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Existence and Uniqueness of Solutions for the Initial Value Problem of Fractional q_k-Difference Equations for Impulsive with Varying Orders*

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Abstract The paper studies the existence and uniqueness for impulsive fractional q_k-difference equations of initial value problems involving Riemann-Liouville fractional q_k-integral and q_k-derivative by defining a new q-shifting operator. In this paper, we obtain existence and uniqueness results for impulsive fractional q_k-difference equations of initial value problems by using the Schaefer's fixed point theorem and Banach contraction mapping principle. In addition, the main result is illustrated with the aid of several examples.

Keywords Impulsive fractional q_k-difference equation, Boundary value problem, Existence, Uniqueness.

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1. Introduction

Fractional calculus is a relatively new research field, and it can describe certain phenomena as well, which has attracted increasing attention in recent years. The quantum calculus is known as the calculus without limits. It substitutes the classical derivative by a difference operator, which allows one to deal with sets of nondifferentiable functions. Quantum difference operators appear in several branches of mathematics, i.e., basic hype-geometric functions, combinatorics, the theory of relativity. For the fundamental concepts of quantum calculus, we refer to the reader to the work by Kac and Cheung [5, 8].

In the recent years, the topic of q-calculus has attracted the attention of several researchers, and a variety of new results can be found in the papers [3, 4, 7, 9, 10, 12–14, 22] and the references therein. In real life, there are many processes and phenomena that are characterized by rapid changes in their state. We usually keep things instantaneous mutations occurred in the process of its development called impulsive phenomena. The phenomenon has been widely appearing in all fields of production and technology research. The most prominent feature of impulsive differential equations is taking the influence of the condition of sudden and abrupt phenomenon into full consideration. It has been extensively used in population

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ecological dynamics systems, infectious disease dynamics as well as descriptions of phenomenon like disease, harvesting and so on.

Impulsive differential equations, in other words, differential equations involving impulsive factors, appear as a natural description of observed evolution phenomenon of several real world problems. For some monographs on impulsive differential equations, we refer to [6, 20, 21].

In [16], the notions of q_k-integral of a function $f : J_k := [t_k, t_{k+1}] \to \mathbb{R}$ have been introduced, and their basic properties were proved and applied, Tariboon et al., investigated the first and second-order initial value problems of impulsive q_k-difference equation respectively, as shown below

$$\begin{aligned}
&D^2_{q_k} x(t) = f(t, x(t)),\ t \in J,\ t \neq t_k,\\
&\Delta x(t_k) = I_k(x(t_k)),\ k = 1, 2, \cdots m,\\
&D_{q_k} x(t_k^+) - D_{q_{k-1}} x(t_k) = I^*_k(x(t_k)),\ k = 1, 2, \cdots m,\\
&x(0) = \alpha,\ D_{q_0} x(0) = \beta,\\
&D_{q_k} x(t) = f(t, x(t)),\ t \in J,\ t \neq t_k,\\
&\Delta x(t_k) = I_k(x(t_k)),\ k = 1, 2, \cdots m,\\
&x(0) = x_0,
\end{aligned}$$

where $x_0 \in \mathbb{R}, \alpha, \beta \in \mathbb{R}, 0 = t_0 < t_1 < t_2 < \cdots < t_k < \cdots < t_m < t_{m+1} = T, f : J \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $I_k \in C(\mathbb{R}, \mathbb{R}), \Delta x(t_k) = x(t_k^+) - x(t_k), k = 1, 2, \cdots m$ and $0 < q_k < 1$ for $k = 0, 1, 2, \cdots m$. In addition, q_k-calculus analogues of some classical integral inequalities, such as Hölder, Hermite-Hadamard, Trapezoid, Ostrowski, Cauchy-Bunyakovsky-Schwarz, Grüss and Grüss-Cebysev, were proved in [17].

In 2015, Agarwal et al., [2] investigated the existence of positive solutions for nonlinear impulsive q_k-difference equations via a monotone iterative method

$$\begin{aligned}
&D_{q_k} u(t) = f(t, u(t)),\ 0 < q_k < 1,\ t \in J',\\
&\Delta u(t_k) = I_k(u(t_k)),\ k = 1, 2, \cdots m,\\
&u(0) = \lambda u(\eta) + d,\ \eta \in J_r,\ r \in \mathbb{Z},
\end{aligned} \tag{1.1}$$

where $f \in C(J \times \mathbb{R}, \mathbb{R}^+),\ I_k \in C(\mathbb{R}, \mathbb{R}^+),\ J = [0, T],\ T > 0,\ 0 = t_0 < t_1 < t_2 < \cdots < t_k < \cdots < t_m < t_{m+1} = T,\ J' = J \setminus \{t_1, t_2, \cdots t_m\},\ J_r = (t_r, T),\ 0 \le \lambda < 1,\ 0 \le r \le m$ and $\Delta u(t_k) = u(t_k^+) - u(t_k^-),\ u(t_k^+)$ and $u(t_k^-)$ denote the right and left limits of $u(t)$ at $t = t_k (k = 1, 2, \cdots m)$ respectively.

In [18], the new concepts of fractional quantum calculus were defined by introducing a new q-shifting operator. After giving the basic properties of the new q-shifting operator, the q-derivative and the q-integral were defined. New definitions of the Riemann-Liouville fractional q-integral and q-difference of an interval $[a, b]$ were given, and their properties were discussed. As applications, the authors obtained the existence and uniqueness results of initial value problems for impulsive fractional q_k-difference equations of the orders $0 < \alpha < 1$ and $1 < \alpha < 2$ respectively, as shown below,

$$\begin{aligned}
&{}_{t_k}D^\alpha_{q_k} x(t) = f(t, x(t)),\ t \in J,\ t \neq t_k,\\
&\widetilde{\Delta}\, x(t_k) = \varphi_k(x(t_k)),\ k = 1, 2, \cdots m,\\
&x(0) = 0,\\
&{}_{t_k}D^\alpha_{q_k} x(t) = f(t, x(t)),\ t \in J,\ t \neq t_k,
\end{aligned}$$

$$\begin{aligned}
&\widetilde{\Delta}\, x(t_k) = \varphi_k(x(t_k)),\ k = 1, 2, \cdots m,\\
&\Delta^* x(t_k) = \varphi_k^* x(t_k),\ k = 1, 2, \cdots m,\\
&x(0) = 0,\ {}_0D_{q_0}^{\alpha-1} x(0) = \beta,
\end{aligned}$$

where $\widetilde{\Delta}\ x(t_k) = {}_{t_k}I_{q_k}^{1-\alpha} x(t_k^+) - {}_{t_{k-1}}I_{q_{k-1}}^{1-\alpha} x(t_k)$ and $\Delta^* x(t_k) = {}_{t_k}I_{q_k}^{2-\alpha} x(t_k^+) - {}_{t_{k-1}}I_{q_{k-1}}^{2-\alpha} x(t_k), k = 1, 2, \cdots m.$

Inspired by the above papers, in this paper, we study the following initial value problem of the impulsive fractional q_k-difference equation

$$\begin{aligned}
&{}_{t_k}D_{q_k}^{\alpha_k} x(t) = f\left(t, x(t)\right),\ t \in J_k = [0, T] \setminus \{t_1, t_2, \cdots t_k\},\\
&\triangle\, \widetilde{x}\ (t_k) = \varphi_k\left(x(t_k)\right),\ k = 1, 2, \cdots m,\\
&\triangle^* x(t_k) = \varphi_k^*\left(x(t_k)\right),\ k = 1, 2, \cdots m,\\
&\triangle^{**} x(t_k) = \varphi_k^{**}\left(x(t_k)\right),\ k = 1, 2, \cdots m,\\
&x(0) = 0,\ {}_0D_{q_0}^{\alpha_0-1} x(0) = \beta,\ {}_0D_{q_0}^{\alpha_0-2} x(0) = \gamma,
\end{aligned} \tag{1.2}$$

where $\beta \in \mathbb{R},\ 0 = t_0 < t_1 < t_2 < \cdots t_m < t_{m+1} = T,\ f : J \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $\varphi_k,\ \varphi_k^*,\ \varphi_k^{**} \in C(\mathbb{R}, \mathbb{R})$ and $x(t)$ are continuous everywhere except for some t_k at which $x(t_k^+)$ and $x(t_k^-)$ exist and $x(t_k^-) = x(t_k)$ for $k = 1, 2, \cdots m$, $0 < q_k < 1$ for $k = 0, 1, 2, \cdots m$, and ${}_{t_k}D_{q_k}^{\alpha_k}$ denotes the Riemann-Liouville q_k-fractional derivative of the order α_k, $J_k = (t_k, t_{k+1}]$, especially $J_0 = [0, t_1]$. Moreover,

$$\begin{aligned}
&\Delta\, \widetilde{x}\ (t_k) = {}_{t_k}D_{q_k}^{\alpha_k-1}\left(x(t_k^+)\right) - {}_{t_{k-1}}D_{q_{k-1}}^{\alpha_{k-1}-1}\left(x(t_k)\right),\\
&\Delta^* x(t_k) = {}_{t_k}D_{q_k}^{\alpha_k-2}\left(x(t_k^+)\right) - {}_{t_{k-1}}D_{q_{k-1}}^{\alpha_{k-1}-2}\left(x(t_k)\right),\\
&\Delta^{**} x(t_k) = {}_{t_k}I_{q_k}^{3-\alpha_k}\left(x(t_k^+)\right) - {}_{t_{k-1}}I_{q_{k-1}}^{3-\alpha_{k-1}}\left(x(t_k)\right).
\end{aligned} \tag{1.3}$$

The main innovation points of this paper are as follows:

1. Compared with [18], our paper has been improved for the order of the equation.

2. The order of the equation studied in most papers is fixed, while the order of the equation we study varies with k, a change in k causes a change in order, which adds to the difficulty of the paper.

