D_{0^+}^{\alpha,\beta}x(t)=0, \quad t\in(0,1), \\ x(0)=0, x(1)=\eta x(\xi) \end{cases} \tag{1.2}$$

has a nontrivial solution $ct^{\gamma-1}$, where $c\in\mathbb{R}, c\neq0$.

Fractional differential equations can describe the objective world more accurately than integer differential equations, so they are widely used in physical mechanics, biomedicine, viscoelastic system, finance and other aspects (see [1,9,22]). There are various definitions for fractional integrals and derivatives, and the most commonly used are Riemann-Liouville derivative and Caputo derivative (see [5,8,10,14,19]).

Hilfer fractional derivative is an interpolation between Riemann-Liouville derivative and Caputo fractional derivative. When $\beta=0$, it corresponds to Riemann-Liuoville fractional derivative. When $\beta=1$, it corresponds to Caputo fractional

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derivative. Therefore, it is important to study Hilfer fractional derivative, which includes both Riemann-Liuoville and Caputo fractional derivative. Recently, many scholars have devoted themselves to the study on fractional differential equations with Hilfer derivative and have obtained abundant results (see [6, 11–13, 16, 20]).

For the special case of Hilfer fractional derivative $\beta = 0$ and $\beta = 1$, Wang studied BVP (1.1) in the sense of Riemann-Liouville fractional derivative (see [18]), and Yang studied BVP (1.1) in the sense of Caputo fractional derivative (see [21]).

Wang investigated the following Hilfer FBVP at non-resonance by fixed point theorem (see [16])

$$\begin{cases} D_{a^+}^{\alpha,\beta} u(t) = f(t, u(t)), \quad t \in (a, b], \\ I_{a^+}^{1-\gamma} u(a^+) = \sum\limits_{i=1}^{m} \lambda_i u(\tau_i), \quad \tau_i \in (a, b], \end{cases}$$

where $0 < \alpha < 1$, $0 \le \beta \le 1$, $\Gamma(\gamma) \neq \sum\limits_{i=1}^{m} \lambda_i (\tau_i - a)^{\gamma-1}$, and $D_{a^+}^{\alpha,\beta}$ is Hilfer fractional derivative.

Yong investigated the following Hilfer FBVP at resonance by upper and lower solutions (see [12])

$$\begin{cases} D_{0^+}^{\alpha,\beta} x(t) = f(t, x(t)), \quad t \in (0, T] = J \setminus \{0\}, \\ I_{0^+}^{1-\gamma} x(0) = \sum\limits_{i=1}^{m} c_i x(\tau_i), \quad \tau_i \in J, \end{cases}$$

where $0 < \alpha < 1$, $0 \le \beta \le 1$, $\Gamma(\gamma) = \sum\limits_{i=1}^{m} c_i (\tau_i)^{\gamma-1}$, and $D_{0^+}^{\alpha,\beta}$ is Hilfer fractional derivative.

First, most of studies on the Hilfer fractional derivative are considered in the case of $0 < \alpha < 1$ (see [6, 11–13, 16, 20]), while in this paper, we consider the case of the higher order $1 < \alpha < 2$. Higher order differential equations can be applied to the establishment of control systems and diffusion systems. For example, Nigmatullin derived fractional diffusion-wave equation (see [17])

$${}_0D_t^{\alpha} u(x, t) = \frac{d^2 u(x, t)}{dx^2}.$$

When $\alpha = 1$, it is the traditional diffusion equation, when $\alpha = 2$, it is the traditional wave equation, and when $1 < \alpha < 2$, it is an intermediate state between diffusion and wave.

Secondly, the studies on the Hilfer FBVPs are limited to the non-resonant cases (see [11, 13, 16, 20]), and there is little research on the resonant case (see [6, 12]). Especially, as far as we know, no relevant results have been obtained for the existence of positive solutions so far. The study of positive solutions has important theoretical significance and practical value in practical problems. For example, the optimal control problem the HIV model can be abstracted into the following fractional differential equations of the same order (see [15])

$$\begin{cases} {}_0^C D_T^{\alpha} x(t) = f(t, x(t), u(t)), \\ x(0) = x_0, \end{cases}$$

where $0 < \alpha < 1$, $x(t)$ is the n-dimensional state vector, $u(t)$ is the m-control vector, and f is the n-dimensional vector function. For a given control function, the existence of optimal control for positive solutions is obtained in the minimum functional sense.

In this paper, we consider the existence of positive solutions for Hilfer FBVPs at resonance. This paper is organized as follows. In Section 2, we review the basic definitions, lemmas and theorems. In Section 3, by fixed point index theorem and spectral theory of linear operators, we study the positive solutions for Hilfer FBVPs at resonance. In Section 4, we give relevant examples to prove our main results. In Section 5, the paper is summarized and prospected.

2. Preliminaries

Definition 2.1 ([7]). The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $x : (0, +\infty) \to \mathbb{R}$ is given by

$$I_{0^+}^{\alpha} x(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} x(s)ds,$$

provided the right-hand side is point-wise defined on $(0, +\infty)$.

