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Existence and Non-Existence of Positive Solutions for a Discrete Fractional Boundary Value Problem

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Abstract In this work, we deal with two-point boundary problem for a finite nabla fractional difference equation. First, we establish an associated Green's function and state some of its properties. Under suitable conditions, we deduce the existence and non-existence of positive solutions to the considered problem. Finally, we construct a few examples to illustrate the established results.

Keywords Nabla fractional difference, dirichlet boundary conditions, boundary value problem, Green's function, fixed-point, positive solution, eigenvalue

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1. Introduction

In 1695, L'Hospital inquired Leibniz on the differential operator $\dfrac{d^n}{dt^n}$, "what if the order is $\frac{1}{2}$", to which Leibniz replied, "it will lead to a paradox from which one-day useful consequences will be drawn". This question gave birth to a branch of mathematics that we know today as the fractional calculus [8, 30]. Although it almost started at the same time as differential calculus, most of the early developments of fractional calculus were confined to the basement for a long time. Today, fractional calculus has been successfully applied in mathematical modelling for medical sciences, computational biology, economics, physics and several areas of engineering. For further applications and historical literature, we refer to a few classical texts on fractional calculus here by Miller and Ross [26], Samko, Kilbas and Mariche [29], Podlubny [28] and Kilbas, Srivastava and Trujillo [23].

On the other hand, discrete fractional calculus deals with arbitrary order differences and sums defined on a discrete domain with a forward (delta) or a backward (nabla) operator. The theory of discrete fractional calculus is relatively new with the most notable works done in the past decade. The notion of fractional difference and sum can be traced back to the work of Gray and Zhang [13] as well as Miller and Ross [27]. In this line, Atici and Eloe [16] developed nabla fractional Riemann–Liouville difference operator, initiated the study of nabla fractional initial value problem and established the exponential law, product rule and nabla Laplace transform. Following their works, the contributions of several mathematicians have made the theory of discrete fractional calculus a fruitful field of research in science

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and engineering. Here, we refer to a recent monograph by Goodrich and Peterson [11] and the references therein, which is an excellent source for all those who wish to work in this field.

The study of boundary value problems (BVPs) has a rich historical background and can be traced back to the work of Euler and Taylor on vibrating strings. On the discrete fractional side, there is a sudden growth of the interest in the development of nabla fractional BVPs. Many authors have studied nabla fractional BVPs recently. To name a few, Ahrendt [2], Goar [10] and Ikram [16] worked with self-adjoint Caputo nabla BVPs. Brackins [7] studied a particular class of self-adjoint Riemann–Liouville nabla BVPs and derived the Green's function associated with it along with a few of its properties. Gholami and Ghanbari [9] obtained the Green's function for a non-homogeneous Riemann–Liouville nabla BVP with Dirichlet boundary conditions. Jonnalagadda [17–22] analysed some qualitative properties to two-point non-linear Riemann–Liouville nabla BVPs associated with a variety of boundary conditions.

Our purpose of this article is to establish sufficient conditions on the existence and non-existence of positive solutions to the following two-point non-linear nabla fractional BVP with parameter $\beta > 0$, using Guo–Krasnoselskii fixed-point theorem [1].

$$\begin{cases} -\left(\nabla^{\alpha}_{\rho(a)}u\right)(t) = \beta f(t,u), \quad t \in \mathbb{N}^{b}_{a+2}, \\ u(a) = 0, \quad u(b) = 0, \end{cases} \tag{1.1}$$

where $a, b \in \mathbb{R}$ with $b - a \in \mathbb{N}_3, 1 < \alpha < 2$ and $f : \mathbb{N}^b_a \times \mathbb{R} \to \mathbb{R}^+ \cup \{0\}$.

The present article is organized as follows. Section 2 contains a few preliminaries on nabla fractional calculus. In Sections 3 and 4, we present the main results on the existence and non-existence of positive solutions to (1.1). Finally, we conclude this article with a few examples to demonstrate the applicability of our main results.

2. Preliminaries

Denote the set of all real numbers and positive integers by $\mathbb{R}$ and $\mathbb{Z}^+$ respectively. We use the following notations, definitions and known results of nabla fractional calculus [11]. Assume that empty sums and products are 0 and 1 respectively.

Definition 2.1. For $a \in \mathbb{R}$, the sets $\mathbb{N}_a$ and $\mathbb{N}^b_a$, where $b - a \in \mathbb{Z}^+$, are defined by

$$\mathbb{N}_a = \{a, a+1, a+2, ...\}, \quad \mathbb{N}^b_a = \{a, a+1, a+2, ..., b\}.$$

Definition 2.2. We define the backward jump operator, $\rho : \mathbb{N}_{a+1} \longrightarrow \mathbb{N}_a$, by

$$\rho(t) = t - 1, \quad t \in \mathbb{N}_{a+1}.$$

Let $u : \mathbb{N}_a \to \mathbb{R}$ and $N \in \mathbb{N}_1$. The first order backward (nabla) difference of u is defined by $\big(\nabla u\big)(t) = u(t) - u(t-1)$, for $t \in \mathbb{N}_{a+1}$, and the N^{th}-order nabla difference of u is defined recursively by $\big(\nabla^N u\big)(t) = \Big(\nabla\big(\nabla^{N-1}u\big)\Big)(t)$, for $t \in \mathbb{N}_{a+N}$.