This paper is organized as follows. In Section 2, we present some preliminaries. Section 3 contains the main results, which are established by means of Schaefer's fixed point theorem and Banach contraction mapping principle, while several examples are illustrated the main results in Section 4.

2. Preliminaries

To get the main results, we present some knowledge of fractional q_k-calculus. The presentation here can be found in [18].

For any positive integer k, the q_k-shifting operator

$${}_a\phi_{q_k}(m) = q_k m + (1 - q_k)a$$

satisfies the relation

$${}_a\phi_{q_k}^k(m) = {}_a\phi_{q_k}^{k-1}({}_a\phi_{q_k}(m)), {}_a\ \phi_{q_k}^0 = m.$$

We define the power of q_k-shifting operator as

$$ {}_a(n-m)_{q_k}^{(0)}=1,\ {}_a(n-m)_{q_k}^{(k)}=\prod_{i=0}^{k-1}(n-_a\phi_{q_k})^i(m)),k\in\mathbb{N}\cup\{\infty\}. $$

If $\gamma\in\mathbb{R}$, then

$$ {}_a(n-m)_{q_k}^{\gamma}=n^{\gamma}\prod_{i=0}^{\infty}\frac{1-\frac{a}{n}\ \phi_{q_k}^{i}(m/n)}{1-\frac{a}{n}\ \phi_{q_k}^{i+\gamma}(m/n)},\ n\neq 0. $$

Lemma 2.1 ([18]). *For any $\gamma,n,m\in\mathbb{R}$ with $n\neq a$ and $k\in\mathbb{N}\bigcup\{\infty\}$, we have*
(i) $(n-m)_a^{(k)}=(n-a)^k(\frac{m-a}{n-a};q)_k$*;*
(ii) $(n-m)_a^{(\gamma)}=(n-a)^{\gamma}\prod_{i=0}^{\infty}\frac{1-\frac{m-a}{n-a}q^i}{1-\frac{m-a}{n-a}q^{\gamma+i}}=(n-a)^{\gamma}\frac{(\frac{m-a}{n-a};q)_\infty}{(\frac{m-a}{n-a}q^{\gamma};q}\,)_\infty$*;*
(iii) $(n-_a\phi_q^k(n))_a^{(\gamma)}=(n-a)^{\gamma}\frac{(q^k;q)_\infty}{(q^{\gamma+k};q)_\infty}$*.*

Definition 2.1. [18] Assume that $f:J_k\to\mathbb{R}$ is a continuous function, and let $t\in J_k$. Then, the expression

$$ \begin{aligned} D_{q_k}f(t)&=\frac{f(t)-f(q_kt+(1-q_k)t_k)}{(1-q_k)(t-t_k)},\ t\neq t_k,\\ D_{q_k}f(t_k)&=\lim_{t\to t_k}D_{q_k}f(t) \end{aligned} \tag{2.1} $$

is called the q_k-derivative of a function f at t.

We say that f is q_k-differentiable on J_k, provided $D_{q_k}f(t)$ exists for all $t\in J_k$. Note that if $t_k=0$ and $q_k=q$ in (2.1), then $D_{q_k}f=D_qf$, where D_q is the q-derivative of the function $f(t)$.

The q_k-integral of a function f defined on the interval $[a,b]$ is given by

$$ ({}_aI_{q_k}f)(t)=\int_a^t f(s)_ad_{q_k}s=(1-q_k)(t-a)\sum_{i=0}^{\infty}q_k^if({}_a\phi_{q_k^i}(t)),\ t\in[a,b], $$

and $({}_aI_{q_k}^kf)(t)={}_aI_{q_k}^{k-1}({}_aI_{q_k}f)(t),\ ({}_aI_{q_k}^0f)(t)=f(t),\ k\in\mathbb{N}.$

The fundamental theorem of q_k calculus applies to these operators ${}_aD_{q_k}$ and ${}_aI_{q_k}$

$$ ({}_aD_{q_k}{}_aI_{q_k}f)(t)=f(t), $$

and if f is continuous at $t=a$, then

$$ ({}_aD_{q_k}{}_aI_{q_k}f)(t)=f(t)-f(a). $$

The formula for q_k-integration by parts on an interval $[a,b]$ is

$$ \int_a^b f(s)_aD_{q_k}g(s)_ad_{q_k}s=(fg)(t)|_a^b-\int_a^b g({}_a\phi_{q_k}(s))_aD_{q_k}f(s)_aD_{q_k}f(s)_ad_{q_k}s. $$

Definition 2.2. [18] Let $v\geq 0$ and f be a function defined on $[a,b]$. The fractional q_k-integral of Riemann-Liouville type is given by

$$ \begin{aligned} &({}_aI_{q_k}^0f)(t)=f(t),\\ &({}_aI_{q_k}^vf)(t)=\frac{1}{\Gamma_{q_k}(v)}\int_a^t{}_a(t-{}_a\phi_{q_k}(s))_{q_k}{}^{(v-1)}f(s)_ad_{q_k}s,\ v>0,\ t\in[a,b]. \end{aligned} $$

Definition 2.3. [18] The fractional q_k-derivatives of Riemann-Liouville type of the order $v \geq 0$ on interval $[a, b]$ is defined by

$$\begin{aligned} &({}_aD^0_{q_k} f)(t) = f(t), \\ &({}_aD^v_{q_k} f)(t) = ({}_aD^l_{q_k}\,{}_aI^{l-v}_{q_k} f)(t),\ v > 0, \end{aligned} \tag{2.2}$$

where l is the smallest integer greater than or equal to v.

Definition 2.4 ([18]). Assume that $f : J_k \to \mathbb{R}$ is a continuous function. Then, the q_k-integral is defined by

$$\int_{t_k}^{t} f(s)d_{q_k}s = (1-q_k)(t-t_k)\sum_{n=0}^{\infty} q_k^n f(q_k^n t + (1-q_k^n)t_k), t \in J_k.$$

Moreover, if $a \in (t_k, t)$, then the definition q_k-integral is defined by

$$\begin{aligned} \int_a^t f(s)d_{q_k}s &= \int_{t_k}^t f(s)d_{q_k}s - \int_{t_k}^a f(s)d_{q_k}s \\ &= (1-q_k)(t-t_k)\sum_{n=0}^{\infty} q_k^n f(q_k^n t + (1-q_k^n)t_k) \\ &\quad - (1-q_k)(a-t_k)\sum_{n=0}^{\infty} q_k^n f(q_k^n a + (1-q_k^n)t_k). \end{aligned}$$

Theorem 2.1 ([18]). *Assume that $f, g : J_k \to \mathbb{R}$ is q_k-differentiable on J_k. Then,*
(1) the sum $f + g : J_k \to \mathbb{R}$ is q_k-differentiable on J_k with

$$D_{q_k}(f(t) + g(t)) = D_{q_k} f(t) + D_{q_k} g(t).$$

(2) for any constant $\alpha, \alpha f : J_k \to \mathbb{R}$ is q_k-differentiable on J_k with

$$D_{q_k}(\alpha f)(t) = \alpha D_{q_k} f(t).$$

(3) the product $fg : J_k \to \mathbb{R}$ is q_k-differentiable on J_k with

$$\begin{aligned} D_{q_k}(fg)(t) &= f(t)D_{q_k}g(t) + g(q_k + (1-q_k)t_k)D_{q_k}f(t) \\ &= g(t)D_{q_k}f(t) + f(q_k + (1-q_k)t_k)D_{q_k}g(t). \end{aligned}$$

(4) if $g(t)g(q_kt + (1-q_k)t_k) \neq 0$, then $\frac{f}{g}$ is q_k-differentiable on J_k with

$$D_{q_k}\left(\frac{f}{g}\right)(t) = \frac{g(t)D_{q_k}f(t) - f(t)D_{q_k}g(t)}{g(t)g(q_kt + (1-q_k)t_k)}.$$

