Journal of Nonlinear Modeling and Analysis http://jnma.ca; http://jnma.ijournal.cn
Volume 5, Number 3, September 2023, 540–548 DOI:10.12150/jnma.2023.540

Positive Periodic Solutions of Functional Difference Equations and Applications in Population Dynamics

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Abstract In this work, we discuss the existence of positive periodic solutions for nonlinear functional difference equations. We illustrate the applicability of our main result by examining the Lasota–Wazewska, the Mackey–Glass and the Nicholson's Blowflies models.

Keywords Nonlinear functional difference equation, positive periodic solution, fixed point, existence, population growth model

MSC(2010) 39A10.

1. Introduction

In this article, we investigate the following first-order nonlinear functional difference equation

$$u(t+1) = -p(t)u(t) + q(t)f\left(u\left(\tau(t)\right)\right), \quad t \in \mathbb{N}_{t_0}, \tag{1.1}$$

where $\mathbb{N}_{t_0} = \{t_0, t_0+1, t_0+2, \cdots\}$, p, $q : \mathbb{N}_{t_0} \to (0,\infty)$, $\tau : \mathbb{N}_{t_0} \to \mathbb{N}_1$, $f : \mathbb{R} \to \mathbb{R}$ are continuous functions, $f(v) > 0$ for $v > 0$, $\tau(t) < t$ and $\tau(t) \to \infty$ as $t \to \infty$. We establish conditions under which (1.1) has a positive ω–periodic solution.

We choose sufficiently large $T \in \mathbb{N}_{t_0}$ such that $\tau(t) \in \mathbb{N}_{t_0}$ for $t \in \mathbb{N}_T$. Denote $\mathbb{N}_{t_0}^T = \{t_0, t_0+1, t_0+2, \cdots, T-1, T\}$. Let $\psi : \mathbb{N}_{t_0}^T \to \mathbb{R}$ be an initial bounded function. We say that $u(t) = u(t, T, \psi)$ is a solution of (1.1), if $u(t) = \psi(t)$ on $\mathbb{N}_{t_0}^T$, and satisfies (1.1) for $t \in \mathbb{N}_T$. Without loss of generality, here we take $\psi(t) \equiv 1$.

It is well-known that (1.1) includes many mathematical, ecological and biological models such as:

1. The Lasota–Wazewska model:

$$u(t+1) = -au(t) + be^{-cu(t-d)}, \quad t \in \mathbb{N}_{t_0}. \tag{1.2}$$

Here, $u(t)$ denotes the number of red blood cells at time t, $a > 0$ is the probability of the death of a red blood cell, b and c are positive constants related to the production of red blood cells per unit time, and $d > 0$ is the time required to produce a red blood cell.

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2. The Mackey–Glass model:

$$u(t+1) = -au(t) + b\frac{1}{1+[u\,(t-d)]^n}, \quad n > 0, \quad t \in \mathbb{N}_{t_0}. \tag{1.3}$$

$$u(t+1) = -au(t) + b\frac{u\,(t-d)}{1+[u\,(t-d)]^n}, \quad n > 0, \quad t \in \mathbb{N}_{t_0}. \tag{1.4}$$

These are appropriate models for the dynamics of hematopoiesis, which describe the process of the production of blood cells. Here, $u(t)$ denotes the density of mature cells in blood circulation at time t, and $d > 0$ is the time delay between the production of immature cells in the bone marrow and their maturation for the release in the circulating bloodstream. It is assumed that the cells are lost from the circulation at a rate $a > 0$, b is a positive constant, and the flux of the cells into the circulation from the stem cell compartment depends on the density of mature cells at the previous time $t - d$.

3. The Nicholson's Blowflies model:

$$u(t+1) = -au(t) + bu\,(t-d)\,e^{-cu(t-d)}, \quad t \in \mathbb{N}_{t_0}. \tag{1.5}$$

Here, $u(t)$ denotes the size of the population of the Australian sheep blowfly at time t, $b > 0$ is the maximum daily egg production per capita, $\frac{1}{c} > 0$ is the size at which the blowfly population reproduces at its maximum rate, $a > 0$ is the daily adult death rate per capita, and $d > 0$ is the generation time.