Definition 2.3. [11] Let $t \in \mathbb{R} \setminus \{\ldots, -2, -1, 0\}$ and $r \in \mathbb{R}$ such that $(t + r) \in \mathbb{R} \setminus \{\ldots, -2, -1, 0\}$. The generalized rising function is defined by

$$t^{\overline{r}} = \frac{\Gamma(t + r)}{\Gamma(t)}.$$

Here, $\Gamma(\cdot)$ denotes the Euler gamma function. Also, if $t \in \{\dots,-2,-1,0\}$ and $r \in \mathbb{R}$ such that $(t+r) \in \mathbb{R} \setminus \{\dots,-2,-1,0\}$, then we use the convention $t^{\overline{r}} = 0$.

Definition 2.4. [11] Let t, $a \in \mathbb{R}$ and $\mu \in \mathbb{R} \setminus \{\dots,-2,-1\}$. The μ^{th}-order nabla fractional Taylor monomial is given by

$$H_\mu(t,a) = \frac{(t-a)^{\overline{\mu}}}{\Gamma(\mu+1)},$$

provided that the right-hand side exists.

We observe the following properties of nabla fractional Taylor monomials.

Lemma 2.1 (Proposition 4.3, [16]). *Let $\mu > -1$ and $s \in \mathbb{N}_a$. Then the following hold:*

1. *If $t \in \mathbb{N}_{\rho(s)}$, then $H_\mu(t,\rho(s)) \geq 0$, and if $t \in \mathbb{N}_s$, then $H_\mu(t,\rho(s)) > 0$.*
2. *If $t \in \mathbb{N}_s$ and $-1 < \mu < 0$, then $H_\mu(t,\rho(s))$ is an increasing function of s.*
3. *If $t \in \mathbb{N}_{s+1}$ and $-1 < \mu < 0$, then $H_\mu(t,\rho(s))$ is a decreasing function of t.*
4. *If $t \in \mathbb{N}_{\rho(s)}$ and $\mu > 0$, then $H_\mu(t,\rho(s))$ is a decreasing function of s.*
5. *If $t \in \mathbb{N}_{\rho(s)}$ and $\mu \geq 0$, then $H_\mu(t,\rho(s))$ is a non-decreasing function of t.*
6. *If $t \in \mathbb{N}_s$ and $\mu > 0$, then $H_\mu(t,\rho(s))$ is an increasing function of t.*
7. *If $0 < v \leq \mu$, then $H_v(t,a) \leq H_\mu(t,a)$, for each fixed $t \in \mathbb{N}_a$.*

Lemma 2.2 (Lemma 2, [12]). *Let a, b be two real numbers such that $0 < a \leq b$ and $1 < \alpha < 2$. Then, $\frac{(a-s)^{\overline{\alpha-1}}}{(b-s)^{\overline{\alpha-1}}}$ is a decreasing function of s for $s \in \mathbb{N}_0^{a-1}$.*

Proof. It is enough to show that $\nabla_s \left(\frac{(a-s)^{\overline{\alpha-1}}}{(b-s)^{\overline{\alpha-1}}}\right) < 0$.

Consider

$$\begin{aligned}
&\nabla_s \left(\frac{(a-s)^{\overline{\alpha-1}}}{(b-s)^{\overline{\alpha-1}}}\right) \\
=&\frac{-(b-s)^{\overline{\alpha-1}}(\alpha-1)(a-\rho(s))^{\overline{\alpha-2}} + (a-s)^{\overline{\alpha-1}}(\alpha-1)(b-\rho(s))^{\overline{\alpha-2}}}{(b-s)^{\overline{\alpha-1}}(b-\rho(s))^{\overline{\alpha-1}}} \\
=&\frac{(\alpha-1)\left(-(b-s)(b-\rho(s))^{\overline{\alpha-2}}(a-\rho(s))^{\overline{\alpha-2}} + (a-s)(a-\rho(s))^{\overline{\alpha-2}}(b-\rho(s))^{\overline{\alpha-2}}\right)}{(b-s)^{\overline{\alpha-1}}(b-\rho(s))^{\overline{\alpha-1}}} \\
=&\frac{(\alpha-1)(b-\rho(s))^{\overline{\alpha-2}}(a-\rho(s))^{\overline{\alpha-2}}(-b+s+a-s)}{(b-s)^{\overline{\alpha-1}}(b-\rho(s))^{\overline{\alpha-1}}} \\
=&\frac{(\alpha-1)(b-\rho(s))^{\overline{\alpha-2}}(a-\rho(s))^{\overline{\alpha-2}}(a-b)}{(b-s)^{\overline{\alpha-1}}(b-\rho(s))^{\overline{\alpha-1}}}.
\end{aligned}$$

Since $b > a$, it follows from Lemma 2.1 that $\nabla_s \left(\frac{(a-s)^{\overline{\alpha-1}}}{(b-s)^{\overline{\alpha-1}}}\right) < 0$. The proof is completed. □

Definition 2.5. [11] Let $u : \mathbb{N}_{a+1} \to \mathbb{R}$ and $\nu > 0$. The ν^{th}-order nabla sum of u is given by

$$\left(\nabla_a^{-\nu} u\right)(t) = \sum_{s=a+1}^{t} H_{\nu-1}(t,\rho(s))u(s), \quad t \in \mathbb{N}_{a+1}.$$

Definition 2.6. [11] Let $u : \mathbb{N}_{a+1} \to \mathbb{R}$, $\nu > 0$ and choose $N \in \mathbb{N}_1$ such that $N-1 < \nu \leq N$. The ν^{th}-order Riemann–Liouville nabla difference of u is given by

$$(\nabla_a^\nu u)(t) = \Big(\nabla^N \big(\nabla_a^{-(N-\nu)} u\big)\Big)(t), \quad t \in \mathbb{N}_{a+N}.$$

Now, we write the expression for the Green's function corresponding to (1.1), and state a few properties which will be used later.

