Journal of Nonlinear Modeling and Analysis
Volume 5, Number 3, September 2023, 565–579
http://jnma.ca; http://jnma.ijournal.cn
DOI:10.12150/jnma.2023.565

Well-Posedness for Fractional (p, q)-Difference Equations Initial Value Problem*

Mi Zhou[1,†]

Abstract In this paper, we investigate a class of the fractional (p, q)-difference initial value problem with the fractional (p, q)-integral boundary conditions with the aid of the method of successive approximations(Picard method) and fractional (p, q)-Gronwall inequality, obtaining sufficient conditions for the existence, uniqueness and continuous dependence results of solutions.

Keywords Well-posedness, fractional (p, q)-difference equation, initial value problem, fractional (p, q)-Gronwall inequality

MSC(2010) 26A33, 34A12, 39A70.

1. Introduction

Fractional calculus is an interesting and old subject that is a generalization of ordinary differentiation and integration. The study of the q-difference equation appeared at the beginning of the 21st century and was initially developed by Jackson [8]and Carmichael [3]. So far, q-difference equations have been infiltrated into various subjects (see [1,6,7]). Some basic definitions and properties of q-difference calculus can be found in [9]. In addition, fractional calculus has proved to be a valuable tool in many fields of science and engineering such as control, fluid flow, mechanics and electrical networks (see the papers [2,17–19] and the references therein).

In recent years, with the development of science and technology and the advancement of fractional order theory, many researchers have developed the theory of quantum calculus based on the two parameters p and q. The fractional-order (p, q)-difference equations have been widely used in physical sciences, Lie groups, special functions, hypergeometric series, Bezier curves and approximations. Some basic results of (p, q)-difference calculus can be found in [4,10,11,13,20,21].

In particular, there are few works that have considered the fractional (p, q)-difference equations. In 2020, Soontharanonl and Sitthiwirattham [22] studied the existence of a class of fractional (p, q)-difference equation. A class of fractional (p, q)-integrodifference equation with periodic fractional (p, q)-integral boundary conditions was considered with Banach and Schauder's fixed point theorems [25]. In 2021, Qin and Sun [15] proved the existence and uniqueness of positive solutions for fractional (p, q)-difference equation using some standard fixed point theorems.

†The corresponding author.
Email address: zhoum_aa@sina.com (M. Zhou)
[1]School of Mathematics and Physics, University of South China, Hengyang, Hunan 421001, China
*The authors were supported by Scientific Research Foundation for the PhD (University of South China, Grant No. 210XQD024).

In this year, Qin and Sun [16] studied the solvability and stability for a class of singular fractional (p, q)-difference equation using Arzela's lemma and fractional (p, q)-Gronwall inequality. In the same year, the boundary value problem of a class of fractional (p, q)-difference Schrödinger equations was studied by Qin and Sun [14]. In 2022, Neang, Nonlaopon and Tariboon [12] investigated the separate local boundary value conditions of fractional (p, q)-difference equation and obtained the existence and uniqueness of solutions based on some standard fixed point theorems. For some developments concerning the existence(and uniqueness) in fractional (p, q)-difference equations, we refer to [5, 23, 24] and references therein.

Inspired by the works mentioned above, we investigate the existence, uniqueness and continuous dependence of the solution to the fractional (p, q)-difference initial value problem(IVP)

$$\begin{cases} D_{p,q}^{\alpha} u(t) = f(p^{\alpha} t, u(p^{\alpha} t)), & t \in (0, 1], \\ I_{p,q}^{1-\alpha} u(t)|_{t=0} = \eta, \end{cases} \tag{1.1}$$

where $0 < \alpha < 1$, $0 < q < p \leq 1$, and $D_{p,q}^{\alpha}$ is an Riemann-Liouville type fractional (p, q)-difference operator.

Few papers have investigated the fractional (p, q)-difference equations, since fractional (p, q)-operator was defined lately. Compared with the papers [12, 14–16] and [5, 22–25], the main novelty of this paper is as follows. We apply the method of successive approximations(Picard method) and fractional (p, q)-Gronwall inequality to study the existence, uniqueness and continuous dependence results of the solution for problem (1.1), which is the first and probably the only work in this direction. To the author's knowledge, there is no result on the continuous dependence of the solution for fractional (p, q)-difference equation. In this paper, we aim to fill this margin to some extent. Thus, our works are new and meaningful.

The organization of our paper is as follows. In Section 2, we present some basic definitions and preliminaries results. In Section 3, we prove the main results of this paper, which include the existence, uniqueness and continuous dependence of the solution to problem (1.1).

2. Preliminaries

In this section, we present some concepts of fractional (p, q)-difference calculus and some necessary basic preliminaries. Let $0 < q < p \leq 1$. Define

$$[k]_q := \begin{cases} \dfrac{1 - q^k}{1 - q}, & k \in \mathbb{N}, \\ 1, & k = 0, \end{cases}$$

$$[k]_{p,q} := \begin{cases} \dfrac{p^k - q^k}{p - q} = p^{k-1} [k]_{\frac{q}{p}}, & k \in \mathbb{N}, \\ 0, & k = 0, \end{cases}$$

and the (p,q)-analogue factorial is defined as

$$[k]_{p,q}! := \begin{cases} [k]_{p,q}[k-1]_{p,q}\cdots[1]_{p,q} = \prod_{i=1}^{k}\frac{p^i-q^i}{p-q}, & k\in\mathbb{N},\\ 1, & k=0.\end{cases}$$

The (p,q)-analogue of the power function $(a-b)^{(n)}_{p,q}$ with $n \in N_0 := \{0,1,2,...\}$ is given by

$$(a-b)^{(0)}_{p,q} := 1, \quad (a-b)^{(n)}_{p,q} := \prod_{k=0}^{n-1}(ap^k - bq^k), \quad a,b\in\mathbb{R}.$$

