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On the Analytical Approach of Codimension-Three Degenerate Bogdanov-Takens (B-T) Bifurcation in Satellite Dynamical System

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Abstract In this paper, we have conducted parametric analysis on the dynamics of satellite complex system using bifurcation theory. At first, five equilibrium points $\mathcal{E}_{0,1,2,3,4}$ are symbolically computed in which $\mathcal{E}_{1,3}$ and $\mathcal{E}_{2,4}$ are symmetric. Then, several theorems are stated and proved for the existence of B-T bifurcation on all equilibrium points with the aid of generalized eigenvectors and practical formulae instead of linearizations. Moreover, a special case $\alpha_2 = 0$ is observed, which confirms all the discussed cases belong to a codimension-three bifurcation along with degeneracy conditions.

Keywords Satellite dynamical system, Bogdanov-Takens bifurcation, normal form, generalized eigenvector

MSC(2010) 37G99, 34C23, 37L10.

1. Introduction

A system of ordinary differential equations obeying the changes in its topology with the variation in involved parameters can lead to the concept of bifurcation. This term is further categorized into local and global bifurcations such as in the study of Alle effect in predator prey models by Deeptajyoti et al., [39, 40] and Prahlad et al., [34]. However, if all parameters except one are set to be fixed, it is considered as a codimension (codim) one bifurcation including Hopf [16, 17, 32, 36, 38], zero-Hopf [18, 31, 37], saddle [3, 42] and Homoclinic/Hetroclinic [14, 15, 19, 28]. The variation in more than one parameter at a time can lead to bifurcations with a higher codimension. It is difficult to determine such bifurcations by using hit and trial methods, and that is why several analytical techniques have been discovered for achieving such a type of bifurcations, in which critical normal form on the center manifold has gained much attention. Bogdanov-Takens is one of the higher codimension bifurcations initiated with the work of Takens [11] and Bogdanov [12], but Arnold [6] and Guckenheimer [21] in 1983 derived the following normal form

$$\begin{aligned}\dot{\eta}_0 &= \eta_1,\\ \dot{\eta}_1 &= \sum_{k\geq 2}\left(\alpha_k\eta_0^k + \beta_k\eta_0^{k-1}\eta_1\right),\end{aligned} \tag{1.1}$$

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on the center manifold and discussed the exact qualitative attitude of trajectories near B-T critical point. This technique created a new way and attracted researchers by presenting the obtained normal form in a more simplified way [2, 4, 20].

In equation (1.1), if $\alpha_2 \times \beta_2 \neq 0$, then codim-2 bifurcation occurs and its bifurcation diagram can be seen in [29]. Cusp [33], Bogdanov-Takens [25, 26] , Double Hopf [23] and Bautin [22] are the main types in codim-2 bifurcation, which include tedious computer based calculations. But when α_2 or β_2 equals to zero in equatiion (1.1), a complex situation termed as codim-3 bifurcation can occur. In 2011, Kuznetsov [30] introduced a new way of getting the normal form by using generalized eigenvectors and practical formulae instead of linearization. This methodology is not limited to numerical computation, but also useful in the analytic of symbolic computation. In 2021, Darabsah [24] used B-T bifurcation from medical point of view in the excitably class of neurons. Mondal et al., [9] used it in ecology for predator-prey system, and the same analytical formulae for B-T bifurcation are used by Mondal et al., [10] in the food chain model with two species. In 2022, Xiang, Lu and Huang [41] explored the B-T bifurcation in host-parasitoid model for the special case that the carrying capacity K equals to $\frac{r_1}{\eta}$. Due to the advancement in technology and craze of exploring cosmic dynamics, systems in mechanical engineering can never lose its importance. In 2000, Sui et al., [1] considered six-dimensional satellite system and reported its failure analysis in controlling chaos. A negligible torque term was added into satellite system as a perturbed parameter and analyzed its transversal line between stable and unstable manifolds by Kuang [27]. Similarly, Aslanov and Yudintsev [7] discussed dynamics of gyrostat in satellite by bringing modification into Melnikov's function. Whereas, for more information relating dynamics and control in spacecraft, one can follow the results given by Liu in his book [43]. In 2018, Chegini, Sadati and Salarieh [13] worked analytically and numerically as well on tri-axial rigid body moving in elliptical orbit. Whereas, Khan et al., [8] modified satellite double wing model in the following form

$$\begin{cases} \dot{x} = \frac{yz}{3} - ax + \frac{z}{\sqrt{6}}, \\ \dot{y} = -xz + by, \\ \dot{z} = xy - cz - \sqrt{6}x. \end{cases} \tag{1.2}$$

Dynamics of system (1.2) shows chaos for parameter values $a = 4$, $b = 0.17$ and $c = 4$ with initial conditions $(0.1, 0.1, 0.1)$.
In our case, while analyzing dynamics of the satellite system, we have observed that

1. Khan et al., [8] fixed all involved parameters for finding its equilibrium points and dynamical analysis;
2. recently, Anam et al., [5] have obtained their multi-scrolls;
3. the above cited literature shows that satellite system is enriched with a qualitative aspect and some research can be found on its stability.

System (1.2) has complex dynamics due to the nonlinearity in each equation and involved parameters. However, relevant literature on its bifurcation analysis has not been identified. This gap has motivated us to work on the occurrence of higher codimension bifurcation in satellite system (1.2) for all equilibrium points. This parametric study leads us to the existence of degeneracy in B-T bifurcation as well, where Hopf, Saddle and Homoclinic orbits meet. Moreover, these analytical results and theorems will not only be helpful from a mathematical perspective, but also be of great interest for engineers.

This work is organized as follows. In Section 2, basic formulae are given for achieving B-T bifurcation. Computation of generalized eigenvectors, normal forms and multilinear functions for the satellite dynamical model on all equilibrium points is discussed in Section 3, while Section 4 contains our concluding remarks.