Definition 2.2 ([7]). The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a function $x : (0, +\infty) \to \mathbb{R}$ is given by

$$D_{0^+}^{\alpha} x(t) = \frac{d^n}{dt^n} (I_{0^+}^{n-\alpha} x)(t),$$

provided the right-hand side is point-wise defined on $(0, +\infty)$, where n is the smallest integer greater than or equal to α.

Definition 2.3 ([3]). The Hilfer fractional derivative of order $1 < \alpha < 2$, and the type $0 \leq \beta \leq 1$ of a function $x \in L^1[0, 1]$ is given by

$$D_{0^+}^{\alpha,\beta} x(t) = (I_{0^+}^{\beta(2-\alpha)} D_{0^+}^{\gamma} x)(t),$$

where $\gamma = \alpha + \beta(2 - \alpha)$. If $\beta = 0$, $D_{0^+}^{\alpha,0} = D_{0^+}^{\alpha}$ represents Riemann-Liouville fractional derivative. If $\beta = 1$, $D_{0^+}^{\alpha,1} = ^c D_{0^+}^{\alpha}$ represents Caputo fractional derivative.

Definition 2.4 ([7]). Mittag-Leffler function with two parameters of order $\alpha > 0$ of a function $z \in \mathbb{C}$ is given by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{+\infty} \frac{z^k}{\Gamma(k\alpha + \beta)}.$$

Remark 2.1 ([7]). Mittag-Leffler function with two parameters is absolutely convergent in the whole complex plane.

Lemma 2.1 ([4]). *Let $1 < \alpha < 2$, $0 \leq \beta \leq 1$, and $\gamma = \alpha + \beta(2 - \alpha)$. If $x \in C[0, 1]$, then the following equations hold:*

$$I_{0^+}^{\gamma} D_{0^+}^{\gamma} x = I_{0^+}^{\alpha} D_{0^+}^{\alpha,\beta} x,$$

$$D_{0^+}^{\gamma} I_{0^+}^{\alpha} x = D_{0^+}^{\beta(2-\alpha)} x,$$

$$D_{0^+}^{\alpha,\beta} I_{0^+}^{\alpha} x = I_{0^+}^{\beta(2-\alpha)} D_{0^+}^{\beta(2-\alpha)} x.$$

Lemma 2.2 ([7]). *Let $1 < \alpha < 2$. If $x \in C[0,1]$, then the following equation holds:*

$$I_{0+}^{\alpha} D_{0+}^{\alpha} x(t) = x(t) - c_1 t^{\alpha-1} - c_2 t^{\alpha-2}, \quad c_1, c_2 \in \mathbb{R}.$$

Lemma 2.3 ([7]). *If $\alpha > 0$, $\beta > 0$, then the following equations hold:*

$$I_{0+}^{\alpha} t^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)} t^{\alpha+\beta-1},$$

$$D_{0+}^{\alpha} t^{\alpha-1} = 0.$$

Theorem 2.1 ([12]). *For any $\alpha > 0$, $\lambda \in \mathbb{R}$. We get*

(1) *for any $x \in C_{1-\gamma}[[0,1],\mathbb{R}]$, $\sum\limits_{k=0}^{+\infty} \lambda^k I_{0+}^{k\alpha} x(t)$ is convergent, and*

$$\sum_{k=0}^{+\infty} \lambda^k I_{0+}^{k\alpha} x(t) = x(t) + \lambda \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(\lambda(t-s)^{\alpha}) x(s) ds;$$

(2) *operator $I - \lambda I_{0+}^{\alpha} : C_{1-\gamma}[[0,1],\mathbb{R}] \to C_{1-\gamma}[[0,1],\mathbb{R}]$ is reversible and continuous, and*

$$(I - \lambda I_{0+}^{\alpha})^{-1} x(t) = \sum_{k=0}^{+\infty} \lambda^k I_{0+}^{k\alpha} x(t).$$

Denote

$$g(t) = \frac{1}{\Gamma(\alpha-2)} + \sum_{k=1}^{+\infty} \frac{t^k}{\Gamma(k\alpha+\alpha-2)}.$$

We can get

$$g'(t) = \sum_{k=1}^{+\infty} \frac{k t^{k-1}}{\Gamma(k\alpha+\alpha-2)} > 0, \quad t \in (0,+\infty);$$

$$g(0) = \frac{1}{\Gamma(\alpha-2)} < 0, \quad \lim_{t\to+\infty} g(t) = +\infty.$$

Then, $g(t)$ has a unique $\lambda^* > 0$ such that $g(\lambda^*) = 0$ on $(0,+\infty)$.

In this paper, we give the following assumptions.

(A_1) $\lambda \in (0, \lambda^*]$.

(A_2) $f : [0,1] \times [0,+\infty) \to \mathbb{R}$ is continuous.