If $\alpha \in \mathbb{R}$, the general form is given by

$$(a-b)^{(\alpha)}_{p,q} = p^{\binom{\alpha}{2}}(a-b)^{(\alpha)}_{\frac{q}{p}} = a^{\alpha}p^{\binom{\alpha}{2}}\prod_{i=0}^{\infty}\frac{a-b(\frac{q}{p})^i}{a-b(\frac{q}{p})^{\alpha+i}}, \quad 0<b<a,$$

where $\binom{\alpha}{2} = \frac{\alpha(\alpha-1)}{2}$. Notice that

$$(a-b)^{(\alpha)}_{p,q} = a^{\alpha}(1-\frac{b}{a})^{(\alpha)}_{p,q} \text{ and } a^{(\alpha)}_{p,q} = p^{\binom{\alpha}{2}}a^{\alpha}, \quad a>0.$$

When $a = 0$, we define $a^{(\alpha)}_{p,q} = 0$. For $0 < q < p \le 1$, the (p,q)-gamma and (p,q)-beta functions are defined by

$$\Gamma_{p,q}(x) = \begin{cases} \frac{(p-q)^{(x-1)}_{p,q}}{(p-q)^{x-1}} = \frac{(1-q/p)^{(x-1)}_{p,q}}{(1-q/p)^{x-1}}, & x\in\mathbb{R}\setminus\{0,-1,-2....\},\\ [x-1]_{p,q}!, & x\in\mathbb{Z},\end{cases}$$

$$B_{p,q}(x,y) := \int_0^1 t^{x-1}(1-qt)^{(y-1)}_{p,q}d_{p,q}t = p^{\frac{1}{2}(y-1)(2x+y-2)}\frac{\Gamma_{p,q}(x)\Gamma_{p,q}(y)}{\Gamma_{p,q}(x+y)}$$

respectively.

Definition 2.1. (see [21]) Let $0 < q < p \le 1$. The (p,q)-derivative of the function f is defined as

$$D_{p,q}f(t) := \frac{f(pt)-f(qt)}{(p-q)t}, \quad t\neq 0$$

and $(D_{p,q}f)(0) = \lim_{t\to 0}(D_{p,q}f)(t)$, provided that f is differentiable at 0. Meanwhile, the high-order $(p,q)-$derivative $D^n_{p,q}f(t)$ is defined by

$$D^n_{p,q}f(t) = \begin{cases} f(t), & n=0,\\ D_{p,q}D^{n-1}_{p,q}f(t), & n\in\mathbb{N}.\end{cases}$$

Definition 2.2. (see [21]) Let $0 < q < p \le 1$, f be an arbitrary function, and let x be a real number. The $(p,q)-$integral of the function f is defined as

$$\int_0^x f(t)d_{p,q}dt = (p-q)x\sum_{k=0}^{\infty}\frac{q^k}{p^{k+1}}f(\frac{q^k}{p^{k+1}}x), \tag{2.1}$$

provided that the series of the right side in (2.1) converges. In this case, f is called (p,q)-integrable on $[0,x]$, and denote

$$I_{p,q}f(x) = \int_0^x f(t)d_{p,q}t.$$

Definition 2.3. (see [21]) Let $0 < q < p \leq 1$, f be an arbitrary function, and let a and b be two real numbers. Then, we define

$$\int_a^b f(t)d_{p,q}t = \int_0^b f(t)d_{p,q}t - \int_0^a f(t)d_{p,q}t.$$

Lemma 2.1. (see [21]) *Let $0 < q < p \leq 1$, and let a and b be two real numbers. Then, the following formulas hold.*

(a) $\int_a^a f(t)d_{p,q}t = 0$;

(b) $\int_a^b \alpha f(t)d_{p,q}t = \alpha \int_a^b f(t)d_{p,q}t, \quad \alpha \in \mathbb{R}$;

(c) $\int_a^b f(t)d_{p,q}t = -\int_b^a f(t)d_{p,q}t$;

(d) $\int_a^b f(t)d_{p,q}t = \int_b^c f(t)d_{p,q}t + \int_c^b f(t)d_{p,q}t, \quad c \in \mathbb{R}$;

(e) $\int_a^b [f(t)+g(t)]d_{p,q}t = \int_a^b f(t)d_{p,q}t + \int_a^b g(t)d_{p,q}t$.

Definition 2.4. (see [22]) Let $\alpha > 0$, $0 < q < p \leq 1$, and let f be an arbitrary function on $[0,+\infty)$. The fractional (p,q)-integral is defined by

$$I_{p,q}^{\alpha}f(t) = \frac{1}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} \int_0^t (t-qs)_{p,q}^{(\alpha-1)} f\left(\frac{s}{p^{\alpha-1}}\right) d_{p,q}s$$

and $I_{p,q}^0 f(t) = f(t)$.

Definition 2.5. (see [22]) Let $\alpha > 0$, $0 < q < p \leq 1$, and let $f : (0,\infty) \to \mathbb{R}$ be an arbitrary function. The fractional (p,q)-difference operator of Riemann-Liouville type of order α is defined by

$$D_{p,q}^{\alpha}f(t) = D_{p,q}^{N} I_{p,q}^{N-\alpha} f(t)$$

and $D_{p,q}^0 f(t) = f(t)$, where N is the smallest integer greater than or equal to α.

Lemma 2.2. (see [22]) *For $\alpha > 0$, $0 < q < p \leq 1$, and $f : (0,\infty) \to \mathbb{R}$, we get*

$$D_{p,q}^{\alpha} I_{p,q}^{\alpha} f(t) = f(t).$$

Lemma 2.3. (see [22]) *For $\alpha \in (N-1, N]$, $N \in \mathbb{N}$, $0 < q < p \leq 1$, and $f : (0,\infty) \to \mathbb{R}$, we get*

$$I_{p,q}^{\alpha} D_{p,q}^{\alpha} f(t) = f(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \ldots + c_N t^{\alpha-N},$$

for some $c_i \in \mathbb{R}$, $i = 1, 2, \ldots., N$.