2. Basic concepts for Bogdanov-Takens bifurcation

This section includes fundamentals for achieving B-T bifurcation in the system of ordinary differential equations with order $n \geq 2$. We start with the expansion of system including ordinary differential equations

$$\dot{x} = \mathfrak{A}x + \mathfrak{F}(x): \quad x \in \mathbb{R}^n \tag{2.1}$$

through Taylor series near equilibria $\mathcal{E}$ as

$$\dot{x} = \mathfrak{A}x + \frac{1}{2!}\mathfrak{B}(x,x) + \frac{1}{3!}\mathfrak{C}(x,x,x), \tag{2.2}$$

where $\mathfrak{B}$ and $\mathfrak{C}$ are multilinear functions and are helpful in getting a normal form of several bifurcations. Analytical formulae for the calculation of multilinear functions are

$$\begin{aligned}\mathfrak{B}(x,y) &= \sum_{j,k=1}^{n}\left(\frac{\partial^2\mathfrak{F}_1(\xi)}{\partial\xi_j\partial\xi_k}\bigg|_{\xi=0}, \frac{\partial^2\mathfrak{F}_2(\xi)}{\partial\xi_j\partial\xi_k}\bigg|_{\xi=0}, \frac{\partial^2\mathfrak{F}_3(\xi)}{\partial\xi_j\partial\xi_k}\bigg|_{\xi=0}\right)^T x_j y_k,\\ \mathfrak{C}(x,y,z) &= \sum_{j,k,l=1}^{n}\left(\frac{\partial^3\mathfrak{F}_1(\xi)}{\partial\xi_j\partial\xi_k\partial\xi_l}\bigg|_{\xi=0}, \frac{\partial^3\mathfrak{F}_2(\xi)}{\partial\xi_j\partial\xi_k\partial\xi_l}\bigg|_{\xi=0}, \frac{\partial^3\mathfrak{F}_3(\xi)}{\partial\xi_j\partial\xi_k\partial\xi_l}\bigg|_{\xi=0}\right)^T x_j y_k z_l.\end{aligned} \tag{2.3}$$

Lemma 2.1 ([35]). *A dynamical system* (2.1) *has two zero eigenvalues if the characteristic equation*

$$\lambda^3 + \sigma_1\lambda^2 + \sigma_2\lambda + \sigma_3$$

of its linearized matrix satisfies the following condition

$$\sigma_1 \neq 0, \;\; \sigma_2 = 0, \;\; \sigma_3 = 0. \tag{2.4}$$

Lemma 2.2 ([35]). *If system* (2.1) *fulfills the condition* (2.4) *given in Lemma 2.1, then there exist four generalized eigenvectors* $\bar{s}_{0,1}$, $\bar{r}_{0,1}$ *such that*

$$\begin{aligned}&< \bar{s}_0, \bar{r}_0 >= 1, \quad < \bar{s}_1, \bar{r}_1 >= 1,\\ &< \bar{s}_0, \bar{r}_1 >= 0, \quad < \bar{s}_1, \bar{r}_0 >= 0.\end{aligned} \tag{2.5}$$

Lemma 2.3 ([35]). *Suppose that system* (2.1) *has double zero eigenvalues with generalized eigenvectors. Then such a system can be converted into the normal form*

$$\begin{aligned}\dot{\eta}_0 &= \eta_1,\\ \dot{\eta}_1 &= \alpha_2\eta_0^2 + \beta_2\eta_0\eta_1 + \alpha_3\eta_0^3 + \beta_3\eta_0^2\eta_1 + \alpha_4\eta_0^4 + \beta_4\eta_0^3\eta_1.\end{aligned} \tag{2.6}$$

In Lemma 2.3, α_2, β_2 are quadratic, α_3, β_3 show cubic and α_4, β_4 present fourth-order terms. In our case, tri-linear and quad-linear terms are zero. Therefore, α_i, β_i for $i = 2, 3, 4$ can be reduced to

$$\begin{aligned} \alpha_2 &= \frac{1}{2}\Big\langle \bar{s}_1, \mathfrak{B}(\bar{r}_0, \bar{r}_0) \Big\rangle, \\ \beta_2 &= \Big\langle \bar{s}_1, \mathfrak{B}(\bar{r}_0, \bar{r}_1) - \bar{h}_{20} \Big\rangle, \end{aligned} \tag{2.7}$$

and $\bar{h}_{20}$ can be obtained by solving singular linearized system

$$\mathfrak{A}\bar{h}_{20} = 2\alpha_2\bar{r}_1 - \mathfrak{B}(\bar{r}_0, \bar{r}_0). \tag{2.8}$$

As the determinant of matrix $\mathfrak{A}$ is zero in each case, the linearized matrices are singular and cannot be solved using an inverse of $\mathfrak{A}$. Coefficients of cubic terms in the normal form can be computed as

$$\begin{aligned} \alpha_3 &= \frac{1}{2}\Big\langle \bar{s}_1, \mathfrak{B}(\bar{h}_{20}, \bar{r}_0) - \alpha_2\bar{h}_{11} \Big\rangle, \\ \beta_3 &= \frac{1}{2}\Big\langle \bar{s}_1, 2\mathfrak{B}(\bar{h}_{11}, \bar{r}_0) + \mathfrak{B}(\bar{h}_{20}, \bar{r}_1) - \bar{h}_{30} - 2\alpha_2\bar{h}_{02} - 2\beta_2\bar{h}_{11} \Big\rangle, \end{aligned} \tag{2.9}$$

where $\bar{h}_{02}$, $\bar{h}_{11}$ and $\bar{h}_{30}$ can be obtained by solving the following set of linearized systems

$$\begin{cases} \mathfrak{A}\bar{h}_{11} = 2\beta_2\bar{r}_1 + \bar{h}_{20} - \mathfrak{B}(\bar{r}_0, \bar{r}_1), \\ \mathfrak{A}\bar{h}_{02} = 2\bar{h}_{11} - \mathfrak{B}(\bar{r}_1, \bar{r}_1), \\ \mathfrak{A}\bar{h}_{30} = 6\bar{r}_1\alpha_3 + 6\bar{h}_{11}\alpha_2 - 3\mathfrak{B}(\bar{h}_{20}, \bar{r}_0). \end{cases} \tag{2.10}$$