The problem of the existence of positive periodic solutions for functional difference equations has generated substantial curiosity in the past two decades. This is due to the fact that such equations have been proposed as models for a variety of real world problems. One important problem related to these models is whether they can support positive periodic solutions. Such a problem has been studied extensively to a greater extent by a number of authors. For example, we refer the readers to [1, 3, 5, 7–12] and the references therein.

In this article, we obtain sufficient conditions on the existence of positive ω–periodic solutions of (1.1). The existence results for these types of equations in the literature are largely based on the assumption that the functions p, q are ω–periodic. An interesting problem which we discuss in this work is to know whether there exists a positive periodic solution of (1.1), when the above conditions are not met.

2. Preliminaries

We shall use the following notations, definitions, and the known results of discrete calculus [2]. Throughout the article, the empty sums and products are taken to be 0 and 1 respectively.

Definition 2.1. [2] Let $u : \mathbb{N}_a \to \mathbb{R}$. The first-order forward (delta) difference of u is defined by

$$\big(\Delta u\big)(t) = u(t+1) - u(t), \quad t \in \mathbb{N}_a.$$

Definition 2.2. [4] The space X is the set of real-valued functions defined on $\mathbb{N}_{t_0}$ where random individual function is bounded with respect to the usual supremum norm

$$\|u\| = \sup_{t\in\mathbb{N}_{t_0}} |u(t)|.$$

It is well-known that under the supremum norm X is a Banach space.

We use the following fixed point theorem to prove the main result in the next section.

Theorem 2.1 ([4])**.** *If Ω is a closed, bounded, and convex subset of a Banach space X, and the mapping $S : \Omega \to \Omega$ is completely continuous, then S has a fixed point in Ω.*

3. Main result

In this section, we establish sufficient conditions on the existence of positive ω–periodic solutions of (1.1).

Lemma 3.1. *Suppose that there exists a function $r : \mathbb{N}_T \to \mathbb{R}^+$ such that*

$$\prod_{s=t}^{t+\omega-1} [-p(s) + q(s)r(s)] = 1, \quad t \in \mathbb{N}_T, \tag{3.1}$$

then the function

$$g(t) = \prod_{s=T}^{t-1} [-p(s) + q(s)r(s)], \quad t \in \mathbb{N}_T$$

is ω–periodic.

Proof. For $t \in \mathbb{N}_T$, consider that

$$\begin{aligned} g(t+\omega) &= \prod_{s=T}^{t+\omega-1} [-p(s) + q(s)r(s)] \\ &= \left[\prod_{s=T}^{t-1} [-p(s) + q(s)r(s)]\right] \left[\prod_{s=t}^{t+\omega-1} [-p(s) + q(s)r(s)]\right] \\ &= \prod_{s=T}^{t-1} [-p(s) + q(s)r(s)] \\ &= g(t), \end{aligned}$$

implying that g is ω–periodic. The proof is completed. □

Theorem 3.1. *Suppose that there exists a function $r : \mathbb{N}_T \to \mathbb{R}^+$ such that* (3.1) *holds and*

$$f\left(\prod_{s=T}^{\tau(t)-1} [-p(s) + q(s)r(s)]\right) \prod_{s=T}^{t-1} \left[\frac{1}{-p(s) + q(s)r(s)}\right] = r(t), \quad \tau(t) \in \mathbb{N}_T. \tag{3.2}$$

Then (1.1) *has a positive ω–periodic solution.*

Proof. With respect to Lemma 3.1, we define

$$M = \sup_{t\in\mathbb{N}_T} g(t), \quad m = \inf_{t\in\mathbb{N}_T} g(t). \tag{3.3}$$

We now define a closed, bounded and convex subset Ω of X as follows:

$$\begin{aligned}\Omega = \{u \in X : u(t+\omega) = u(t), \quad t\in\mathbb{N}_T; \qquad m \le u(t) \le M, \quad t \in \mathbb{N}_T;\\ r(t)u(t) = f\left(u\left(\tau(t)\right)\right), \quad t\in\mathbb{N}_T; \qquad u(t) = 1, \quad t\in\mathbb{N}_{t_0}^T\}.\end{aligned} \tag{3.4}$$