Lemma 2.4. (see [22]) *For $\alpha,\ \beta > 0$, and $0 < q < p \leq 1$, $(p,q)-$integral and $(p,q)-$difference operator have the following properties.*

(a) $I_{p,q}^{\alpha}[I_{p,q}^{\beta} f(x)] = I_{p,q}^{\beta}[I_{p,q}^{\alpha} f(x)] = I_{p,q}^{\alpha+\beta} f(x)$;

(b) $D_{p,q} I_{p,q} f(x) = f(x)$, *and* $I_{p,q} D_{p,q} f(x) = f(x) - f(0)$.

Lemma 2.5. *Let* $0<\alpha<1$ *and* $0<q<p\le 1$. *Then,*

$$\int_0^t (t-qs)_{p,q}^{(\alpha-1)} d_{p,q}s = t^\alpha \int_0^1 (1-q\tau)_{p,q}^{(\alpha-1)} d_{p,q}\tau = t^\alpha B(1,\alpha) \tag{2.2}$$

and

$$\int_0^t (t-qs)_{p,q}^{(\alpha-1)} s^{-\alpha} d_{p,q}s = \int_0^1 (1-q\tau)_{p,q}^{(\alpha-1)} \tau^{-\alpha} d_{p,q}\tau = B_{p,q}(\alpha,1-\alpha). \tag{2.3}$$

Proof. By Definition 2.4 and the definition of (p,q)-beta function, we can obtain

$$\begin{aligned}
\int_0^t (t-qs)_{p,q}^{(\alpha-1)} d_{p,q}s =&(p-q)t\sum_{k=0}^{\infty}\frac{q^k}{p^{k+1}}\left(t-q\frac{q^k}{p^{k+1}}t\right)_{p,q}^{(\alpha-1)}\\
=&(p-q)t\sum_{k=0}^{\infty}\frac{q^k}{p^{k+1}}t^{\alpha-1}\left(1-q\frac{q^k}{p^{k+1}}\right)_{p,q}^{(\alpha-1)}\\
=&t^\alpha(p-q)\sum_{k=0}^{\infty}\frac{q^k}{p^{k+1}}\left(1-q\frac{q^k}{p^{k+1}}\right)_{p,q}^{(\alpha-1)}\\
=&t^\alpha\int_0^1 (1-q\tau)_{p,q}^{(\alpha-1)} d_{p,q}\tau = t^\alpha B(1,\alpha).
\end{aligned}$$

Thus, equality (2.2) holds. Furthermore, from Definition 2.4 and the definition of (p,q)-beta function, we can obtain

$$\begin{aligned}
\int_0^t (t-qs)_{p,q}^{(\alpha-1)} s^{-\alpha} d_{p,q}s =&(p-q)t\sum_{k=0}^{\infty}\frac{q^k}{p^{k+1}}\left(t-q\frac{q^k}{p^{k+1}}t\right)_{p,q}^{(\alpha-1)}\left(\frac{q^k}{p^{k+1}}t\right)^{-\alpha}\\
=&(p-q)t\sum_{k=0}^{\infty}\frac{q^k}{p^{k+1}}t^{\alpha-1}\left(1-q\frac{q^k}{p^{k+1}}\right)_{p,q}^{(\alpha-1)}t^{-\alpha}\left(\frac{q^k}{p^{k+1}}\right)^{-\alpha}\\
=&(p-q)\sum_{k=0}^{\infty}\frac{q^k}{p^{k+1}}\left(1-q\frac{q^k}{p^{k+1}}\right)_{p,q}^{(\alpha-1)}\left(\frac{q^k}{p^{k+1}}\right)^{-\alpha}\\
=&\int_0^1 (1-q\tau)_{p,q}^{(\alpha-1)} t^{-\alpha} d_{p,q}\tau = B(\alpha,1-\alpha).
\end{aligned}$$

Thus, equality (2.3) holds. □

3. Main results

In this section, we establish some sufficient conditions for the existence, uniqueness and continuous dependence of the solution to initial value problem (1.1). Denote $\mathbb{R}^+=[0,+\infty)$.

Lemma 3.1. *Assume that* $0<\alpha<1$. *Then,*

$$I_{p,q}^{1-\alpha}t^{\alpha-1}=\Delta_{p,q},\ t\in\mathbb{R},$$

where

$$\Delta_{p,q}=\frac{p^{-(\alpha-1)(\frac{3}{2}\alpha-1)}}{p^{\binom{1-\alpha}{2}}}\Gamma_{p,q}(\alpha).$$

Proof. Since $0 < \alpha < 1$, $-1 < (1-\alpha) - 1 < 0$, $-1 < \alpha - 1 < 0$. Thus, from Definition 2.4, Lemma 2.5(2.3) and the definition of $(p,q)-$beta function, we have

$$\begin{aligned}
I_{p,q}^{1-\alpha} t^{\alpha-1} =& \frac{1}{p^{\binom{1-\alpha}{2}}\Gamma_{p,q}(1-\alpha)} \int_0^t (t-qs)_{p,q}^{(-\alpha)} \left(\frac{s}{p^{\alpha-1}}\right)^{\alpha-1} d_{p,q}s \\
=& \frac{p^{-(\alpha-1)^2}}{p^{\binom{1-\alpha}{2}}\Gamma_{p,q}(1-\alpha)} \int_0^t (t-qs)_{p,q}^{((1-\alpha)-1)} s^{\alpha-1} d_{p,q}s \\
=& \frac{p^{-(\alpha-1)^2}}{p^{\binom{1-\alpha}{2}}\Gamma_{p,q}(1-\alpha)} \int_0^1 (1-q\tau)_{p,q}^{((1-\alpha)-1)} \tau^{\alpha-1} d_{p,q}\tau \\
=& \frac{p^{-(\alpha-1)^2}}{p^{\binom{1-\alpha}{2}}\Gamma_{p,q}(1-\alpha)} B_{p,q}(\alpha, 1-\alpha) \\
=& \frac{p^{-(\alpha-1)(\frac{3}{2}\alpha-1)}}{p^{\binom{1-\alpha}{2}}} \Gamma_{p,q}(\alpha) = \Delta_{p,q}.
\end{aligned}$$

□

Lemma 3.2. *Let $0 < \alpha < 1$. Then, $D_{p,q}^{\alpha} t^{\alpha-1} = 0$, for $t \in \mathbb{R}$.*