Lemma 2.2 ([18]). *Let $\alpha, \beta \in \mathbb{R}^+$ and f be a continuous function on $[a, b]$, $a \geq 0$, and the Riemann-Liouville fractional q_k-integral has the following semi-group property*

$${}_aI^{\beta}_{q_k}\,{}_aI^{\alpha}_{q_k} f(t) = {}_aI^{\alpha}_{q_k}\,{}_aI^{\beta}_{q_k} f(t) = {}_aI^{\alpha+\beta}_{q_k} f(t).$$

Lemma 2.3 ([18]). *Let f be a q_k-integrable function on $[a, b]$. Then, the following equality holds*

$${}_aD^{\alpha}_{q_k}\,{}_aI^{\alpha}_{q_k} f(t) = f(t),\ \alpha > 0,\ t \in [a, b].$$

Lemma 2.4 ([18]). *Let $\alpha > 0$ and p be a positive integer. Then, for $t \in [a,b]$, the following equality holds*

$$ {}_aI^{\alpha}_{q_k}\,{}_aD^{p}_{q_k} f(t) = {}_aD^{p}_{q_k}\,{}_aI^{\alpha}_{q_k} - \sum_{k=0}^{p-1} \frac{(t-a)^{\alpha-p+k}}{\Gamma_{q_k}(\alpha+k-p+1)}\,{}_aD^{k}_{q_k} f(a). \tag{2.3}$$

From [18], we have the following formulas

$$ {}_aD^{\alpha}_{q_k}(t-a)^{\beta} = \frac{\Gamma_{q_k}(\beta+1)}{\Gamma_{q_k}(\beta-\alpha+1)}(t-a)^{\beta-\alpha}, \tag{2.4}$$

$$ {}_aI^{\alpha}_{q_k}(t-a)^{\beta} = \frac{\Gamma_{q_k}(\beta+1)}{\Gamma_{q_k}(\beta+\alpha+1)}(t-a)^{\beta+\alpha}. \tag{2.5}$$

Lemma 2.5 ([1]). *Let $\alpha > 0$. If $u \in C(0,1)\bigcap L(0,1)$ and $D^{\alpha}_q u \in C(0,1)\bigcap L(0,1)$, then there exists $c_i \in \mathbb{R}, i = 1,2,\cdots,n$, satisfying*

$$ I^{\alpha}_q D^{\alpha}_q u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n}. \tag{2.6}$$

Theorem 2.2 ([19]). *Let X be a Banach space and $W \subset PC(J,X)$. If the following conditions are satisfied,*

(i) W is uniformly bounded subset of $PC(J,X)$;

(ii) W is equicontinuous in $(t_k,t_{k+1}), k = 0,1,2,\cdots m$, where $t_0 = 0, t_{m+1} = T$;

(iii) $W(t) = \{u(t)|u \in W, t \in J\ \{t_1,t_2,\cdots t_m\}\}, W(t_k^+) = \{u(t_k^+)|u \in W\}$ and $W(t_k^-) = \{u(t_k^-)|u \in W\}$ is a relatively compact subsets of X.

Then, W is a relatively compact subset of $PC(J,X)$.

Theorem 2.3 ([15]). *(Schaefer's fixed point theorem) Let A be a continuous and compact mapping of a Banach space X into itself, such that the set $E = \{x \in X : x = \lambda Ax, for\ some\ 0 \le \lambda \le 1\}$ is bounded. Then, A has a fixed point.*

Theorem 2.4 ([11]). *(q-Gronwall Inequality) Suppose $u(t)$ is a non-negative, locally q-integrable on $[0,b]$, and satisfies $u(t) \le a(t) + g(t)\int_0^t (t-qs)_q^{\alpha-1}u(s)d_qs$, where $a > 0, a(t)$ is non-negative and the local time measure q-integrable on $[0,b]$, the function $g(t)$ is non-negative, non-decreasing function on $[0,b]$, and satisfies $g(t) \le M, M > 0$. Then, we have*

$$ u(t) \le a(t) + \int_0^t \sum_{n=1}^{\infty} \frac{(g(t)\Gamma_q(\alpha))^n}{\Gamma_q(n\alpha)}(t-qs)^{\alpha-1}a(s)d_qs.$$

Lemma 2.6. *Let $f \in C(J,\mathbb{R})$. Then, the unique solution of*

$$\begin{aligned}
&{}_{t_k}D^{\alpha_k}_{q_k}x(t) = f(t,x(t)),\ t \in J_k = [0,T]\setminus\{t_1,t_2,\cdots t_k\},\\
&\Delta\,\widetilde{x}\,(t_k) = \varphi_k\left(x(t_k)\right),\ k = 1,2,\cdots m,\\
&\Delta^{*}x(t_k) = \varphi_k^{*}\left(x(t_k)\right),\ k = 1,2,\cdots m,\\
&\Delta^{**}x(t_k) = \varphi_k^{**}\left(x(t_k)\right),\ k = 1,2,\cdots m,\\
&x(0) = 0,\ {}_0D^{\alpha_0-1}_{q_0}x(0) = \beta,\ {}_0D^{\alpha_0-2}_{q_0}x(0) = \gamma
\end{aligned} \tag{2.7}$$

is given by

$$x(t)=\begin{cases}\frac{\gamma}{\Gamma_{q_0}(\alpha_0-1)}t^{\alpha_0-2}+\frac{\beta}{\Gamma_{q_0}(\alpha_0)}t^{\alpha_0-1}+{}_0I_{q_0}^{\alpha_0}f(t,x(t)),\ t\in[0,t_1],\\ \frac{(t-t_k)^{\alpha_k-1}}{\Gamma_{q_k}(\alpha_k)}\left[\beta+\left[\sum\limits_{0<t_k<t}\varphi_k\left(x(t_k)\right)+{}_{t_{k-1}}I_{q_{k-1}}f\left(t_k,x(t_k)\right)\right.\right.\\ \left.\left.-{}_{t_k}I_{q_k}f\left(t_k,x(t_k)\right)\right]\right]+\frac{(t-t_k)^{\alpha_k-2}}{\Gamma_{q_k}(\alpha_k-1)}\left[\gamma+\beta t_{k-1}+\sum\limits_{0<t_k<t}\left[\varphi_k^*\left(x(t_k)\right)\right.\right.\\ \left.\left.-{}_{t_{k-1}}I^2_{q_{k-1}}f\left(t_k,x(t_k)\right)-{}_{t_k}I^2_{q_k}f\left(t_k,x(t_k)\right)\right]\right]-\sum\limits_{0<t_k<t}(t_k-t_{k-1})_{t_{k-1}}\\ I_{q_{k-1}}f\left(t_k,x(t_k)\right)+\sum\limits_{0<t_k<t}\sum\limits_{0<t_j<t_k}\left[{}_{t_{j-1}}I_{q_{j-1}}f\left(t_j,x(t_j)\right)\right]\\ +\sum\limits_{0<t_k<t}\varphi_j\left(x(t_j)\right)+\frac{(t-t_k)^{\alpha_k-3}}{\Gamma_{q_k}(\alpha_k-2)}\left[\gamma t_{k-1}+\beta t_{k-1}^2+\sum\limits_{0<t_k<t}\sum\limits_{0<t_j<t_k}(t_k-t_{k-1})\right.\\ \left[{}_{tj-1}I^2_{q_{j-1}}f\left(t_j,x(t_j)\right)-{}_{t_j}I^2_{q_j}f\left(t_j,x(t_j)\right)+\gamma+\beta t_{k-1}+\varphi^*_{k-1}(x(t_{k-1}))\right]\\ +\sum\limits_{0<t_k<t}\varphi_k^*\left(x(t_k)\right)+\sum\limits_{0<t_k<t}\sum\limits_{0<t_j<t_k}\left[{}_{tj-1}I^3_{q_{j-1}}f\left(t_j,x(t_j)\right)\right.\\ \left.-{}_{tj}I^3_{q_j}f\left(t_j,x(t_j)\right)\right]+\sum\limits_{0<t_k<t}\sum\limits_{0<t_j<t_k}(t_k-t_{k-1})^2\left[\beta+\varphi_{k-1}(x(t_{k-1}))\right.\\ \left.\left.+{}_{tj-1}I_{q_{j-1}}f\left(t_j,x(t_j)\right)-{}_{tj}I_{q_j}f\left(t_j,x(t_j)\right)\right]\right]+{}_{t_k}I_{q_k}^{\alpha_k}f\left(t,x(t)\right),\\ t\in(t_k,t_{k+1}].\end{cases}\tag{2.8}$$

Proof. For $t\in[0,t_1]$, taking the Riemann-Liouville fractional q_0-integral of the order α_0 for the first equation of (2.7) and using (2.6), we can get

$$x(t)=C_{10}t^{\alpha_0-1}+C_{20}t^{\alpha_0-2}+C_{30}t^{\alpha_0-3}+{}_0I_{q_0}^{\alpha_0}f(t,x(t)).\tag{2.9}$$

According to the initial conditions, $x(0)=0$, we know $C_{30}=0$.

