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Dynamics of a Discrete Two-Species Competitive Model with Michaelies-Menten Type Harvesting in the First Species*

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Abstract In this paper, we use a semidiscretization method to derive a discrete two-species competitive model with Michaelis-Menten type harvesting in the first species. First, the existence and local stability of fixed points of the system are investigated by employing a key lemma. Subsequently, the transcritical bifurcation, period-doubling bifurcation and pitchfork bifurcation of the model are investigated by using the Center Manifold Theorem and bifurcation theory. Finally, numerical simulations are presented to illustrate corresponding theoretical results.

Keywords Competitive model with Michaelis-Menten type harvesting, semidiscretization method, transcritical bifurcation, period-doubling bifurcation, pitchfork bifurcation

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1. Introduction and preliminaries

In the past few decades, more and more investigators have begun to pay attention to investigating competitive systems [1, 2, 4–6, 9–12, 15, 19, 24–26, 29, 30, 32–34], and many excellent results concerned with the extinction and global attractivity of competitive systems have been obtained.

Murray [17] investigated the competitive system of traditional two-species Lotka-Volterra model

$$\begin{cases} \frac{dx_1}{dt} = x_1(b_1 - a_{11}x_1 - a_{12}x_2), \\ \frac{dx_2}{dt} = x_2(b_2 - a_{21}x_1 - a_{22}x_2), \end{cases} \tag{1.1}$$

where x_1 and x_2 denote the population density of the two species at time t respectively, and $b_i, a_{ij}, i, j = 1, 2$, are positive constants.

In addition, when human activity is the main cause which leads to the extinction of endangered species, the study of resource-management, including fisheries, forestry, and wildlife management, has great importance. It is sometimes necessary

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to harvest some populations, but harvesting should be regulated so that both the ecological sustainability and conservation of the species can be implemented in a long running. In order to further understand the scientific management of renewable resources and make the meaning of a model more realistic, many scholars are devoted to establishing suitable biological models. Among them, Chen [3] studied the following model

$$\begin{cases} \frac{dx}{dt} = r_1 x(1 - \frac{x}{k_1} - \alpha \frac{y}{k_1}) - \frac{qEx}{m_1 E + m_2 x}, \\ \frac{dy}{dt} = r_2 y(1 - \frac{y}{k_2}), \end{cases} \tag{1.2}$$

where x and y denote the population density of the first and second species at time t respectively, q denotes the fishing coefficient of the first species, E denotes the fishing effort, and $r_1, r_2, k_1, k_2, \alpha, m_1, m_2$ are all positive constants. The function $h(x) = \frac{qEx}{m_1 E + m_2 x}$ is called Michaelis-Menten type harvesting, which was proposed by Clark and Mangel [7]. In other pieces of literature, $h(x)$ may also take qEx, $\frac{qE}{m}$ or $\frac{qx}{m}$.

Later, in [31], based on model (1.2), Yu, Zhu and Li considered the following system:

$$\begin{cases} \frac{dx}{dt} = r_1 x(1 - \frac{x}{k_1}) - \alpha_1 xy - \frac{q_1 Ex}{m_1 E + h_1 x}, \\ \frac{dy}{dt} = r_2 y(1 - \frac{y}{k_2}) - \alpha_2 xy, \end{cases} \tag{1.3}$$

where $r_1, r_2, k_1, k_2, \alpha_1, \alpha_2, q_1, m_1, h_1$ and E are all positive. For simplicity, the authors made the following nondimensional scheme:

$$\bar{t} = r_1 t, \bar{x} = \frac{1}{k_1} x, \bar{y} = \frac{1}{k_2} y.$$

Dropping the bars, system (1.3) becomes

$$\begin{cases} \frac{dx}{dt} = x(1 - x - a_1 y - \frac{b}{c+x}), \\ \frac{dy}{dt} = \rho y(1 - y - a_2 x), \end{cases} \tag{1.4}$$

where $a_1 = \frac{\alpha_1 k_2}{r_1}, b = \frac{q_1 E}{k_1 r_1 h_1}, c = \frac{m_1 E}{h_1 k_1}, \rho = \frac{r_2}{r_1}, a_2 = \frac{k_1 \alpha_2}{r_2}$.

Generally speaking, it is impossible to obtain an exact solution for a complex differential equation system. Therefore, one usually derives its approximate solution by using computer. Then, we should study its corresponding discrete model. For a given system, there are many discretization methods including Euler forward difference scheme, Euler backward difference scheme, semidiscretization methods and etc. In this article, we use the semidiscretization method, which has been applied in many studies ([8, 13, 14, 21]). For the related work, please also see [16, 18, 20, 27, 28].

The discrete version of system (1.4) has not been found to be investigated yet. Now, we use the semidiscretization method to derive its discrete model. For this, suppose that $[t]$ denotes the greatest integer not exceeding t. We consider the average change rate of system (1.4) at integer number points

$$\begin{cases} \frac{1}{x(t)} \frac{dx(t)}{dt} = 1 - x([t]) - a_1 y([t]) - \frac{b}{c + x([t])}, \\ \frac{1}{y(t)} \frac{dy(t)}{dt} = \rho(1 - y([t]) - a_2 x([t])). \end{cases} \tag{1.5}$$

It is easy to see that system (1.5) has piecewise constant arguments, and that a solution $(x(t), y(t))$ of system (1.5) for $t \in [0, +\infty)$ possesses the following characteristics:
1. on the interval $[0, +\infty)$, $x(t)$ and $y(t)$ are continuous;
2. when $t \in [0, +\infty)$, except for the points $t \in \{0, 1, 2, 3, \cdots\}$, $\frac{dx(t)}{dt}$ and $\frac{dy(t)}{dt}$ exist everywhere.

The following system can be obtained by integrating system (1.5) with the interval [n,t] for any $t \in [n, n+1)$ and $n = 0, 1, 2, \cdots$

$$\begin{cases} x(t) = x_n e^{1-x_n-a_1y_n-\frac{b}{c+x_n}}(t-n), \\ y(t) = y_n e^{\rho(1-y_n-a_2x_n)}(t-n), \end{cases} \tag{1.6}$$

where $x_n = x(n)$ and $y_n = y(n)$.

Letting $t \to (n+1)^-$ in (1.6), it produces

$$\begin{cases} x_{n+1} = x_n e^{1-x_n-a_1y_n-\frac{b}{c+x_n}}, \\ y_{n+1} = y_n e^{\rho(1-y_n-a_2x_n)}, \end{cases} \tag{1.7}$$

where $a_1, a_2, b, c, \rho > 0$, are the same as those in (1.4).

This paper is organized as follows: In Section 2, we analyze the existence of fixed points of system (1.7). In Section 3, we investigate the local stability of fixed points of system (1.7). In Section 4, we derive the sufficient conditions for the occurence of the transcritical bifurcation, pitchfork bifurcation and period-doubling bifurcation of system (1.7). In Section 5, we present some numerical simulations to verify the corresponding theoretical results. Finally, we draw some conclusions and discussions in Section 6.

Before we analyze the fixed points of system (1.7), we recall the following lemma (see [22, p422]).

Lemma 1.1. *Let $F(\lambda) = \lambda^2 + B\lambda + C$, where B and C are two real constants. Suppose that λ_1 and λ_2 are two roots of $F(\lambda) = 0$. Then, the following statements hold.*

(i) If $F(1) > 0$, then

$(i.1)$ $|\lambda_1| < 1$ and $|\lambda_2| < 1$, if and only if $F(-1) > 0$ and $C < 1$;

$(i.2)$ $\lambda_1 = -1$ and $\lambda_2 \neq -1$, if and only if $F(-1) = 0$ and $B \neq 2$;

$(i.3)$ $|\lambda_1| < 1$ and $|\lambda_2| > 1$, if and only if $F(-1) < 0$;

$(i.4)$ $|\lambda_1| > 1$ and $|\lambda_2| > 1$, if and only if $F(-1) > 0$ and $C > 1$;

$(i.5)$ λ_1 and λ_2 are a pair of conjugate complex roots, and $|\lambda_1| = |\lambda_2| = 1$, if and only if $-2 < B < 2$ and $C = 1$;

$(i.6)$ $\lambda_1 = \lambda_2 = -1$, if and only if $F(-1) = 0$ and $B = 2$.

(ii) If $F(1) = 0$, namely, 1 is one root of $F(\lambda) = 0$, then the another root λ satisfies $|\lambda| = (<, >)1$, if and only if $|C| = (<, >)1$.

(iii) If $F(1) < 0$, then $F(\lambda) = 0$ has one root lying in $(1, \infty)$. Moreover,

$(iii.1)$ the other root λ satisfies $\lambda < (=) - 1$, if and only if $F(-1) < (=)0$;

$(iii.2)$ the other root $-1 < \lambda < 1$, if and only if $F(-1) > 0$.

2. The existence of fixed points

The fixed points of system (1.7) satisfy the following equations:

$$x = xe^{1-x-a_1y-\frac{b}{c+x}},\ y = ye^{\rho(1-y-a_2x)},$$

i.e.,

$$\begin{aligned} & x\left(1-x-a_1y-\frac{b}{c+x}\right)=0, \\ & y\left(1-y-a_2x\right)=0. \end{aligned} \tag{2.1}$$

We only consider nonnegative fixed points due to the biological meanings of system (1.7). Obviously, system (1.7) always has two boundary fixed points $E_0(0,0)$ and $E_1(0,1)$ for all parameters. For other boundary fixed points and positive fixed points, we discuss the following cases.

1. When $x \neq 0, y=0$, the other fixed points of system (1.7) are determined by the following conditions: x is nonnegative and satisfies the equation

$$x^2-(1-c)\,x+b-c=0, \tag{2.2}$$

and $y=0$. Let Δ_1 denote the discriminant of equation (2.2), i.e.,

$$\Delta_1=(1+c)^2-4b.$$

Then

$$\Delta_1>(=,<)0 \Leftrightarrow b<(=,>)\frac{(1+c)^2}{4}.$$

If the other fixed points for system (1.7) exist, then $\Delta_1 \geq 0$, i.e., $b \leq \frac{(1+c)^2}{4}$.

Thereout,

$$x_{21}=\frac{1-c-\sqrt{\Delta_1}}{2}, x_{22}=\frac{1-c+\sqrt{\Delta_1}}{2}.$$

Besides, we notice that $c \leq \frac{(1+c)^2}{4}$ and $c=\frac{(1+c)^2}{4}$ if and only if $c=1$.

Therefore,we can get the following results.

(1) If $0<b<c, x_{21}<0, x_{22}>0$.

(2) If $b=c$, when $0<c<1, x_{21}=0, x_{22}>0$; when $c=1, x_{21}=x_{22}=0$; when $c>1, x_{21}<0, x_{22}=0$.

(3) If $c<b<\frac{(1+c)^2}{4}$, when $0<c<1, x_{21}>0, x_{22}>0$; when $c>1, x_{21}<0, x_{22}<0$.

(4) If $b=\frac{(1+c)^2}{4}$, when $0<c<1, x_{23}:=x_{21}=x_{22}=\frac{1-c}{2}>0$; when $c=1, x_{23}:=x_{21}=x_{22}=0$; when $c>1, x_{23}:=x_{21}=x_{22}<0$.

(5) If $b>\frac{(1+c)^2}{4}$, system (1.7) has no other boundary fixed points.

2. When $x \neq 0,\ y \neq 0$, the possible positive fixed points of system (1.7) satisfy the following equation:

$$\begin{aligned} & 1-x-a_1y-\frac{b}{c+x}=0, \\ & 1-y-a_2x=0, \end{aligned} \tag{2.3}$$

i.e., x is a positive root of the equation:

$$Ax^2-Bx+C=0, \tag{2.4}$$

where $A = a_1a_2 - 1,\ B = a_1 + c - a_1a_2c - 1,\ C = c - a_1c - b$, and $y = 1 - a_2x > 0$.

Let the discriminant of (2.4) be denoted by Δ_2, i.e.,

$$\Delta_2 = B^2 - 4AC = (-cA - a_1 + 1)^2 + 4bA.$$

It is obvious that $\Delta_2 > 0$, if $A > 0$.