Consider the following FBVP which is equivalent to (1.1)

$$\begin{cases} -D_{0+}^{\alpha,\beta} x(t) + \lambda x(t) = f(t, x(t)) + \lambda x(t), & t \in (0,1), \\ x(0) = 0, x(1) = \eta x(\xi). \end{cases} \tag{2.1}$$

Theorem 2.2. *If (A_1) holds and $y \in C[0,1]$, then the unique solution to problem*

$$\begin{cases} -D_{0+}^{\alpha,\beta} x(t) + \lambda x(t) = y(t), & t \in (0,1), \\ x(0) = 0, x(1) = \eta x(\xi) \end{cases} \tag{2.2}$$

is

$$x(t) = \int_0^1 K(t,s) y(s) ds, \tag{2.3}$$

where

$$
\begin{aligned}
&K(t,s) = K_0(t,s) + H_\lambda(t)w(s),\\
&K_0(t,s) = \begin{cases} \dfrac{G_\lambda(1-s)H_\lambda(t) - G_\lambda(t-s)H_\lambda(1)}{H_\lambda(1)}, & 0 \le s < t \le 1,\\ \dfrac{G_\lambda(1-s)H_\lambda(t)}{H_\lambda(1)}, & 0 \le t \le s \le 1, \end{cases}\\
&w(s) = \frac{\eta K_0(\xi,s)}{H_\lambda(1) - \eta H_\lambda(\xi)},\\
&G_\lambda(t) = t^{\alpha-1}E_{\alpha,\alpha}(\lambda t^\alpha),\\
&H_\lambda(t) = t^{\gamma-1}E_{\alpha,\gamma}(\lambda t^\alpha).
\end{aligned}
$$

Proof. (i) If x is the solution to (2.2), then it can be expressed as (2.3). Applying I^α_{0+} to both sides of (2.2),

$$-I^\alpha_{0+}D^{\alpha,\beta}_{0+}x(t) + \lambda I^\alpha_{0+}x(t) = I^\alpha_{0+}y(t).$$

By Lemma 2.1, we have

$$-I^\gamma_{0+}D^\gamma_{0+}x(t) + \lambda I^\alpha_{0+}x(t) = I^\alpha_{0+}y(t).$$

From Lemma 2.2, we get

$$-x(t) + c_1t^{\gamma-1} + c_2t^{\gamma-2} + \lambda I^\alpha_{0+}x(t) = I^\alpha_{0+}y(t). \tag{2.4}$$

By the boundary condition $x(0) = 0$, we have $c_2 = 0$. Then,

$$(I - \lambda I^\alpha_{0+})x(t) = -I^\alpha_{0+}y(t) + c_1t^{\gamma-1}.$$

From Theorem 2.1 and Lemma 2.3, we obtain

$$
\begin{aligned}
x(t) =&(I - \lambda I^\alpha_{0+})^{-1}(-I^\alpha_{0+}y(t) + c_1t^{\gamma-1})\\
=&\sum_{k=0}^{+\infty}\lambda^k I^{k\alpha}_{0+}(-I^\alpha_{0+}y(t) + c_1t^{\gamma-1})\\
=&-\sum_{k=0}^{+\infty}\lambda^k I^{k\alpha+\alpha}_{0+}y(t) + c_1\sum_{k=0}^{+\infty}\lambda^k I^{k\alpha}_{0+}t^{\gamma-1}\\
=&-\int_0^t\sum_{k=0}^{+\infty}\lambda^k\frac{(t-s)^{k\alpha+\alpha-1}}{\Gamma(k\alpha+\alpha)}y(s)ds + c_1\sum_{k=0}^{+\infty}\lambda^k\frac{\Gamma(\gamma)t^{k\alpha+\gamma-1}}{\Gamma(k\alpha+\gamma)}\\
=&-\int_0^t(t-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t-s)^\alpha)y(s)ds + ct^{\gamma-1}E_{\alpha,\gamma}(\lambda t^\alpha)\\
=&-\int_0^t G_\lambda(t-s)y(s)ds + cH_\lambda(t), \quad c = c_1\Gamma(\gamma).
\end{aligned}
$$

Therefore,

$$x(1) = -\int_0^1 G_\lambda(1-s)y(s)ds + cH_\lambda(1),$$

$$x(\xi) = -\int_0^{\xi} G_\lambda(\xi - s)y(s)ds + cH_\lambda(\xi).$$

By the boundary condition $x(1) = \eta x(\xi)$, we have

$$c = \frac{\int_0^1 G_\lambda(1-s)y(s)ds - \eta\int_0^{\xi} G_\lambda(\xi - s)y(s)ds}{H_\lambda(1) - \eta H_\lambda(\xi)}.$$

Then, $x(t)$ can be expressed as follows.