Taking the Riemann-Liouville fractional q_0-derivative of the order α_0-2 for (2.5) on J_0, we have

$${}_0D_{q_0}^{\alpha_0-2}x(t)=C_{20}\Gamma_{q_0}(\alpha_0-1)+tC_{10}\Gamma_{q_0}(\alpha_0)+{}_0I_{q_0}^2f(t,x(t)).$$

For ${}_0D_{q_0}^{\alpha_0-2}x(0)=\gamma$, then $C_{20}=\frac{\gamma}{\Gamma_{q_0}(\alpha_0-1)}$.

Therefore,

$$x(t)=\frac{\gamma}{\Gamma_{q_0}(\alpha_0-1)}t^{\alpha_0-2}+C_{10}t^{\alpha_0-1}+{}_0I_{q_0}^{\alpha_0}f(t,x(t)).\tag{2.10}$$

Taking the Riemann-Liouville fractional q_0-derivative of the order α_0-1 for (2.10), according to (2.2) and (2.5), then we have

$${}_0D_{q_0}^{\alpha-1}x(t)=\Gamma_{q_0}C_{10}+{}_0I_{q_0}f(t,x(t)).\tag{2.11}$$

The third initial condition of (2.7) with (2.11) yields $C_{10}=\frac{\beta}{\Gamma_{q_0}(\alpha_0)}$. Therefore, (2.9) can be written as

$$x(t)=\frac{\gamma}{\Gamma_{q_0}(\alpha_0-1)}t^{\alpha_0-2}+\frac{\beta}{\Gamma_{q_0}(\alpha_0)}t^{\alpha_0-1}+{}_0I_{q_0}^{\alpha_0}f(t,x(t)).\tag{2.12}$$

Applying the Riemann-Liouville fractional q_0-derivative of the orders $\alpha_0-1, \alpha_0-2, 3-\alpha_0$, for (2.12) at $t=t_1$, we have

$$\begin{aligned}
&{}_0D_{q_0}^{\alpha_0-1}x(t_1)=\beta+{}_0I_{q_0}^{1}f\left(t_1,x(t_1)\right),\\
&{}_0D_{q_0}^{\alpha_0-2}x(t_1)=\gamma+\beta t_1+{}_0I_{q_0}^{2}f\left(t_1,x(t_1)\right),\\
&{}_0I_{q_0}^{3-\alpha_0}x(t_1)=\gamma t_1+\frac{\beta}{\Gamma_{q_0}(3)}t_1^2+{}_0I_{q_0}^{3}f\left(t_1,x(t_1)\right).
\end{aligned}$$

For $t\in J_1=(t_1,t_2]$, from (2.6), we obtain

$$x(t)=C_{11}(t-t_1)^{\alpha_1-1}+C_{21}(t-t_1)^{\alpha_1-2}+C_{31}(t-t_1)^{\alpha_1-3}+{}_{t_1}I_{q_1}^{\alpha_1}f(t,x(t)). \tag{2.13}$$

Applying the Riemann-Liouville fractional q_1-derivative of the orders $\alpha_1-1, \alpha_1-2, 3-\alpha_1$, for (2.13), we have

$$\begin{aligned}
{}_{t_1}D_{q_1}^{\alpha_1-1}x(t)&=C_{11}\Gamma_{q_1}(\alpha_1)+{}_{t_1}I_{q_1}^{1}f(t,x(t)),\\
{}_{t_1}D_{q_1}^{\alpha_1-2}x(t)&=C_{11}\Gamma_{q_1}(\alpha_1)(t-t_1)+C_{21}\Gamma_{q_1}(\alpha_1-1)+{}_{t_1}I_{q_1}^{2}f(t,x(t)),\\
{}_{t_1}I_{q_1}^{3-\alpha_1}x(t)&=C_{11}\frac{\Gamma_{q_1}(\alpha_1)}{\Gamma_{q_1}(3)}(t-t_1)^2+C_{21}\Gamma_{q_1}(\alpha_1-1)(t-t_1)\\
&\quad+C_{31}\Gamma_{q_1}(\alpha_1-2)+{}_{t_1}I_{q_1}^{3}f(t,x(t)).
\end{aligned}$$

According to (1.3), we obtain

$$\Gamma_{q_1}(\alpha_1)C_{11}+{}_{t_1}I_{q_1}^{1}f(t_1,x(t_1))-\beta-{}_0I_{q_0}^{1}f(t_1,x(t_1))=\varphi_1(x(t_1)).$$

That is,

$$C_{11}=\frac{\varphi_1(x(t_1))+{}_0I_{q_0}^{1}f(t_1,x(t_1))+\beta-{}_{t_1}I_{q_1}^{1}f(t_1,x(t_1))}{\Gamma_{q_1}(\alpha_1)},$$

$$C_{21}\Gamma_{q_1}(\alpha_1-1)+{}_{t_1}I_{q_1}^{2}f(t_1,x(t_1))-\beta t_1-\gamma-{}_0I_{q_0}^{2}f(t_1,x(t_1))=\varphi_1^{*}(x(t_1)).$$

That is,

$$C_{21}=\frac{\varphi_1^{*}(x(t_1))+{}_0I_{q_0}^{2}f(t_1,x(t_1))+\gamma+\beta t_1-{}_{t_1}I_{q_1}^{2}f(t_1,x(t_1))}{\Gamma_{q_1}(\alpha_1-1)},$$

$$C_{31}\Gamma_{q_1}(\alpha_1-2)+{}_{t_1}I_{q_1}^{3}f(t_1,x(t_1))-\frac{\beta}{\Gamma_{q_0}(3)}t_1^2-\gamma t_1-{}_0I_{q_0}^{3}f(t_1,x(t_1))=\varphi_1^{**}(x(t_1)).$$

That is,

$$C_{31}=\frac{\varphi_1^{**}(x(t_1))+{}_0I_{q_0}^{3}f(t_1,x(t_1))+\frac{\beta}{\Gamma_{q_0}(3)}t_1^2+\gamma t_1-{}_{t_1}I_{q_1}^{3}f(t_1,x(t_1))}{\Gamma_{q_1}(\alpha_1-2)}.$$

We can get the solution of expression at $(t_1, t_2]$,

$$\begin{aligned}x(t) =&\frac{\varphi_1(x(t_1)) + {}_0I_{q_0}f(t_1, x(t_1)) + \beta - {}_{t_1}I_{q_1}f(t_1, x(t_1))}{\Gamma_{q_1}(\alpha_1)}(t-t_1)^{\alpha_1-1}\\ &+\frac{\varphi_1^*(x(t_1)) + {}_0I_{q_0}^2 f(t_1, x(t_1)) + \gamma + \beta t_1 - {}_{t_1}I_{q_1}^2 f(t_1, x(t_1))}{\Gamma_{q_1}(\alpha_1-1)}(t-t_1)^{\alpha_1-2}\\ &+\frac{\varphi_1^{**}(x(t_1)) + {}_0I_{q_0}^3 f(t_1, x(t_1)) + \frac{\beta}{\Gamma_{q_0}(3)}t_1^2 + \gamma t_1 - {}_{t_1}I_{q_1}^3 f(t_1, x(t_1))}{\Gamma_{q_1}(\alpha_1-2)}(t-t_1)^{\alpha_1-3}\\ &+{}_{t_1}I_{q_1}^{\alpha_1} f(t, x(t)).\end{aligned}$$