When $\Delta_2 \geq 0$, there exist positive fixed points of system (1.7), and

$$x_{31} = \frac{B - \sqrt{\Delta_2}}{2A},\ x_{32} = \frac{B + \sqrt{\Delta_2}}{2A}. \tag{2.5}$$

(1) If $\Delta_2 > 0$, we consider the following cases:

Case 1: $A > 0, C < 0$. Then, $x_{31} < 0, x_{32} > 0$ and system (1.7) has only one positive fixed point $E_{32}(x_{32}, y_{32}) = (x_{32}, 1 - a_2x_{32})$, if $x_{32} < \frac{1}{a_2}$.

Case 2: $A < 0, C > 0$. Then, $x_{31} > 0, x_{32} < 0$ and system (1.7) has only one positive fixed point $E_{31}(x_{31}, y_{31}) = (x_{31}, 1 - a_2x_{31})$, if $x_{31} < \frac{1}{a_2}$.

Case 3: $A < 0, B < 0, C < 0$. Then, $x_{31} > x_{32} > 0$, or $A > 0, B > 0, C > 0$, then $x_{32} > x_{31} > 0$ and system (1.7) has two positive fixed points:

$$E_{31}(x_{31}, y_{31}) = (x_{31}, 1 - a_2x_{31})$$

and

$$E_{32}(x_{32}, y_{32}) = (x_{32}, 1 - a_2x_{32}).$$

Both E_{31} and E_{32} exist, if $max\,\{x_{31}, x_{32}\} < \frac{1}{a_2}$.

Case 4: $A > 0, B < 0, C = 0$. Then, $x_{32} = 0 > x_{31}$. Or $A < 0, B > 0, C = 0$, then $x_{31} = 0 > x_{32}$ and system (1.7) has no positive fixed point.

Case 5: $A < 0, B < 0, C = 0$. Then, $x_{31} > 0 = x_{32}$ and system (1.7) only has one positive fixed point $E_{31}(x_{31}, y_{31}) = (x_{31}, 1 - a_2x_{31})$, if $x_{31} < \frac{1}{a_2}$.

Case 6: $A > 0, B > 0, C = 0$. Then, $x_{32} > 0 = x_{31}$ and system (1.7) has only one positive fixed point $E_{32}(x_{32}, y_{32}) = (x_{32}, 1 - a_2x_{32})$, if $x_{32} < \frac{1}{a_2}$.

(2) If $\Delta_2 = 0, B < 0$, then $x_{33} := x_{31} = x_{32} = \frac{B}{2A} > 0$ and system (1.7) has only one positive fixed point $E_{33}(x_{33}, y_{33}) = (\frac{B}{2A}, 1 - a_2\frac{B}{2A})$, if $\frac{B}{2A} < \frac{1}{a_2}$.

(3) If $\Delta_2 < 0$, then system (1.7) has no positive fixed point.

From what have discussed above, we can get the following results.

Theorem 2.1. *System* (1.7) *always has two boundary fixed points* $E_0(0, 0)$ *and* $E_1(0, 1)$ *for all parameters. The other possible boundary fixed points and positive fixed points are as follows.*

1. *For other possible boundary fixed points:*

(1) *if* $0 < b < c$, *system* (1.7) *has only one additional boundary fixed point* $E_{22}(x_{22}, 0) = (\frac{1-c+\sqrt{(1+c)^2-4b}}{2}, 0)$*;*

(2) *if* $b = c$ *and* $0 < c < 1$, *system* (1.7) *has only one additional boundary fixed point* $E_{22}(x_{22}, 0) = (\frac{1-c+\sqrt{(1+c)^2-4b}}{2}, 0)$*;*

(3) *if* $c < b < \frac{(1+c)^2}{4}$ *and* $0 < c < 1$, *system* (1.7) *has two additional boundary fixed points* $E_{21}(x_{21}, 0) = (\frac{1-c-\sqrt{(1+c)^2-4b}}{2}, 0)$ *and* $E_{22}(x_{22}, 0) = (\frac{1-c+\sqrt{(1+c)^2-4b}}{2}, 0)$*;*

(4) *if* $b = \frac{(1+c)^2}{4}$ *and* $0 < c < 1$, *system* (1.7) *has only one additional boundary fixed point* $E_{23}(x_{23}, 0) = (\frac{1-c}{2}, 0)$*;*

(5) *if* $b > \frac{(1+c)^2}{4}$, *system* (1.7) *has no additional boundary fixed point.*

2. *For possible positive fixed points:*

(1) *when* $\Delta_2 > 0$, *we have the following results.*

(1.1) *If* $A < 0, C > 0$ *or* $A < 0, B < 0, C = 0$, *then system* (1.7) *has only one positive fixed point* $E_{31}(x_{31}, y_{31})$ *for* $x_{31} < \frac{1}{a_2}$.

(1.2) *If* $A > 0, C < 0$ *or* $A > 0, B > 0, C = 0$, *then system* (1.7) *has only one positive fixed point* $E_{32}(x_{32}, y_{32})$ *for* $x_{32} < \frac{1}{a_2}$.

(1.3) *If* $A < 0, B < 0, C < 0$ *or* $A > 0, B > 0, C > 0$, *then system* (1.7) *has two positive fixed point* $E_{31}(x_{31}, y_{31})$ *and* $E_{32}(x_{32}, y_{32})$ *for max* $\{x_{31}, x_{32}\} < \frac{1}{a_2}$.

(2) *When* $\Delta_2 = 0$, *then system* (1.7) *has only one positive fixed points* $E_{33}(x_{33}, y_{33})$ *for* $x_{33} < \frac{1}{a_2}$.

(3) *When* $\Delta_2 < 0$, *then system* (1.7) *has no positive fixed point.*

3. Stability of fixed points

The Jacobian matrix of system (1.7) at any fixed point $E(x, y)$ takes the following form

$$J(E) = \begin{pmatrix} \left(\frac{bx}{(c+x)^2} - x + 1\right)e^{1-x-a_1y-\frac{b}{c+x}} & -a_1xe^{1-x-a_1y-\frac{b}{c+x}} \\ -a_2\rho ye^{\rho(1-y-a_2x)} & (1-\rho y)e^{\rho(1-y-a_2x)} \end{pmatrix}.$$

The characteristic polynomial of Jacobian matrix $J(E)$ reads as

$$F(\lambda) = \lambda^2 - p\lambda + q,$$

where

$$p = Tr(J(E)), q = Det(J(E)).$$

Now, we formulate some results for the stability of the fixed points in the following theorems.

Theorem 3.1. *The following statements about the boundary fixed points* $E_0(0, 0)$ *and* $E_1(0, 1)$ *of system* (1.7) *are true.*

1. *For* $E_0(0, 0)$, *we have the following results:*
   1) *If* $b < c$, *then* E_0 *is an unstable node;*
   2) *If* $b = c$, *then* E_0 *is non-hyperbolic;*
   3) *If* $b > c$, *then* E_0 *is a saddle.*
2. *For* $E_1(0, 1)$, *we have the following results:*
   1) *When* $0 < \rho < 2$,
      (1.1) if $0 < a_1 < 1 - \frac{b}{c}$, *then* E_1 *is a saddle;*
      (1.2) if $a_1 = 1 - \frac{b}{c}$, *then* E_1 *is non-hyperbolic;*
      (1.3) if $a_1 > 1 - \frac{b}{c}$, *then* E_1 *is a stable node.*
   2) *When* $\rho = 2$, E_1 *is non-hyperbolic.*
   3) *If* $\rho > 2$,
      (3.1) if $0 < a_1 < 1 - \frac{b}{c}$, *then* E_1 *is an unstable node;*
      (3.2) if $a_1 = 1 - \frac{b}{c}$, *then* E_1 *is non-hyperbolic;*
      (3.3) if $a_1 > 1 - \frac{b}{c}$, *then* E_1 *is a saddle.*

Proof. 1. The Jacobian matrix of system (1.7) at $E_0 = (0, 0)$ is

$$J(E_0) = \begin{pmatrix} e^{1-\frac{b}{c}} & 0 \\ 0 & e^{\rho} \end{pmatrix}.$$

Obviously, $\lambda_1 = e^{1-\frac{b}{c}}$ and $\lambda_2 = e^{\rho}$.

Note that $|\lambda_2| > 1$ is always true. If $b < c$, then $|\lambda_1| > 1$. Therefore, E_0 is an unstable node, i.e., a source; if $b = c$, then $|\lambda_1| = 1$, so E_0 is non-hyperbolic; if $b > c$, implying $|\lambda_1| < 1$, then E_0 is a saddle.

2. The Jacobian matrix of system (1.7) at $E_1 = (0, 1)$ can be simplified as follows:

$$J(E_1) = \begin{pmatrix} e^{1-a_1-\frac{b}{c}} & 0 \\ -a_2\rho & 1-\rho \end{pmatrix}.$$

Obviously, $\lambda_1 = e^{1-a_1-\frac{b}{c}}$ and $\lambda_2 = 1 - \rho$.

When $0 < \rho < 2$, $|\lambda_2| < 1$. If $0 < a_1 < 1 - \frac{b}{c}$, it means $|\lambda_1| > 1$, then E_1 is a saddle; if $a_1 = 1 - \frac{b}{c}$, then $|\lambda_1| = 1$, so E_1 is non-hyperbolic; if $a_1 > 1 - \frac{b}{c}$, then $|\lambda_1| < 1$. Therefore, E_1 is a stable node, i.e., a sink.

When $\rho = 2$, we imply $|\lambda_2| = 1$, then E_1 is non-hyperbolic.

When $\rho > 2$, $|\lambda_2| > 1$. If $0 < a_1 < 1 - \frac{b}{c}$, it means $|\lambda_1| > 1$, then E_1 is an unstable node; if $a_1 = 1 - \frac{b}{c}$, then $|\lambda_1| = 1$, so E_1 is non-hyperbolic; if $a_1 > 1 - \frac{b}{c}$, then $|\lambda_1| < 1$. Therefore, E_1 is a saddle.

This completes the proof. □

Theorem 3.2. *For the boundary fixed points E_{21}, E_{22} and E_{23} of system* (1.7), *we have the following results:*

1. *Assume $c < b < \frac{(1+c)^2}{4}$ and $0 < c < 1$, then E_{21} exists, and we have the following results:*
   1) *If $0 < a_2 < \frac{2}{1-c-\sqrt{(1+c)^2-4b}}$, then E_{21} is an unstable node;*
   2) *If $a_2 = \frac{2}{1-c-\sqrt{(1+c)^2-4b}}$, then E_{21} is non-hyperbolic;*
   3) *If $a_2 > \frac{2}{1-c-\sqrt{(1+c)^2-4b}}$, then E_{21} is a saddle.*
2. *Assume $0 < b < c$ or $c \le b < \frac{(1+c)^2}{4}$ and $0 < c < 1$, then E_{22} exists, and we have the following results:*
   1) *If $0 < a_2 < \frac{2}{1-c+\sqrt{(1+c)^2-4b}}$, then E_{22} is a saddle;*
   2) *If $a_2 = \frac{2}{1-c+\sqrt{(1+c)^2-4b}}$, then E_{22} is non-hyperbolic;*
   3) *If $a_2 > \frac{2}{1-c+\sqrt{(1+c)^2-4b}}$, then E_{22} is a stable node.*
3. *Assume $b = \frac{(1+c)^2}{4}$ and $0 < c < 1$, then E_{23} exists, and it is always non-hyperbolic.*

Proof. The boundary fixed points satisfy

$$1 - x_{2i} - a_1 y_{2i} - \frac{b}{c + x_{2i}} = 0, y_{2i} = 0,$$

where, $i = 1, 2, 3$. The Jacobian matrix of system (1.7) at E_{2i} can be written as

$$J(E_{2i}) = \begin{pmatrix} \frac{2bx_{2i}+bc}{(c+x_{2i})^2} & -a_1x_{2i} \\ 0 & e^{\rho(1-a_2x_{2i})} \end{pmatrix},$$

where, $i = 1, 2, 3$.

1. It is easy to get that the eigenvalues of $J(E_{21})$ are $\lambda_1 = \frac{2bx_{21}+bc}{(c+x_{21})^2}$ and $\lambda_2 = e^{\rho(1-a_2x_{21})}$.