Proof. From Definition 2.5, we have

$$D_{p,q}^{\alpha} t^{\alpha-1} = D_{p,q} I_{p,q}^{1-\alpha} t^{\alpha-1}.$$

According to Definition 2.1 and Lemma 3.1, we get

$$D_{p,q}^{\alpha} t^{\alpha-1} = D_{p,q} \Delta_{p,q} = 0.$$

□

Lemma 3.3. *Let $0 < q < p \leq 1$, $0 < \alpha < 1$, and let $f \in C([0,1] \times \mathbb{R}^+)$ be a given nonnegative function. Then, $x(t)$ is a solution to (1.1), if and only if it is the solution of the following integral equation*

$$u(t) = \frac{\eta}{\Delta_{p,q}} t^{\alpha-1} + \frac{1}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} \int_0^t (t-qs)_{p,q}^{(\alpha-1)} f(ps, u(ps)) d_{p,q}s. \tag{3.1}$$

Proof. Assume that $u(t)$ is a solution to (1.1). Taking the operator $I_{p,q}^{\alpha}$ on both sides of the equation of problem (1.1), by Definition (2.4) and Lemma 2.3, we have

$$u(t) = c_1 t^{\alpha-1} + I_{p,q}^{\alpha} f(p^{\alpha}t, u(p^{\alpha}t)), \tag{3.2}$$

for some constants $c_1 \in \mathbb{R}$. Taking the operator $I_{p,q}^{1-\alpha}$ on both sides of the equation of problem (3.2), from the boundary condition $I_{p,q}^{1-\alpha} u(t)|_{t=0} = \eta$ and Lemma 3.1, we have

$$c_1 = \frac{\eta}{\Delta_{p,q}}.$$

Hence,

$$\begin{aligned}
u(t) =& \frac{\eta}{\Delta_{p,q}} t^{\alpha-1} + I_{p,q}^{\alpha} f(p^{\alpha}t, u(p^{\alpha}t)) \\
=& \frac{\eta}{\Delta_{p,q}} t^{\alpha-1} + \frac{1}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} \int_0^t (t-qs)_{p,q}^{(\alpha-1)} f(ps, u(ps)) d_{p,q}s.
\end{aligned}$$

Next, we prove the sufficiency of this lemma. Assume that $u(t)$ satisfies (3.1). Applying the operator $D_{p,q}^{\alpha}$ to both sides of (3.1), by Lemmas 2.2 and 3.2, we have

$$D_{p,q}^{\alpha}u(t) = D_{p,q}^{\alpha}I_{p,q}^{\alpha}f(p^{\alpha}t, u(p^{\alpha}t)) = f(p^{\alpha}t, u(p^{\alpha}t)).$$

□

3.1. Existence and uniqueness

In this part, we will obtain the existence and uniqueness of the solution to (1.1) by Picard method.

Theorem 3.1. *Let $0 < q < p \leq 1$, $0 < \alpha < 1$, and let $f \in C([0,1] \times \mathbb{R}^+)$ be a given nonnegative function satisfying the following conditions.*

(H$_1$) *There exists a constant $M > 0$ such that*

$$M = \sup_{(t,u)\in[0,1]\times\mathbb{R}^+} |f(t,u)|.$$

(H$_2$) *f satisfies Lipschitz condition with Lipschitz constant L such that*

$$|f(t,u) - f(t,v)| \leq L|u - v|.$$

(H$_3$)

$$L < \Gamma_{p,q}(\alpha + 1).$$

Then, there exists a unique solution $u(t)$ for problem (1.1).

Proof. Set Picards's sequence functions

$$\begin{aligned} u_0(t) &= \frac{\eta}{\Delta_{p,q}} t^{\alpha-1}, \\ u_n(t) &= u_0(t) + \frac{1}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} \int_0^t (t - qs)_{p,q}^{(\alpha-1)} f(ps, u_{n-1}(ps)) d_{p,q}s, \end{aligned} \tag{3.3}$$

where $n = 1, 2, \cdots$.

The function $u_n (n = 1, 2, \cdots)$ are continuous, and u_n can be written as a sum of successive differences

$$u_n = u_0 + \sum_{i=1}^{n} (u_i - u_{i-1}).$$

It means that the convergence of the sequence u_n is equivalent to the convergence of the infinite series $\sum_{i=1}^{n}(u_i - u_{i-1})$, and the solution will be

$$u(t) = \lim_{n\to\infty} u_n(t),$$

i.e., if the infinite series $\sum_{i=1}^{n}(u_i - u_{i-1})$ converges, then the sequence $u_n(t)$ will converge to $u(t)$. To prove the uniform convergence of $\{u_n(t)\}$, we consider the associated series

$$\sum_{n=1}^{\infty} (u_n(t) - u_{n-1}(t)).$$

When $n=1$, from (3.3), we have

$$\begin{aligned}|u_1(t)-u_0(t)| =&\left|\frac{1}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)}\int_0^t (t-qs)_{p,q}^{(\alpha-1)}f(ps,u_0(ps))d_{p,q}s\right|\\ =&\frac{1}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)}\left|\int_0^t (t-qs)_{p,q}^{(\alpha-1)}f(ps,u_0(ps))d_{p,q}s\right|.\end{aligned}$$

Noticing by Definition 2.2, we know

$$\begin{aligned}&\left|\int_0^t (t-qs)_{p,q}^{(\alpha-1)}f(ps,u_0(ps))d_{p,q}s\right|\\ =&\left|(p-q)t\sum_{k=0}^{\infty}\frac{q^k}{p^{k+1}}(t-q\frac{q^k}{p^{k+1}}t)_{p,q}^{\alpha-1}f(p\frac{q^k}{p^{k+1}}t,u_0(p\frac{q^k}{p^{k+1}}t))\right|\\ \leq&(p-q)t^{\alpha}\sum_{k=0}^{\infty}\left|\frac{q^k}{p^{k+1}}(1-(\frac{q}{p})^{k+1})_{p,q}^{\alpha-1}f(\frac{q^k}{p^k}t,u_0(\frac{q^k}{p^k}t))\right|.\end{aligned}\tag{3.4}$$

Since $M=\sup_{(t,u)\in[0,1]\times\mathbb{R}}|f(t,u)|$, $\sum_{k=0}^{\infty}\left|\frac{q^k}{p^{k+1}}(1-(\frac{q}{p})^{k+1})_{p,q}^{\alpha-1}\right|$ is convergent. Hence,

$$\sum_{k=0}^{\infty}\left|\frac{q^k}{p^{k+1}}(1-(\frac{q}{p})^{k+1})_{p,q}^{\alpha-1}f(\frac{q^k}{p^k}t,u_0(\frac{q^k}{p^k}t))\right|$$

is also convergent.