Define the operator $S : \Omega \to X$ as follows:

$$(Su)(t) = \begin{cases} \displaystyle\prod_{s=T}^{t-1}\left[-p(s) + q(s)\frac{f\left(u\left(\tau(s)\right)\right)}{u(s)}\right], \quad t\in\mathbb{N}_T,\\ 1, \quad t\in\mathbb{N}_{t_0}^T. \end{cases} \tag{3.5}$$

Clearly, S is continuous. First, we show that $S(\Omega)$ is contained in Ω. Take $u \in \Omega$. For $t\in\mathbb{N}_T$, we have

$$\begin{aligned}(Su)(t) &= \prod_{s=T}^{t-1}\left[-p(s) + q(s)\frac{f\left(u\left(\tau(s)\right)\right)}{u(s)}\right]\\ &= \prod_{s=T}^{t-1}\left[-p(s) + q(s)r(s)\right]\\ &\le M\end{aligned}$$

and

$$\begin{aligned}(Su)(t) &= \prod_{s=T}^{t-1}\left[-p(s) + q(s)\frac{f\left(u\left(\tau(s)\right)\right)}{u(s)}\right]\\ &= \prod_{s=T}^{t-1}\left[-p(s) + q(s)r(s)\right]\\ &\ge m,\end{aligned}$$

implying

$$m \le (Su)(t) \le M, \quad t\in\mathbb{N}_T. \tag{3.6}$$

It follows from the definition of S that

$$(Su)(t) = 1, \quad t\in\mathbb{N}_{t_0}^{T-1}. \tag{3.7}$$

Also, for $t\in\mathbb{N}_T$, we have

$$\begin{aligned}&f\Big((Su)(\tau(t))\Big)\\ =&f\left(\prod_{s=T}^{\tau(t)-1}\left[-p(s) + q(s)\frac{f\left(u\left(\tau(s)\right)\right)}{u(s)}\right]\right)\\ =&f\left(\prod_{s=T}^{\tau(t)-1}\left[-p(s) + q(s)r(s)\right]\right)\end{aligned}$$

$$
\begin{aligned}
&= f\left(\prod_{s=T}^{\tau(t)-1}[-p(s)+q(s)r(s)]\right)\prod_{s=T}^{t-1}\left[\frac{1}{-p(s)+q(s)r(s)}\right]\prod_{s=T}^{t-1}[-p(s)+q(s)r(s)]\\
&= r(t)\prod_{s=T}^{t-1}\left[-p(s)+q(s)\frac{f\left(u\left(\tau(s)\right)\right)}{u(s)}\right]\\
&= r(t)(Su)(t).
\end{aligned}
\tag{3.8}
$$

Further, for $t \in \mathbb{N}_T$, we have

$$
\begin{aligned}
(Su)(t+\omega) &= \prod_{s=T}^{t+\omega-1}[-p(s)+q(s)r(s)]\\
&= \left[\prod_{s=T}^{t-1}[-p(s)+q(s)r(s)]\right]\left[\prod_{s=t}^{t+\omega-1}[-p(s)+q(s)r(s)]\right]\\
&= \prod_{s=T}^{t-1}[-p(s)+q(s)r(s)]\\
&= \prod_{s=T}^{t-1}\left[-p(s)+q(s)\frac{f\left(u\left(\tau(s)\right)\right)}{u(s)}\right]\\
&= (Su)(t).
\end{aligned}
\tag{3.9}
$$

Thus, from (3.6) - (3.9), we conclude that $(Su) \in \Omega$ implying that $S(\Omega)$ is contained in Ω. Moreover, the functions belonging to $S(\Omega)$ are uniformly bounded on $\mathbb{N}_T$. Finally, we show that the functions of $S(\Omega)$ are equicontinuous. Without loss of generality, suppose that $t_1 < t_2$. With respect to (3.3), for $t \in \mathbb{N}_T$, we have

$$
\begin{aligned}
&(Su)(t_1) - (Su)(t_1+1)\\
=&\prod_{s=T}^{t_1-1}\left[-p(s)+q(s)\frac{f\left(u\left(\tau(s)\right)\right)}{u(s)}\right] - \prod_{s=T}^{t_1}\left[-p(s)+q(s)\frac{f\left(u\left(\tau(s)\right)\right)}{u(s)}\right]\\
=&\prod_{s=T}^{t_1-1}[-p(s)+q(s)r(s)] - \prod_{s=T}^{t_1}[-p(s)+q(s)r(s)],
\end{aligned}
$$