In order to compare the quantity λ_1 with 1, noticing that the numerator and the denominator of λ_1 are positive, we only need to consider the sign of $2bx_{21} + bc - (c + x_{21})^2$. Notice

$$2bx_{21} + bc - (c + x_{21})^2 = \frac{\sqrt{\Delta_1}(1 + c - \sqrt{\Delta_1} - 2b)}{2},$$

and

$$\begin{aligned} 1 + c - \sqrt{\Delta_1} - 2b &= 2b(\frac{2}{1 + c + \sqrt{\Delta_1}} - 1) \\ &> 2b(\frac{2}{1 + c + (1 - c)} - 1) = 0, \end{aligned}$$

in which we have used the fact that $c < b$ and $0 < c < 1$.

The above analysis shows that $\lambda_1 > 1$. If $0 < a_2 < \frac{1}{x_{21}}$, then $|\lambda_2| > 1$. Therefore, E_{21} is an unstable node; if $a_2 = \frac{1}{x_{21}}$, then $|\lambda_2| = 1$, so E_{21} is non-hyperbolic; if $a_2 > \frac{1}{x_{21}}$, we imply $|\lambda_1| < 1$, then E_{21} is a saddle.

2. The eigenvalues of $J(E_{22})$ are $\lambda_1 = \frac{2bx_{22}+bc}{(c+x_{22})^2}$ and $\lambda_2 = e^{\rho(1-a_2x_{22})}$. Similarly, we have

$$\begin{aligned} 2bx_{22} + bc - (c + x_{22})^2 &= -\frac{\sqrt{\Delta_1}(1 + c + \sqrt{\Delta_1} - 2b)}{2} \\ &= -b\sqrt{\Delta_1}(\frac{2}{1 + c - \sqrt{\Delta_1}} - 1). \end{aligned}$$

From Theorem (2.1), we know that the conditions for the existence of E_{22} are $0 < b < c$ or $c \leq b < \frac{(1+c)^2}{4}$ and $0 < c < 1$. Let $N(b) = 1 + c - \sqrt{\Delta_1} = 1 + c - \sqrt{(1 + c)^2 - 4b}$, and note that $N(b)$ is monotonically increasing with respect to b in the interval $(0, \frac{(1+c)^2}{4})$. Therefore, when $0 < b < c$, we have

$$N(b) < N(c) = 1 + c - |1 - c| < 2.$$

When $c \leq b < \frac{(1+c)^2}{4}$, noticing $0 < c < 1$, we have

$$N(b) < N(\frac{(1 + c)^2}{4}) = 1 + c < 2.$$

Accordingly, we can conclude that $N(b) < 2$ is always true when E_{22} exists, which implies $0 < \lambda_1 < 1$.

If $0 < a_2 < \frac{1}{x_{22}}$, then $|\lambda_2| > 1$. Therefore, E_{22} is a saddle; if $a_2 = \frac{1}{x_{22}}$, then $|\lambda_2| = 1$, so E_{22} is non-hyperbolic; if $a_2 > \frac{1}{x_{22}}$, we imply $|\lambda_1| < 1$, then E_{22} is a stable node.

3. The eigenvalues of $J(E_{23})$ are $\lambda_1 = \frac{2bx_{23}+bc}{(c+x_{23})^2}$ and $\lambda_2 = e^{\rho(1-a_2x_{23})}$. It is clear that

$$2bx_{23} + bc = b(1-c) + bc = b$$

and

$$(c + x_{23})^2 = (\frac{1+c}{2})^2 = b.$$

Therefore, $\lambda_1 = 1$ and E_{23} is non-hyperbolic. The proof is completed. □

Theorem 3.3. *For the positive fixed points of system* (1.7)*, one has the following consequences.*

1. *Assume* $\Delta_2 > 0$. *If* $A < 0, C > 0$ *or* $A < 0, B < 0, C = 0$ *or* $A < 0, B < 0, C < 0$ *or* $A > 0, B > 0, C > 0$*, then* E_{31} *exists for* $x_{31} < \frac{1}{a_2}$*. Let*

$$\rho_s = 2\left(\frac{b\left(2x_{31}+c\right)}{\left(c+x_{31}\right)^2} + a_1y_{31} + 1\right) \Big/ \left(\frac{by_{31}\left(2x_{31}+c\right)}{\left(c+x_{31}\right)^2} + y_{31}(a_1+1)\right)$$

and

$$\rho_t = \left(\frac{b\left(2x_{31}+c\right)}{\left(c+x_{31}\right)^2} + a_1y_{31} - 1\right) \Big/ \left(\frac{by_{31}\left(2x_{31}+c\right)}{\left(c+x_{31}\right)^2} + a_1y_{31}\right).$$

The following results hold:

1) E_{31} *is a source if* $\rho < min\left\{\rho_s, \rho_t\right\}$;

2) E_{31} *is non-hyperbolic if* $\rho = \rho_s$;

3) E_{31} *is a saddle if* $\rho > \rho_s$.

2. *Assume* $\Delta_2 > 0$. *If* $A > 0, C < 0$ *or* $A > 0, B > 0, C = 0$ *or* $A < 0, B < 0, C < 0$ *or* $A > 0, B > 0, C > 0$*, then* E_{32} *exists for* $x_{32} < \frac{1}{a_2}$*. Let*

$$\rho_u = 2\left(\frac{b\left(2x_{32}+c\right)}{\left(c+x_{32}\right)^2} + a_1y_{32} + 1\right) \Big/ \left(\frac{by_{32}\left(2x_{32}+c\right)}{\left(c+x_{32}\right)^2} + y_{32}(a_1+1)\right).$$

The following results hold:

1) If $\rho < \rho_u$*, then* E_{32} *is a saddle;*

2) If $\rho = \rho_u$*, then* E_{32} *is non-hyperbolic;*

3) If $\rho > \rho_u$*, then* E_{32} *is a source.*

3. *Assume* $\Delta_2 = 0$ *and* $\frac{B}{2A} < \frac{1}{a_2}$*, then* E_{33} *exists, and it is always non-hyperbolic.*

Proof. The positive fixed points satisfy

$$1 - x_{3i} - a_1y_{3i} - \frac{b}{c + x_{3i}} = 0,\ 1 - y_{3i} - a2x_{3i} = 0,$$

where, $i = 1, 2, 3$. Therefore, the Jacobian matrix of system (1.7) at E_{3i} can be written as

$$J(E_{3i}) = \begin{pmatrix} \frac{b(2x_{3i}+c)}{(c+x_{3i})^2} + a_1y_{3i} & -a_1x_{3i} \\ -a_2\rho y_{3i} & 1 - \rho y_{3i} \end{pmatrix},$$

where, $i = 1, 2, 3$.

The characteristic polynomial of Jacobian matrix $J(E_{3i})$ is

$$F(\lambda) = \lambda^2 - p\lambda + q,$$

where

$$p = Tr(J(E_{3i})) = \frac{b\left(2x_{3i} + c\right)}{\left(c + x_{3i}\right)^2} + (a_1 - \rho)y_{3i} + 1,$$

$$q = Det(J(E_{3i})) = \frac{b\left(2x_{3i} + c\right)}{\left(c + x_{3i}\right)^2}(1 - \rho y_{3i}) + (1 - \rho)a_1 y_{3i}.$$

We have

$$\begin{aligned} F(1) &= 1 - Tr(J(E_{3i})) + Det(J(E_{3i})) \\ &= \rho y_{3i}\left(1 - \frac{b\left(2x_{3i} + c\right)}{\left(c + x_{3i}\right)^2} - a_1\right) \\ &= -\frac{\rho x_{3i} y_{3i}}{x_{3i} + c}\left(2Ax_{3i} - B\right), \end{aligned} \tag{3.1}$$

where, $i = 1, 2, 3$.

1. Substituting $x_{31} = \frac{B-\sqrt{\Delta_2}}{2A}$ into the equation (3.1), we can get

$$F(1) = \frac{\rho x_{31} y_{31}\sqrt{\Delta_2}}{x_{31} + c} > 0.$$

Besides,

$$\begin{aligned} F(-1) &= 1 + Tr(J(E_{31})) + Det(J(E_{31})) \\ &= \frac{b\left(2x_{31} + c\right)}{\left(c + x_{31}\right)^2}\left(2 - \rho y_{31}\right) + 2a_1 y_{31} - (a_1 + 1)\rho y_{31} + 2, \end{aligned}$$

$$F(-1) > (=, <)0 \Leftrightarrow \rho < (=, >)\rho_s,$$

and

$$\begin{aligned} q &= Det(J(E_{31})) \\ &= \frac{b\left(2x_{31} + c\right)}{\left(c + x_{31}\right)^2}(1 - \rho y_{31}) + (1 - \rho)a_1 y_{31}, \end{aligned}$$

$$q - 1 > (=, <)0 \Leftrightarrow \rho < (=, >)\rho_t.$$

By Lemma (1.1), when $\rho < min\left\{\rho_s, \rho_t\right\}$, $|\lambda_1| > 1$, and $|\lambda_2| > 1$. Therefore, E_{31} is a source.

When $\rho = \rho_s$, $F(-1) = 0$, therefore E_{31} is non-hyperbolic.

When $\rho > \rho_s$, $|\lambda_1| < 1$, and $|\lambda_2| > 1$, then E_{31} is a saddle.

2. Substituting $x_{32} = \frac{B+\sqrt{\Delta_2}}{2A}$ into the equation (3.1), we can get

$$F(1) = -\frac{\rho x_{32} y_{32}\sqrt{\Delta_2}}{x_{32} + c} < 0.$$

By Lemma (1.1), we have $|\lambda_1| > 1$.

Besides,

$$\begin{aligned} F(-1) &= 1 + Tr(J(E_{32})) + Det(J(E_{32})) \\ &= \frac{b\,(2x_{32}+c)}{(c+x_{32})^2}\,(2-\rho y_{32}) + 2a_1 y_{32} - (a_1+1)\rho y_{32} + 2, \end{aligned}$$

$$F(-1) > (=,<) 0 \Leftrightarrow \rho < (=,>) \rho_u.$$

By Lemma (1.1), if $\rho < \rho_u$, $|\lambda_2| < 1$, then E_{32} is a saddle; if $\rho = \rho_u$, $\lambda_2 = -1$, so E_{32} is non-hyperbolic; if $\rho > \rho_u$, $\lambda_2 < -1$ and $|\lambda_2| > 1$, therefore E_{32} is a source.

3. Similarly, we have $F(1)$ of $J(E_{33})$ is equal to 0, i.e., $F(1) = 0$. Therefore, from Lemma (1.1), E_{33} is always non-hyperbolic.

The proof is finished. □

4. Bifurcation analysis

In this section, we are in a position to use the Center Manifold Theorem and bifurcation theorem to analyze the local bifurcation problems of the fixed points E_0, E_1, E_{21} and E_{22}. The study on E_{23}, E_{31}, E_{32} and E_{33} is left as our future work. For the related work, we refer to [16,18,20,22,27,28].

4.1. For fixed point $E_0 = (0,0)$

Theorem (3.1) shows that a bifurcation of E_0 may occur in the space of parameters $(a_1, a_2, b, c, \rho) \in S_{E_+} = \{(a_1, a_2, b, c, \rho) \in R^5_+ | a_1 > 0, a_2 > 0, b > 0, c > 0, \rho > 0.\}$.

Theorem 4.1. *Set the parameters* $(a_1, a_2, b, c, \rho) \in S_{E_+} = \{(a_1, a_2, b, c, \rho) \in R^5_+ | a_1 > 0, a_2 > 0, b > 0, c > 0, \rho > 0.\}$. *Let* $b_0 = c$. *If* $c \neq 1$, *then system* (1.7) *undergoes a transcritical bifurcation at* E_0, *when the parameter* b *varies in a small neighborhood of critical value* b_0. *If* $c = 1$, *then system* (1.7) *undergoes a pitchfork bifurcation at* E_0, *when the parameter* b *varies in a small neighborhood of critical value* b_0.

Proof. In order to show the detailed process, we proceed according to the following steps.