Thus,

$$\begin{aligned}&\left|\int_0^t (t-qs)_{p,q}^{(\alpha-1)}f(ps,u_0(ps))d_{p,q}s\right|\\ \leq&(p-q)t^{\alpha}\sum_{k=0}^{\infty}\left|\frac{q^k}{p^{k+1}}(1-(\frac{q}{p})^{k+1})_{p,q}^{\alpha-1}f(\frac{q^k}{p^k}t,u_0(\frac{q^k}{p^k}t))\right|\\ =&\int_0^t\left|(t-qs)_{p,q}^{(\alpha-1)}f(ps,u_0(ps))\right|d_{p,q}s.\end{aligned}\tag{3.5}$$

Then, from (H_1) and Lemma 2.5, we get

$$\begin{aligned}|u_1(t)-u_0(t)| \leq&\frac{1}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)}\int_0^t\left|(t-qs)_{p,q}^{(\alpha-1)}f(ps,u_0(ps))\right|d_{p,q}s\\ \leq&\frac{M}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)}\int_0^t (t-qs)_{p,q}^{(\alpha-1)}d_{p,q}s\\ =&\frac{Mt^{\alpha}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)}\int_0^1 (1-q\tau)_{p,q}^{(\alpha-1)}d_{p,q}\tau\\ =&\frac{MB_{p,q}(1,\alpha)}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)}t^{\alpha}\leq\frac{M}{\Gamma_{p,q}(\alpha+1)}.\end{aligned}\tag{3.6}$$

Now, we shall obtain an estimate for $u_n(t)-u_{n-1}(t)$, $n\geq 2$,

$$\begin{aligned}|u_n(t)-u_{n-1}(t)|=&\frac{1}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)}\bigg|\int_0^t (t-qs)_{p,q}^{(\alpha-1)}f(ps,u_{n-1}(ps))d_{p,q}s\\&-\int_0^t (t-qs)_{p,q}^{(\alpha-1)}f(ps,u_{n-2}(ps))d_{p,q}s\bigg|.\end{aligned}$$

By the above analysis and condition (H_2), we have

$$|u_n(t)-u_{n-1}(t)|\leq\frac{L}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)}\int_0^t (t-qs)_{p,q}^{(\alpha-1)}|u_{n-1}(ps)-u_{n-2}(ps)|d_{p,q}s.$$

Taking $n=2$, then using (3.6) and Lemma 2.5, we have

$$\begin{aligned}|u_2(t)-u_1(t)|\leq&\frac{L}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)}\int_0^t (t-qs)_{p,q}^{(\alpha-1)}|u_1(ps)-u_0(ps)|d_{p,q}s\\\leq&\frac{L}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)}\int_0^t (t-qs)_{p,q}^{(\alpha-1)}\frac{M}{\Gamma_{p,q}(\alpha+1)}d_{p,q}s\\=&\frac{LMB_{p,q}(1,\alpha)}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)\Gamma_{p,q}(\alpha+1)}t^{\alpha}\\\leq&\frac{LM}{(\Gamma_{p,q}(\alpha+1))^2}.\end{aligned}$$

Then,

$$|u_2-u_1|\leq\frac{LM}{(\Gamma_{p,q}(\alpha+1))^2}. \tag{3.7}$$

Taking $n=3$, then using (3.7) and Lemma 2.5, we have

$$\begin{aligned}|u_3-u_2|\leq&\frac{L}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)}\int_0^t (t-qs)_{p,q}^{(\alpha-1)}|u_2(ps)-u_1(ps)|d_{p,q}s\\\leq&\frac{L}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)}\int_0^t (t-qs)_{p,q}^{(\alpha-1)}\frac{LM}{(\Gamma_{p,q}(\alpha+1))^2}d_{p,q}s\\=&\frac{L^2MB_{p,q}(1,\alpha)}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)(\Gamma_{p,q}(\alpha+1))^2}t^{\alpha}\\\leq&\frac{L^2M}{(\Gamma_{p,q}(\alpha+1))^3}.\end{aligned}$$

Repeating this technique, we obtain the general estimate for the terms of the series

$$|u_n-u_{n-1}|\leq\frac{L^{n-1}M}{(\Gamma_{p,q}(\alpha+1))^n}.$$

Taking into account

$$L<\Gamma_{p,q}(\alpha+1),$$

we can choose an enough large n, such that

$$\frac{L^{n-1}M}{(\Gamma_{p,q}(\alpha+1))^n}<\frac{1}{2},$$

which means that the uniform convergence of

$$\sum_{n=1}^{\infty}(u_n(t)-u_{n-1}(t))$$

is proved. Therefore, the sequence $u_n(t)$ is uniformly convergent.

Since $f\in C([0,1]\times\mathbb{R}^+)$, then

$$\begin{aligned}u(t)=&\lim_{n\to\infty}\frac{1}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)}\int_0^t(t-qs)_{p,q}^{(\alpha-1)}f(ps,u_n(ps))d_{p,q}s\\ =&\frac{1}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)}\int_0^t(t-qs)_{p,q}^{(\alpha-1)}f(ps,u(ps))d_{p,q}s.\end{aligned}$$

Thus, the existence of a solution is proved.