implying that

$$
\begin{aligned}
(Su)(t_1) - (Su)(t_2) &= \sum_{s=t_1}^{t_2-1}[(Su)(s) - (Su)(s+1)]\\
&= \sum_{s=t_1}^{t_2-1}\left[\prod_{\xi=T}^{s-1}[-p(\xi)+q(\xi)r(\xi)] - \prod_{\xi=T}^{s}[-p(\xi)+q(\xi)r(\xi)]\right]\\
&\le \sum_{s=t_1}^{t_2-1}(M-m)\\
&= (M-m)(t_2-t_1).
\end{aligned}
$$

Also, for $t \in \mathbb{N}_{t_0}^T$, we have $(Su)(t_1) - (Su)(t_2) = 0$. This shows that the functions of $S(\Omega)$ are equicontinuous. Hence, S is completely continuous. Therefore, by

Theorem 2.1, there exists a $u_0 \in \Omega$ such that $Su_0 = u_0$. That is, (1.1) has a positive ω–periodic solution. The proof is completed. $\square$

Now, consider the following first-order functional difference equation

$$u(t+1) = -p(t)u(t) + q(t)u\left(\tau(t)\right), \quad t \in \mathbb{N}_{t_0}, \tag{3.10}$$

where $\mathbb{N}_{t_0} = \{t_0, t_0+1, t_0+2, \cdots\}$, p, $q : \mathbb{N}_{t_0} \to (0,\infty)$, $\tau : \mathbb{N}_{t_0} \to \mathbb{N}_1$, $\tau(t) < t$ and $\tau(t) \to \infty$ as $t \to \infty$. We establish conditions under which (3.10) has a positive ω–periodic solution.

We choose sufficiently large $T \in \mathbb{N}_{t_0}$ such that $\tau(t) \in \mathbb{N}_{t_0}$ for $t \in \mathbb{N}_T$. Denote $\mathbb{N}_{t_0}^T = \{t_0, t_0+1, t_0+2, \cdots, T-1, T\}$. Let $\psi : \mathbb{N}_{t_0}^T \to \mathbb{R}$ be an initial bounded function. We say that $u(t) = u(t, T, \psi)$ is a solution of (3.10), if $u(t) = \psi(t)$ on $\mathbb{N}_{t_0}^T$, and satisfies (3.10) for $t \in \mathbb{N}_T$. Without loss of generality, here we take $\psi(t) \equiv 1$.

Theorem 3.2. *Suppose that there exists a function $r : \mathbb{N}_T \to \mathbb{R}^+$ such that* (3.1) *holds and*

$$\prod_{s=\tau(t)}^{t-1} \left[\frac{1}{-p(s)+q(s)r(s)}\right] = r(t), \quad \tau(t) \in \mathbb{N}_T. \tag{3.11}$$

Then (3.10) *has a positive ω–periodic solution.*

Proof. The proof is similar to the proof of Theorem 3.1. So, we omit it. $\square$

4. Examples

In this section, we provide two examples to demonstrate the applicability of our main results.

Example 4.1. Consider the first order functional difference equation

$$u(t+1) = -(1.5)u(t) + (\cos \pi t + 1.5)u\left(t-4\right), \quad t \in \mathbb{N}_0. \tag{4.1}$$

Here $p(t) = (1.5)$, $q(t) = (\cos \pi t + 1.5)$ and $\tau(t) = t-4$ for $t \in \mathbb{N}_0$. We choose $T = 4$ such that $\tau(t) \in \mathbb{N}_0$ for $t \in \mathbb{N}_T$. Take $r(t) = 1$ for $t \in \mathbb{N}_T$. For $t \in \mathbb{N}_T$ and $\omega = 4$, we have

$$\begin{aligned}\prod_{s=t}^{t+\omega-1} [-p(s)+q(s)r(s)] &= \prod_{s=t}^{t+4-1} [-(1.5)+(\cos \pi s + 1.5)] \\ &= \prod_{s=t}^{t+3} \cos \pi s \\ &= [\cos \pi t]\,[\cos \pi(t+1)]\,[\cos \pi(t+2)]\,[\cos \pi(t+3)] \\ &= (-1)^t(-1)^{t+1}(-1)^{t+2}(-1)^{t+3} = 1,\end{aligned}$$