Step 1. Giving a small perturbation b^* of the parameter b around the critical value b_0, i.e., $b^* = b - b_0$, with $0 < |b^*| \ll 1$, system (1.7) is perturbed into

$$\begin{cases} x_{n+1} = x_n e^{1 - x_n - a_1 y - \frac{b^* + b_0}{c + x_n}}, \\ y_{n+1} = y_n e^{\rho(1 - y_n - a_2 x_n)}. \end{cases} \tag{4.1}$$

Letting $b^*_{n+1} = b^*_n = b^*$, system (4.1) can be written as

$$\begin{cases} x_{n+1} = x_n e^{1 - x_n - a_1 y_n - \frac{b^*_n + b_0}{c + x_n}}, \\ y_{n+1} = y_n e^{\rho(1 - y_n - a_2 x_n)}, \\ b^*_{n+1} = b^*_n. \end{cases} \tag{4.2}$$

Step 2. Taylor expanding of system (4.2) at $(x_n, y_n, b_n^*) = (0,0,0)$ takes the form

$$\begin{cases} x_{n+1} = a_{100}x_n + a_{010}y_n + a_{001}b_n^* + a_{200}x_n^2 + a_{020}y_n^2 \\ \qquad +a_{002}{b_n^*}^2 + a_{110}a_ny_n + a_{101}x_nb_n^* + a_{011}y_nb_n^* \\ \qquad +a_{300}x_n^3 + a_{030}y_n^3 + a_{003}{b_n^*}^3 + a_{210}x_n^2y_n \\ \qquad +a_{120}x_ny_n^2 + a_{021}y_n^2b_n^* + a_{201}x_n^2b_n^* + a_{102}x_n{b_n^*}^2 \\ \qquad +a_{012}y_n{b_n^*}^2 + a_{111}x_ny_nb_n^* + o(\rho_1^3), \\ y_{n+1} = b_{100}x_n + b_{010}y_n + b_{200}x_n^2 + b_{020}y_n^2 + b_{110}x_ny_n \\ \qquad +b_{300}x_n^3 + b_{030}y_n^3 + b_{210}x_n^2y_n + b_{120}x_ny_n^2 + o(\rho_1^3), \\ b_{n+1}^* = b_n^*, \end{cases} \tag{4.3}$$

where

$$\begin{aligned} &\rho_1 = \sqrt{x_n^2 + y_n^2 + (b_n^*)^2}, \\ &a_{010} = a_{001} = a_{020} = a_{002} = a_{011} = a_{030} = a_{003} = a_{021} = a_{012} = 0, a_{100} = 1, \\ &a_{200} = \frac{1}{c} - 1, a_{110} = -a_1, a_{101} = -\frac{1}{c}, a_{300} = \frac{c^2 - 2c - 1}{2c^2}, \\ &a_{210} = \frac{a_1(c-1)}{c}, a_{120} = \frac{a_1^2}{2}, a_{201} = \frac{1}{c}, a_{102} = \frac{1}{2c^2}, a_{111} = \frac{a_1}{c}, \\ &b_{100} = b_{200} = b_{300} = 0, b_{010} = e^\rho, b_{020} = -\rho e^\rho, b_{110} = -a_2\rho e^\rho, b_{030} = \frac{\rho^2 e^\rho}{2}, \\ &b_{210} = \frac{a_2^2\rho^2 e^\rho}{2}, b_{120} = a_2\rho^2 e^\rho. \end{aligned}$$

Let

$$J(E_0) = \begin{pmatrix} a_{100} & a_{010} & 0 \\ b_{100} & b_{010} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad i.e., J(E_0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^\rho & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Therefore, we rewrite system (4.3) as the following form

$$\begin{cases} x_{n+1} = x_n + F(x_n, y_n, b_n^*) + o(\rho_1^3), \\ y_{n+1} = e^\rho y_n + G(x_n, y_n, b_n^*) + o(\rho_1^3), \\ b_{n+1}^* = b_n^*, \end{cases} \tag{4.4}$$

where

$$\begin{aligned} F(x_n, y_n, b_n^*) =& a_{200}x_n^2 + a_{020}y_n^2 + a_{002}{b_n^*}^2 + a_{110}x_ny_n \\ &+ a_{101}x_nb_n^* + a_{011}y_nb_n^* + a_{300}x_n^3 + a_{030}y_n^3 \\ &+ a_{003}{b_n^*}^3 + a_{210}x_n^2y_n + a_{120}x_ny_n^2 + a_{021}y_n^2b_n^* \\ &+ a_{201}x_n^2b_n^* + a_{102}x_n{b_n^*}^2 + a_{012}y_n{b_n^*}^2 + a_{111}x_ny_nb_n^*, \end{aligned}$$

$$\begin{aligned}G(x_n,y_n,b_n^*)=&b_{200}x_n^2+b_{020}y_n^2+b_{110}x_ny_n+b_{300}x_n^3\\&+b_{030}y_n^3+b_{210}x_n^2y_n+b_{120}x_ny_n^2.\end{aligned}$$

Step 3. Suppose that on the center manifold

$$y_n=h(x_n,b_n^*)=h_{20}x_n^2+h_{11}x_nb_n^*+h_{02}b_n^{*2}+o(\rho_2^2),$$

where $\rho_2=\sqrt{x_n^2+b_n^{*2}}$. Then, according to

$$y_{n+1}=e^{\rho}h(x_n,b_n^*)+G(x_n,h(x_n,b_n^*),b_n^*)+o(\rho_2^3),$$

$$\begin{aligned}h(x_{n+1},b_{n+1}^*)=&h_{20}x_{n+1}^2+h_{11}x_{n+1}b_{n+1}^*+h_{02}(b_{n+1}^*)^2+o(\rho_2^2)\\=&h_{20}(x_n+F(x_n,h(x_n,b_n^*),b_n^*))^2+h_{11}(x_n+F(x_n,h(x_n,b_n^*),b_n^*))b_n^*\\&+h_{02}b_n^{*2}+o(\rho_2^2)\end{aligned}$$

and $y_{n+1}=h(x_{n+1},b_{n+1}^*)$, we obtain the center manifold equation

$$\begin{aligned}e^{\rho}h(x_n,b_n^*)+G(x_n,h(x_n,b_n^*),b_n^*)=&h_{20}(x_n+F(x_n,h(x_n,b_n^*),b_n^*))^2\\&+h_{11}(x_n+F(x_n,h(x_n,b_n^*),b_n^*))b_n^*+h_{02}b_n^{*2}.\end{aligned}$$

Comparing the corresponding coefficients of terms with the same orders in the above center manifold equation, we get

$$h_{20}=h_{11}=h_{02}=0.$$

Hence, system (4.4) restricted to the center manifold takes as

$$\begin{aligned}x_{n+1}=f_1(x_n,b_n^*):=&x_n+F(x_n,h(x_n,b_n^*),b_n^*)+o(\rho_2^2)\\=&x_n+\left(\frac{1}{c}-1\right)x_n^2-\frac{1}{c}x_nb_n^*+\frac{c^2-2c-1}{2c^2}x_n^3\\&+\frac{1}{c}x_n^2b_n^*+\frac{1}{2c^2}x_nb_n^{*2}+o(\rho_2^3).\end{aligned}$$

Therefore, one has

$$\begin{aligned}&f_1(x_n,b_n^*)|_{(0,0)}=0,\ \left.\frac{\partial f_1}{\partial x_n}\right|_{(0,0)}=1,\ \left.\frac{\partial f_1}{\partial b_n^*}\right|_{(0,0)}=0,\ \left.\frac{\partial^2 f_1}{\partial x_n\partial b_n^*}\right|_{(0,0)}=-\frac{1}{c}\neq 0,\\&\left.\frac{\partial^2 f_1}{\partial x_n^2}\right|_{(0,0)}=2\left(\frac{1}{c}-1\right),\ \left.\frac{\partial^3 f_1}{\partial x_n^3}\right|_{(0,0)}=\frac{3\left(c^2-2c-1\right)}{c^2}.\end{aligned}$$

According to (21.1.42)-(21.1.46) in [23, p507], if $c\neq 1$, then $\left.\frac{\partial^2 f}{\partial x_n^2}\right|_{(0,0)}\neq 0$. All the conditions for the occurrence of the transcritical bifurcation are established. Hence, it is valid for the occurrence of the transcritical bifurcation in the fixed point E_0.

When $c=1$, it is clear that $\left.\frac{\partial^2 f_1}{\partial x_n^2}\right|_{(0,0)}=0$ and $\left.\frac{\partial^3 f_1}{\partial x_n^3}\right|_{(0,0)}=-6\neq 0$. From (21.1.70)-(21.1.75) in [23, p511], system (1.7) undergoes a pitchfork bifurcation at E_0. □

4.2. For fixed point $E_1 = (0, 1)$

The fixed point $E_1(0, 1)$ always exists regardless of what values all the parameters take. When $a_1 = a_{10} := 1 - \frac{b}{c}$ or $\rho = 2$, Theorem (3.1) shows that E_1 is a non-hyperbolic fixed point. As soon as the parameter a_1 or ρ goes through corresponding critical values, the dimensional numbers for the stable manifold and the unstable manifold of the fixed point E_1 vary. Therefore, a bifurcation probably occurs. Now, the considered parameter case is divided into the following three subcases:
Case I: $a_1 = a_{10}, \rho \neq 2$;
Case II: $a_1 \neq a_{10}, \rho = 2$;
Case III: $a_1 = a_{10}, \rho = 2$.

First, we consider Case I: $a_1 = a_{10}, \rho \neq 2$, i.e., the parameters $(a_1, a_2, b, c, \rho) \in \Omega_1 = \{(a_1, a_2, b, c, \rho) \in R_+^5 | a_1 > 0, a_2 > 0, 0 < b < c, \rho \neq 2.\}$, and let $a_{10} = 1 - \frac{b}{c}$. Thereout, the following result is obtained.

Theorem 4.2. *Assume the parameters* $(a_1, a_2, b, c, \rho) \in \Omega_1 = \{(a_1, a_2, b, c, \rho) \in R_+^5 | a_1 > 0, a_2 > 0, 0 < b < c, \rho \neq 2.\}$. *Let* $a_{10} = 1 - \frac{b}{c}$. *If* $a_2 c \neq 1$, *then system* (1.7) *undergoes a transcritical bifurcation at* E_1, *when the parameter* a_1 *goes through the critical value* a_{10}.

Proof. Let $l_n = x_n - 0, m_n = y_n - 1$, which transforms $E_1(0, 1)$ to the origin $O(0, 0)$ and system (1.7) into

$$\begin{cases} l_{n+1} = l_n e^{1 - l_n - a_1(m_n + 1) - \frac{b}{c + l_n}}, \\ m_{n+1} = (m_n + 1)e^{\rho(-m_n - a_2 l_n)} - 1. \end{cases} \tag{4.5}$$

Giving a small perturbation a_1^* of the parameter a_1 around the critical value a_{10}, i.e., $a_1^* = a_1 - a_{10}$, with $0 < |a_1^*| \ll 1$, system (4.5) is perturbed into

$$\begin{cases} l_{n+1} = l_n e^{1 - l_n - (a_1^* + a_{10})(m_n + 1) - \frac{b}{c + l_n}}, \\ m_{n+1} = (m_n + 1)e^{\rho(-m_n - a_2 l_n)} - 1. \end{cases} \tag{4.6}$$

Letting $(a_1^*)_{n+1} = (a_1^*)_n = a_1^*$, (4.6) can be regarded as

$$\begin{cases} l_{n+1} = l_n e^{1 - l_n - ((a_1^*)_n + a_{10})(m_n + 1) - \frac{b}{c + l_n}}, \\ m_{n+1} = (m_n + 1)e^{\rho(-m_n - a_2 l_n)} - 1, \\ (a_1^*)_{n+1} = (a_1^*)_n. \end{cases} \tag{4.7}$$