$$\begin{aligned}
&x(t)\\
=&-\int_0^t G_\lambda(t-s)y(s)ds + \frac{\int_0^1 G_\lambda(1-s)y(s)ds - \eta\int_0^{\xi} G_\lambda(\xi-s)y(s)ds}{H_\lambda(1) - \eta H_\lambda(\xi)}H_\lambda(t)\\
=&\frac{-\int_0^t G_\lambda(t-s)H_\lambda(1)y(s)ds + \int_0^1 G_\lambda(1-s)H_\lambda(t)y(s)ds}{H_\lambda(1)} - \frac{\int_0^1 G_\lambda(1-s)H_\lambda(t)y(s)ds}{H_\lambda(1)}\\
&+\frac{\int_0^1 G_\lambda(1-s)y(s)ds}{H_\lambda(1) - \eta H_\lambda(\xi)}H_\lambda(t) - \frac{\eta\int_0^{\xi} G_\lambda(\xi-s)y(s)ds}{H_\lambda(1) - \eta H_\lambda(\xi)}H_\lambda(t)\\
=&\int_0^1 K_0(t,s)y(s)ds + \frac{\eta\int_0^1 G_\lambda(1-s)H_\lambda(\xi)y(s)ds}{H_\lambda(1)[H_\lambda(1) - \eta H_\lambda(\xi)]}H_\lambda(t) - \frac{\eta\int_0^{\xi} G_\lambda(\xi-s)H_\lambda(1)y(s)ds}{H_\lambda(1)[H_\lambda(1) - \eta H_\lambda(\xi)]}H_\lambda(t)\\
=&\int_0^1 K_0(t,s)y(s)ds + \frac{\eta[\int_0^1 G_\lambda(1-s)H_\lambda(\xi)y(s)ds - \int_0^{\xi} G_\lambda(\xi-s)H_\lambda(1)y(s)ds]}{H_\lambda(1)[H_\lambda(1) - \eta H_\lambda(\xi)]}H_\lambda(t)\\
=&\int_0^1 \left[K_0(t,s) + H_\lambda(t)w(s)\right]y(s)ds\\
=&\int_0^1 K(t,s)y(s)ds.
\end{aligned}$$

(ii) We proof that (2.3) is the solution to (2.2).

Applying D_{0+}^{γ} to both sides of (2.4),

$$-D_{0+}^{\gamma}x(t) + c_1 D_{0+}^{\gamma}t^{\gamma-1} + c_2 D_{0+}^{\gamma}t^{\gamma-2} + \lambda D_{0+}^{\gamma}I_{0+}^{\alpha}x(t) = D_{0+}^{\gamma}I_{0+}^{\alpha}y(t).$$

By Lemmas 2.1 and 2.3, we get

$$-D_{0+}^{\gamma}x(t) + \lambda D_{0+}^{\beta(2-\alpha)}x(t) = D_{0+}^{\beta(2-\alpha)}y(t). \tag{2.5}$$

Applying $I_{0+}^{\beta(2-\alpha)}$ to both sides of (2.5),

$$-I_{0+}^{\beta(2-\alpha)}D_{0+}^{\gamma}x(t) + \lambda I_{0+}^{\beta(2-\alpha)}D_{0+}^{\beta(2-\alpha)}x(t) = I_{0+}^{\beta(2-\alpha)}D_{0+}^{\beta(2-\alpha)}y(t).$$

From Lemma 2.2, we have

$$-D_{0+}^{\alpha,\beta}x(t) + \lambda x(t) = y(t).$$

This completes the proof.

Definition 2.5. Function $x \in C[0,1]$ satisfying (2.3) is called the solution to $FBVP$ (2.2).

Lemma 2.4. *If (A_1) holds, then the following inequalities hold:*

(1) $K(t,s) > 0$, $\forall t, s \in (0,1)$;

(2) $m_2(s)t^{\gamma-1} \le K(t,s) \le m_1(s)t^{\gamma-1}$, $\forall t, s \in (0,1)$, *where* $m_1(s) = G_\lambda(1-s) + H_\lambda(1)w(s)$, *and* $m_2(s) = \frac{w(s)}{\Gamma(\gamma)}$.

Proof. (i) By the definitions of $G_\lambda(t)$ and $H_\lambda(t)$, we get

$$G'_\lambda(t)=\sum_{k=0}^{+\infty}\frac{\lambda^k t^{k\alpha+\alpha-2}}{\Gamma(k\alpha+\alpha-1)}>0,\quad t\in(0,1].$$

Since $G_\lambda(t)$ is continuous at $t=0$ and increasing on $(0,1]$, monotonicity can be extended to endpoints. Then, $G_\lambda(t)$ is increasing on $[0,1]$ and $G_\lambda(t)>0$ on $(0,1]$.

$$H'_\lambda(t)=\sum_{k=0}^{+\infty}\frac{\lambda^k t^{k\alpha+\gamma-2}}{\Gamma(k\alpha+\gamma-1)}>0,\quad t\in(0,1].$$

Then, $H_\lambda(t)$ is increasing on $[0,1]$ and $H_\lambda(t)>0$ on $(0,1]$.