Theorem 2.1 ([7, 9, 18]). *Let $1 < \alpha < 2$ and $f : \mathbb{N}_a^b \times \mathbb{R} \to \mathbb{R}^+ \cup \{0\}$. The equivalent form of (1.1) is given by*

$$u(t) = \beta \sum_{s=a+2}^{b} G(t,s) f(s, u(s)), \quad t \in \mathbb{N}_a^b, \tag{2.1}$$

where the Green's function is given by

$$G(t,s) = \begin{cases} G_1(t,s) = \dfrac{H_{\alpha-1}(t,a)}{H_{\alpha-1}(b,a)} H_{\alpha-1}(b, \rho(s)), & t \in \mathbb{N}_a^{s-1}, \\ G_2(t,s) = \dfrac{H_{\alpha-1}(t,a)}{H_{\alpha-1}(b,a)} H_{\alpha-1}(b, \rho(s)) - H_{\alpha-1}(t, \rho(s)), & t \in \mathbb{N}_s^b. \end{cases} \tag{2.2}$$

Theorem 2.2 ([7, 9, 18]). *The Green's function $G(t,s)$ defined in (2.2) satisfies the following properties:*

1. $G(a,s) = G(b,s) = 0$, *for all* $s \in \mathbb{N}_{a+1}^b$.
2. $G(t, a+1) = 0$, *for all* $t \in \mathbb{N}_a^b$.
3. $G(t,s) > 0$, *for all* $(t,s) \in \mathbb{N}_{a+1}^{b-1} \times \mathbb{N}_{a+2}^b$.
4. $\max_{t \in \mathbb{N}_{a+1}^{b-1}} G(t,s) = G(s-1, s)$, *for all* $s \in \mathbb{N}_{a+2}^b$.
5. $\sum_{s=a+1}^b G(t,s) \leq \lambda$, *for all* $(t,s) \in \mathbb{N}_a^b \times \mathbb{N}_{a+1}^b$, *where*

$$\lambda = \Big(\frac{b-a-1}{\alpha\Gamma(\alpha+1)}\Big)\Big(\frac{(\alpha-1)(b-a)+1}{\alpha}\Big)^{\overline{\alpha-1}}. \tag{2.3}$$

The following theorem is useful to obtain the main results of this article.

Theorem 2.3 (Lemma 6, [12]). *There exists a number $\gamma \in (0,1)$ such that*

$$\min_{t \in \mathbb{N}_c^d} G(t,s) \geq \gamma \max_{t \in \mathbb{N}_a^b} G(t,s) = \gamma G(s-1, s), \tag{2.4}$$

where $c, d \in \mathbb{N}_{a+1}^{b-1}$ such that $c = a + \left\lceil \frac{b-a+1}{4} \right\rceil$ and $d = a + 3\left\lfloor \frac{b-a+1}{4} \right\rfloor$.

Proof. We use the properties of Taylor monomials and Green's function from Definition 2.4, Lemma 2.1 and Theorem 2.2 respectively.

Consider, for $s \in \mathbb{N}_{a+2}^b$,

$$\frac{G(t,s)}{G(s-1,s)} = \begin{cases} \frac{(t-a)^{\overline{\alpha-1}}}{(s-a-1)^{\overline{\alpha-1}}}, & \text{for } s > t, \\ \frac{(t-a)^{\overline{\alpha-1}}}{(s-a-1)^{\overline{\alpha-1}}} - \frac{(t-s+1)^{\overline{\alpha-1}}(b-a)^{\overline{\alpha-1}}}{(b-s+1)^{\overline{\alpha-1}}(s-a-1)^{\overline{\alpha-1}}}, & \text{for } s \leq t. \end{cases}$$

Now, for $s > t$ and $c \leq t \leq d$, $G_1(t,s)$ is an increasing function with respect to t. Then, we have

$$\begin{aligned}\min_{t\in\mathbb{N}_c^d} G_1(t,s) =& G_1(c,s)\\ =& \frac{(c-a)^{\overline{\alpha-1}}(b-s+1)^{\overline{\alpha-1}}}{(b-a)^{\overline{\alpha-1}}\Gamma(\alpha)}.\end{aligned}$$

For $t > s$ and $c \leq t \leq d$, $G_2(t,s)$ is an decreasing function with respect to t. Then, we have

$$\begin{aligned}\min_{t\in\mathbb{N}_c^d} G_2(t,s) =& G_2(d,s)\\ =& \frac{(d-a)^{\overline{\alpha-1}}(b-s+1)^{\overline{\alpha-1}}}{(b-a)^{\overline{\alpha-1}}\Gamma(\alpha)} - \frac{(d-s+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}.\end{aligned}$$

Thus,

$$\begin{aligned}\min_{t\in\mathbb{N}_c^d} G(t,s) &= \begin{cases} G_2(d,s), & \text{for } s \in \mathbb{N}_{a+2}^{c},\\ \min\{G_2(d,s), G_1(c,s)\}, & \text{for } s \in \mathbb{N}_{c+1}^{d-1},\\ G_1(c,s), & \text{for } s \in \mathbb{N}_{d}^{b},\end{cases}\\ &= \begin{cases} G_2(d,s), & \text{for } s \in \mathbb{N}_{a+2}^{r},\\ G_1(c,s), & \text{for } s \in \mathbb{N}_{r}^{b},\end{cases}\end{aligned}$$

where $c < r < d$.