Repeating the above process, for $t \in J_k$, we obtain

$$x(t) = \begin{cases} \frac{\gamma}{\Gamma_{q_0}(\alpha_0-1)}t^{\alpha_0-2} + \frac{\beta}{\Gamma_{q_0}(\alpha_0)}t^{\alpha_0-1} + {}_0I_{q_0}^{\alpha_0} f(t, x(t)),\ t \in [0, t_1], \\ \frac{(t-t_k)^{\alpha_k-1}}{\Gamma_{q_k}(\alpha_k)}\left[\beta + \left[\sum\limits_{0<t_k<t} \varphi_k\left(x(t_k)\right) + {}_{t_{k-1}}I_{q_{k-1}} f\left(t_k, x(t_k)\right)\right.\right. \\ \left.\left.-{}_{t_k}I_{q_k} f\left(t_k, x(t_k)\right)\right]\right] + \frac{(t-t_k)^{\alpha_k-2}}{\Gamma_{q_k}(\alpha_k-1)}\left[\gamma + \beta t_{k-1} + \sum\limits_{0<t_k<t}\left[\varphi_k^*\left(x(t_k)\right)\right.\right. \\ \left.\left.-{}_{t_{k-1}}I_{q_{k-1}}^2 f\left(t_k, x(t_k)\right) - {}_{t_k}I_{q_k}^2 f\left(t_k, x(t_k)\right)\right]\right] - \sum\limits_{0<t_k<t}(t_k - t_{k-1})_{t_{k-1}} \\ I_{q_{k-1}} f\left(t_k, x(t_k)\right) + \sum\limits_{0<t_k<t}\sum\limits_{0<t_j<t_k}\left[{}_{t_{j-1}}I_{q_{j-1}} f\left(t_j, x(t_j)\right)\right] \\ + \sum\limits_{0<t_k<t}\varphi_j\left(x(t_j)\right) + \frac{(t-t_k)^{\alpha_k-3}}{\Gamma_{q_k}(\alpha_k-2)}\left[\gamma t_{k-1} + \beta t_{k-1}^2 + \sum\limits_{0<t_k<t}\sum\limits_{0<t_j<t_k}(t_k - t_{k-1})\right. \\ \left[{}_{tj-1}I_{q_{j-1}}^2 f\left(t_j, x(t_j)\right) - {}_{t_j}I_{q_j}^2 f\left(t_j, x(t_j)\right) + \gamma + \beta t_{k-1} + \varphi_{k-1}^*(x(t_{k-1}))\right] \\ + \sum\limits_{0<t_k<t}\varphi_k^*\left(x(t_k)\right) + \sum\limits_{0<t_k<t}\sum\limits_{0<t_j<t_k}\left[{}_{tj-1}I_{q_{j-1}}^3 f\left(t_j, x(t_j)\right)\right. \\ \left.-{}_{tj}I_{q_j}^3 f\left(t_j, x(t_j)\right)\right] + \sum\limits_{0<t_k<t}\sum\limits_{0<t_j<t_k}(t_k - t_{k-1})^2\left[\beta + \varphi_{k-1}(x(t_{k-1}))\right. \\ \left.\left.+{}_{tj-1}I_{q_{j-1}} f\left(t_j, x(t_j)\right) - {}_{tj}I_{q_j} f\left(t_j, x(t_j)\right)\right]\right] + {}_{t_k}I_{q_k}^{\alpha_k} f\left(t, x(t)\right), \\ t \in (t_k, t_{k+1}], \end{cases} \tag{2.14}$$

where $\sum\limits_{0<0}(\cdot) = 0$, which we complete the proof.

3. Main results

In this section, we will prove the existence and uniqueness of solutions for the following initial value problem for impulsive fractional q_k-difference equation of order $2 < \alpha_k \leq 3$, and we use the Banach contraction mapping principle to accomplish the result. In order to achieve the goal, we are supposed to introduce the space: for $\gamma \in \mathbb{R}_+$, $C_{\gamma,k}(J_k, \mathbb{R}) = \{x : J_k \to \mathbb{R} : (t-t_k)^\gamma x(t) \in C(J_k, \mathbb{R})\}$ with the norm $\|x\|_{C_{\gamma,k}} = \sup\limits_{t\in J_k}|(t-t_k)^\gamma x(t)|$ and $PC_\gamma = \{x : J \to \mathbb{R}$: for each $t \in J_k$ and $(t-t_k)^\gamma x(t) \in C(J_k, \mathbb{R}), k = 0, 1, 2\cdots m\}$ with the norm: $\|x\|_{PC_\gamma} = \max\limits_{0\leq k\leq m}\{\sup\limits_{t\in J_k}|(t-t_k)^\gamma x(t)|\}$ and $x(t_k^+), x(t_k^-)$ exist, and $x(t_k^-) = x(t_k)$, where $k = 0, 1, 2\cdots m$. Obviously, PC_γ is a Banach space.

For the sake of convenience, we set the following constants

$$\begin{aligned}\psi_1 = \frac{\widetilde{T}}{\widetilde{\Gamma}}\Bigg\{&(M+2M^*)m + Lt_m + \sum_{j=1}^{m}\frac{(t_j-t_{j-1})^3}{\Gamma_{q_{j-1}}(4)}L + \sum_{j=1}^{m}(t_j-t_{j-1})^2M(m-1)\\ &+\sum_{j=1}^{m}(t_j-t_{j-1})^3L + \sum_{j=1}^{m}L(t_j-t_{j-1})(1-t_{j-1}) - \sum_{j=1}^{m}\frac{(t_j-t_{j-1})^2}{1+q_{j-1}}L\\ &+\sum_{j=1}^{m}(j-1)M + \sum_{j=1}^{m}\frac{(t_j-t_{j-1})^3}{1+q_{j-1}}L + \sum_{j=1}^{m}(t_j-t_{j-1})(j-1)M^* + L\Bigg\},\end{aligned} \tag{3.1}$$

$$\begin{aligned}\psi_2 = \frac{\widetilde{T}}{\widetilde{\Gamma}}\Bigg\{&(\Omega_2+2\Omega_3)m + \beta(t_{m-1}^2+1-t_m) + \gamma(1+t_{m-1}) + \Omega_1 t_m\\ &-\sum_{j=1}^{m}\frac{(t_j-t_{j-1})^2}{1+q_{j-1}}\Omega_1 + \sum_{j=1}^{m}(t_j-t_{j-1})(1-t_{j-1})\Omega_1 + \sum_{j=1}^{m}\frac{(t_j-t_{j-1})^3}{1+q_{j-1}}\Omega_1\\ &+\sum_{j=1}^{m}(t_j-t_{j-1})\left[\gamma+\beta t_{j-1}+(j-1)\Omega_3\right] + \sum_{j=1}^{m}(j-1)\Omega_2 + \sum_{j=1}^{m}\frac{(t_j-t_{j-1})^3}{\Gamma_{q_{j-1}}(4)}\Omega_1\\ &+\sum_{j=1}^{m}(t_j-t_{j-1})^2\left[\beta+\Omega_2(j-1)+(t_j-t_{j-1})\Omega_1\right] + \Omega_1\Bigg\},\end{aligned} \tag{3.2}$$

where $\widetilde{T}= \max\{T^{\gamma+\alpha_k-3},\ T^{\gamma+\alpha_k-2}, T^{\gamma+\alpha_k-1}\}$, $\widetilde{\Gamma}= \min\{\Gamma_{q_k}(\alpha_k-2),\ \Gamma_{q_k}(\alpha_k-1),\ \Gamma_{q_k}(\alpha_k),\ k=0,1,2,\cdots m\}$, and $\gamma+\alpha_k>3$.

Theorem 3.1. *Assume that the following assumptions hold.*

(H_1) $f : J\times\mathbb{R}\to\mathbb{R}$ *is a continuous function, and satisfies*

$$|f(t,x)-f(t,y)| \le L|x-y|, L>0, \forall t\in J, x,y\in\mathbb{R}.$$

(H_2) $\varphi_k : \mathbb{R}\to\mathbb{R},\ k=1,2,\cdots m,$ *are continuous functions, and satisfy*

$$|\varphi_k(x)-\varphi_k(y)| \le M|x-y|, M>0, \forall x,y\in\mathbb{R}.$$

(H_3) $\varphi_k^* : \mathbb{R}\to\mathbb{R},\ k=1,2,\cdots m,$ *are continuous functions, and satisfy*

$$|\varphi_k^*(x)-\varphi_k^*(y)| \le M^*|x-y|,\ M^*>0, \forall x,y\in\mathbb{R}.$$

(H_4) $\varphi_k^{**} : \mathbb{R}\to\mathbb{R},\ k=1,2,\cdots m,$ *are continuous functions, and satisfy*

$$|\varphi_k^{**}(x)-\varphi_k^{**}(y)| \le M^{**}|x-y|,\ M^{**}>0, \forall x,y\in\mathbb{R}.$$

If

$$\psi_1 \le \delta < 1,$$

where ψ_1 *is defined by* (3.1)*, then the initial value problem* (2.7) *has a unique solution on* J.

In addition, we define a ball $B_R = \{x\in PC_r(J,\mathbb{R}) : ||x||_{PC_r} \le R\}$ with $R \ge \frac{\psi_2}{1-\varepsilon}$. Setting

$$\sup_{t\in J}|f(t,0)| = \Omega_1,\ \max\{\varphi_k(0) : k=1,2,\cdots m\} = \Omega_2,$$

$$\max\{\varphi_k^*(0) : k=1,2,\cdots m\} = \Omega_3,\ \max\{\varphi_k^{**}(0) : k=1,2,\cdots m\} = \Omega_4,$$

we will prove that $AB_R \subset B_R$, where $B_R = \{x \in PC_r(J,\mathbb{R}) : ||x||_{PC_r} \leq R\}$, and a constant R satisfies

$$R \geq \frac{\psi_2}{1-\varepsilon},$$

where ψ_2 is defined by (3.2) and $\delta < \varepsilon < 1$.