Taylor expanding (4.7) at $(l_n, m_n, (a_1^*)_n) = (0, 0, 0)$ gets

$$\begin{pmatrix} l_{n+1} \\ m_{n+1} \\ (a_1^*)_{n+1} \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ -a_2\rho & 1 - \rho & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} l_n \\ m_n \\ (a_1^*)_n \end{pmatrix} + \begin{pmatrix} g_1\left(l_n, m_n, (a_1^*)_n\right) + o(\rho_3^3) \\ g_2\left(l_n, m_n, (a_1^*)_n\right) + o(\rho_3^3) \\ 0 \end{pmatrix}, \tag{4.8}$$

where $\rho_3 = \sqrt{l_n^2 + m_n^2 + (a_1^*)_n^2}$,

$$g_1\left(l_n, m_n, (a_1^*)_n\right) = (\frac{b}{c^2} - 1)l_n^2 + (\frac{b}{c} - 1)l_n m_n - l_n(a_1^*)_n$$

$$
\begin{aligned}
&+\left[\frac{1}{2}(\frac{b}{c^2}-1)^2-\frac{b}{c^3}\right]l_n^3+(\frac{b}{c}-1)(\frac{b}{c^2}-1)l_n^2m_n\\
&+(1-\frac{b}{c^2})l_n^2(a_1^*)_n+\frac{1}{2}(\frac{b}{c}-1)^2l_nm_n^2+\frac{1}{2}l_n(a_1^*)_n^2\\
&-\frac{b}{c}l_nm_n(a_1^*)_n,
\end{aligned}
$$

$$
\begin{aligned}
g_2\left(l_n,m_n,(a_1^*)_n\right)=&\frac{a_2^2\rho^2}{2}l_n^2+(a_2\rho^2-a_2\rho)l_nm_n+\left(\frac{\rho^2}{2}-\rho\right)m_n^2\\
&-\frac{a_2^3\rho^3}{6}+\frac{(a_2^2\rho^2-a_2^2\rho^3)}{2}l_n^2m_n+\left(a_2\rho^2-\frac{a_2\rho^3}{2}\right)l_nm_n^2\\
&+\frac{3\rho^2-\rho^3}{6}.
\end{aligned}
$$

Let $A=\begin{pmatrix}1&0&0\\-a_2\rho&1-\rho&0\\0&0&1\end{pmatrix}$. Then, we derive the three eigenvalues of A as

$$\lambda_1=1,\qquad \lambda_2=1-\rho,\qquad \lambda_3=1,$$

and the corresponding eigenvectors

$$(\xi_1,\eta_1,\varphi_1)^T=(1,-a_2,0)^T,(\xi_2,\eta_2,\varphi_2)^T=(0,1,0)^T,(\xi_3,\eta_3,\varphi_3)^T=(0,0,1)^T$$

respectively. Notice that $0<\rho\neq 2$ implies that $|\lambda_2|\neq 1$.

Take $T=\begin{pmatrix}\xi_1&\xi_2&\xi_3\\\eta_1&\eta_2&\eta_3\\\varphi_1&\varphi_2&\varphi_3\end{pmatrix}$, namely,

$$T=\begin{pmatrix}1&0&0\\-a_2&1&0\\0&0&1\end{pmatrix},\ \text{then}\ \ T^{-1}=\begin{pmatrix}1&0&0\\a_2&1&0\\0&0&1\end{pmatrix}.$$

The transformation $\begin{pmatrix}l_n\\m_n\\(a_1^*)_n\end{pmatrix}=T\begin{pmatrix}u_n\\v_n\\\delta_n\end{pmatrix}$ changes system (4.7) into

$$\begin{pmatrix}u_{n+1}\\v_{n+1}\\\delta_{n+1}\end{pmatrix}=\begin{pmatrix}1&0&0\\0&1-\rho&0\\0&0&1\end{pmatrix}\begin{pmatrix}u_n\\v_n\\\delta_n\end{pmatrix}+\begin{pmatrix}g_3\left(u_n,v_n,\delta_n\right)+o(\rho_4^3)\\g_4\left(u_n,v_n,\delta_n\right)+o(\rho_4^3)\\0\end{pmatrix},\tag{4.9}$$

where

$$\rho_4=\sqrt{u_n^2+v_n^2+\delta_n^2},$$

$$g_3(u_n, v_n, \delta_n) = g_1(u_n, -a_2 u_n + v_n, \delta_n),$$
$$g_4(u_n, v_n, \delta_n) = a_2 g_1(u_n, -a_2 u_n + v_n, \delta_n) + g_2(u_n, -a_2 u_n + v_n, \delta_n).$$

Assume that on the center manifold

$$v_n = h(u_n, \delta_n) = a_{20} u_n^2 + a_{11} u_n \delta_n + a_{02} \delta_n^2 + o(\rho_5^2),$$

where $\rho_5 = \sqrt{u_n^2 + \delta_n^2}$. Then, from

$$\begin{aligned} v_{n+1} =&(1-\rho) h(u_n, \delta_n) + a_2 g_1(u_n, -a_2 u_n + v_n, \delta_n) \\ &+ g_2(u_n, -a_2 u_n + v_n, \delta_n) + o(\rho_5^2), \\ h(u_{n+1}, \delta_{n+1}) =&a_{20} u_{n+1}^2 + a_{11} u_{n+1} \delta_n + a_{02} \delta_{n+1}^2 + o(\rho_5^2) \\ =&a_{20} (u_n + g_1(u_n, -a_2 u_n + v_n, \delta_n))^2 \\ &+ a_{11} (u_n + g_1(u_n, -a_2 u_n + v_n, \delta_n)) \delta_n + a_{02} \delta_n^2 + o(\rho_5^2) \end{aligned}$$

and $v_{n+1} = h(u_{n+1}, \delta_{n+1})$, we obtain the center manifold equation

$$\begin{aligned} (1-\rho) h(u_n, \delta_n) &+ a_2 g_1(u_n, -a_2 u_n + v_n, \delta_n) \\ &+ g_2(u_n, -a_2 u_n + v_n, \delta_n) + o(\rho_5^2) \\ =&a_{20} (u_n + g_1(u_n, -a_2 u_n + v_n, \delta_n))^2 \\ &+ a_{11} (u_n + g_1(u_n, -a_2 u_n + v_n, \delta_n)) \delta_n + a_{02} \delta_n^2 + o(\rho_5^2). \end{aligned}$$

Comparing the corresponding coefficients of terms with the same order in the above center manifold equation, it is easy to derive that

$$a_{20} = \frac{a_2}{\rho} \left(\frac{b}{c^2} - 1 \right), a_{11} = -\frac{a_2}{\rho}, a_{02} = 0.$$

Therefore, system (4.9) restricted to the center manifold is given by

$$u_{n+1} = f_2(u_n, \delta_n) := u_n + \frac{(1 - a_2 c)(b - c)}{c_1^2} u_n^2 - u_n \delta_n + o(\rho_5^2).$$

Hence, the following results are derived:

$$\begin{aligned} f_2(u_n, \delta_n)|_{(0,0)} = 0, \left.\frac{\partial f_2}{\partial u_n}\right|_{(0,0)} = 1, \left.\frac{\partial f_2}{\partial \delta_n}\right|_{(0,0)} = 0, \\ \left.\frac{\partial^2 f_2}{\partial u_n \partial \delta_n}\right|_{(0,0)} = -1 \neq 0, \left.\frac{\partial^2 f_2}{\partial u_n^2}\right|_{(0,0)} = 2\frac{(1 - a_2 c)(b - c)}{c^2} \neq 0. \end{aligned}$$

According to (21.1.42)-(21.1.46) in [23, p507], when $a_2 c \neq 1$, all the conditions for the occurrence of the transcritical bifurcation are satisfied. Hence, system (1.7) undergoes a transcritical bifurcation at the fixed point E_1. The proof is over. □

Next, one studies Case II: $a_1 \neq a_{10}, \rho = 2$. By the Theorem (3.1), one can see that $|\lambda_1| \neq 1$ and $\lambda_2 = -1$, when $a_1 \neq a_{10}, \rho = 2$. Thereout, the following result can be derived.

Theorem 4.3. *Suppose that the parameters* $(a_1, a_2, b, c, \rho) \in \Omega_2 = \{(a_1, a_2, b, c, \rho) \in R_+^5 | a_1 > 0, a_2 > 0, 0 < b < c, a_1 \neq 1 - \frac{b}{c}, \rho > 0\}$. *Let* $\rho_0 = 2$. *If the parameter* ρ *goes through the critical value* ρ_0, *then system* (1.7) *undergoes a period-doubling bifurcation at* E_1. *Moreover, the period-two orbit bifurcated from* E_1 *lies on the right of* ρ_0 *and is stable.*

Proof. Shifting $E_1 = (0, 1)$ to the origin $O = (0, 0)$ and giving a small perturbation ρ^* of the parameter ρ at the critical value ρ_0 with $0 < |\rho^*| \ll 1$, system (4.5) is transformed into the following form

$$\begin{cases} l_{n+1} = l_n e^{1-l_n-a_1(m_n+1)-\frac{b}{c+l_n}}, \\ m_{n+1} = (m_n+1)e^{(\rho^*+\rho_0)(-m_n-a_2l_n)} - 1. \end{cases} \tag{4.10}$$

Set $\rho^*_{n+1} = \rho^*_n = \rho^*$. Then (4.10) can be seen as

$$\begin{cases} l_{n+1} = l_n e^{1-l_n-a_1(m_n+1)-\frac{b}{c+l_n}}, \\ m_{n+1} = (m_n+1)e^{(\rho^*_n+\rho_0)(-m_n-a_2l_n)} - 1, \\ \rho^*_{n+1} = \rho^*_n. \end{cases} \tag{4.11}$$

Taylor expanding of system (4.11) at $(l_n, m_n, \rho^*_n) = (0, 0, 0)$ takes the form

$$\begin{cases} l_{n+1} = & c_{100}l_n + c_{010}m_n + c_{200}l_n^2 + c_{020}m_n^2 + c_{110}l_nm_n \\ & +c_{300}l_n^3 + c_{030}m_n^3 + c_{210}l_n^2m_n + c_{120}l_nm_n^2 + o(\rho_6^3), \\ m_{n+1} = & d_{100}l_n + d_{010}m_n + d_{001}\rho^*_n + d_{200}l_n^2 + d_{020}m_n^2 \\ & +d_{002}{\rho^*_n}^2 + d_{110}l_nm_n + d_{101}l_n\rho^*_n + d_{011}m_n\rho^*_n \\ & +d_{300}l_n^3 + d_{030}m_n^3 + d_{003}{\rho^*_n}^3 + d_{210}l_n^2m_n \\ & +d_{120}m_nl_n^2 + d_{021}m_n^2\rho^*_n + d_{201}l_n^2\rho^*_n + d_{102}l_n{\rho^*_n}^2 \\ & +d_{012}m_n{\rho^*_n}^2 + d_{111}l_nm_n\rho^*_n + o(\rho_6^3), \\ \rho^*_{n+1} = & \rho^*_n, \end{cases} \tag{4.12}$$

where

$$\begin{aligned} &\rho_6 = \sqrt{l_n^2 + m_n^2 + (\rho^*_n)^2}, \\ &c_{010} = c_{020} = c_{030} = 0, c_{100} = e^{1-\frac{b}{c}-a_1}, c_{200} = \left(\frac{b}{c^2} - 1\right)e^{1-\frac{b}{c}-a_1}, \\ &c_{110} = -a_1e^{1-\frac{b}{c}-a_1}, c_{300} = \left(\frac{1}{2}\left(\frac{b}{c^2}-1\right)^2 - \frac{b}{c^3}\right)e^{1-\frac{b}{c}-a_1}, \\ &c_{210} = a_1\left(1-\frac{b}{c^2}\right)e^{1-\frac{b}{c}-a_1}, c_{120} = \frac{a_1^2}{2}e^{1-\frac{b}{c}-a_1}, \\ &d_{001} = d_{020} = d_{002} = d_{003} = d_{120} = d_{102} = d_{012} = 0, d_{100} = -2a_2, \\ &d_{010} = d_{011} = -1, d_{200} = d_{201} = 2a_2^2, d_{110} = 2a_2, d_{101} = -a_2, \\ &d_{300} = -\frac{4}{3}a_2^3, d_{030} = \frac{2}{3}, d_{210} = -2a_2^2, d_{021} = 1, d_{111} = 3a_2. \end{aligned}$$