Consider

$$\frac{\min_{t\in\mathbb{N}_c^d} G(t,s)}{G(s-1,s)} = \begin{cases} \frac{(d-a)^{\overline{\alpha-1}}}{(s-a-1)^{\overline{\alpha-1}}} - \frac{(d-s+1)^{\overline{\alpha-1}}(b-a)^{\overline{\alpha-1}}}{(b-s+1)^{\overline{\alpha-1}}(s-a-1)^{\overline{\alpha-1}}}, & \text{for } s \in \mathbb{N}_{a+2}^{r},\\ \frac{(c-a)^{\overline{\alpha-1}}}{(s-a-1)^{\overline{\alpha-1}}}, & \text{for } s \in \mathbb{N}_{r}^{b}.\end{cases}$$

Thus,

$$\min_{t\in\mathbb{N}_c^d} G(t,s) \geq \gamma(s) \max_{t\in\mathbb{N}_a^b} G(t,s), \tag{2.5}$$

where

$$\gamma(s) = \min\left[\frac{(c-a)^{\overline{\alpha-1}}}{(s-a-1)^{\overline{\alpha-1}}}, \frac{(d-a)^{\overline{\alpha-1}}}{(s-a-1)^{\overline{\alpha-1}}} - \frac{(d-s+1)^{\overline{\alpha-1}}(b-a)^{\overline{\alpha-1}}}{(b-s+1)^{\overline{\alpha-1}}(s-a-1)^{\overline{\alpha-1}}}\right].$$

For $s \in \mathbb{N}_r^b$, denote

$$\begin{aligned}\gamma_1(s) =& \frac{(c-a)^{\overline{\alpha-1}}}{(s-a-1)^{\overline{\alpha-1}}}\\ \geq& \frac{(c-a)^{\overline{\alpha-1}}}{(b-a-1)^{\overline{\alpha-1}}}.\end{aligned}$$

Similarly, for $s \in \mathbb{N}_{a+2}^r$, we take

$$\gamma_2(s) = \frac{1}{(s-a-1)^{\overline{\alpha-1}}} \left[(d-a)^{\overline{\alpha-1}} - \frac{(d-s+1)^{\overline{\alpha-1}}(b-a)^{\overline{\alpha-1}}}{(b-s+1)^{\overline{\alpha-1}}} \right].$$

By Lemma 2.2, we see that $\frac{(d-s+1)^{\overline{\alpha-1}}}{(b-s+1)^{\overline{\alpha-1}}}$ is a decreasing function for $s \in \mathbb{N}_{a+2}^r$.

Then,

$$\begin{aligned}\gamma_2(s) \geq & \frac{1}{(s-a-1)^{\overline{\alpha-1}}} \left[(d-a)^{\overline{\alpha-1}} - \frac{(d-a-1)^{\overline{\alpha-1}}(b-a)^{\overline{\alpha-1}}}{(b-a-1)^{\overline{\alpha-1}}} \right] \\ > & \frac{1}{(d-a)^{\overline{\alpha-1}}} \left[(d-a)^{\overline{\alpha-1}} - \frac{(d-a-1)^{\overline{\alpha-1}}(b-a)^{\overline{\alpha-1}}}{(b-a-1)^{\overline{\alpha-1}}} \right].\end{aligned}$$

Thus,

$$\min_{t \in \mathbb{N}_c^d} G(t,s) \geq \gamma \max_{t \in \mathbb{N}_a^b} G(t,s), \tag{2.6}$$

where

$$\gamma = \min \left[\frac{(c-a)^{\overline{\alpha-1}}}{(b-a-1)^{\overline{\alpha-1}}}, 1 - \frac{(d-a-1)^{\overline{\alpha-1}}(b-a)^{\overline{\alpha-1}}}{(b-a-1)^{\overline{\alpha-1}}(d-a)^{\overline{\alpha-1}}} \right].$$

Since $G_1(c,s) > 0$ and $G_2(d,s) > 0$, we have $\gamma(s) > 0$ for all $s \in \mathbb{N}_{a+2}^b$, implying $\gamma > 0$. It would be sufficient to prove that one of the terms $\frac{(c-a)^{\overline{\alpha-1}}}{(b-a-1)^{\overline{\alpha-1}}}, 1 - \frac{(d-a-1)^{\overline{\alpha-1}}(b-a)^{\overline{\alpha-1}}}{(b-a-1)^{\overline{\alpha-1}}(d-a)^{\overline{\alpha-1}}}$ is less than 1. It follows from Lemma 2.1 that

$$\frac{(c-a)^{\overline{\alpha-1}}}{(b-a-1)^{\overline{\alpha-1}}} < 1.$$

Therefore, we conclude $\gamma \in (0,1)$. The proof is completed. □

3. Existence

In this section, we establish sufficient conditions on the existence of positive solutions of (1.1) using Guo–Krasnoselskii fixed-point theorem on a conical shell.

Definition 3.1. Let $\mathcal{B}$ be a Banach space over $\mathbb{R}$. A closed non-empty subset K of $\mathcal{B}$ is said to be a cone provided,
(i) $au + bv \in K$, for all $u, v \in K$ and all $a, b \geq 0$,
(ii) $u \in K$ and $-u \in K$ imply $u = 0$.