By the definition of $H_\lambda(t)$, we have

$$\frac{t^{\gamma-1}}{\Gamma(\gamma)}\le H_\lambda(t)=\sum_{k=0}^{+\infty}\frac{\lambda^k t^{k\alpha+\gamma-1}}{\Gamma(k\alpha+\gamma)}\le\sum_{k=0}^{+\infty}\frac{\lambda^k t^{\gamma-1}}{\Gamma(k\alpha+\gamma)}=t^{\gamma-1}H_\lambda(1),\quad t\in[0,1].$$

When $0<t\le s<1$, we obtain

$$\begin{aligned}K(t,s)=&\frac{G_\lambda(1-s)H_\lambda(t)}{H_\lambda(1)}+H_\lambda(t)w(s)\\ \le&G_\lambda(1-s)t^{\gamma-1}+t^{\gamma-1}H_\lambda(1)w(s)=m_1(s)t^{\gamma-1}.\end{aligned}$$

When $0<s<t<1$, we get

$$\begin{aligned}K(t,s)=&\frac{G_\lambda(1-s)H_\lambda(t)-G_\lambda(t-s)H_\lambda(1)}{H_\lambda(1)}+H_\lambda(t)w(s)\\ \le&\frac{G_\lambda(1-s)H_\lambda(t)}{H_\lambda(1)}+H_\lambda(t)w(s)\\ \le&m_1(s)t^{\gamma-1}.\end{aligned}$$

(ii) By the property of $g(t)$, we have

$$\begin{aligned}G''_\lambda(t)=&\sum_{k=0}^{+\infty}\frac{\lambda^k t^{k\alpha+\alpha-3}}{\Gamma(k\alpha+\alpha-2)}=t^{\alpha-3}\Big(\frac{1}{\Gamma(\alpha-2)}+\sum_{k=1}^{+\infty}\frac{\lambda^k t^{k\alpha}}{\Gamma(k\alpha+\alpha-2)}\Big)\\ =&t^{\alpha-3}g(\lambda t^\alpha)\le t^{\alpha-3}g(\lambda)\le t^{\alpha-3}g(\lambda^*)=0,\quad t\in(0,1).\end{aligned}$$

Then, $G'_\lambda(t)$ is decreasing on $(0,1]$.

Therefore,

$$\begin{aligned}&\frac{\partial}{\partial s}[G_\lambda(1-s)H_\lambda(t)-G_\lambda(t-s)H_\lambda(1)]\\ =&-G'_\lambda(1-s)H_\lambda(t)+G'_\lambda(t-s)H_\lambda(1)\\ \ge&G'_\lambda(1-s)[H_\lambda(1)-H_\lambda(t)].\end{aligned}$$

Integrating both sides with respect to s, we obtain

$$\begin{aligned}&G_\lambda(1-s)H_\lambda(t)-G_\lambda(t-s)H_\lambda(1)\\ \ge&\int_0^s G'_\lambda(1-\tau)[H_\lambda(1)-H_\lambda(t)]d\tau\\ =&[G_\lambda(1)-G_\lambda(1-s)][H_\lambda(1)-H_\lambda(t)]>0.\end{aligned}$$

That is to say, when $0 < s < t < 1$, we have

$$K_0(t,s) = \frac{G_\lambda(1-s)H_\lambda(t) - G_\lambda(t-s)H_\lambda(1)}{H_\lambda(1)} > 0.$$

When $0 < t \le s < 1$, we get

$$K_0(t,s) = \frac{G_\lambda(1-s)H_\lambda(t)}{H_\lambda(1)} > 0.$$

Therefore,

$$\begin{aligned} K(t,s) &= K_0(t,s) + H_\lambda(t)w(s) \ge H_\lambda(t)w(s) \\ &\ge \frac{t^{\gamma-1}}{\Gamma(\gamma)} w(s) = m_2(s)t^{\gamma-1}. \end{aligned}$$

(iii) By $\eta\xi^{\gamma-1} = 1$ and the definition of Mittag-Leffler function, we have

$$E'_{\alpha,\gamma}(t) = \sum_{k=0}^{+\infty} \frac{kt^{k-1}}{\Gamma(k\alpha+\gamma)} > 0, \quad t \in (0,1]$$

and

$$H_\lambda(1) - \eta H_\lambda(\xi) = E_{\alpha,\gamma}(\lambda) - \eta\xi^{\gamma-1}E_{\alpha,\gamma}(\lambda\xi^\alpha) = E_{\alpha,\gamma}(\lambda) - E_{\alpha,\gamma}(\lambda\xi^\alpha) > 0.$$

Therefore,

$$w(s) = \frac{\eta K_0(\xi,s)}{H_\lambda(1) - \eta H_\lambda(\xi)} > 0.$$

By (ii), we get

$$K(t,s) \ge \frac{t^{\gamma-1}}{\Gamma(\gamma)} w(s) > 0.$$

This completes the proof.