Let $x \in B_R$. For each $t \in J_k, k = 0, 1, 2, \cdots m$, we have

$$\begin{aligned}
&|(Ax)(t)| \\
&= \frac{(t-t_k)^{\alpha_k-1}}{\Gamma_{q_k}(\alpha_k)} \Bigg[|\beta| + \Bigg[\sum_{0<t_k<t} \varphi_k(x(t_k)) + {}_{t_{k-1}}I_{q_{k-1}} f(t_k, x(t_k)) \\
&-{}_{t_k}I_{q_k} f(t_k, x(t_k))]] + \frac{(t-t_k)^{\alpha_k-2}}{\Gamma_{q_k}(\alpha_k-1)}[|\gamma| + |\beta|t_{k-1} + \sum_{0<t_k<t}[\varphi_k^*(x(t_k)) \\
&-{}_{t_{k-1}}I^2_{q_{k-1}} f(t_k, x(t_k)) - {}_{t_k}I^2_{q_k} f(t_k, x(t_k)\Big] + \sum_{0<t_k<t}(t_k - t_{k-1})_{t_{k-1}} \\
&I_{q_{k-1}} f(t_k, x(t_k)) + \sum_{0<t_k<t}\sum_{0<t_j<t_k}\left[{}_{t_{j-1}}I_{q_{j-1}} f(t_j, x(t_j))\right] + \sum_{0<t_k<t}\varphi_j(x(t_j)) \\
&+ \frac{(t-t_k)^{\alpha_k-3}}{\Gamma_{q_k}(\alpha_k-2)}\Bigg[|\gamma t_{k-1}| + |\beta t^2_{k-1}| + \sum_{0<t_k<t}\sum_{0<t_j<t_k}(t_k - t_{k-1}) \\
&\cdot\Bigg[{}_{tj-1}I^2_{q_{j-1}} f(t_j, x(t_j)) - {}_{t_j}I^2_{q_j} f(t_j, x(t_j)) + |\gamma| + |\beta t_{k-1}| + \varphi^*_{k-1}(x(t_{k-1}))\Bigg] \\
&+ \sum_{0<t_k<t}\varphi_k^*(x(t_k)) + \sum_{0<t_k<t}\sum_{0<t_j<t_k}\left[{}_{tj-1}I^3_{q_{j-1}} f(t_j, x(t_j)) - {}_{tj}I^3_{q_j} f(t_j, x(t_j))\right] \\
&+ \sum_{0<t_k<t}\sum_{0<t_j<t_k}(t_k - t_{k-1})^2\left[|\beta| + \varphi_{k-1}(x(t_{k-1})) + {}_{tj-1}I_{q_{j-1}} f(t_j, x(t_j))\right. \\
&\left. - {}_{tj}I_{q_j} f(t_j, x(t_j))\right]\Bigg] + {}_{t_k}I^{\alpha_k}_{q_k} f(t, x(t)).
\end{aligned}$$

Let $\{x_n\}$ be a sequence in $x \in B_R$ converging to a point $x \in B_R$. Then, by Lebesgue's dominated convergence theorem, we have

$$\begin{aligned}
&|(Ax_n)(t)| \\
&= \frac{(t-t_k)^{\alpha_k-1}}{\Gamma_{q_k}(\alpha_k)} \Bigg[|\beta| + \Bigg[\sum_{0<t_k<t} \varphi_k(x_n(t_k)) + {}_{t_{k-1}}I_{q_{k-1}} f(t_k, x_n(t_k)) \\
&-{}_{t_k}I_{q_k} f(t_k, x_n(t_k))]] + \frac{(t-t_k)^{\alpha_k-2}}{\Gamma_{q_k}(\alpha_k-1)}[|\gamma| + |\beta|t_{k-1} + \sum_{0<t_k<t}[\varphi_k^*(x_n(t_k)) \\
&- {}_{t_{k-1}}I^2_{q_{k-1}} f(t_k, x_n(t_k)) - {}_{t_k}I^2_{q_k} f\,(t_k, x_n(t_k)] + \sum_{0<t_k<t}(t_k - t_{k-1})_{t_{k-1}} \\
&I_{q_{k-1}} f(t_k, x_n(t_k)) + \sum_{0<t_k<t}\sum_{0<t_j<t_k}\left[{}_{t_{j-1}}I_{q_{j-1}} f(t_j, x_n(t_j))\right] \\
&+ \sum_{0<t_k<t}\varphi_j(x_n(t_j)) + \frac{(t-t_k)^{\alpha_k-3}}{\Gamma_{q_k}(\alpha_k-2)}\Bigg[|\gamma t_{k-1}| + |\beta t^2_{k-1}| \\
&+ \sum_{0<t_k<t}\sum_{0<t_j<t_k}(t_k - t_{k-1}) \cdot \Bigg[{}_{tj-1}I^2_{q_{j-1}} f(t_j, x_n(t_j)) - {}_{t_j}I^2_{q_j} f(t_j, x_n(t_j)) \\
&+ |\gamma| + |\beta t_{k-1}| + \varphi^*_{k-1}(x_n(t_{k-1}))\Bigg] + \sum_{0<t_k<t}\varphi_k^*(x_n(t_k))
\end{aligned}$$

$$+\sum_{0<t_k<t}\sum_{0<t_j<t_k}\left[{}_{tj-1}I^3_{q_{j-1}}f(t_j,x_n(t_j))-{}_{tj}I^3_{q_j}f(t_j,x_n(t_j))\right]$$
$$+\sum_{0<t_k<t}\sum_{0<t_j<t_k}(t_k-t_{k-1})^2\Big[|\beta|+\varphi_{k-1}(x_n(t_{k-1}))+{}_{tj-1}I_{q_{j-1}}f(t_j,x_n(t_j))$$
$$-{}_{tj}I_{q_j}f(t_j,x_n(t_j))\Big]\Bigg]+{}_{t_k}I^{\alpha_k}_{q_k}f(t,x_n(t))$$
$$=\frac{(t-t_k)^{\alpha_k-1}}{\Gamma_{q_k}(\alpha_k)}\left[|\beta|+\left[\sum_{0<t_k<t}\varphi_k(x(t_k))+{}_{t_{k-1}}I_{q_{k-1}}f(t_k,x(t_k))\right.\right.$$
$$-{}_{t_k}I_{q_k}f(t_k,x(t_k))]]+\frac{(t-t_k)^{\alpha_k-2}}{\Gamma_{q_k}(\alpha_k-1)}[|\gamma|+|\beta|t_{k-1}+\sum_{0<t_k<t}[\varphi^*_k(x(t_k))$$
$$-{}_{t_{k-1}}I^2_{q_{k-1}}f(t_k,x(t_k))-{}_{t_k}I^2_{q_k}f\,(t_k,x(t_k)]+\sum_{0<t_k<t}(t_k-t_{k-1})_{t_{k-1}}$$
$$I_{q_{k-1}}f(t_k,x(t_k))+\sum_{0<t_k<t}\sum_{0<t_j<t_k}\left[{}_{t_{j-1}}I_{q_{j-1}}f(t_j,x(t_j))\right]+\sum_{0<t_k<t}\varphi_j(x(t_j))$$
$$+\frac{(t-t_k)^{\alpha_k-3}}{\Gamma_{q_k}(\alpha_k-2)}\Bigg[|\gamma t_{k-1}|+|\beta t^2_{k-1}|+\sum_{0<t_k<t}\sum_{0<t_j<t_k}(t_k-t_{k-1})$$
$$\cdot\left[{}_{tj-1}I^2_{q_{j-1}}f(t_j,x(t_j))-{}_{t_j}I^2_{q_j}f(t_j,x(t_j))+|\gamma|+|\beta t_{k-1}|+\varphi^*_{k-1}(x(t_{k-1}))\right]$$
$$+\sum_{0<t_k<t}\varphi^*_k(x(t_k))+\sum_{0<t_k<t}\sum_{0<t_j<t_k}\left[{}_{tj-1}I^3_{q_{j-1}}f(t_j,x(t_j))-{}_{tj}I^3_{q_j}f(t_j,x(t_j))\right]$$
$$+\sum_{0<t_k<t}\sum_{0<t_j<t_k}(t_k-t_{k-1})^2\Big[|\beta|+\varphi_{k-1}(x(t_{k-1}))+{}_{tj-1}I_{q_{j-1}}f(t_j,x(t_j))$$
$$-{}_{tj}I_{q_j}f(t_j,x(t_j))\Big]\Bigg]+{}_{t_k}I^{\alpha_k}_{q_k}f(t,x(t)).$$

This shows that A is convergence in $x\in B_R$.