We can think of system (4.12) as the following form

$$\begin{pmatrix} l_{n+1} \\ m_{n+1} \\ \rho^*_{n+1} \end{pmatrix} \to \begin{pmatrix} e^{1-\frac{b}{c}-a_1} & 0 & 0 \\ -2a_2 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} l_n \\ m_n \\ \rho^*_n \end{pmatrix} + \begin{pmatrix} g_5(l_n, m_n, \rho^*_n) + o(\rho_6^3) \\ g_6(l_n, m_n, \rho^*_n) + o(\rho_6^3) \\ 0 \end{pmatrix}, \tag{4.13}$$

where

$$\begin{aligned} g_5\left(l_n, m_n, \rho_n^*\right) =& c_{200} l_n^2 + c_{020} m_n^2 + c_{110} l_n m_n + c_{300} l_n^3 \\ & + c_{030} m_n^3 + c_{210} l_n^2 m_n + c_{120} l_n m_n^2, \\ g_6\left(l_n, m_n, \rho_n^*\right) =& d_{200} l_n^2 + d_{020} m_n^2 + d_{002} {\rho_n^*}^2 + d_{110} l_n m_n \\ & + d_{101} l_n \rho_n^* + d_{011} m_n \rho_n^* + d_{300} l_n^3 + d_{030} m_n^3 \\ & + d_{003} {\rho_n^*}^3 + d_{210} l_n^2 m_n + d_{120} l_n m_n^2 + d_{021} m_n^2 \rho_n^* \\ & + d_{201} l_n^2 \rho_n^* + d_{102} l_n {\rho_n^*}^2 + d_{012} m_n {\rho_n^*}^2 + d_{111} l_n m_n \rho_n^*. \end{aligned}$$

It is not difficult to derive the three eigenvalues of the matrix

$$A = \begin{pmatrix} e^{1-\frac{b}{c}-a_1} & 0 & 0 \\ -2a_2 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

to be

$$\lambda_1 = e^{1-\frac{b}{c}-a_1}, \lambda_2 = -1 \text{ and } \lambda_3 = 1,$$

with corresponding eigenvectors

$$\begin{pmatrix} \xi_1 \\ \eta_1 \\ \varphi_1 \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{-2a_2}{e^{1-\frac{b}{c}-a_1}+1} \\ 0 \end{pmatrix}, \begin{pmatrix} \xi_2 \\ \eta_2 \\ \varphi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} \xi_3 \\ \eta_3 \\ \varphi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

The condition $a_1 \neq 1 - \frac{b}{c}$ shows that $\lambda_1 \neq 1$.

Set $T = (\xi_1, \eta_1, \varphi_1)$,

$$\text{i.e., } T = \begin{pmatrix} 1 & 0 & 0 \\ \frac{-2a_2}{e^{1-\frac{b}{c}-a_1}+1} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \text{ then } T^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ \frac{2a_2}{e^{1-\frac{b}{c}-a_1}+1} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Taking the transformation

$$\begin{pmatrix} l_n \\ m_n \\ \rho_n^* \end{pmatrix} = T \begin{pmatrix} u_n \\ v_n \\ \delta_n \end{pmatrix},$$

system (4.13) is changed into

$$\begin{pmatrix} u_{n+1} \\ v_{n+1} \\ \delta_{n+1} \end{pmatrix} = \begin{pmatrix} e^{1-\frac{b}{c}-a_1} & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u_n \\ v_n \\ \delta_n \end{pmatrix} + \begin{pmatrix} g_7\left(u_n, v_n, \delta_n\right) + o(\rho_7^3) \\ g_8\left(u_n, v_n, \delta_n\right) + o(\rho_7^3) \\ 0 \end{pmatrix}, \tag{4.14}$$

where

$$\begin{aligned}\rho_7 =& \sqrt{u_n^2 + v_n^2 + (\delta_n)^2},\\ g_7\left(u_n, v_n, \delta_n\right) =& g_5\left(u_n, \frac{-2a_2}{e^{1-\frac{b}{c}-a_1}+1}u_n + v_n, \delta_n\right),\\ g_8\left(u_n, v_n, \delta_n\right) =& \frac{2a_2}{e^{1-\frac{b}{c}-a_1}+1} g_5\left(u_n, \frac{-2a_2}{e^{1-\frac{b}{c}-a_1}+1}u_n + v_n, \delta_n\right)\\ &+ g_6\left(u_n, \frac{-2a_2}{e^{1-\frac{b}{c}-a_1}+1}u_n + v_n, \delta_n\right).\end{aligned}$$

Suppose that on the center manifold

$$u_n = h(v_n, \delta_n) = b_{20}u_n^2 + b_{11}u_n\delta_n + b_{02}\delta_n^2 + o(\rho_8^2),$$

where $\rho_8 = \sqrt{v_n^2 + \delta_n^2}$, which must satisfy

$$u_{n+1} = h(v_{n+1}, \delta_{n+1}) = e^{1-\frac{b}{c}-a_1}h(v_n, \delta_n) + g_7\left(h(v_n, \delta_n), v_n, \delta_n\right) + o(\rho_8^3).$$

Similar to Case I, one can establish the corresponding center manifold equation. Comparing the corresponding coefficients of terms with the same type in the equation produces

$$b_{20} = 0, b_{11} = \frac{1}{e^{1-\frac{b}{c}-a_1}+1}, b_{02} = 0.$$

Hence, system (4.14) restricted to the center manifold is given by

$$v_{n+1} = f_3(v_n, \delta_n) := -v_n - v_n\delta_n + s_{21}v_n^2\delta_n + s_{12}v_n\delta_n^2 + \frac{2}{3}v_n^3 + o(\rho_8^3),$$

where

$$\begin{aligned}s_{21} &= \frac{2a_2}{e^{1-\frac{b}{c}-a_1}+1}\left(1 - \frac{a_1 e^{1-\frac{b}{c}-a_1}}{e^{1-\frac{b}{c}-a_1}+1}\right) + 1,\\ s_{12} &= \frac{2a_2}{\left(e^{1-\frac{b}{c}-a_1}+1\right)^2} - \frac{a_2}{e^{1-\frac{b}{c}-a_1}+1}.\end{aligned}$$

Next, we calculate the following quantities to judge the occurrence of a period-doubling bifurcation according to (21.2.17)-(21.2.22) in [23, p516].

One has

$$f_3^2(v_n, \delta_n) = v_n + 2v_n\delta_n + (1 - 2s_{12})v_n\delta_n^2 - \frac{4}{3}v_n^3 + o(\rho_8^3).$$

Thereout, the following results are derived:

$$\begin{aligned}&f_3(v_n, \delta_n)|_{(0,0)} = 0, \left.\frac{\partial f_3}{\partial v_n}\right|_{(0,0)} = -1, \left.\frac{\partial f_3^2}{\partial \delta_n}\right|_{(0,0)} = 0,\\ &\left.\frac{\partial^2 f_3^2}{\partial v_n^2}\right|_{(0,0)} = 0, \left.\frac{\partial^2 f_3^2}{\partial v_n \partial \delta_n}\right|_{(0,0)} = 2 \neq 0, \left.\frac{\partial^3 f_3^2}{\partial v_n^3}\right|_{(0,0)} = -8 \neq 0.\end{aligned}$$

Hence, system (1.7) undergoes a period-doubling bifurcation at E_1. Again,

$$\left(-\frac{\partial^3 f_3^2}{\partial v_n^3}\Big/\frac{\partial^2 f_3^2}{\partial v_n \partial \delta_n}\right)\Big|_{(0,0)} = 4 > 0.$$

Therefore, the period-two orbit bifurcated from E_1 lies on the right of $\rho_0 = 2$.

In addition, one can also compute the following two quantities, which are the transversal condition and non-degenerate condition respectively for judging the occurrence and stability of a period-doubling bifurcation (see [8, 16, 18, 20, 22, 24–28, 33]),

$$\alpha_1 = \left(\frac{\partial^2 f_3}{\partial v_n \partial \delta_n} + \frac{1}{2}\frac{\partial f_3}{\partial \delta_n}\frac{\partial^2 f_3}{\partial v_n^2}\right)\Big|_{(0,0)},$$
$$\alpha_2 = \left(\frac{1}{6}\frac{\partial^3 f_3}{\partial v_n^3} + \left(\frac{1}{2}\frac{\partial^2 f_3}{\partial v_n^2}\right)^2\right)\Big|_{(0,0)}.$$

It is clear that $\alpha_1 = -1$ and $\alpha_2 = \frac{1}{9}$. Due to $\alpha_2 > 0$, the period-two orbit bifurcated from E_1 is stable. The proof is completed. □

Finally, considering the Case III: $a_1 = a_{10}, \rho = 2$, one can easily get the two eigenvalues of the linearized matrix at this fixed point E_1 to be $\lambda_1 = 1$ and $\lambda_2 = -1$. A fold-flip bifurcation may occur and the bifurcation problem is very complex. This is left as our future work.

4.3. For fixed point $E_{21}(x_{21}, 0)$ and $E_{22}(x_{22}, 0)$

By Theorem (3.2), it is clear that a bifurcation of E_{21} may occur in the space of parameters $(a_1, a_2, b, c, \rho) \in \Omega_3 = \{(a_1, a_2, b, c, \rho) \in R_+^5 | a_1 > 0, a_2 > 0, 0 < c < b < \frac{(1+c)^2}{4}, \rho > 0.\}$. One has the following consequence.

Theorem 4.4. *Assume the parameters* $(a_1, a_2, b, c, \rho) \in \Omega_3 = \{(a_1, a_2, b, c, \rho) \in R_+^5 | a_1 > 0, a_2 > 0, 0 < c < b < \frac{(1+c)^2}{4}, \rho > 0.\}$. *Set* $a_{20} = \frac{1}{x_{21}} = \frac{2}{1-c-\sqrt{(1+c)^2-4b}}$. *Then, system* (1.7) *undergoes a transcritical bifurcation at* E_{21}, *when the parameter* a_2 *varies in a small neighborhood of critical value* a_{20}.

Proof. Let $l_n = x_n - x_{21}, v_n = y_n - 0$, which transforms the fixed point E_{21} to the origin $O(0,0)$, and system (1.7) into

$$\begin{cases} l_{n+1} = (l_n + x_{21})e^{1-(l_n+x_{21})-a_1 m_n - \frac{b}{c+l_n+x_{21}}} - x_{21}, \\ m_{n+1} = m_n e^{\rho(1-m_n-a_2(l_n+x_{21}))}. \end{cases} \tag{4.15}$$

Giving a small perturbation a_2^* of the parameter a_2 around the critical value a_{20}, i.e., $a_2^* = a_2 - \frac{1}{x_{21}}$, with $0 < |a_2^*| \ll 1$, system (4.15) is perturbed into

$$\begin{cases} l_{n+1} = (l_n + x_{21})e^{1-(l_n+x_{21})-a_1 m_n - \frac{b}{c+l_n+x_{21}}} - x_{21}, \\ m_{n+1} = m_n e^{\rho\left(1-m_n-\left(a_2^*+\frac{1}{x_{21}}\right)(l_n+x_{21})\right)}. \end{cases} \tag{4.16}$$

Setting $(a_2^*)_{n+1} = (a_2^*)_n = a_2^*$, system (4.16) can be written as

$$\begin{cases} l_{n+1} = (l_n + x_{21})e^{1-(l_n+x_{21})-a_1 m_n - \frac{b}{c+l_n+x_{21}}} - x_{21}, \\ m_{n+1} = m_n e^{\rho\left(1-m_n-\left((a_2^*)_n+\frac{1}{x_{21}}\right)(l_n+x_{21})\right)}, \\ (a_2^*)_{n+1} = (a_2^*)_n. \end{cases} \tag{4.17}$$