To prove the uniqueness, let $v(t)$ be a continuous solution of (1.1). Then,

$$v(t)=\frac{\eta}{\Delta_{p,q}}t^{\alpha-1}+\frac{1}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)}\int_0^t(t-qs)_{p,q}^{(\alpha-1)}f(ps,v(ps))d_{p,q}s$$

and

$$\begin{aligned}|v(t)-u_n(t)|=&\frac{1}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)}\bigg|\int_0^t(t-qs)_{p,q}^{(\alpha-1)}f(ps,v(ps))d_{p,q}s\\ &-\int_0^t(t-qs)_{p,q}^{(\alpha-1)}f(ps,u_{n-1}(ps))d_{p,q}s\bigg|.\end{aligned}$$

From (3.4)-(3.5) and (H_2), we have

$$\begin{aligned}|v(t)-u_n(t)|\le&\frac{1}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)}\int_0^t\bigg|(t-qs)_{p,q}^{(\alpha-1)}f(ps,v(ps))d_{p,q}s\\ &-\int_0^t(t-qs)_{p,q}^{(\alpha-1)}f(ps,u_{n-1}(ps))\bigg|d_{p,q}s\\ \le&\frac{L}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)}\int_0^t(t-qs)_{p,q}^{(\alpha-1)}|v(ps)-u_{n-1}(ps)|d_{p,q}s.\end{aligned}\tag{3.8}$$

However,

$$|v(t)-u_0(t)|\le\frac{M}{\Gamma_{p,q}(\alpha+1)},$$

and using (3.8), we can obtain

$$|v(t)-u_n(t)|\le\frac{L^{n-1}M}{(\Gamma_{p,q}(\alpha+1))^n}.$$

Therefore,

$$\lim_{n\to\infty}u_n(t)=v(t)=u(t).$$

□

3.2. Continuous dependence

In this part, first, we study the continuous dependence of the solution of Cauchy-type problem for fractional $(p, q)-$difference equation using $(p, q)-$Gronwall inequality as a handy tool. To present the dependence of the solution in its order, let us consider solutions to two IVPs with the neighbouring orders. We need the following lemma.

Lemma 3.4. *(The (p, q)-Gronwall inequality) Suppose that $a(t) \geq 0$, and $\lambda \geq 0$. Assume that function $u : [0, \infty) \to \mathbb{R}^+$ is continuous and satisfies*

$$u(t) \leq a(t) + \frac{\lambda}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} \int_0^t (t - qs)_{p,q}^{(\alpha-1)} u(ps) d_{p,q}s.$$

Then, the following inequality

$$u(t) \leq E_{p,q}(\lambda, t) \sup_{0 \leq s \leq t} a(s), \quad t \in [0, \infty)$$

holds, where $E_{p,q}(\lambda, t) = \sum_{k=0}^{\infty} \frac{\lambda^k t^{\alpha k}}{\Gamma_{p,q}(k\alpha+1)}$.

Theorem 3.2. *Let $\alpha > 0$, $\varepsilon > 0$ such that $0 < \alpha - \varepsilon < \alpha < 1$. Let f be a continuous function satisfying condition (H_2). For $0 \leq t \leq h < 1$, assume that u and v are the solutions of the IVP (1.1) and*

$$\begin{cases} D_{p,q}^{\alpha-\varepsilon} v(t) = f(p^\alpha t, v(p^\alpha t)), \quad t \in (0, 1], \\ I_{p,q}^{1-\alpha+\varepsilon} v(t)|_{t=0} = \bar{\eta}, \end{cases} \tag{3.9}$$

respectively. Then, for $0 < t \leq h$, the following inequality

$$|v(t) - u(t)| \leq E_{p,q}(L, t) \sup_{0 \leq s \leq t \leq h} a(s)$$

holds, where

$$\begin{aligned} a(t) =& \left| \frac{\bar{\eta}}{\bar{\Delta}_{p,q}} t^{\alpha-\varepsilon-1} - \frac{\eta}{\Delta_{p,q}} t^{\alpha-1} \right| + \|f\| \left| \frac{t^{\alpha-\varepsilon}}{\Gamma_{p,q}(\alpha - \varepsilon + 1)} - \frac{t^{\alpha-\varepsilon} B(1, \alpha - \varepsilon)}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} \right| \\ &+ \|f\| \left| \frac{t^{\alpha-\varepsilon} B(1, \alpha - \varepsilon)}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} - \frac{t^\alpha}{\Gamma_{p,q}(\alpha + 1)} \right|, \\ E_{p,q}(L, t) =& \sum_{k=0}^{\infty} \frac{L^k t^{(\alpha-\varepsilon)k}}{\Gamma_{p,q}(k(\alpha - \varepsilon) + 1)}. \end{aligned} \tag{3.10}$$

Proof. The solutions to the IVPs (1.1) and (3.9) are

$$u(t) = \frac{\eta}{\Delta_{p,q}} t^{\alpha-1} + \frac{1}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} \int_0^t (t - qs)_{p,q}^{(\alpha-1)} f(ps, u(ps)) d_{p,q}s$$

and

$$v(t) = \frac{\bar{\eta}}{\bar{\Delta}_{p,q}} t^{\alpha-\varepsilon-1} + \frac{1}{p^{\binom{\alpha-\varepsilon}{2}}\Gamma_{p,q}(\alpha - \varepsilon)} \int_0^t (t - qs)_{p,q}^{(\alpha-\varepsilon-1)} f(ps, v(ps)) d_{p,q}s$$

respectively, where

$$\bar{\Delta}_{p,q} = \frac{p^{-(\alpha-\varepsilon-1)^2}}{p^{\binom{1-\alpha+\varepsilon}{2}}}\Gamma_{p,q}(\alpha-\varepsilon).$$