implying that (3.1) holds. For $\tau(t) \in \mathbb{N}_T$ and $\omega = 4$, we have

$$\begin{aligned}\prod_{s=\tau(t)}^{t-1} \left[\frac{1}{-p(s)+q(s)r(s)}\right] &= \prod_{s=t-4}^{t-1} \frac{1}{[-(1.5)+(\cos \pi s + 1.5)]} \\ &= \prod_{s=t-4}^{t-1} \frac{1}{\cos \pi s}\end{aligned}$$

$$= \frac{1}{[\cos\pi(t-4)]\,[\cos\pi(t-3)]\,[\cos\pi(t-2)]\,[\cos\pi(t-1)]}$$
$$= \frac{1}{(-1)^{t-4}(-1)^{t-3}(-1)^{t-2}(-1)^{t-1}} = 1 = r(t),$$

implying that (3.11) holds. All conditions of Theorem 3.2 are satisfied. Thus, (4.1) has a positive 4–periodic solution

$$u(t) = \begin{cases} \prod_{s=T}^{t-1} \cos\pi s, & t \in \mathbb{N}_T, \\ 1, & t \in \mathbb{N}_0^T. \end{cases}$$

Example 4.2. Consider the functional difference equation

$$u(t+1) = -(t^2+1)u(t) + \left(e^{-2\cos\pi t} + t^2 + 1\right) u\left(t-2\right), \quad t \in \mathbb{N}_0. \tag{4.2}$$

Here $p(t) = t^2 + 1$, $q(t) = e^{-2\cos\pi t} + t^2 + 1$ and $\tau(t) = t - 2$ for $t \in \mathbb{N}_0$. We choose $T = 2$ such that $\tau(t) \in \mathbb{N}_0$ for $t \in \mathbb{N}_T$. Take $r(t) = 1$ for $t \in \mathbb{N}_T$. For $t \in \mathbb{N}_T$ and $\omega = 2$, we have

$$\begin{aligned}
\prod_{s=t}^{t+\omega-1} [-p(s) + q(s)r(s)] &= \prod_{s=t}^{t+2-1} \left[-(s^2+1) + \left(e^{-2\cos\pi s} + s^2 + 1\right)\right] \\
&= \prod_{s=t}^{t+1} e^{-2\cos\pi s} \\
&= \left(e^{-2\cos\pi t}\right)\left(e^{-2\cos\pi(t+1)}\right) \\
&= \left(e^{-2\cos\pi t}\right)\left(e^{2\cos\pi t}\right) = 1,
\end{aligned}$$

implying that (3.1) holds. For $\tau(t) \in \mathbb{N}_T$ and $\omega = 2$, we have

$$\begin{aligned}
\prod_{s=\tau(t)}^{t-1} \left[\frac{1}{-p(s) + q(s)r(s)}\right] &= \prod_{s=t-2}^{t-1} \frac{1}{[-(s^2+1) + (e^{-2\cos\pi s} + s^2 + 1)]} \\
&= \prod_{s=t-2}^{t-1} e^{2\cos\pi s} \\
&= e^{2\cos\pi(t-2)} e^{2\cos\pi(t-1)} \\
&= e^{2\cos\pi t} e^{-2\cos\pi t} \\
&= 1 = r(t),
\end{aligned}$$

implying that (3.11) holds. All conditions of Theorem 3.2 are satisfied. Thus, (4.2) has a positive 2–periodic solution

$$u(t) = \begin{cases} \prod_{s=T}^{t-1} e^{-2\cos\pi s}, & t \in \mathbb{N}_T, \\ 1, & t \in \mathbb{N}_0^T. \end{cases}$$

5. Applications

In this section, we illustrate the applicability of our main results by examining the population growth models (1.2)-(1.5).