Let $E = C[0,1]$ with $\|x\| = \max\limits_{0\le t\le 1} |x(t)|$. Then E is a Banach space. Denote $P_r = \{x \in E : \|x\| < r\}$. Define a cone

$$P = \{x \in E : x(t) \ge 0, t \in [0,1]\}.$$

Let

$$\begin{aligned} Ax(t) &= \int_0^1 K(t,s)[f(s,x(s)) + \lambda x(s)]ds, \\ Lx(t) &= \int_0^1 K(t,s)x(s)ds. \end{aligned}$$

Lemma 2.5 ([18]). *If (A_1) holds, then the first eigenvalue of L is λ, and the corresponding eigenfunction is $x_0(t) = t^{\gamma-1}$. That is, $x_0 = \lambda L x_0$.*

Lemma 2.6 ([2]). *Let E be a Banach space. P is a cone, and P_r is a bounded open set in E. $A : \overline{P_r} \cap P \to P$ is a completely continuous operator.*

(1) *If $\exists x_0 \in P \setminus \{\theta\}$ such that $x - Ax \ne \mu x_0$, $\forall \mu \ge 0$, and $x \in \partial P_r \cap P$, then $i(A, P_r \cap P, P) = 0$.*

(2) *If $Ax \ne \mu x$, $\forall \mu \ge 1$, and $x \in \partial P_r \cap P$, then $i(A, P_r \cap P, P) = 1$.*

3. Main results

Theorem 3.1. *If* (A_1), (A_2) *and the following inequalities hold*

$$\liminf_{x\to 0^+} \min_{0\le t\le 1} \frac{f(t,x)}{x} > 0, \tag{3.1}$$

$$\limsup_{x\to +\infty} \max_{0\le t\le 1} \frac{f(t,x)}{x} < 0, \tag{3.2}$$

then FBVP (1.1) *has at least one positive solution.*

Proof. (i) By (3.1), there exists $r_1 > 0$ such that

$$f(t,x) \ge 0, \quad \forall (t,x) \in [0,1] \times [0,r_1].$$

$$Ax(t) = \int_0^1 K(t,s)[f(s,x(s)) + \lambda x(s)]ds \ge \int_0^1 K(t,s)\lambda x(s)ds = \lambda Lx(t). \tag{3.3}$$

Suppose that A has no fixed point on $\partial P_{r_1} \cap P$. We prove that

$$x - Ax \ne \mu x_0, \quad \forall \mu \ge 0, x \in \partial P_{r_1} \cap P. \tag{3.4}$$

Otherwise, there exists $\mu_0 \ge 0$ and $x_1 \in \partial P_{r_1} \cap P$ such that

$$x_1 - Ax_1 = \mu_0 x_0.$$

By Lemma 2.4, we obtain

$$x_1 \ge \mu_0 x_0.$$

Let

$$\mu^* = \sup\{\mu : x_1 \ge \mu x_0\}.$$

Then, $x_1 \ge \mu^* x_0$. By (3.3) and Lemma 2.5, we get

$$x_1 = Ax_1 + \mu_0 x_0 \ge \lambda Lx_1 + \mu_0 x_0 \ge \mu^* \lambda Lx_0 + \mu_0 x_0 = \mu^* x_0 + \mu_0 x_0 = (\mu^* + \mu_0)x_0.$$

It contradicts the definition of μ^*. Then, (3.4) holds, and by Lemma 2.6,

$$i(A, P_{r_1} \cap P, P) = 0. \tag{3.5}$$

(ii) By (3.2), there exists $r_2 > r_1 > 0$ and $0 < \delta < 1$ such that

$$f(t,x) \le -\delta\lambda x, \quad \forall (t,x) \in [0,1] \times [r_2, +\infty). \tag{3.6}$$

Let

$$\Omega = \{x \in P : x = \mu Ax, 0 \le \mu \le 1\}.$$

Now, we present that Ω is bounded. For $x \in \Omega$, let $\widetilde{x}(t) = \min\{x(t), r_2\}$. By (3.6) and (A_2),

$$f(t,x) \le -\delta\lambda x + f(t, \widetilde{x}(t)) + \lambda \widetilde{x}(t). \tag{3.7}$$

By Lemma 2.4, we get

$$\begin{aligned}
A\widetilde{x}(t) &= \int_0^1 K(t,s)[f(s,\widetilde{x}(s)) + \lambda\widetilde{x}(s)]ds \\
&\le \int_0^1 m_1(s)t^{\gamma-1}[f(s,\widetilde{x}(s)) + \lambda\widetilde{x}(s)]ds \\
&\le \max_{(t,x)\in[0,1]\times[0,r_2]}\{f(t,x)+\lambda x\}\int_0^1 m_1(s)ds = M.
\end{aligned} \tag{3.8}$$

By (3.7) and (3.8), we have

$$\begin{aligned}
x(t) = \mu Ax(t) \le Ax(t) &= \int_0^1 K(t,s)[f(s,x(s)) + \lambda x(s)]ds \\
&\le \int_0^1 K(t,s)[-\delta\lambda x(s) + f(s,\widetilde{x}(s)) + \lambda\widetilde{x}(s) + \lambda x(s)]ds \\
&= A\widetilde{x}(t) + \lambda(1-\delta)Lx(t) \\
&\le M + Tx(t),
\end{aligned}$$

where $T = \lambda(1-\delta)L$. Then,

$$(I-T)x(t) \le M.$$

By Lemma 2.5, we get $r(T) = 1-\delta$. Then, the inverse operator of $I-T$ exists, and

$$\|x\| \le M\|(I-T)^{-1}\|.$$

Therefore, Ω is bounded.