In the similar way,

$$\alpha_4 = \frac{1}{2 \times 4!}\Big\langle \bar{s}_1, 8\mathfrak{B}(\bar{h}_{30}, \bar{r}_0) + 6\mathfrak{B}(\bar{h}_{20}, \bar{h}_{20}) - \alpha_2\bar{h}_{21} \Big\rangle \tag{2.11}$$

and

$$\begin{aligned} \beta_4 = {} & \frac{1}{6}\Big\langle \bar{s}_1, 3\mathfrak{B}\left(\bar{h}_{21}, \bar{r}_0\right) + 3\mathfrak{B}\left(\bar{h}_{11}, \bar{h}_{20}\right) + \mathfrak{B}\left(\bar{h}_{30}, \bar{r}_1\right) \Big\rangle \\ & - \Big\langle \bar{s}_1, \frac{\bar{h}_{40}}{6} + \frac{\beta_2\bar{h}_{21}}{2} \Big\rangle - \Big\langle \bar{s}_1, \alpha_2\bar{h}_{12} + \alpha_3\bar{h}_{02} + \beta_3\bar{h}_{11} \Big\rangle \end{aligned} \tag{2.12}$$

can be achieved by solving the corresponding singular linearized systems given in Eq. (2.13)

$$\begin{cases} \mathfrak{A}\bar{h}_{12} = 2\bar{h}_{21} + 2\beta_2\bar{h}_{02} - \mathfrak{B}(\bar{h}_{02}, \bar{r}_0) - 2\mathfrak{B}(\bar{h}_{11}, \bar{r}_1), \\ \mathfrak{A}\bar{h}_{21} = \bar{h}_{30} + 2\beta_3\bar{r}_1 + 2\alpha_2\bar{h}_{02} + 2\beta_2\bar{h}_{11} - 2\mathfrak{B}(\bar{h}_{11}, \bar{r}_0) - \mathfrak{B}(\bar{h}_{20}, \bar{r}_1), \\ \mathfrak{A}\bar{h}_{40} = 24\alpha_4\bar{r}_1 + 12\alpha_2\bar{h}_{21} + 24\alpha_3\bar{h}_{11} - 4\mathfrak{B}(\bar{h}_{30}, \bar{r}_0) - 3\mathfrak{B}(\bar{h}_{20}, \bar{h}_{20}). \end{cases} \tag{2.13}$$

3. Existence of Bogdanov-Takens bifurcation in satellite dynamical system (1.2)

In bifurcation theory, discussion mostly starts with the stationary point of dynamical systems. Solving system (1.2) yields

$$\begin{cases}\mathcal{E}_0 = (0,0,0),\\ \mathcal{E}_{1,3} = \left(\pm\frac{\sqrt{2bM_2}(M_1+3)}{8a\sqrt{c}}, \frac{\sqrt{6}(M_1+1)}{4}, \pm\frac{\sqrt{3bM_2}}{2\sqrt{c}}\right),\\ \mathcal{E}_{2,4} = \left(\mp\frac{\sqrt{2bM_3}(M_1-3)}{8a\sqrt{c}}, -\frac{\sqrt{6}(M_1-1)}{4}, \pm\frac{\sqrt{3bM_3}}{2\sqrt{c}}\right),\end{cases} \tag{3.1}$$

in which $M_1 = \sqrt{8ac+9}$, $M_2 = \sqrt{4ac+3-M_1}$ and $M_3 = \sqrt{4ac+3+M_1}$.

Theorem 3.1. *System* (1.2) *exhibits double zero eigenvalues at equilibria* $\mathcal{E}_0$*, for* $a = 1$*,* $b \neq 0$ *and* $c = -1$*.*

Proof. The Jacobian matrix of system (1.2) is

$$\mathfrak{J} = \begin{pmatrix} -a & \frac{z}{3} & \frac{y}{3}+\frac{1}{\sqrt{6}} \\ -z & b & -x \\ y-\sqrt{6} & x & -c \end{pmatrix}, \tag{3.2}$$

and the Jacobian matrix (3.2) at $\mathcal{E}_0$ gives

$$\mathfrak{J}\Big|_{\mathcal{E}_0} = \begin{pmatrix} -a & 0 & \frac{1}{\sqrt{6}} \\ 0 & b & 0 \\ -\sqrt{6} & 0 & -c \end{pmatrix}. \tag{3.3}$$

The characteristic equation of matrix (3.3) is

$${\lambda_0}^3 + \sigma_{01}{\lambda_0}^2 + \sigma_{02}\lambda_0 + \sigma_{03}, \tag{3.4}$$

where

$$\begin{cases}\sigma_{01} = a-b+c,\\ \sigma_{02} = -ab+ac-bc+1,\\ \sigma_{03} = -\left(abc+b\right).\end{cases} \tag{3.5}$$