Definition 3.2. An operator is called completely continuous, if it is continuous and maps bounded sets into precompact sets.

Theorem 3.1 ([1]). *[Guo–Krasnoselskii fixed-point theorem] Let $\mathcal{B}$ be a Banach space and $\mathcal{K} \subseteq \mathcal{B}$ be a cone. Assume that Ω_1 and Ω_2 are open sets contained in $\mathcal{B}$ such that $0 \in \Omega_1$ and $\overline{\Omega}_1 \subseteq \Omega_2$. Assume further that $T : \mathcal{K} \cap (\overline{\Omega}_2 \setminus \Omega_1) \longrightarrow \mathcal{K}$ is a completely continuous operator. If, either*

1. *$\|Tu\| \leq \|u\|$ for $u \in \mathcal{K} \cap \partial\Omega_1$ and $\|Tu\| \geq \|u\|$ for $u \in \mathcal{K} \cap \partial\Omega_2$; or*
2. *$\|Tu\| \geq \|u\|$ for $u \in \mathcal{K} \cap \partial\Omega_1$ and $\|Tu\| \leq \|u\|$ for $u \in \mathcal{K} \cap \partial\Omega_2$*

holds, then T has at least one fixed-point in $\mathcal{K} \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

Denote

$$\mathcal{B} = \{u : \mathbb{N}_a^b \to \mathbb{R} \mid u(a) = u(b) = 0\} \subseteq \mathbb{R}^{b-a+1}.$$

Clearly, $\mathcal{B}$ is a Banach space equipped with the maximum norm, i.e.,

$$||u|| = \max_{t \in \mathbb{N}_a^b} |u(t)|.$$

Define the operator $T_\beta : \mathcal{B} \to \mathcal{B}$ by

$$\left(T_\beta u\right)(t) = \beta \sum_{s=a+2}^{b} G(t,s) f(s, u(s)), \quad t \in \mathbb{N}_a^b. \tag{3.1}$$

Since T_β is defined on a discrete finite domain, it is trivially completely continuous. We also observe from (2.1) and (3.1) that u is a fixed-point of T_β, if and only if u is a solution of (1.1).

Define the cone

$$\mathcal{K} = \{u \in \mathcal{B} : u(t) \geq 0, \text{ for } t \in \mathbb{N}_a^b \text{ and } \min_{t \in \mathbb{N}_c^d} u(t) \geq \gamma ||u||\}.$$

First, we show that $T_\beta : \mathcal{K} \to \mathcal{K}$.

Let $u \in \mathcal{K}$. Clearly, $\left(T_\beta u\right)(t) \geq 0$, for $t \in \mathbb{N}_a^b$. Consider that

$$\begin{aligned}
\min_{t \in \mathbb{N}_c^d} \left(T_\beta u\right)(t) =& \min_{t \in \mathbb{N}_c^d} \left[\beta \sum_{s=a+2}^{b} G(t,s) f(s,u(s))\right] \\
\geq& \beta \sum_{s=a+2}^{b} \min_{t \in \mathbb{N}_c^d} \left[G(t,s)\right] f(s,u(s)) \\
\geq& \beta \sum_{s=a+2}^{b} \gamma \max_{t \in \mathbb{N}_a^b} \left[G(t,s)\right] f(s,u(s)) \\
\geq& \gamma \max_{t \in \mathbb{N}_a^b} \beta \sum_{s=a+2}^{b} G(t,s) f(s,u(s)) \\
=& \gamma \max_{t \in \mathbb{N}_a^b} \left|\beta \sum_{s=a+2}^{b} G(t,s) f(s,u(s))\right| \\
=& \gamma ||T_\beta u||.
\end{aligned}$$

Thus, we have $T_\beta : \mathcal{K} \to \mathcal{K}$.

Here, we state the following hypotheses, which will be used later.

(F1) $f(t,u) = h(t)g(u)$ where $h : \mathbb{N}_a^b \to \mathbb{R}^+$ and $g : \mathbb{R} \to \mathbb{R}^+ \cup \{0\}$;

(F2) $\lim\limits_{u \to 0^+} \frac{g(u)}{u} = 0$ and $\lim\limits_{u \to +\infty} \frac{g(u)}{u} = \infty$;

(F3) $\lim\limits_{u\to 0^+} \frac{g(u)}{u} = \infty$ and $\lim\limits_{u\to +\infty} \frac{g(u)}{u} = 0$;

(F4) $g_0 = \lim\limits_{u\to 0^+} \sup \dfrac{g(u)}{u}$ and $g_\infty = \lim\limits_{u\to +\infty} \sup \dfrac{g(u)}{u}$;

(F5) $g_0^* = \lim\limits_{u\to 0^+} \inf \dfrac{g(u)}{u}$ and $g_\infty^* = \lim\limits_{u\to +\infty} \inf \dfrac{g(u)}{u}$.