Let $r_3 = M\|(I-T)^{-1}\| + r_2$, and by Lemma 2.6,

$$i(A, P_{r_3}\cap P, P) = 1. \tag{3.9}$$

By (3.8) and (3.9), we obtain

$$i(A, (P_{r_3}\setminus P_{r_1})\cap P, P) = 1.$$

Then, A has a fixed point on $(P_{r_3}\setminus P_{r_1})\cap P$. That is to say, FBVP (1.1) has at least one positive solution. This completes the proof.

Theorem 3.2. *If* (A_1), (A_2) *and the following inequality hold*

$$\limsup_{x\to 0^+}\max_{0\le t\le 1}\frac{f(t,x)}{x} < 0, \tag{3.10}$$

then $FBVP$ (1.1) *has at least one positive solution.*

Proof. By (3.10), there exists $R_1 > 0$ such that

$$f(t,x) \le 0, \quad \forall (t,x)\in[0,1]\times[0,R_1]. \tag{3.11}$$

Suppose that A has no fixed point on $\partial P_{R_1}\cap P$. We testify that

$$Ax \ne \mu x, \quad \forall \mu > 1, x\in\partial P_{R_1}\cap P. \tag{3.12}$$

Otherwise, there exists $\mu_0 > 1$ and $x_1 \in \partial P_{R_1} \cap P$ such that

$$Ax_1 = \mu_0 x_1.$$

By (3.11), we get

$$Ax_1 = \int_0^1 K(t,s)[f(s,x_1(s)) + \lambda x_1(s)]ds \le \int_0^1 K(t,s)\lambda x_1(s)ds = T_1 x_1,$$

where $T_1 = \lambda L$, we obtain $r(T_1) = 1$. Then,

$$Ax_1 = \mu_0 x_1 \le T_1 x_1.$$

By induction, we obtain

$$\mu_0^n x_1 \le T_1^n x_1 \le \|T_1^n\| \|x_1\|, \quad n = 1, 2, \cdots .$$

Therefore,

$$r(T_1) = \lim_{n\to+\infty} \sqrt[n]{\|T_1^n\|} \ge \mu_0 > 1.$$

It contradicts with $r(T_1) = 1$. Hence, (3.12) holds, and by Lemma 2.6

$$i(A, P_{R_1} \cap P, P) = 1. \tag{3.13}$$

Then, A has a fixed point on $P_{R_1} \cap P$. That is to say, FBVP (1.1) has at least one positive solution. This completes the proof.

4. Examples

Example 4.1. Consider the following FBVP

$$\begin{cases} D_{0^+}^{\frac{3}{2},\frac{1}{3}} x(t) + f(t,x(t)) = 0, \quad t \in (0,1), \\ x(0) = 0, x(1) = 4x(\frac{1}{8}), \end{cases} \tag{4.1}$$

where $\alpha = \frac{3}{2}$, $\beta = \frac{1}{3}$, $\gamma = \frac{5}{3}$, $\eta = 4$, $\xi = \frac{1}{8}$, and $\eta\xi^{\gamma-1} = 1$.

Then, (4.1) with $f(t,x) = \frac{x}{2} - x^2 \cos t$ satisfies the following inequalities

$$\liminf_{x\to 0^+} \min_{0\le t\le 1} \frac{f(t,x)}{x} = \frac{1}{2} > 0,$$

$$\limsup_{x\to+\infty} \max_{0\le t\le 1} \frac{f(t,x)}{x} = -\infty < 0.$$

By Theorem 3.1, FBVP (4.1) has at least one positive solution.

Example 4.2. Consider the following FBVP

$$\begin{cases} D_{0^+}^{\frac{5}{2},\frac{1}{2}} x(t) + f(t,x(t)) = 0, \quad t \in (0,1), \\ x(0) = 0, x(1) = 128x(\frac{1}{16}), \end{cases} \tag{4.2}$$

where $\alpha = \frac{5}{2}$, $\beta = \frac{1}{2}$, $\gamma = \frac{11}{4}$, $\eta = 128$, $\xi = \frac{1}{16}$, and $\eta\xi^{\gamma-1} = 1$.