According to Lemma 2.1, there exist double zero eigenvalues in system (1.2) for $\mathcal{E}_0$, if $\sigma_{01} \neq 0$, $\sigma_{02} = 0$ and $\sigma_{03} = 0$, which yields $a = -1$, $b \neq 0$ and $c = 1$. Substituting these values back into equations (3.4)−(3.5) yields

$$\lambda_{01} = 0, \quad \lambda_{02} = 0, \quad \lambda_{03} = -b, \tag{3.6}$$

which is our desired result. □ □

Theorem 3.2. *There exist four generalized eigenvectors*

$$\bar{\rho}_0 = \left(\frac{\sqrt{7}}{7}, 0, -\frac{\sqrt{42}}{7}\right), \quad \bar{\rho}_1 = \left(\sqrt{7}, 0, \frac{\sqrt{42}}{6}\right),$$
$$\bar{\zeta}_0 = \left(\frac{\sqrt{7}}{7}, 0, -\frac{\sqrt{42}}{7}\right), \quad \bar{\zeta}_1 = \left(\frac{6\sqrt{7}}{49}, 0, \frac{\sqrt{42}}{49}\right) \tag{3.7}$$

in system (1.2) *at* $\mathcal{E}_0$*, for* $a=-1$*,* $b\neq 0$ *and* $c=1$*.*

Proof. In Theorem 3.1, it is proved that double zero eigenvalues exist in the considered satellite system. Next, we need to find generalized eigenvectors which satisfy orthogonal conditions (2.5) given in Lemma 2.2. However, these vectors are not unique, but one can follow the technique given by Kuznetsov [30]. For this purpose, let us suppose that $\mathfrak{A}_0=\mathfrak{J}\big|_{\mathcal{E}_0}$. Then, $\bar{\zeta}_0$ and $\bar{\zeta}_1$ in equation (3.7) can be achieved by solving

$$\begin{pmatrix} -a & 0 & \frac{1}{\sqrt{6}} \\ 0 & b & 0 \\ -\sqrt{6} & 0 & -c \end{pmatrix}\begin{pmatrix}\zeta_{01}\\ \zeta_{02}\\ \zeta_{03}\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix},\quad \begin{pmatrix} -a & 0 & \frac{1}{\sqrt{6}} \\ 0 & b & 0 \\ -\sqrt{6} & 0 & -c \end{pmatrix}\begin{pmatrix}\zeta_{11}\\ \zeta_{12}\\ \zeta_{13}\end{pmatrix}=\begin{pmatrix}\zeta_{01}\\ \zeta_{02}\\ \zeta_{03}\end{pmatrix}, \tag{3.8}$$

where $\bar{\zeta}_0=(\zeta_{01},\zeta_{02},\zeta_{03})^T$ and $\bar{\zeta}_1=(\zeta_{11},\zeta_{12},\zeta_{13})^T$. In a similar fashion for $\bar{\rho}_0=(\rho_{01},\rho_{02},\rho_{03})^T$ and $\bar{\rho}_1=(\rho_{11},\rho_{12},\rho_{13})^T$ with the transpose of $\mathfrak{A}_0$, the following matrix equalities can be solved

$$\begin{pmatrix} -a & 0 & \frac{1}{\sqrt{6}} \\ 0 & b & 0 \\ -\sqrt{6} & 0 & -c \end{pmatrix}^T\begin{pmatrix}\rho_{11}\\ \rho_{12}\\ \rho_{13}\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix},\quad \begin{pmatrix} -a & 0 & \frac{1}{\sqrt{6}} \\ 0 & b & 0 \\ -\sqrt{6} & 0 & -c \end{pmatrix}^T\begin{pmatrix}\rho_{01}\\ \rho_{02}\\ \rho_{03}\end{pmatrix}=\begin{pmatrix}\rho_{11}\\ \rho_{12}\\ \rho_{13}\end{pmatrix}. \tag{3.9}$$

Vectors achieved from the solution of equations (3.8) and (3.9) are generalized eigenvectors, and can satisfy the orthonormality condition given in Lemma 2.2. □

Theorem 3.3. *System* (1.2) *exhibits double zero eigenvalues at equilibria* $\mathcal{E}_1$*, for* $a=-\frac{9}{8c}$*,* $b=c$ *and* $c=\frac{3}{\sqrt{8}}$*.*

Proof. Jacobian matrix of system (1.2) at $\mathcal{E}_1$ gives

$$\mathfrak{J}\big|_{\mathcal{E}_1}=\begin{pmatrix} -a & \frac{\sqrt{3bM_2}}{6\sqrt{c}} & \frac{(M_1+3)}{2\sqrt{6}} \\ -\frac{\sqrt{3bM_2}}{2\sqrt{c}} & b & -\frac{\sqrt{2bM_2}(M_1+3)}{8a} \\ \frac{3(M_1-3)}{2\sqrt{6}} & \frac{\sqrt{2bM_2}(M_1+3)}{8a} & -c \end{pmatrix}. \tag{3.10}$$

The characteristic equation of matrix (3.10) is

$$\lambda_1{}^3+\sigma_{11}\lambda_1{}^2+\sigma_{12}\lambda_1+\sigma_{13}, \tag{3.11}$$

where

$$\begin{cases}\sigma_{11}=a-b+c,\\ \sigma_{12}=\frac{-32c(b-c)a^3+((-32c^2+8M_2)b+(-4M_1^2+36)c)a^2+(M_1+3)^2bM_2}{32a^2c},\\ \sigma_{13}=-96\left(a^2c^2-\frac{c(M_1^2+2M_2-9)a}{8}-\frac{(3M_2(M_1+3)(M_1+1))}{32}\right)ab.\end{cases} \tag{3.12}$$

According to Lemma 2.1, we have to solve σ_{12} and σ_{13} by equating zero. First, we use $b=c$, $M_1=0$ which implies $a=-\frac{9}{8c}$, and $M_2=0$ in equation (3.12) to obtain

$$\begin{cases}\sigma_{11}=a,\\ \sigma_{12}=-c^2+\frac{9}{8},\\ \sigma_{13}=0.\end{cases} \tag{3.13}$$