From (3.4)-(3.5) and (H_2), we have

$$\begin{aligned}
|v(t)-u(t)| =& \Bigg| \frac{\bar{\eta}}{\bar{\Delta}_{p,q}} t^{\alpha-\varepsilon-1} - \frac{\eta}{\Delta_{p,q}} t^{\alpha-1} \\
&+ \frac{1}{p^{\binom{\alpha-\varepsilon}{2}}\Gamma_{p,q}(\alpha-\varepsilon)} \int_0^t (t-qs)_{p,q}^{(\alpha-\varepsilon-1)} f(ps, v(ps)) d_{p,q}s \\
&- \frac{1}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} \int_0^t (t-qs)_{p,q}^{(\alpha-1)} f(ps, u(ps)) d_{p,q}s \Bigg| \\
\le& \Bigg| \frac{\bar{\eta}}{\bar{\Delta}_{p,q}} t^{\alpha-\varepsilon-1} - \frac{\eta}{\Delta_{p,q}} t^{\alpha-1} \Bigg| \\
&+ \Bigg| \int_0^t \left[\frac{(t-qs)_{p,q}^{(\alpha-\varepsilon-1)}}{p^{\binom{\alpha-\varepsilon}{2}}\Gamma_{p,q}(\alpha-\varepsilon)} - \frac{(t-qs)_{p,q}^{(\alpha-\varepsilon-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} \right] f(ps, v(ps)) d_{p,q}s \Bigg| \\
&+ \Bigg| \int_0^t \frac{(t-qs)_{p,q}^{(\alpha-\varepsilon-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} |f(ps, v(ps)) - f(ps, u(ps))| d_{p,q}s \Bigg| \\
&+ \Bigg| \int_0^t \frac{(t-qs)_{p,q}^{(\alpha-\varepsilon-1)} - (t-qs)_{p,q}^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} f(ps, u(ps)) d_{p,q}s \Bigg| \\
\le& \Bigg| \frac{\bar{\eta}}{\bar{\Delta}_{p,q}} t^{\alpha-\varepsilon-1} - \frac{\eta}{\Delta_{p,q}} t^{\alpha-1} \Bigg| \\
&+ \|f\| \Bigg| \int_0^t \left[\frac{(t-qs)_{p,q}^{(\alpha-\varepsilon-1)}}{p^{\binom{\alpha-\varepsilon}{2}}\Gamma_{p,q}(\alpha-\varepsilon)} - \frac{(t-qs)_{p,q}^{(\alpha-\varepsilon-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} \right] d_{p,q}s \Bigg| \\
&+ \int_0^t \frac{(t-qs)_{p,q}^{(\alpha-\varepsilon-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} |f(ps, v(ps)) - f(ps, u(ps))| d_{p,q}s \\
&+ \|f\| \Bigg| \int_0^t \frac{(t-qs)_{p,q}^{(\alpha-\varepsilon-1)} - (t-qs)_{p,q}^{(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} d_{p,q}s \Bigg| \\
\le& a(t) + \frac{L}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} \int_0^t (t-qs)_{p,q}^{(\alpha-\varepsilon-1)} |v(ps) - u(ps)| d_{p,q}s,
\end{aligned}$$

where

$$\begin{aligned}
a(t) =& \Bigg| \frac{\bar{\eta}}{\bar{\Delta}_{p,q}} t^{\alpha-\varepsilon-1} - \frac{\eta}{\Delta_{p,q}} t^{\alpha-1} \Bigg| + \|f\| \Bigg| \frac{t^{\alpha-\varepsilon}}{\Gamma_{p,q}(\alpha-\varepsilon+1)} - \frac{t^{\alpha-\varepsilon}B(1,\alpha-\varepsilon)}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} \Bigg| \\
&+ \|f\| \Bigg| \frac{t^{\alpha-\varepsilon}B(1,\alpha-\varepsilon)}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} - \frac{t^{\alpha}}{\Gamma_{p,q}(\alpha+1)} \Bigg|.
\end{aligned}$$

An application of Lemma 3.4 yields

$$|v(t)-u(t)| \le E_{p,q}(L,t) \sup_{0\le s\le t\le h} a(s),$$

where $a(t)$ and $E_{p,q}(L,t)$ are defined by (3.10). $\square$

Next, we discuss the fractional $(p,q)-$difference equation (1.1) with the small change in the initial condition, i.e.,

$$I_{p,q}^{1-\alpha}u(t)|_{t=0} = \eta + \varepsilon, \tag{3.11}$$

where ε is an arbitrary constant. Now, we state and prove the result as follows.

Theorem 3.3. *Assume that $0 < \alpha < 1$ and ε is an arbitrary constant. Let f be a continuous function satisfying condition (H2). Suppose that $u(t)$ and $v(t)$ are the solutions of the IVP (1.1) and*

$$\begin{cases} D_{p,q}^{\alpha}v(t) = f(p^{\alpha}t, v(p^{\alpha}t)), \quad t \in (0,1], \\ I_{p,q}^{1-\alpha}v(t)|_{t=0} = \eta + \varepsilon, \end{cases} \tag{3.12}$$

respectively. Then,

$$|v(t) - u(t)| \leq \frac{|\varepsilon|}{\Delta_{p,q}} E_{p,q}(L,t). \tag{3.13}$$

Proof. The solutions of the IVPs (1.1) and (3.12) are

$$u(t) = \frac{\eta}{\Delta_{p,q}} t^{\alpha-1} + \frac{1}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} \int_0^t (t-qs)_{p,q}^{(\alpha-1)} f(ps, u(ps)) d_{p,q}s \tag{3.14}$$

and

$$v(t) = \frac{\eta+\varepsilon}{\Delta_{p,q}} t^{\alpha-1} + \frac{1}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} \int_0^t (t-qs)_{p,q}^{(\alpha-1)} f(ps, v(ps)) d_{p,q}s \tag{3.15}$$

respectively. Then, (3.14) minus (3.15),

$$\begin{aligned} &|v(t) - u(t)| \\ =&\left| \frac{\varepsilon}{\Delta_{p,q}} t^{\alpha-1} + \frac{1}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} \int_0^t (t-qs)_{p,q}^{(\alpha-1)} (f(ps, v(ps)) - f(ps, u(ps))) d_{p,q}s \right| \\ \leq& \frac{|\varepsilon|}{\Delta_{p,q}} t^{\alpha-1} + \frac{1}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} \int_0^t (t-qs)_{p,q}^{(\alpha-1)} \Big| (f(ps, v(ps)) - f(ps, u(ps))) \Big| d_{p,q}s. \end{aligned}$$