Denote

$$G = \max_{(t,s)\in \mathbb{N}_{a+1}^{b-1}\times \mathbb{N}_{a+2}^{b}} G(t,s),$$

$$H = \max_{t\in\mathbb{N}_a^b} h(t) \text{ and } h = \min_{t\in\mathbb{N}_a^b} h(t).$$

Theorem 3.2. *Assume that* $(F1)$ *and* $(F2)$ *hold. If there exists a sufficiently small positive constant* δ *and a sufficiently large constant* M *such that* $H\delta < hM$*, then for each*

$$\beta \in \left[(Gh(b-a-1)M)^{-1}, (GH(b-a-1)\delta)^{-1}\right], \tag{3.2}$$

(1.1) *has at least one positive solution.*

Proof. By condition $(F2)$, there exists $r_1 > 0$ and a sufficiently small constant $\delta > 0$ such that

$$g(u) \le \delta r_1 \quad \text{whenever} \quad 0 < u \le r_1. \tag{3.3}$$

Set $\Omega_1 = \{u \in \mathcal{B} : \|u\| < r_1\}$. Thus, for $u \in K$ with $\|u\| = r_1$, by (3.2) and (3.3), we have

$$\begin{aligned}
\|T_\beta u\| &= \max_{t\in\mathbb{N}_b^a} \left|\beta \sum_{s=a+2}^{b} G(t,s) f(s,u(s))\right| \\
&\le \beta \sum_{s=a+2}^{b} \max_{t\in\mathbb{N}_b^a} [G(t,s)]\, h(s) g(u(s)) \\
&\le \beta G H \delta r_1 (b-a-1) \\
&\le r_1 = \|u\|.
\end{aligned}$$

Therefore, $\|T_\beta y\| \le \|u\|$ for $u \in \mathcal{K} \cap \partial\Omega_1$. Similarly, by condition $(F2)$, we can find $0 < r_1 < r_2$ and a sufficient large constant M such that

$$g(u) \ge \frac{M r_2}{\gamma^2} \text{ for } u \ge r_2. \tag{3.4}$$

Set $r_2^* = \frac{r_2}{\gamma} > r_2$ and $\Omega_2 = \{u \in \mathcal{B} : \|u\| < r_2^*\}$. Then, for $u \in K$ with $\|u\| = r_2^*$, we have

$$\min_{t\in\mathbb{N}_c^d} u(t) \ge \gamma\|u\| = \gamma r_2^*,$$

implying $u(t) \ge r_2$ for $t \in \mathbb{N}_a^b$. Therefore, by (3.2) and (3.4), we have

$$\begin{aligned}
\|T_\beta u\| &\ge \min_{t\in\mathbb{N}_c^d} |T_\beta u(t)| \\
&= \min_{t\in\mathbb{N}_c^d} \beta \sum_{s=a+2}^{b} G(t,s) f(s,u(s))
\end{aligned}$$

$$\begin{aligned}
&\geq \beta \sum_{s=a+2}^{b} \min_{t\in\mathbb{N}_c^d}[G(t,s)]f(s,u(s))\\
&\geq \gamma\beta \sum_{s=a+2}^{b} \max_{t\in\mathbb{N}_b^a}[G(t,s)]\,h(s)g(u(s))\\
&\geq \gamma\beta Gh\frac{Mr_2}{\gamma^2}(b-a-1)\\
&\geq r_2^* = \|u\|\,.
\end{aligned}$$

Thus, we conclude $\|T_\beta u\| \geq \|u\|$ for $u \in \partial\Omega_2 \cap \mathcal{K}$. By part (1) of Theorem 3.1, we conclude that T_β has a fixed-point u_0 in $\mathcal{K} \cap (\overline{\Omega}_2 \setminus \Omega_1)$, satisfying $r_1 < \|u_0\| < r_2^*$. The proof is completed. □

Theorem 3.3. *Assume that* $(F1)$ *and* $(F3)$ *hold. If there exists a sufficiently large constant* L *such that* $H < hL$*, then for each*

$$\beta \in \left[(Gh(b-a-1)L)^{-1}, (GH(b-a-1))^{-1}\right], \tag{3.5}$$

(1.1) *has at least one positive solution.*

Proof. By condition $(F3)$, there exists $r_3 > 0$ and a sufficiently large constant $L > 0$ such that $g(u) \geq \frac{Lr_3}{\gamma}$ for $0 < u \leq r_3$. Set $\Omega_1 = \{u \in \mathcal{B} : \|u\| < r_3\}$. Then, for $u \in \Omega_1$, we have

$$\begin{aligned}
\|T_\beta u\| &\geq \min_{t\in\mathbb{N}_c^d} |T_\beta u(t)|\\
&= \min_{t\in\mathbb{N}_c^d} \beta \sum_{s=a+2}^{b} G(t,s)f(s,u(s))\\
&\geq \beta \sum_{s=a+2}^{b} \min_{t\in\mathbb{N}_c^d}[G(t,s)]f(s,u(s))\\
&\geq \gamma\beta \sum_{s=a+2}^{b} \max_{t\in\mathbb{N}_b^a}[G(t,s)]\,h(s)g(u(s))\\
&\geq \gamma\beta Gh\frac{Lr_3}{\gamma}(b-a-1)\\
&\geq r_3 = \|u\|\,.
\end{aligned}$$

This implies $\|T_\beta u\| \geq \|u\|$, for $u \in \partial\Omega_1 \cap \mathcal{K}$. Now, we consider two cases for the construction of Ω_2.