Now, we only need to select an appropriate value for c such that σ_{12} becomes zero. Therefore, we select $c = \frac{3}{\sqrt{8}}$. Hence, our desired result

$$\lambda_{11} = 0, \quad \lambda_{12} = 0, \quad \lambda_{13} = a \tag{3.14}$$

of double zero eigenvalues in system (1.2) is achieved. □ □

Corollary 3.1. *Equilibrium point $\mathcal{E}_3$ shows the symmetry with $\mathcal{E}_1$, and that is why double zero eigenvalues condition can be fulfilled in satellite system* (1.2) *at $a = -\frac{9}{8c}$, $b = c$ and $c = \frac{3}{\sqrt{8}}$.*

Theorem 3.4. *There exist four generalized eigenvectors*

$$\begin{aligned} \bar{\varrho}_0 &= \left(\frac{1}{2}, 0, -\frac{\sqrt{3}}{2}\right), \quad \bar{\varrho}_1 = \left(\frac{3}{\sqrt{2}}, 0, \frac{\sqrt{6}}{2}\right), \\ \bar{\varphi}_0 &= \left(\frac{1}{2}, 0, -\frac{\sqrt{3}}{2}\right), \quad \bar{\varphi}_1 = \left(\frac{\sqrt{2}}{4}, 0, \frac{\sqrt{6}}{12}\right) \end{aligned} \tag{3.15}$$

in system (1.2) *at $\mathcal{E}_1$, for $a = -\frac{9}{8c}$, $b = c$ and $c = \frac{3}{\sqrt{8}}$.*

Proof. In Theorem 3.3, it can be observed that there exist double zero eigenvalues on equilibria $\mathcal{E}_1$. Now, we need to find its generalized eigenvectors which satisfy orthogonal conditions (2.5) given in Lemma 2.2. For this purpose, let us consider $\mathfrak{A}_1 = \mathfrak{J}\big|_{\mathcal{E}_1}$. Then, $\bar{\varphi}_0$ and $\bar{\varphi}_1$ in equation (3.15) can be achieved by solving

$$\begin{pmatrix} \frac{3\sqrt{2}}{4} & 0 & \frac{\sqrt{6}}{4} \\ 0 & \frac{3\sqrt{2}}{4} & 0 \\ -\frac{3\sqrt{6}}{4} & 0 & -\frac{3\sqrt{2}}{4} \end{pmatrix} \begin{pmatrix} \varphi_{01} \\ \varphi_{02} \\ \varphi_{03} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} \frac{3\sqrt{2}}{4} & 0 & \frac{\sqrt{6}}{4} \\ 0 & \frac{3\sqrt{2}}{4} & 0 \\ -\frac{3\sqrt{6}}{4} & 0 & -\frac{3\sqrt{2}}{4} \end{pmatrix} \begin{pmatrix} \varphi_{11} \\ \varphi_{12} \\ \varphi_{13} \end{pmatrix} = \begin{pmatrix} \varphi_{01} \\ \varphi_{02} \\ \varphi_{03} \end{pmatrix}, \tag{3.16}$$

where $\bar{\varphi}_0 = (\varphi_{01}, \varphi_{02}, \varphi_{03})^T$ and $\bar{\varphi}_1 = (\varphi_{11}, \varphi_{12}, \varphi_{13})^T$. In a similar fashion for $\bar{\varrho}_0 = (\varrho_{01}, \varrho_{02}, \varrho_{03})^T$ and $\bar{\varrho}_1 = (\varrho_{11}, \varrho_{12}, \varrho_{13})^T$ with the transpose of $\mathfrak{A}_1$, the following matrix equalities can be solved

$$\begin{pmatrix} \frac{3\sqrt{2}}{4} & 0 & \frac{\sqrt{6}}{4} \\ 0 & \frac{3\sqrt{2}}{4} & 0 \\ -\frac{3\sqrt{6}}{4} & 0 & -\frac{3\sqrt{2}}{4} \end{pmatrix}^T \begin{pmatrix} \varrho_{11} \\ \varrho_{12} \\ \varrho_{13} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} \frac{3\sqrt{2}}{4} & 0 & \frac{\sqrt{6}}{4} \\ 0 & \frac{3\sqrt{2}}{4} & 0 \\ -\frac{3\sqrt{6}}{4} & 0 & -\frac{3\sqrt{2}}{4} \end{pmatrix}^T \begin{pmatrix} \varrho_{01} \\ \varrho_{02} \\ \varrho_{03} \end{pmatrix} = \begin{pmatrix} \varrho_{11} \\ \varrho_{12} \\ \varrho_{13} \end{pmatrix}. \tag{3.17}$$

Vectors achieved from the solution of equations (3.16) and (3.17) are generalized eigenvectors and can satisfy the orthonormality condition given in Lemma 2.2. □

Theorem 3.5. *System* (1.2) *has double zero eigenvalues at equilibria $\mathcal{E}_2$, for $a = -\frac{9}{8c}$, $b = c$ and $c = \frac{3}{\sqrt{8}}$.*

Proof. The Jacobian matrix of system (1.2) at $\mathcal{E}_2$ gives

$$\mathfrak{J}\big|_{\mathcal{E}_2} = \begin{pmatrix} -a & \frac{\sqrt{3bM_3}}{6\sqrt{c}} & -\frac{(M_1-3)}{2\sqrt{6}} \\ -\frac{\sqrt{3bM_3}}{2\sqrt{c}} & b & \frac{\sqrt{2bM_3}(M_1-3)}{8a\sqrt{c}} \\ \frac{-3(M_1+3)}{2\sqrt{6}} & -\frac{\sqrt{2bM_3}(M_1-3)}{8a\sqrt{c}} & -c \end{pmatrix}. \tag{3.18}$$