Notice that the above inequality can be easily proved by a similar method to (3.4)-(3.5). Thus, by condition (H2) and Lemma 3.4, we have

$$\begin{aligned} |v(t) - u(t)| \leq& \frac{|\varepsilon|}{\Delta_{p,q}} t^{\alpha-1} + \frac{L}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} \int_0^t (t-qs)_{p,q}^{(\alpha-1)} \Big| v(ps) - u(ps) \Big| d_{p,q}s \\ \leq& \frac{|\varepsilon|}{\Delta_{p,q}} E_{p,q}(L,t). \end{aligned}$$

$\square$

Remark 3.1. From (3.13), we know $u(\tau) = v(\tau) = 0$, as $\varepsilon \to 0$. Thus, we can see that the solution of (1.1) is unique with respect to the initial value under condition (H_2).

Remark 3.2. We can acquire from Theorem 3.2 and Theorem 3.3 that a small change in order the and initial condition (3.11) will cause a change in the solution on $[k,1]$ for the k between 0 to 1, which does not contain initial point 0.

Acknowledgements

The author is very grateful to the anonymous reviewers and editors for their careful reading and valuable suggestions, which have notably improved the quality of this paper.

References

[1] W. H. Abdi, *Certain inversion and representation formulae for q-Laplace transforms*, Mathematische Zeitschrift, 1964, 83(3), 238–249.

[2] G. Bangerezako, *Variational q-calculus*, Journal of Mathematical Analysis and Applications, 2004, 289(2), 650–665.

[3] R. D. Carmichael, *The General theory of linear q-difference equations*, American Journal of Mathematics, 1912, 34(2), 147–168.

[4] R. Chakrabarti and R. Jagannathan, *A (p, q)-oscillator realization of two-parameter quantum algebras*, Journal of Physics A: Mathematical and General, 1991, 24(13), 5683–5701.

[5] T. Dumrongpokaphan, S. K. Ntouyas and T. Sitthiwirattham, *Separate fractional (p, q)-integrodifference equations via nonlocal fractional (p, q)-integral boundary conditions*, Symmetry, 2021, 13(11), 2212, 15 pages.

[6] T. Ernst, *The History of q-Calculus and A New Method*, Department of Mathematics, Uppsala University, 2000.

[7] R. Finkelstein and E. Marcus, *Transformation theory of the q-oscillator*, Journal of Mathematical Physics, 1995, 36(6), 2652–2672.

[8] F. H. Jackson, *On q-difference equations*, American Jorunal of Mathematics, 1910, 32(4), 305–314.

[9] V. Kac and P. Cheung, *Quantum Calculus*, Springer Science and Business Media, New York, 2001.

[10] N. Kamsrisuk, C. Promsakon, S. K. Ntouyas and J. Tariboon, *Nonlocal boundary value problems for (p, q)-difference equations*, Differential Equations & Applications, 2018, 10(2), 183–195.

[11] M. Mursaleen, K. J. Ansari and A. Khan, *Some approximation results by (p, q)-analogue of Bernstein–Stancu operators*, Applied Mathematics and Computation, 2015, 264, 392–402.

[12] P. Neang, K. Nonlaopon and J. Tariboon, *Existence and uniqueness results for fractional (p, q)-difference equations with separated boundary conditions*, Mathematics, 2022, 10(5), 767–782.

[13] C. Promsakon, N. Kamsrisuk, S. K. Ntouyas and J. Tariboon, *On the second-order quantum (p, q)-Difference equations with separated boundary conditions*, Advances in Mathematical Physics, 2018, 92, Article ID 9089865, 9 pages.

[14] Z. Qin and S. Sun, *On a nonlinear fractional (p, q)-difference Schrödinger equation*, Journal of Applied Mathematics Computing, 2021, 68(3), 1685–1698.

[15] Z. Qin and S. Sun, *Positive solutions for fractional (p, q)-difference boundary value problems*, Journal of Applied Mathematics Computing, 2021, 68(4), 2571–2588.

[16] Z. Qin and S. Sun, *Solvability and stability for singular fractional (p, q)-difference equations*, Journal of Nonlinear Modeling and Analysis, 2021, 3(4), 647–661.

[17] B. Ross and I. Petrá, *Fractional Calculus and Its Applications*, John Wiley & Sons Limited, London, 2013.

[18] B. Ross and B. K. Sachdeva, *The solution of certain integral equations by means of operators of arbitrary order*, The American Mathematical Monthly, 1990, 97(6), 498–503.

[19] J. Sabatier, O. P. Agrawal and J. A. Tenreiro Machado, *Advances in Fractional Calculus: Theoretical Developments and Applications in Physics and Engineering*, Springer, London, 2007.

[20] P. N. Sadjang, *On the (p, q)-gamma and the (p, q)-beta functions*, arXiv, 2015. https://doi.org/10.48550/arXiv.1506.07394

[21] P. N. Sadjang, *On the fundamental theorem of (p, q)-calculus and some (p, q)-Taylor formulas*, Results in Mathematics, 2018, 73, 39, 21 pages.

[22] J. Soontharanon and T. Sitthiwirattham, *On fractional (p, q)-calculus*, Advances in Difference Equations, 2020, 35, 18 pages.

[23] J. Soontharanon and T. Sitthiwirattham, *Existence results of nonlocal Robin boundary value problems for fractional (p, q)-integrodifference equations*, Advances in Difference Equations, 2020, 342, 17 pages.

[24] J. Soontharanon and T. Sitthiwirattham, *On sequential fractional Caputo (p, q)-integrodifference equations via three-point fractional Riemann-Liouville (p, q)-difference boundary condition*, AIMS Mathematics, 2021, 7(1), 704–722.

[25] J. Soontharanon and T. Sitthiwirattham, *On periodic fractional (p, q)-integral boundary value problems for sequential fractional (p, q)-integrodifference equations*, Axioms, 2021, 10(4), 264, 16 pages.