Corollary 5.1. *Suppose that there exists a function* $r : \mathbb{N}_T \to \mathbb{R}^+$ *such that* (3.1) *and* (3.2) *hold. Then,* (1.2) *has a positive* ω*–periodic solution.*

Proof. Here $p(t) = a > 0$, $q(t) = b > 0$ and $\tau(t) = t - d < t$ for all $t \in \mathbb{N}_{t_0}$. We choose sufficiently large $T \in \mathbb{N}_{t_0}$ such that $\tau(t) \in \mathbb{N}_{t_0}$ for $t \in \mathbb{N}_T$. Also, $f(v) = e^{-cv} > 0$ for all $v > 0$ and $\tau(t) \to \infty$ as $t \to \infty$. Then, by Theorem 3.1, (1.2) has a positive ω–periodic solution

$$u(t) = \begin{cases} \displaystyle\prod_{s=T}^{t-1} [-a + br(s)], & t \in \mathbb{N}_T, \\ 1, \quad t \in \mathbb{N}_{t_0}^T. \end{cases} \tag{5.1}$$

The proof is completed. □

Corollary 5.2. *Suppose that there exists a function* $r : \mathbb{N}_T \to \mathbb{R}^+$ *such that* (3.1) *and* (3.2) *hold. Then,* (1.3) *has a positive* ω*–periodic solution.*

Proof. Here $p(t) = a > 0$, $q(t) = b > 0$ and $\tau(t) = t - d < t$ for all $t \in \mathbb{N}_{t_0}$. We choose sufficiently large $T \in \mathbb{N}_{t_0}$ such that $\tau(t) \in \mathbb{N}_{t_0}$ for $t \in \mathbb{N}_T$. Also, $f(v) = \frac{1}{1+v^n} > 0$ for all $v > 0$ and $\tau(t) \to \infty$ as $t \to \infty$. Then, by Theorem 3.1, (1.3) has a positive ω–periodic solution

$$u(t) = \begin{cases} \displaystyle\prod_{s=T}^{t-1} [-a + br(s)], & t \in \mathbb{N}_T, \\ 1, \quad t \in \mathbb{N}_{t_0}^T. \end{cases} \tag{5.2}$$

The proof is completed. □

Corollary 5.3. *Suppose that there exists a function* $r : \mathbb{N}_T \to \mathbb{R}^+$ *such that* (3.1) *and* (3.2) *hold. Then,* (1.4) *has a positive* ω*–periodic solution.*

Proof. Here $p(t) = a > 0$, $q(t) = b > 0$ and $\tau(t) = t - d < t$ for all $t \in \mathbb{N}_{t_0}$. We choose sufficiently large $T \in \mathbb{N}_{t_0}$ such that $\tau(t) \in \mathbb{N}_{t_0}$ for $t \in \mathbb{N}_T$. Also, $f(v) = \frac{v}{1+v^n} > 0$ for all $v > 0$ and $\tau(t) \to \infty$ as $t \to \infty$. Then, by Theorem 3.1, (1.4) has a positive ω–periodic solution

$$u(t) = \begin{cases} \displaystyle\prod_{s=T}^{t-1} [-a + br(s)], & t \in \mathbb{N}_T, \\ 1, \quad t \in \mathbb{N}_{t_0}^T. \end{cases} \tag{5.3}$$

The proof is completed. □

Corollary 5.4. *Suppose that there exists a function* $r : \mathbb{N}_T \to \mathbb{R}^+$ *such that* (3.1) *and* (3.2) *hold. Then,* (1.5) *has a positive* ω*–periodic solution.*

Proof. Here $p(t) = a > 0$, $q(t) = b > 0$ and $\tau(t) = t - d < t$ for all $t \in \mathbb{N}_{t_0}$. We choose sufficiently large $T \in \mathbb{N}_{t_0}$ such that $\tau(t) \in \mathbb{N}_{t_0}$ for $t \in \mathbb{N}_T$. Also, $f(v) = ve^{-cv} > 0$ for all $v > 0$ and $\tau(t) \to \infty$ as $t \to \infty$. Then, by Theorem 3.1, (1.5) has a positive ω–periodic solution

$$u(t) = \begin{cases} \displaystyle\prod_{s=T}^{t-1} [-a + br(s)], & t \in \mathbb{N}_T, \\ 1, \quad t \in \mathbb{N}_{t_0}^T. \end{cases} \tag{5.4}$$

The proof is completed. □

Acknowledgements

The author is grateful to the reviewers and the editors for their helpful comments and suggestions that have helped improve our paper.

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