The characteristic equation of matrix (3.18) is

$$\lambda_2{}^3 + \sigma_{21}\lambda_2{}^2 + \sigma_{22}\lambda_2 + \sigma_{23}, \tag{3.19}$$

where

$$\begin{cases} \sigma_{21} = a - b + c, \\ \sigma_{22} = \frac{-32c(b-c)a^3+((-32c^2+8M_3)b+(-4M_1^2+36)c)a^2+(M_1-3)^2bM_3}{32a^2c}, \\ \sigma_{23} = -b\left(ac - \frac{(M_1^2+2M_3-9)}{8} - \frac{(3M_3(M_1-3)(M_1-1))}{32ac}\right)ab. \end{cases} \tag{3.20}$$

According to Lemma 2.1, we have to solve σ_{22} and σ_{23} for achieving suitable parameter values. First, we use $b = c$, $M_1 = 0$ which implies $a = -\frac{9}{8c}$, and $M_3 = 0$ in equation (3.20) to obtain

$$\begin{cases} \sigma_{21} = a, \\ \sigma_{22} = -c^2 + \frac{9}{8}, \\ \sigma_{23} = 0. \end{cases} \tag{3.21}$$

Then, we need to select an appropriate value for c such that σ_{22} become zero. Therefore, we select $c = \frac{3}{\sqrt{8}}$. Hence, our desired result

$$\lambda_{21} = 0, \quad \lambda_{22} = 0, \quad \lambda_{23} = a \tag{3.22}$$

is obtained that system (1.2) exhibits double zero eigenvalues at equilibria $\mathcal{E}_2$ for suitable values of a, b and c. □ □

Corollary 3.2. *Equilibrium point $\mathcal{E}_2$ shows the symmetry with $\mathcal{E}_4$, and that is why double zero eigenvalues condition can be fulfilled in satellite system* (1.2) *at $\mathcal{E}_4$ for $a = -\frac{9}{8c}$, $b = c$ and $c = \frac{3}{\sqrt{8}}$.*

Remark 3.1. The generalized eigenvectors for system (1.2) at equilibria $\mathcal{E}_2$ are the same as those given in equation (3.15), because the Jacobian matrix at $\mathcal{E}_2$, for specific values of a, b and c, yields $\mathfrak{A}_2 = \mathfrak{A}_1$ as well.

3.1. Normal form for B-T bifurcation

Theorem 3.6. *Let us suppose that there exist generalized eigenvectors $\bar{\rho}_0$, $\bar{\rho}_1$, $\bar{\zeta}_0$ and $\bar{\zeta}_1$ in system* (1.2). *Then the normal form for B-T bifurcation is*

$$\begin{cases} \dot{\eta}_0 = \eta_1, \\ \dot{\eta}_1 = \beta_2\eta_0\eta_1 + \alpha_3\eta_0^3 + \beta_3\eta_0^2\eta_1 + \alpha_4\eta_0^4 + \beta_4\eta_0^3\eta_1, \end{cases} \tag{3.23}$$

where

$$\begin{cases} \beta_2 = -\sqrt{7} + \frac{60}{343b}, \\ \alpha_3 = \frac{1}{7b}, \\ \beta_3 = \frac{102900\sqrt{7}b+823543b^2+352947b+213698}{823543b^2}, \\ \alpha_4 = \frac{0.000011244(-45619b+1905)}{b^2}, \\ \beta_4 = \frac{1.3229\left(-9.9252b^4+3.2048b^3-0.038148b^2-0.54120b+0.037782\right)}{b^4}. \end{cases}$$

Proof. Jacobian matrix of system (1.2) at $\mathcal{E}_0$ can be found in equation (3.3), whereas, its generalized eigenvectors are calculated in Theorem 3.2. Using analytical formula of multilinear functions (2.3) for satellite dynamical system (1.2) gives

$$\mathfrak{B}(x,y) = \left(\frac{x_2y_3 + x_3y_2}{3}, -\left(x_1y_3 + x_3y_1\right), x_1y_2 + x_2y_1\right)^T, \tag{3.24}$$

$$\mathfrak{C} = (0,0,0)^T \quad \text{and} \quad \mathfrak{D} = (0,0,0)^T. \tag{3.25}$$

After that, we need coefficients of quadratic, cubic and fourth-order terms. Hence, for α_2, we need to substitute $\bar{\zeta}_0$ and $\bar{\zeta}_1$ into equation (3.24)

$$\mathfrak{B}(\bar{\zeta}_0,\bar{\zeta}_0) = \left(0, \frac{2\sqrt{6}}{7}, 0\right)^T, \quad \mathfrak{B}(\bar{\zeta}_0,\bar{\zeta}_1) = \left(0, \frac{5\sqrt{6}}{49}, 0\right)^T, \tag{3.26}$$

and its dot product with $\bar{\rho}_1$ yields orthogonality such that

$$\alpha_2 = 0. \tag{3.27}$$

Substituting α_2 and equation (3.26) into equation (2.8), we get

$$\bar{h}_{20} = \left(1, -\frac{2\sqrt{6}}{7b}, -\sqrt{6}\right)^T. \tag{3.28}$$

Whereas, substituting $\bar{h}_{20}$, α_2 and $\mathfrak{B}(\bar{\zeta}_0,\bar{\zeta}_1)$ into equation (2.7) gives

$$\beta_2 = -\sqrt{7} + \frac{60}{343b}. \tag{3.29}$$

Since $\alpha_2 * \beta_2 = 0$, this case refers to the existence of codim-3 bifurcation in satellite system. Hence, we need cubic and fourth-order terms to check non-degeneracy as well. For cubic and fourth-order terms, we find $\mathfrak{B}(\bar{h}_{20},\bar{\zeta}_0)$, $\mathfrak{B}(\bar{h}_{20},\bar{\zeta}_1)$ and $\mathfrak{B}(\bar{h}_{11},\bar{\zeta}_0)$:

$$\begin{cases} \mathfrak{B}(\bar{h}_{20},\bar{\zeta}_0) = \left(\frac{4\sqrt{7}}{49b}, \frac{2\sqrt{42}}{7}, -\frac{2\sqrt{42}}{49b}\right)^T, \\ \mathfrak{B}(\bar{h}_{20},\bar{\zeta}_1) = \left(-\frac{4\sqrt{7}}{343b}, \frac{5\sqrt{42}}{49}, -\frac{12\sqrt{42}}{343b}\right)^T, \\ \mathfrak{B}(\bar{h}_{11},\bar{\zeta}_0) = \left(\frac{2(5b+14)\sqrt{7}}{343b^2}, -\frac{\sqrt{42}(-31213b+360\sqrt{7})}{117649b}, -\frac{(5b+14)\sqrt{42}}{343b^2}\right)^T. \end{cases} \tag{3.30}$$