According to $(H_1)-(H_4)$, we can get

$$|(Ax)(t)|$$
$$\le\frac{(t-t_k)^{\alpha_k-1}}{\Gamma_{q_k}(\alpha_k)}\Big[|\beta|+(MR+\Omega_2)k+(LR+\Omega_1)t_k\Big]$$
$$+\frac{(t-t_k)^{\alpha_k-2}}{\Gamma_{q_k}(\alpha_k-1)}\Bigg[|\gamma|+|\beta|t_{k-1}+(M^*R+\Omega_3)k$$
$$-\sum_{j=1}^{k}\frac{(t_j-t_{j-1})^2}{1+q_{j-1}}(LR+\Omega_1)+\sum_{j=1}^{k}(LR+\Omega_1)(t_j-t_{j-1})(1-t_{j-1})$$
$$+\sum_{j=1}^{k}(MR+\Omega_2)(j-1)\Bigg]+\frac{(t-t_k)^{\alpha_k-3}}{\Gamma_{q_k}(\alpha_k-2)}\Bigg[|\beta|t^2_{k-1}+|\gamma|t_{k-1}$$
$$+\sum_{j=1}^{k}(t_j-t_{j-1})\left[\frac{(t_j-t_{j-1})^2}{1+q_{j-1}}(LR+\Omega_1)+|\gamma|+|\beta|t_{j-1}+(M^*R+\Omega_3)(j-1)\right]$$
$$+(M^*R+\Omega_3)k+\sum_{j=1}^{k}\frac{(t_j-t_{j-1})^3}{\Gamma_{q_{j-1}}(4)}(LR+\Omega_1)$$

$$+\sum_{j=1}^{k}(t_j-t_{j-1})^2\big[|\beta|+(MR+\Omega_2)(j-1)+(t_j-t_{j-1})(LR+\Omega_1)\big]\Bigg]$$
$$+\frac{(t-t_k)^{\alpha_k}}{\Gamma_{q_k}(\alpha_k+1)}(LR+\Omega_1).$$

Multiplying both sides of the above inequality by $(t-t_k)^\gamma$, for $t\in J_k$, we have

$$\begin{aligned}
&(t-t_k)^\gamma|(Ax)(t)|\\
&\leq\frac{(t-t_k)^{\alpha_k-1+\gamma}}{\Gamma_{q_k}(\alpha_k)}\Bigg[|\beta|+(MR+\Omega_2)k+(LR+\Omega_1)t_k\Bigg]\\
&+\frac{(t-t_k)^{\alpha_k-2+\gamma}}{\Gamma_{q_k}(\alpha_k-1)}\Bigg[|\gamma|+|\beta|t_{k-1}+(M^*R+\Omega_3)k-\sum_{j=1}^{k}\frac{(t_j-t_{j-1})^2}{1+q_{j-1}}(LR+\Omega_1)\\
&+\sum_{j=1}^{k}(LR+\Omega_1)(t_j-t_{j-1})(1-t_{j-1})+\sum_{j=1}^{k}(MR+\Omega_2)(j-1)\Bigg]\\
&+\frac{(t-t_k)^{\alpha_k-3+\gamma}}{\Gamma_{q_k}(\alpha_k-2)}\Bigg[|\beta|t_{k-1}^2+|\gamma|t_{k-1}+\sum_{j=1}^{k}(t_j-t_{j-1})\Bigg[\frac{(t_j-t_{j-1})^2}{1+q_{j-1}}(LR+\Omega_1)\\
&+|\gamma|+|\beta|t_{j-1}+(M^*R+\Omega_3)(j-1)\Bigg]+(M^*R+\Omega_3)k+\sum_{j=1}^{k}\frac{(t_j-t_{j-1})^3}{\Gamma_{q_{j-1}}(4)}\\
&(LR+\Omega_1)+\sum_{j=1}^{k}(t_j-t_{j-1})^2\big[|\beta|+(MR+\Omega_2)(j-1)+(t_j-t_{j-1})(LR+\Omega_1)\big]\Bigg]\\
&+\frac{(t-t_k)^{\alpha_k+\gamma}}{\Gamma_{q_k}(\alpha_k+1)}(LR+\Omega_1)\leq\psi_1R+\psi_2\leq(\delta+1-\varepsilon)\leq R,
\end{aligned}$$

which yields $\|Ax\|\leq R$. Then, we get $AB_R\subseteq B_R$.

For any $x,y\in PC_\gamma(J,R)$ and for each $t\in J$, we have

$$\begin{aligned}
&|(Ax)(t)-(Ay)(t)|\\
&\leq\frac{(t-t_k)^{\alpha_k-1}}{\Gamma_{q_k}(\alpha_k)}\left[kM\|x-y\|+L\|x-y\|t_k\right]+\frac{(t-t_k)^{\alpha_k-2}}{\Gamma_{q_k}(\alpha_k-1)}\\
&\Bigg[kM^*\|x-y\|-\sum_{j=1}^{k}\frac{(t_j-t_{j-1})^2}{1+q_{j-1}}L\|x-y\|-\sum_{j=1}^{k}(t_j-t_{j-1})L\|x-y\|t_{j-1}+\\
&\sum_{j=1}^{k}L\|x-y\|t_{j-1}+kM\|x-y\|\Bigg]+\frac{(t-t_k)^{\alpha_k-3}}{\Gamma_{q_k}(\alpha_k-2)}\Bigg[\sum_{j=1}^{k}(t_j-t_{j-1})\frac{(t_j-t_{j-1})^2}{1+q_{j-1}}L\|x-y\|\\
&+M^*\|x-y\|(j-1)+M^*\|x-y\|k+\sum_{j=1}^{k}\frac{(t_j-t_{j-1})^3}{\Gamma_{q_{j-1}(4)}}L\|x-y\|+\sum_{j=1}^{k}(t_j-t_{j-1})^2\\
&[M\|x-y\|+L\|x-y\|\Bigg]+\frac{(t-t_k)^{\alpha_k}}{\Gamma_{q_k(\alpha_k+1)}}L\|x-y\|.
\end{aligned}$$

Again, multiplying both sides of the above inequality by $(t-t_k)^\gamma$, for $t\in J_k$, we have

$$
\begin{aligned}
&|(t-t_k)^{\gamma}(Ax)(t)-(t-t_k)^{\gamma}(Ay)(t)| \\
&\le \frac{(t-t_k)^{\alpha_k-1+\gamma}}{\Gamma_{q_k}(\alpha_k)}[kM\|x-y\|+L\|x-y\|t_k]+\frac{(t-t_k)^{\alpha_k-2+\gamma}}{\Gamma_{q_k}(\alpha_k-1)}\Big[kM^*\|x-y\| \\
&-\sum_{j=1}^{k}\frac{(t_j-t_{j-1})^2}{1+q_{j-1}}L\|x-y\|-\sum_{j=1}^{k}(t_j-t_{j-1})L\|x-y\|t_{j-1}+\sum_{j=1}^{k}L\|x-y\|t_{j-1} \\
&+kM\|x-y\|\Big]+\frac{(t-t_k)^{\alpha_k-3+\gamma}}{\Gamma_{q_k}(\alpha_k-2)}\Big[\sum_{j=1}^{k}(t_j-t_{j-1})\frac{(t_j-t_{j-1})^2}{1+q_{j-1}}L\|x-y\| \\
&+M^*\|x-y\|(j-1)+M^*\|x-y\|k+\sum_{j=1}^{k}]\frac{(t_j-t_{j-1})^3}{\Gamma_{q_{j-1}(4)}}L\|x-y\| \\
&+\sum_{j=1}^{k}(t_j-t_{j-1})^2[M\|x-y\|+L\|x-y\|\Big]+\frac{(t-t_k)^{\alpha_k+\gamma}}{\Gamma_{q_k(\alpha_k+1+\gamma)}}L\|x-y\|,
\end{aligned}
$$

which implies $\|x-y\|\le\psi_1\|x-y\|$.

As $\psi_1<1$, by the Banach contraction mapping principle, we can draw the conclusion that A has a fixed point, which is a unique solution of (1.2) on J.

We consider another Banach space $PC(J,\mathbb{R})$ with the norm $\|x\|=\|x\|_\infty$ and $\|x\|_\infty=\sup\{|\cdot|,t\in J,t\neq t_k\},x\in PC(J,\mathbb{R})$, where $PC(J,\mathbb{R})=\{x:J\to\mathbb{R},x(t)$ is continuous everywhere except for some t_k, at which $x(t_k^+)$ and $x(t_k^-)$ exist and $x(t_k^-)=x(t_k),k=1,2,\cdots,m\}$.

Define an integral operator $A:PC(J,\mathbb{R})\to PC(J,\mathbb{R})$.

Lemma 3.1. *Assume that*

(H_5) *there exist continuous functions* $a(t),b(t)$, *such that* $|f(t,x)|\le a(t)+b(t)|x|,(t,x)\in J\times\mathbb{R}$ *with* $\sup_{t\in J}|a(t)|=a_1,\sup_{t\in J}|b(t)|=b_1$*;*

(H_6) *there exist nonnegative constants* a_k,b_k *such that* $|\varphi_k(x)|\le a_k|x|,|\varphi_k^*(x)|\le b_k|x|,|\varphi_k^{**}(x)|\le c_k|x|,\forall x\in\mathbb{R},k=1,2,\cdots,m$, *and note* $a=\sum_{k=1}^m a_k,b=\sum_{k=1}^m b_k,c=\sum_{k=1}^m c_k$.

Then, the operator A is completely continuous.

Proof. The proof consists of several steps.

(i) By the continuity of f,I_k,I_k^*, it is easy to get A is continuous.