Then, (4.2) with $f(t,x) = x^2 \cos t - \frac{x}{2}$ satisfies the following inequality

$$\limsup_{x\to 0^+} \max_{0\le t\le 1} \frac{f(t,x)}{x} = -\frac{1}{2} < 0.$$

By Theorem 3.2, FBVP (4.2) has at least one positive solution.

5. Conclusions

In this paper, by using fixed point index theorem and spectral theory of linear operators, we obtain the existence of positive solutions for Hilfer FBVP.

We consider the case $1 < \alpha < 2$. However, when $0 < \alpha < 1$, it is not difficult to find that some constraints should be given to α and β to avoid singularity and to ensure the existence of positive solutions. In addition, we need to study what other conditions are worthy of further exploration and research.

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References

[1] I. Ameen and P. Novati, *The solution of fractional order epidemic model by implicit Adams methods*, Applied Mathematical Modelling, 2017, 43, 78–84.

[2] D. Guo, *Nonlinear Functional Analysis*, Shandong Science and Technology Press, Jinan, 1985.

[3] R. Hilfer, *Applications of Fractional Calculus in Physics*, World Scientific, Singapore, 2000.

[4] R. Hilfer, Y. Luchko and Z. Tomovski, *Operational method for the solution of fractional differential equations with generalized Riemann-Liouville fractional derivatives*, Fractional Calculus and Applied Analysis, 2009, 12(3), 299–318.

[5] W. Jiang, *Solvability for a coupled system of fractional differential equations at resonance*, Nonlinear Analysis: Real World Applications, 2012, 13(5), 2285–2292.

[6] T. U. Khan, M. A. Khan and Y. Chu, *A new generalized Hilfer-type fractional derivative with applications to space-time diffusion equation*, Results in Physics, 2021, 22, Article ID 103953, 11 pages.

[7] A. A. Kilbas, H. H. Srivastava and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier, Amsterdam, 2006.

[8] N. Kosmatov and W. Jiang, *Resonant functional problems of fractional order*, Chaos, Solitons & Fractals, 2016, 91, 573–579.

[9] J. S. Leszczynski and T. Blaszczyk, *Modeling the transition between stable and unstable operation while emptying a silo*, Granular Matter, 2011, 13, 429–438.

[10] Y. Liu and Q. Li, *Existence and uniqueness of solutions and Lyapunov-type inequality for a mixed fractional boundary value problem*, Journal of Nonlinear Modeling and Analysis, 2022, 4(2), 207–219.

[11] V. H. Ngo and D. O'Regan, *A remark on ψ-Hilfer fractional differential equations with non-instantaneous impulses*, Mathematical Methods in the Applied Sciences, 2020, 43(6), 3354–3368.

[12] Y. D. Ri, H. C. Choi and K. J. Chang, *Constructive existence of solutions of multi-point boundary value problem for Hilfer fractional differential equation at resonance*, Mediterranean Journal of Mathematics, 2020, 17, 95, 20 pages.

[13] D. Vivek, K. Kanagarajan and S. Sivasundaram, *On the Behavior of Solutions of Fractional Differential Equations on Time Scale Via Hilfer Fractional Derivatives*, Fractional Calculus and Applied Analysis, 2018, 21(4), 1120–1138.

[14] D. Wang and J. Jiang, *Existence and Multiplicity of Positive Solutions for a Singular Nonlinear High Order Fractional Differential Problem with Multipoint Boundary Conditions*, Journal of Nonlinear Modeling and Analysis, 2021, 3(4), 603–616.

[15] H. Wang, *Fractional Differential Equations and Their Applications in Infectious Diseases*, Taiyuan University of Technology, Taiyuan, 2019.

[16] J. Wang and Y. Zhang, *Nonlocal initial value problems for differential equations with Hilfer fractional derivative*, Applied Mathematics and Computation, 2015, 266, 850–859.

[17] X. Wang, *Analysis and Application of Fractional Order Control System*, North China Electric Power University, Beijing, 2011.

[18] Y. Wang and L. Liu, *Positive solutions for a class of fractional 3-point boundary value problems at resonance*, Advances in Difference Equations, 2017, 7, 13 pages.

[19] Y. Wang and H. Wang, *Triple positive solutions for fractional differential equation boundary value problems at resonance*, Applied Mathematics Letters, 2020, 106, Article ID 106376, 7 pages.

[20] A. Wongcharoen, B. Ahmad, S. K. Ntouyas and J. Tariboon, *Three-Point Boundary Value Problems for the Langevin Equation with the Hilfer Fractional Derivative*, Advances in Mathematical Physics, 2020, Article ID 9606428, 11 pages.

[21] A. Yang and H. Wang, *Positive solutions of two-point boundary value problems of nonlinear fractional differential equation at resonance*, Electronic Journal of Qualitative Theory of Differential Equations, 2011, 71, 1–15.

[22] X. Zhang, L. Liu, Y. Wu and B. Wiwatanapataphee, *Nontrivial solutions for a fractional advection dispersion equation in anomalous diffusion*, Applied Mathematics Letters, 2017, 66, 1–8.