Substituting equation (3.30) back into equation (2.9), we get

$$\begin{cases} \alpha_3 = \frac{1}{7b}, \\ \beta_3 = \frac{102900\sqrt{7}b+823543b^2+352947b+213698}{823543b^2}. \end{cases} \tag{3.31}$$

Vectors $\bar{h}_{11}$, $\bar{h}_{30}$ and $\bar{h}_{02}$

$$\begin{cases} \bar{h}_{02} = \left(1, \frac{2\sqrt{6}(6b^2-35b-98)}{343b^3}, \sqrt{6}\right)^T, \\ \bar{h}_{11} = \left(1, -\frac{\sqrt{6}(5b+14)}{49b^2}, \frac{6(-2401b+60\sqrt{7})\sqrt{6}}{16807b}\right)^T, \\ \bar{h}_{30} = \left(1, -\frac{6\sqrt{42}}{7b}, -\frac{(48\sqrt{7}+343b)\sqrt{6}}{343b}\right)^T \end{cases} \tag{3.32}$$

are involved in finding equation (3.31), which are calculated by solving a set of linearized systems given in equation (2.10). Similarly, using equations (2.11) and (2.12), one can get

$$\begin{cases} \alpha_4 = \frac{0.000011244(-45619b+1905)}{b^2}, \\ \beta_4 = \frac{1.3229\left(-9.9252b^4+3.2048b^3-0.038148b^2-0.54120b+0.037782\right)}{b^4} \end{cases} \tag{3.33}$$

with

$$\bar{h}_{12} = \begin{pmatrix} 1 \\ -\frac{58(b(b^3+\frac{229}{29b^2}+\frac{154}{29b}-\frac{392}{29})\sqrt{7}-\frac{2160(b-2)(\frac{7}{3}+b)^2}{9947})\sqrt{6}}{343b^5} \\ \frac{(-4802b^3+44b^2-84b-392)\sqrt{42}+2401b^2(b+\frac{120}{343})\sqrt{6}}{2401b^3} \end{pmatrix},$$

$$\bar{h}_{21} = \begin{pmatrix} 1 \\ \frac{-10633b\sqrt{6}(b^2+\frac{32}{31b}-\frac{28}{31})\sqrt{7}+720(b-2)(b+\frac{7}{6})\sqrt{6}}{16807b^4} \\ \frac{(-708246698b^2+2352980b-4023968)\sqrt{42}+22761480b\sqrt{6}}{40353607b^2} \end{pmatrix}, \tag{3.34}$$

$$\bar{h}_{40} = \begin{pmatrix} 1 \\ -\frac{2\sqrt{6}(196\sqrt{7}b^3+1029b^3+96b^2+60b+168)}{343b^4} \\ -\frac{(823543b^2+51840\sqrt{7}+2362584b)\sqrt{6}}{823543b^2} \end{pmatrix}$$

as resultant vectors by solving equation (2.13). Finally, we get our desired result by substituting equations (3.27)−(3.34) into normal form (1.1). □

Corollary 3.3. *System* (1.2) *exhibits degenerate Bogdanov-Takens bifurcation at equilibria* $\mathcal{E}_0$ *for* $b \neq \frac{60}{343\sqrt{7}}$ *and solution of* $41160\sqrt{7}b-823543b^2-151263b-3600 \neq 0$.

Proof. A case arises for non-degenerate Bogdanov-Takens bifurcation, when $\alpha_2 = 0$ but $\beta_2(\beta_2^2 + 9\alpha_3) \neq 0$. In Theorem 3.6, it is observed that $\alpha_2 = 0$ which confirms the existence of non-degeneracy in codim-3 bifurcation. Hence,

$$\beta_2(\beta_2^2 + 9\alpha_3) = \frac{(343\sqrt{7}b - 60)(41160\sqrt{7}b - 823543b^2 - 151263b - 3600)}{40353607b^3} \tag{3.35}$$

is non-zero, when $b \neq \frac{60}{343\sqrt{7}}$ and $41160\sqrt{7}b - 823543b^2 - 151263b - 3600 \neq 0$. □

Theorem 3.7. *Let us suppose that there exist generalized eigenvectors* $\bar{\varrho}_0$, $\bar{\varrho}_1$, $\bar{\zeta}_0$ *and* $\bar{\zeta}_1$ *in system* (1.2). *Then the normal form for B-T bifurcation is*

$$\begin{cases} \dot{\eta}_0 = \eta_1, \\ \dot{\eta}_1 = (-\frac{3\sqrt{2}}{2} + \frac{1}{6})\eta_0\eta_1 + \left(\frac{77\sqrt{(2)}}{72} + \frac{1}{6}\right)\eta_0^2\eta_1 + \left(\frac{35\sqrt{(2)}}{8} - \frac{2021}{216}\right)\eta_0^3\eta_1. \end{cases} \tag{3.36}$$