Taylor's expansion of system (4.17) at $(l_n, m_n, (a_2^*)_n) = (0, 0, 0)$ takes the form

$$\begin{cases} l_{n+1} = & e_{100}l_n + e_{010}m_n + e_{200}l_n^2 + e_{020}m_n^2 + e_{110}l_n m_n \\ & +e_{300}l_n^3 + e_{030}m_n^3 + e_{210}l_n^2 m_n + e_{120}l_n m_n^2 + o(r_1^3), \\ m_{n+1} = & f_{100}l_n + f_{010}m_n + f_{001}(a_2^*)_n + f_{200}l_n^2 + f_{020}m_n^2 \\ & +f_{002}{(a_2^*)_n}^2 + f_{110}l_n m_n + f_{101}l_n(a_2^*)_n + f_{011}m_n(a_2^*)_n \\ & +f_{300}l_n^3 + f_{030}m_n^3 + f_{003}{(a_2^*)_n}^3 + f_{210}l_n^2 m_n \\ & +f_{120}m_n l_n^2 + f_{021}m_n^2(a_2^*)_n + f_{201}l_n^2(a_2^*)_n + f_{102}l_n{(a_2^*)_n}^2 \\ & +f_{012}m_n{(a_2^*)_n}^2 + f_{111}l_n m_n(a_2^*)_n + o(r_1^3), \\ (a_2^*)_{n+1} = & (a_2^*)_n, \end{cases} \tag{4.18}$$

where $r_1 = \sqrt{l_n^2 + m_n^2 + ((a_2^*)_n)^2}$,

$$\begin{aligned} e_{100} =& 1 + \left(\frac{b}{(c+x_{21})^2} - 1\right)x_{21}, e_{010} = -a_1 x_{21}, \\ e_{200} =& \frac{1}{2}\left[2\left(\frac{b}{(c+x_{21})^2} - 1\right) + \left(\frac{b}{(c+x_{21})^2} - 1\right)^2 x_{21} - \frac{2bx_{21}}{(c+x_{21})^3}\right], \\ e_{020} =& \frac{1}{2}a_1^2 x_{21}, e_{110} = -a_1\left(\left(\frac{b}{(c+x_{21})^2} - 1\right)x_{21} + 1\right), \\ e_{300} =& \frac{1}{2}\left(\frac{b}{(c+x_{21})^2} - 1\right)^2 - \frac{b}{(c+x_{21})^3} + \frac{1}{6}\left(\frac{b}{(c+x_{21})^2} - 1\right)^3 x_{21} \\ &+ \frac{bx_{21}}{(c+x_{21})^4} - \frac{bx_{21}}{(c+x_{21})^3}\left(\frac{b}{(c+x_{21})^2} - 1\right), \\ e_{210} =& \frac{a_1 b x_{21}}{(c+x_{21})^3} - \frac{a_1 x_{21}}{2}\left(\frac{b}{(c+x_{21})^2} - 1\right)^2 - a_1\left(\frac{b}{(c+x_{21})^2} - 1\right), \\ e_{030} =& -\frac{1}{6}a_1^3 x_{21}, e_{120} = \frac{a_1^2}{2}\left[\left(\frac{b}{(c+x_{21})^2} - 1\right)x_{21} + 1\right], \end{aligned}$$

$$\begin{aligned} f_{100} =& f_{001} = f_{200} = f_{002} = f_{101} = f_{300} = f_{003} = f_{201} = f_{102} = 0, \\ f_{010} =& 1, f_{020} = -\rho, f_{110} = \frac{\rho}{x_{21}}, f_{011} = -\rho x_{21}, f_{030} = \frac{\rho^2}{2}, \\ f_{210} =& \frac{\rho^2}{2x_{21}^2}, f_{120} = \frac{\rho^2}{x_{21}}, f_{021} = \rho^2 x_{21}, f_{012} = \frac{\rho^2 x_{21}^2}{2}, f_{111} = \rho^2 - \rho. \end{aligned}$$

It is simple to compute

$$\frac{b}{(c+x_{21})^2}-1=\frac{\sqrt{\Delta_1}}{c+x_{21}},$$

and system (4.18) can be seen as the form

$$\begin{pmatrix} l_{n+1}\\ m_{n+1}\\ (a_2^*)_{n+1}\end{pmatrix}\to\begin{pmatrix}1+\frac{\sqrt{\Delta_1}x_{21}}{c+x_{21}} & -a_1x_{21} & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix} l_n\\ m_n\\ (a_2^*)_n\end{pmatrix}+\begin{pmatrix}h_1\left(l_n,m_n,(a_2^*)_n\right)+o(r_1^3)\\ h_2\left(l_n,m_n,(a_2^*)_n\right)+o(r_1^3)\\ 0\end{pmatrix}, \tag{4.19}$$

where

$$\begin{aligned} h_1\left(l_n,m_n,(a_2^*)_n\right)=&e_{200}l_n^2+e_{020}m_n^2+e_{110}l_nm_n+e_{300}l_n^3\\ &+e_{030}m_n^3+e_{210}l_n^2m_n+e_{120}l_nm_n^2,\\ h_2\left(l_n,m_n,(a_2^*)_n\right)=&f_{200}l_n^2+f_{020}m_n^2+f_{002}{(a_2^*)_n}^2+f_{110}l_nm_n\\ &+f_{101}l_n(a_2^*)_n+f_{011}m_n(a_2^*)_n+f_{300}l_n^3+f_{030}m_n^3\\ &+f_{003}{(a_2^*)_n}^3+f_{210}l_n^2m_n+f_{120}l_nm_n^2+f_{021}m_n^2(a_2^*)_n\\ &+f_{201}l_n^2(a_2^*)_n+f_{102}l_n{(a_2^*)_n}^2+f_{012}m_n{(a_2^*)_n}^2+f_{111}l_nm_n(a_2^*)_n.\end{aligned}$$

It is easy to derive the three eigenvalues of matrix

$$A=\begin{pmatrix}1+\frac{\sqrt{\Delta_1}x_{21}}{c+x_{21}} & -a_1x_{21} & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$$

to be

$$\lambda_1=1+\frac{\sqrt{\Delta_1}x_{21}}{c+x_{21}},\lambda_{2,3}=1$$

with corresponding eigenvectors

$$\begin{pmatrix}\xi_1\\ \eta_1\\ \varphi_1\end{pmatrix}=\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}\xi_2\\ \eta_2\\ \varphi_2\end{pmatrix}=\begin{pmatrix}\frac{a_1(c+x_{21})}{\sqrt{\Delta_1}}\\1\\0\end{pmatrix},\begin{pmatrix}\xi_3\\ \eta_3\\ \varphi_3\end{pmatrix}=\begin{pmatrix}0\\0\\1\end{pmatrix}$$

respectively.

$$\text{Set }\ T=\begin{pmatrix}1 & \frac{a_1(c+x_{21})}{\sqrt{\Delta_1}} & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix},\ \text{then }\ T^{-1}=\begin{pmatrix}1 & -\frac{a_1(c+x_{21})}{\sqrt{\Delta_1}} & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}.$$

Taking the transformation

$$\begin{pmatrix} l_n\\ m_n\\ (a_2^*)_n\end{pmatrix}=T\begin{pmatrix}u_n\\ v_n\\ \delta_n\end{pmatrix},$$

system (4.19) is changed into

$$\begin{pmatrix} u_{n+1} \\ v_{n+1} \\ \delta_{n+1} \end{pmatrix} = \begin{pmatrix} 1+\frac{\sqrt{\Delta_1}x_{21}}{c+x_{21}} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u_n \\ v_n \\ \delta_n \end{pmatrix} + \begin{pmatrix} h_3(u_n, v_n, \delta_n) + o(r_2^3) \\ h_4(u_n, v_n, \delta_n) + o(r_2^3) \\ 0 \end{pmatrix}, \tag{4.20}$$

where $r_2 = \sqrt{u_n^2 + v_n^2 + (\delta_n)^2}$,

$$\begin{aligned} h_3(u_n, v_n, \delta_n) =& h_1\left(u_n + \frac{a_1(c+x_{21})}{\sqrt{\Delta_1}} v_n, v_n, \delta_n\right) \\ & - \frac{a_1(c+x_{21})}{\sqrt{\Delta_1}} h_2\left(u_n + \frac{a_1(c+x_{21})}{\sqrt{\Delta_1}} v_n, v_n, \delta_n\right), \\ h_4(u_n, v_n, \delta_n) =& h_2\left(u_n + \frac{a_1(c+x_{21})}{\sqrt{\Delta_1}} v_n, v_n, \delta_n\right). \end{aligned}$$

Putting on the center manifold $u_n = m_{20}v_n^2 + m_{11}v_n\delta_n + m_{02}\delta_n^2 + o(r_3^2)$, where $r_3 = \sqrt{v_n^2 + (\delta_n)^2}$, it is easy to derive

$$\begin{aligned} m_{02} = 0, m_{20} =& \frac{c+x_{21}}{\sqrt{\Delta_1}x_{21}}\Bigg(\frac{a_1(a_1-\rho)(c+x_{21})}{\sqrt{\Delta_1}} - \frac{a_1\rho^2(c+x_{21})}{\Delta_1 x_{21}} \\ & - \frac{a_1^2(\sqrt{\Delta_1}(c+x_{21})^2 - b_1x_{21})}{(c+x_{21})\Delta_1}\Bigg), m_{11} = -\frac{a_1\rho(c+x_{21})^2}{\Delta_1}. \end{aligned}$$

Hence, system (4.20) restricted to the center manifold is given by

$$v_{n+1} = f_4(v_n, \delta_n) := v_n - \rho\left(1 + \frac{a_1(c+x_{21})}{\sqrt{\Delta_1}x_{21}}\right)v_n^2 - \rho x_{21}v_n\delta_n + o(r_3^2).$$

Therefore, one has

$$f_4(v_n, \delta_n)|_{(0,0)} = 0, \left.\frac{\partial f_4}{\partial v_n}\right|_{(0,0)} = 1, \left.\frac{\partial f_4}{\partial \delta_n}\right|_{(0,0)} = 0,$$

$$\left.\frac{\partial^2 f_4}{\partial v_n \partial \delta_n}\right|_{(0,0)} = -\rho x_{21} \neq 0, \left.\frac{\partial^2 f_4}{\partial v_n^2}\right|_{(0,0)} = -2\rho\left(1 + \frac{a_1(c+x_{21})}{\sqrt{\Delta_1}x_{21}}\right) \neq 0.$$

According to (21.1.42)-(21.1.46) in [23, p507], all the conditions for the occurrence of the transcritical bifurcation hold. Hence, system (1.7) undergoes a transcritical bifurcation at the fixed point E_{21}. The proof is over. □

Next, we consider the situation for the existence of the fixed point E_{22}. By Theorem (3.2), it is clear that a bifurcation of system (1.7) at the fixed point E_{22} may occur in the space of parameters $(a_1, a_2, b, c, \rho) \in \Omega_4 = \{(a_1, a_2, b, c, \rho) \in R_+^5 | a_1 > 0, a_2 > 0, \rho > 0, 0 < b < c \text{ or } 0 < c \leq b < \frac{(1+c)^2}{4} < 1.\}$.

Theorem 4.5. *Assume that the parameters* $(a_1, a_2, b, c, \rho) \in \Omega_4 = \{(a_1, a_2, b, c, \rho) \in R_+^5 | a_1 > 0, a_2 > 0, \rho > 0, 0 < b < c \text{ or } 0 < c \leq b < \frac{(1+c)^2}{4} < 1.\}$. *Let* $a_{21} = \frac{1}{x_{22}} = \frac{2}{1-c+\sqrt{(1+c)^2-4b}}$. *If* $a_1(c+x_{22}) \neq \sqrt{\Delta_1}x_{22}$, *system* (1.7) *undergoes a transcritical bifurcation at* E_{22}, *when the parameter* a_2 *varies in a small neighborhood of critical value* a_{21}.