Case 1. Suppose that g is bounded. Then, there exists $R_1 \geq r_2$ such that $g(u) \leq R_1$ for $r_2 \leq u \leq \frac{r_2}{\gamma}$. From (3.5), we know that $\beta \leq (GH(b-a-1))^{-1}$. Thus, we have

$$\begin{aligned}
\|T_\beta u\| &= \max_{t\in\mathbb{N}_b^a} \left|\beta \sum_{s=a+2}^{b} G(t,s)f(s,u(s))\right|\\
&\leq \beta \sum_{s=a+2}^{b} \max_{t\in\mathbb{N}_b^a}[G(t,s)]\,h(s)g(u(s))\\
&\leq \beta GHR_1(b-a-1)
\end{aligned}$$

$$\leq R_1 = \|u\|.$$

Case 2. Suppose that g is unbounded. Then, there exists some constant R_2 and a sufficiently small δ_2 such that $g(u) \leq \delta_2 u$ for $u \geq R_2$ and for $0 < u \leq R_2,\ g(u) \leq g(R_2)$. Let $R = \max\{R_1, R_2\}$. Now, we assume that $\Omega_2 = \{u \in \beta : \|u\| < R\}$. H $g(u) \leq \delta_2 R$. Thus, by (3.5), we have $\beta \leq (GH(b-a-1))^{-1} < (GH\delta_2(b-a-1))^{-1}$, implying

$$\begin{aligned}
\|T_\beta u\| &= \max_{t\in\mathbb{N}_b^a}\left|\beta \sum_{s=a+2}^{b} G(t,s)f(s,u(s))\right| \\
&\leq \beta \sum_{s=a+2}^{b} \max_{t\in\mathbb{N}_b^a} [G(t,s)]\, h(s)g(u(s)) \\
&\leq \beta GH\delta_2 R(b-a-1) \\
&\leq R = \|u\|.
\end{aligned}$$

Therefore, we have $\|T_\beta u\| \leq \|u\|$ in both cases for $u \in \partial\Omega_2 \cap \mathcal{K}$. By part (ii) of Theorem 3.1, we conclude that T_β has a fixed-point $u \in \mathcal{K} \cap (\overline{\Omega}_2 \setminus \Omega_1)$ satisfying $r_3 < \|u_0\| < R$. The proof is completed. □

4. Non-existence

In this section, we establish sufficient conditions on the non-existence of positive solutions for (1.1).

Theorem 4.1. *Assume that* $(F1)-(F4)$ *hold. If* $g_0 < \infty$ *and* $g_\infty < \infty$*, then there exists a* β_0 *such that for all* $0 < \beta < \beta_0$*,* (1.1) *has no positive solution.*

Proof. Since $g_0 < \infty$ and $g_\infty < \infty$, there exist positive numbers m_1, m_2, r_1 and r_2, such that $r_1 < r_2$, $g(u) \leq m_1 u$ for $u \in [0, r_1]$ and $g(u) \leq m_2 u$ for $u \in [0, \infty)$. Let

$$m = \max\left\{m_1, m_2, \max_{r_1\leq u\leq r_2} \frac{g(u)}{u}\right\}.$$

Then, we have $g(u) \leq mu$. Suppose that u_0 is a positive solution of (1.1). We will show that this leads to a contradiction for $0 < \beta < \beta_0 = (GH(b-a-1)m)^{-1}$. Since $T_\beta u_0(t) = u_0(t)$ for $t \in \mathbb{N}_a^b$, we have

$$\begin{aligned}
\|u_0\| = \|T_\beta u_0\| &= \max_{t\in\mathbb{N}_a^b}\left|\beta \sum_{s=a+2}^{b} G(t,s)f(s,u_0(s))\right| \\
&\leq \beta \sum_{s=a+2}^{b} \max_{t\in\mathbb{N}_a^b}[G(t,s)]h(s)g(u_0(s)) \\
&\leq \beta GH \sum_{s=a+2}^{b} g(u_0(s)) \\
&\leq \beta GHmu_o(b-a-1) < \|u_0\|.
\end{aligned}$$

This is a contradiction. Therefore, (1.1) has no positive solution. The proof is completed. □

Theorem 4.2. *Assume that* $(F1)-(F5)$ *hold. If* $g_0^*>0$ *and* $g_\infty^*>0$*, then there exists a* β_0 *such that for all* $\beta>\beta_0$*, (1.1) has no positive solution.*

Proof. Since $g_0^*>0$ and $g_\infty^*>0$, there exist positive numbers n_1,n_2,r_1 and r_2 such that $r_1<r_2$ and $g(u)\geq n_1u$ for $u\in[0,r_1]$, $g(u)\geq n_2u$ for $u\in[0,\infty)$. Let

$$n=\min\left\{n_1,n_2,\min_{r_1\leq u\leq r_2}\frac{g(u)}{u}\right\}>0.$$

Then, we have $g(u)\geq nu$. Suppose that u_1 is a positive solution of (1.1). We will show that this leads to a contradiction for $\beta>\beta_0=(\gamma Gh(b-a-1)n)^{-1}$. Since $T_\beta u_1(t)=u_1(t)$, for $t\in\mathbb{N}_a^b$, we have

$$\begin{aligned}\|u_1\|=\|T_\beta u_1\|\geq&\min_{t\in\mathbb{N}_c^d}\left|\beta\sum_{s=a+2}^{b}G(t,s)f(s,u_1(s))\right|\\ \geq&\beta\sum_{s=a+2}^{b}\min_{t\in\mathbb{N}_c^d}[G(t,s)]f(s,u_1(s))\\ \geq&\gamma\beta\sum_{s=a+2}^{b}\max_{t\in\mathbb{N}_c^d}[G(t,s)]h(s)g(u_1(s))\\ \geq&\beta\gamma Gh\sum_{s=a+2}^{b}g(u_1(s))\\ \geq&\beta\gamma Ghnu_1(b-a-1)>\|u_1\|.\end{aligned}$$

This is a contradiction. Therefore, (1.1) has no positive solution. The proof is completed. □

5. Examples

Finally, in this section, we conclude this article by a few examples to illustrate the applicability of our main results from previous sections.