Proof. Jacobian matrix of system (1.2) at $\mathcal{E}_1$ can be found in equation (3.10). Whereas, its generalized eigenvectors and multilinear functions are calculated in Theorem 3.4 and equation (2.3) respectively. After that, we need coefficients of quadratic, cubic and fourth-order terms. Hence, for α_2, we need to substitute $\bar{\varphi}_0$

and $\bar{\varphi}_1$ into a bilinear function obtained in equation (3.24)

$$\mathfrak{B}(\bar{\varphi}_0, \bar{\varphi}_0) = \left(0, \frac{\sqrt{3}}{2}, 0\right)^T, \quad \mathfrak{B}(\bar{\varphi}_0, \bar{\varphi}_1) = \left(0, \frac{\sqrt{6}}{12}, 0\right)^T, \tag{3.37}$$

and its dot product with $\bar{\varrho}_1$ yields orthogonality such that

$$\alpha_2 = 0. \tag{3.38}$$

Substituting α_2 and equation (3.37) into equation (2.8), we obtain

$$\bar{h}_{20} = \left(1, -\sqrt{\frac{2}{3}}, -\sqrt{3}\right)^T. \tag{3.39}$$

Whereas, substituting $\bar{h}_{20}$, α_2 and $\mathfrak{B}(\bar{\varphi}_0, \bar{\varphi}_1)$ into equation (2.7) yields

$$\beta_2 = -\frac{3\sqrt{2}}{2} + \frac{1}{6} = -1.9546. \tag{3.40}$$

Since $\alpha_2 * \beta_2 = 0$, this case refers to the existence of codim-3 bifurcation in satellite system for $\mathcal{E}_1$. Further, we need cubic and fourth order terms to check nondegeneracy as well. For cubic and fourth-order terms, we find $\mathfrak{B}(\bar{h}_{20}, \bar{\varphi}_0)$, $\mathfrak{B}(\bar{h}_{20}, \bar{\varphi}_1)$ and $\mathfrak{B}(\bar{h}_{11}, \bar{\varphi}_0)$:

$$\begin{cases} \mathfrak{B}(\bar{h}_{20}, \bar{\varphi}_0) = \left(\frac{\sqrt{2}}{6}, \sqrt{3}, -\frac{1}{\sqrt{6}}\right)^T, \\ \mathfrak{B}(\bar{h}_{20}, \bar{\varphi}_1) = \left(-\frac{1}{18}, \frac{1}{\sqrt{6}}, -\frac{\sqrt{3}}{6}\right)^T, \\ \mathfrak{B}(\bar{h}_{11}, \bar{\varphi}_0) = \left(\frac{5}{18}, \frac{\sqrt{3}(35-3\sqrt{2})}{36}, -\frac{5\sqrt{3}}{18}\right)^T. \end{cases} \tag{3.41}$$

Substituting equation (3.41) back into equation (2.9), we get

$$\begin{cases} \alpha_3 = 0, \\ \beta_3 = \frac{77\sqrt{2}}{72} + \frac{1}{6} = 1.6790. \end{cases} \tag{3.42}$$

Vectors $\bar{h}_{11}$, $\bar{h}_{30}$ and $\bar{h}_{02}$

$$\begin{cases} \bar{h}_{02} = (1, -1.6784, 1.5340)^T, \\ \bar{h}_{11} = (1, -0.96229, -1.2276)^T, \\ \bar{h}_{30} = \left(1, -2\sqrt{(6)}, -\frac{5}{\sqrt{3}}\right)^T \end{cases} \tag{3.43}$$

are involved in finding equation (3.42), which are calculated by solving the set of linearized systems given in equation (2.10). Similarly, using equations (2.11) and (2.12), one can get

$$\begin{cases} \alpha_4 = 0, \\ \beta_4 = \frac{35\sqrt{2}}{8} - \frac{2021}{216} = -3.1694 \end{cases} \tag{3.44}$$

with

$$\begin{cases} \bar{h}_{12} = (1, -2.3502, -5.4271)^T, \\ \bar{h}_{21} = (1, -4.2478, -5.3607)^T, \\ \bar{h}_{40} = (1, -18.507, -15.589)^T \end{cases} \tag{3.45}$$

as resultant vectors by solving equation (2.13). Finally, we get our desired result by substituting equations (3.38)−(3.45) into normal form (1.1). □

Corollary 3.4. *System* (1.2) *exhibits degenerate Bogdanov-Takens bifurcation at equilibria* $\mathcal{E}_1$ *with the case* $\alpha_2 = 0$ *and* $\beta_2(\beta_2^2 + 9\alpha_3) = -7.4675 \neq 0$.

Corollary 3.5. *Degenerate Bogdanov-Takens bifurcation for equilibria* $\mathcal{E}_3$ *is the same as that in Theorem 3.7 for obeying the symmetric property in equilibrium points* $\mathcal{E}_1$ *and* $\mathcal{E}_3$.

Corollary 3.6. *Theorem 3.1 reveals that generalized eigenvectors of equilibria* $\mathcal{E}_2$ *are the same as those obtained for* $\mathcal{E}_1$*. Therefore, all conditions related to Bogdanov-Takens bifurcation will also be the same.*

4. Conclusion

In this study, satellite system was considered for bifurcation analysis. A special case, $\alpha_2 = 0$ was raised, which confirmed the existence of codim-3 bifurcation for all equilibrium points in system (1.2) for the first time. These symbolic computations were carried out through generalized eigenvectors and multilinear functions, for which several theorems and remarks were stated and proved. A symmetry in nonzero equilibrium points was observed, which made it more feasible and helpful in the analytical results obtained for such a type of bifurcation. Finally, several conditions were obtained for non-degeneracy in the existence of Bogdanov-Takens bifurcation. In the future, our aim is to work on other codim-2 bifurcations such as Bautin and Cusp. We have also targeted its numerical approach by using Matcont to apply such analytical work in engineering-based applications.

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