Proof. Similar to the situation of E_{21}, by shifting E_{22} to the origin, giving a small perturbation a_2^*, as well as appending the dependent variable $(a_2^*)_n$ to the phase space and performing Taylor expansion, system (1.7) is changed into the following form

$$\begin{cases} l_{n+1} = & e_{100}l_n + e_{010}m_n + e_{200}l_n^2 + e_{020}m_n^2 + e_{110}l_nm_n \\ & +e_{300}l_n^3 + e_{030}m_n^3 + e_{210}l_n^2m_n + e_{120}l_nm_n^2 + o(r_4^3), \\ m_{n+1} = & f_{100}l_n + f_{010}m_n + f_{001}(a_2^*)_n + f_{200}l_n^2 + f_{020}m_n^2 \\ & +f_{002}{(a_2^*)_n}^2 + f_{110}l_nm_n + f_{101}l_n(a_2^*)_n + f_{011}m_n(a_2^*)_n \\ & +f_{300}l_n^3 + f_{030}m_n^3 + f_{003}{(a_2^*)_n}^3 + f_{210}l_n^2m_n \\ & +f_{120}m_nl_n^2 + f_{021}m_n^2(a_2^*)_n + f_{201}l_n^2(a_2^*)_n + f_{102}l_n{(a_2^*)_n}^2 \\ & +f_{012}m_n{(a_2^*)_n}^2 + f_{111}l_nm_n(a_2^*)_n + o(r_4^3), \\ (a_2^*)_{n+1} = (a_2^*)_n, \end{cases} \tag{4.21}$$

where $r_4 = \sqrt{l_n^2 + m_n^2 + ((a_2^*)_n)^2}$,

$$\begin{aligned} e_{100} =& 1 + \left(\frac{b}{(c+x_{22})^2} - 1\right)x_{22}, e_{010} = -a_1x_{22}, \\ e_{200} =& \frac{1}{2}\left[2\left(\frac{b}{(c+x_{22})^2} - 1\right) + \left(\frac{b}{(c+x_{22})^2} - 1\right)^2 x_{22} - \frac{2bx_{22}}{(c+x_{22})^3}\right], \\ e_{020} =& \frac{1}{2}a_1^2x_{22}, e_{110} = -a_1\left(\left(\frac{b}{(c+x_{22})^2} - 1\right)x_{22} + 1\right), \\ e_{300} =& \frac{1}{2}\left(\frac{b}{(c+x_{22})^2} - 1\right)^2 - \frac{b}{(c+x_{22})^3} + \frac{1}{6}\left(\frac{b}{(c+x_{22})^2} - 1\right)^3 x_{22} \\ & + \frac{bx_{22}}{(c+x_{22})^4} - \frac{bx_{22}}{(c+x_{22})^3}\left(\frac{b}{(c+x_{22})^2} - 1\right), \\ e_{210} =& \frac{a_1bx_{22}}{(c+x_{22})^3} - \frac{a_1x_{22}}{2}\left(\frac{b}{(c+x_{22})^2} - 1\right)^2 - a_1\left(\frac{b}{(c+x_{22})^2} - 1\right), \\ e_{030} =& -\frac{1}{6}a_1^3x_{22}, e_{120} = \frac{a_1^2}{2}\left[\left(\frac{b}{(c+x_{22})^2} - 1\right)x_{22} + 1\right], \\ f_{100} =& f_{001} = f_{200} = f_{002} = f_{101} = f_{300} = f_{003} = f_{201} = f_{102} = 0, \\ f_{010} =& 1, f_{020} = -\rho, f_{110} = \frac{\rho}{x_{22}}, f_{011} = -\rho x_{22}, f_{030} = \frac{\rho^2}{2}, \\ f_{210} =& \frac{\rho^2}{2x_{22}^2}, f_{120} = \frac{\rho^2}{x_{22}}, f_{021} = \rho^2x_{22}, f_{012} = \frac{\rho^2x_{22}^2}{2}, f_{111} = \rho^2 - \rho, \end{aligned}$$

in which we only need to replace x_{21} with x_{22} in equation (4.18).

It is easy to derive

$$\frac{b}{(c+x_{22})^2} - 1 = -\frac{\sqrt{\Delta_1}}{c+x_{22}},$$

and system (4.21) can be seen as the form

$$\begin{pmatrix} l_{n+1} \\ m_{n+1} \\ (a_2^*)_{n+1} \end{pmatrix} \to \begin{pmatrix} 1-\frac{\sqrt{\Delta}_1 x_{22}}{c+x_{22}} & -a_1x_{22} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} l_n \\ m_n \\ (a_2^*)_n \end{pmatrix} + \begin{pmatrix} h_5\left(l_n, m_n, (a_2^*)_n\right) + o(r_4^3) \\ h_6\left(l_n, m_n, (a_2^*)_n\right) + o(r_4^3) \\ 0 \end{pmatrix}, \tag{4.22}$$

where

$$\begin{aligned} h_5\left(l_n, m_n, (a_2^*)_n\right) =& e_{200}l_n^2 + e_{020}m_n^2 + e_{110}l_nm_n + e_{300}l_n^3 \\ & + e_{030}m_n^3 + e_{210}l_n^2m_n + e_{120}l_nm_n^2, \\ h_6\left(l_n, m_n, (a_2^*)_n\right) =& f_{200}l_n^2 + f_{020}m_n^2 + f_{002}{(a_2^*)_n}^2 + f_{110}l_nm_n \\ & + f_{101}l_n(a_2^*)_n + f_{011}m_n(a_2^*)_n + f_{300}l_n^3 + f_{030}m_n^3 \\ & + f_{003}{(a_2^*)_n}^3 + f_{210}l_n^2m_n + f_{120}l_nm_n^2 + f_{021}m_n^2(a_2^*)_n \\ & + f_{201}l_n^2(a_2^*)_n + f_{102}l_n{(a_2^*)_n}^2 + f_{012}m_n{(a_2^*)_n}^2 + f_{111}l_nm_n(a_2^*)_n. \end{aligned}$$

Then, the three eigenvalues of matrix

$$A = \begin{pmatrix} 1-\frac{\sqrt{\Delta}_1 x_{22}}{c+x_{22}} & -a_1x_{22} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

are

$$\lambda_1 = 1 - \frac{\sqrt{\Delta}_1 x_{22}}{c+x_{22}}, \lambda_{2,3} = 1$$

with corresponding eigenvectors

$$\begin{pmatrix} \xi_1 \\ \eta_1 \\ \varphi_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \xi_2 \\ \eta_2 \\ \varphi_2 \end{pmatrix} = \begin{pmatrix} -\frac{a_1(c+x_{22})}{\sqrt{\Delta}_1} \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} \xi_3 \\ \eta_3 \\ \varphi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

respectively.

Set $T = (\xi_1, \eta_1, \varphi_1)$,

$$\text{i.e., } T = \begin{pmatrix} 1 & -\frac{a_1(c+x_{22})}{\sqrt{\Delta}_1} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \text{ Then } T^{-1} = \begin{pmatrix} 1 & \frac{a_1(c+x_{22})}{\sqrt{\Delta}_1} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Taking the transformation

$$\begin{pmatrix} l_n \\ m_n \\ (a_2^*)_n \end{pmatrix} = T \begin{pmatrix} u_n \\ v_n \\ \delta_n \end{pmatrix},$$

system (4.22) is changed into

$$\begin{pmatrix} u_{n+1} \\ v_{n+1} \\ \delta_{n+1} \end{pmatrix} = \begin{pmatrix} 1-\frac{\sqrt{\Delta_1}x_{22}}{c+x_{22}} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u_n \\ v_n \\ \delta_n \end{pmatrix} + \begin{pmatrix} h_7(u_n, v_n, \delta_n) + o(r_5^3) \\ h_8(u_n, v_n, \delta_n) + o(r_5^3) \\ 0 \end{pmatrix}, \tag{4.23}$$

where $r_5 = \sqrt{u_n^2 + v_n^2 + (\delta_n)^2}$,

$$\begin{aligned} h_7(u_n, v_n, \delta_n) =& h_5\left(u_n - \frac{a_1(c+x_{22})}{\sqrt{\Delta_1}}v_n, v_n, \delta_n\right) \\ &+ \frac{a_1(c+x_{22})}{\sqrt{\Delta_1}}h_6\left(u_n - \frac{a_1(c+x_{22})}{\sqrt{\Delta_1}}v_n, v_n, \delta_n\right), \\ h_8(u_n, v_n, \delta_n) =& h_6\left(u_n - \frac{a_1(c+x_{22})}{\sqrt{\Delta_1}}v_n, v_n, \delta_n\right). \end{aligned}$$

Putting on the center manifold $u_n = l_{20}v_n^2 + l_{11}v_n\delta_n + l_{02}\delta_n^2 + o(r_6^2)$, where $r_6 = \sqrt{v_n^2 + (\delta_n)^2}$, it is easy to derive

$$\begin{aligned} l_{02} = 0, l_{20} = \frac{c+x_{22}}{\sqrt{\Delta_1}x_{22}}&\left(\frac{-\rho a_1(c+x_{22})}{\sqrt{\Delta_1}} + \frac{a_1^2\rho(c+x_{22})^2}{\Delta_1 x_{22}}\right. \\ &\left. - \frac{bx_{22}a_1^2}{(c+x_{22})\Delta_1}\right), l_{11} = -\frac{a_1\rho(c+x_{22})^2}{\Delta_1}. \end{aligned}$$

Hence, system (4.23) restricted to the center manifold is given by

$$v_{n+1} = f_5(v_n, \delta_n) := v_n + \rho\left(\frac{a_1(c+x_{22})}{\sqrt{\Delta_1}x_{22}} - 1\right)v_n^2 - \rho x_{22}v_n\delta_n + o(r_6^2).$$

Therefore, one has

$$f_5(v_n, \delta_n)|_{(0,0)} = 0, \left.\frac{\partial f_5}{\partial v_n}\right|_{(0,0)} = 1, \left.\frac{\partial f_5}{\partial \delta_n}\right|_{(0,0)} = 0,$$

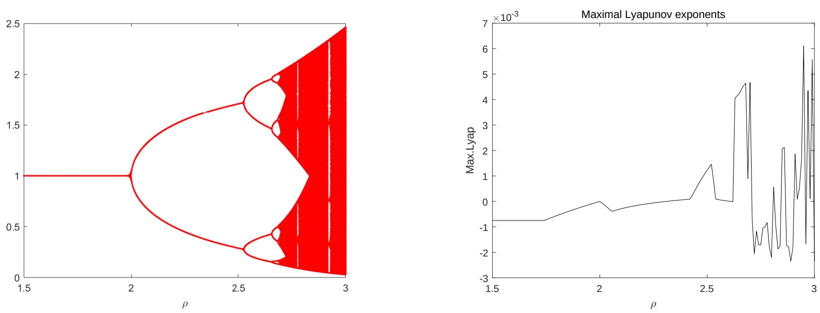
$$\left.\frac{\partial^2 f_5}{\partial v_n \partial \delta_n}\right|_{(0,0)} = -\rho x_{22} \neq 0, \left.\frac{\partial^2 f_5}{\partial v_n^2}\right|_{(0,0)} = 2\rho\left(\frac{a_1(c+x_{22})}{\sqrt{\Delta_1}x_{22}} - 1\right).$$

According to (21.1.42)-(21.1.46) in [23, p507], when $a_1(c+x_{22}) \neq \sqrt{\Delta_1}x_{22}$, we have $\left.\frac{\partial^2 f_5}{\partial v_n^2}\right|_{(0,0)} \neq 0$, and all the conditions for the occurrence of the transcritical bifurcation are true. Therefore, system (1.7) undergoes a transcritical bifurcation at the fixed point E_{22}. □

5. Numerical simulation

In this section, the bifurcation diagrams and Lyapunov exponents of system (1.7) with the specific parameter values are presented by Matlab software, which verify our theoretical results and reveal some new dynamical behaviors in system (1.7).

We choose the parameters $a_1 = 0.5, a_2 = 1, b = 0.4, c = 0.5$, let the parameter ρ vary in the interval $(1.5, 3)$ and take the initial values $(x_0, y_0) = (0.1, 0.1)$ for E_1. Since the bifurcation diagram of (ρ, x)-plane is similar to that of (ρ, y)-plane, we will only show the latter. Then, we can obtain Figure $1(a)$ and observe the existence of period-doubling bifurcation, when $\rho = \rho_0 = 2$, which is in accordance with the result in Theorem (3.3). Figure $1(b)$ means the spectrum of maximum Lyapunov exponent of system (1.7), which displays that the maximum Lyapunov exponent is positive for ρ greater than some critical value ρ_0. This implies the birth of chaos, which is consistent with Figure $1(a)$.



(a) $\rho \in (1.5, 3)$ (b) $\rho \in (1.5, 3)$

Figure 1. Bifurcation of system (1.7) in (a, y)–plane and maximal Lyapunov exponent

6. Discussion and conclusion

In this paper, we discuss the dynamical behaviors of a discrete two-species competitive model with Michaelies-Menten type harvesting in the first species. Under the given parametric conditions, we show the existence and stability of the nonnegative equilibria $E_0 = (0, 0)$, $E_1 = (0, 1)$, E_{2i} and E_{3i}, where $i = 1, 2, 3$. Then, we derive the sufficient conditions for transcritical bifurcation, pitchfork bifurcation and period-doubling bifurcation to occur. Case III for the bifurcation analysis of fixed point $E(0, 1)$ and the bifurcation analysis of E_{23}, E_{3i} are left as our further work, where $i = 1, 2, 3$. Finally, numerical simulation confirms the theoretical analysis results. Our analysis displays that the dynamical behaviors of system (1.7) are very complex: the tiny changes of some parameters lead to the essential varies of the structural rule of system (1.7).

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