Example 5.1. Let $\alpha=1.5,a=1$ and $b=11$. Consider the boundary value problem

$$\begin{cases}-\left(\nabla_{\rho(1)}^{1.5}u\right)(t)=\beta tu^2, & t\in\mathbb{N}_3^{11},\\ u(1)=u(11)=0.\end{cases}\tag{5.1}$$

We have $h(t)=t$, $g(u)=u^2$ for $u\in\mathbb{R}^+$. Take $M=100,\delta=\frac{1}{1000}$. By routine calculation, we get $G\approxeq 1.718,H=11,h=1$, and $\lim\limits_{u\to0^+}\frac{g(u)}{u}=\lim\limits_{u\to0^+}u=0$ and $\lim\limits_{u\to+\infty}\frac{g(u)}{u}=\lim\limits_{u\to+\infty}u=\infty$. Then, $H(b-a-1)\delta=11\times9\times\frac{1}{1000}=\frac{99}{1000}$, $h(b-a-1)M=1\times9\times100=900$. We see $H\delta<hM$. Therefore, all the conditions of Theorem 3.2 are satisfied. Then, the boundary value problem (5.1) has at least one positive solution for each $\beta\in[0.00064,5.879]$.

Example 5.2. Let $\alpha=1.5,a=1$ and $b=11$. Consider the boundary value problem

$$\begin{cases}-\left(\nabla_{\rho(1)}^{1.5}u\right)(t)=\beta(t+2)e^{-u}, & t\in\mathbb{N}_3^{11},\\ u(1)=u(11)=0.\end{cases}\tag{5.2}$$

We have $h(t) = t+2,\ g(u) = e^{-u}$ for $u \in \mathbb{R}^+$. Take $L = 100$. Also, we have $G \approxeq 1.718, H = 13, h = 3$, and $\lim_{u\to 0^+} \frac{g(u)}{u} = \lim_{u\to 0^+} \frac{e^{-u}}{u} = \infty$ and $\lim_{u\to+\infty} \frac{g(u)}{u} = \lim_{u\to+\infty} \frac{e^{-u}}{u} = 0$. Then, $H(b-a-1) = 13 \times 9 = 117,\ h(b-a-1)L = 3 \times 9 \times 100 = 2700$. We see $H < hL$. Therefore, all the conditions of Theorem 3.3 are satisfied. Then, the boundary value problem (5.2) has at least one positive solution for each $\beta \in [0.00021, 0.00497]$.

Example 5.3. Let $\alpha = 1.5, a = 1$ and $b = 11$. Consider the boundary value problem

$$\begin{cases} -\left(\nabla^{1.5}_{\rho(1)}u\right)(t) = \beta(t+3)u, & t \in \mathbb{N}_3^{11}, \\ u(1) = u(11) = 0. \end{cases} \tag{5.3}$$

We have $h(t) = t+3,\ g(u) = u$ for $u \in \mathbb{R}^+$. Take $m_1, m_2 = 3$. Also, we have $G \approxeq 1.718, H = 14, h = 4$, and $m = \max\left\{m_1, m_2, \max_{r_1\le u\le r_2} \frac{g(u)}{u}\right\} = 3$. Then, $g_0 = \lim_{u\to 0^+} \sup \dfrac{g(u)}{u} = 1 < \infty$ and $g_\infty = \lim_{u\to+\infty} \sup \dfrac{g(u)}{u} = 1 < \infty$. Hence, $\beta_0 = (GH(b-a-1)m)^{-1} = 0.00153$. Therefore, all the conditions of Theorem 4.1 are satisfied. Then, the boundary value problem (5.3) has no positive solution for $0 < \beta < \beta_0$.

Example 5.4. Let $\alpha = 1.5, a = 1$ and $b = 11$. Consider the boundary value problem

$$\begin{cases} -\left(\nabla^{1.5}_{\rho(1)}u\right)(t) = \beta(t+3)u, & t \in \mathbb{N}_3^{11}, \\ u(1) = u(11) = 0. \end{cases} \tag{5.4}$$

We have $h(t) = t+3,\ g(u) = u$ for $u \in \mathbb{R}^+$. Take $n_1, n_2 = \frac{1}{3}$. Also, we have $G \approxeq 1.718, H = 14, h = 4$ and $n = \min\left\{n_1, n_2, \min_{r_1\le u\le r_2} \frac{g(u)}{u}\right\} = \frac{1}{3}$. Then,

$$g_0^* = \lim_{u\to 0^+} \inf \frac{g(u)}{u} = 1 > 0,$$

$$g_\infty^* = \lim_{u\to+\infty} \inf \frac{g(u)}{u} = 1 > 0$$

and

$$\gamma = \min\left[\frac{(c-1)^{\overline{0.5}}}{(9)^{\overline{0.5}}}, 1 - \frac{(d-2)^{\overline{0.5}}(10)^{\overline{0.5}}}{(9)^{\overline{0.5}}(d-1)^{\overline{0.5}}}\right],$$

where $c = a + \left\lceil \frac{b-a+1}{4} \right\rceil = 4$ and $d = a + 3\left\lfloor \frac{b-a+1}{4} \right\rfloor = 7$. Substituting c and d above, we get $\gamma = 0.04043$. Hence, $\beta_0 = (\gamma Gh(b-a-1)n)^{-1} = 1.1997$. Therefore, all the conditions of Theorem 4.2 are satisfied. Then, the boundary value problem (5.4) has no positive solution for $\beta > \beta_0$.

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