(ii) A maps bounded sets into bounded sets in $PC(J,\mathbb{R})$. Let $B_R=\{x\in PC(J,\mathbb{R}):\|x\|\le R\}$ be a bounded set in $PC(J,\mathbb{R})$, $t\in(t_k,t_{k+1})$ and $x\in B_R$. Then, we have

$$
\begin{aligned}
&|(Ax)(t)| \\
&=\frac{(t-t_k)^{\alpha_k-1}}{\Gamma_{q_k}(\alpha_k)}\Bigg[|\beta|+\Bigg[\sum_{0<t_k<t}\varphi_k(x(t_k))+{}_{t_{k-1}}I_{q_{k-1}}f(t_k,x(t_k))-{}_{t_k}I_{q_k}f(t_k,x(t_k))\Bigg]\Bigg] \\
&+\frac{(t-t_k)^{\alpha_k-2}}{\Gamma_{q_k}(\alpha_k-1)}[|\gamma|+|\beta|t_{k-1}+\sum_{0<t_k<t}\Big[\varphi_k^*(x(t_k))-{}_{t_{k-1}}I^2_{q_{k-1}}f(t_k,x(t_k)) \\
&-{}_{t_k}I^2_{q_k}f\,(t_k,x(t_k)]+\sum_{0<t_k<t}(t_k-t_{k-1})_{t_{k-1}}I_{q_{k-1}}f(t_k,x(t_k))
\end{aligned}
$$

$$+\sum_{0<t_k<t}\sum_{0<t_j<t_k}\left[{}_{t_{j-1}}I_{q_{j-1}}f(t_j,x(t_j))\right]+\sum_{0<t_k<t}\varphi_j(x(t_j))+\frac{(t-t_k)^{\alpha_k-3}}{\Gamma_{q_k}(\alpha_k-2)}$$

$$\left[|\gamma t_{k-1}|+|\beta t_{k-1}^2|+\sum_{0<t_k<t}\sum_{0<t_j<t_k}(t_k-t_{k-1})\left[{}_{tj-1}I^2_{q_{j-1}}f(t_j,x(t_j))\right.\right.$$

$$\left.-{}_{t_j}I^2_{q_j}f(t_j,x(t_j))+|\gamma|+|\beta t_{k-1}|+\varphi^*_{k-1}x(t_{k-1})\right]+\sum_{0<t_k<t}\varphi^*_k(x(t_k))+$$

$$\sum_{0<t_k<t}\sum_{0<t_j<t_k}\left[{}_{tj-1}I^3_{q_{j-1}}f(t_j,x(t_j))-{}_{tj}I^3_{q_j}f(t_j,x(t_j))\right]+\sum_{0<t_k<t}\sum_{0<t_j<t_k}(t_k-t_{k-1})^2$$

$$\left.\left[|\beta|+\varphi_{k-1}x(t_{k-1})+{}_{tj-1}I_{q_{j-1}}f(t_j,x(t_j))-{}_{tj}I_{q_j}f(t_j,x(t_j))\right]\right]+{}_{t_k}I^{\alpha_k}_{q_k}f(t,x(t))$$

$$\le\frac{(t-t_k)^{\alpha_k-1}}{\Gamma_{q_k}(\alpha_k)}\left[|\beta|+aR+{}_{t_{k-1}}I_{q_{k-1}}(a_1+b_1R)+{}_{t_k}I_{q_k}(a_1+b_1R)\right]$$

$$+\frac{(t-t_k)^{\alpha_k-2}}{\Gamma_{q_k}(\alpha_k-1)}\left[|\gamma|+|\beta|_{t_{k-1}}+b_1R+{}_{t_{k-1}}I^2_{q_{k-1}}(a_1+b_1R)+{}_{t_k}I^2_{q_k}(a_1+b_1R)\right.$$

$$\left.+(t_k-t_{k-1})_{t_{k-1}}I_{q_{k-1}}(a_1+b_1R)+aR\right]+\frac{(t-t_k)^{\alpha_k-3}}{\Gamma_{q_k}(\alpha_k-2)}(|\gamma t_{k-1}|+|\beta t_{k-1}^2|)$$

$$+(t_k-t_{k-1})\left[{}_{t_{j-1}}I^2_{q_{j-1}}(a_1+b_1R)+{}_{t_j}I^2_{q_j}(a_1+b_1R)+|\gamma|+|\beta t_{k-1}|+bR\right]$$

$$+bR+\left[{}_{t_{j-1}}I^3_{q_{j-1}}(a_1+b_1R)+{}_{t_j}I^3_{q_j}(a_1+b_1R)\right]$$

$$+(t_k-t_{k-1})^2\left[|\beta|+aR+{}_{t_j-1}I_{q_{j-1}}(a_1+b_1R)+{}_{t_j}I_{q_j}(a_1+b_1R)\right]$$

$$+{}_{t_k}I^{\alpha_k}_{q_k}(a_1+b_1R)$$

$$\le\frac{(t-t_k)^{\alpha_k-1}}{\Gamma_{q_k}(\alpha_k)}\left[|\beta|+aR+2_{t_k}I_{q_k}(a_1+b_1R)\right]+\frac{(t-t_k)^{\alpha_k-2}}{\Gamma_{q_k}(\alpha_k-1)}\left[|\gamma|+|\beta|+b_1R\right.$$

$$\left.+2_{t_k}I^2_{q_k}(a_1+b_1R)+2_{t_{k-1}}I_{q_{k-1}}(a_1+b_1R)\right]$$

$$+\frac{(t-t_k)^{\alpha_k-3}}{\Gamma_{q_k}(\alpha_k-2)}\left[2|\gamma|+2|\beta|+(2_{t_j}I^2_{q_j}(a_1+b_1R)+2bR)+2_{t_j}I^3_{q_j}(a_1+b_1R)\right.$$

$$\left.+|\beta|+aR+2_{t_j}I_{q_j}(a_1+b_1R)\right]+{}_{t_k}I^{\alpha_k}_{q_k}(a_1+b_1R)$$

$$\le\frac{T'}{\Gamma'}\left[5|\beta|+2aR+4_{t_k}I_{q_k}(a_1+b_1R)+3|\gamma|+3bR+4_{t_k}I^2_{q_k}(a_1+b_1R)\right.$$

$$\left.+2_{t_{k-1}}I_{q_{k-1}}I^3_{q_k}(a_1+b_1R)+{}_{t_k}I^{\alpha_k}_{q_k}(a_1+b_1R)\right]$$

$$=\frac{T'}{\Gamma'}\left[5|\beta|+3|\gamma|+(2a+3b)R+4(t_k-t_{k-1})(a_1+b_1R)+\frac{4(t_k-t_{k-1})^2}{1+q_{k-1}}(a_1+b_1R)\right.$$

$$\left.+2(t_k-t_{k-1})(a_1+b_1R)+\frac{2(t_k-t_{k-1})^3}{\Gamma_{q_{k-1}}(4)}(a_1+b_1R)+\frac{(t-t_k)^{\alpha_k}}{\Gamma_{q_k}(\alpha_k+1)}(a_1+b_1R)\right]$$

$$=\frac{T'}{\Gamma'}\left[5|\beta|+3|\gamma|+4a_1(t_k-t_{k-1})+\frac{4a_1(t_k-t_{k-1})^2}{1+q_{k-1}}+2a_1(t_k-t_{k-1})\right.$$

$$
\begin{aligned}
&+\frac{2a_1(t_k-t_{k-1})^3}{\Gamma_{q_{k-1}}(4)}+\frac{a_1(t-t_k)^{\alpha_k}}{\Gamma_{q_k}(\alpha_k+1)}\Bigg]+\Bigg[2a+3b+6b_1(t_k-t_{k-1})+\frac{4b_1(t_k-t_{k-1})^2}{1+q_{k-1}}\\
&+\frac{2b_1(t_k-t_{k-1})^3}{\Gamma_{q_{k-1}}(4)}+\frac{b_1(t-t_k)^{\alpha_k}}{\Gamma_{q_k}(\alpha_k+1)}\Bigg]R\\
&=A_1+B_1R:=M,
\end{aligned}
$$

where $T'=\max\{T^{\alpha_k-3},\ T^{\alpha_k-2},T^{\alpha_k-1}\},\ \Gamma'=\min\{\Gamma_{q_k}(\alpha_k-2),\ \Gamma_{q_k}(\alpha_k-1),\ \Gamma_{q_k}(\alpha_k),\ k=0,1,2,\cdots m\}$, and $\gamma+\alpha_k>3$,

$$
\begin{aligned}
A_1=&5|\beta|+3|\gamma|+4a_1(t_k-t_{k-1})+\frac{4a_1(t_k-t_{k-1})^2}{1+q_{k-1}}+2a_1(t_k-t_{k-1})\\
&+\frac{2a_1(t_k-t_{k-1})^3}{\Gamma_{q_{k-1}}(4)}+\frac{a_1(t-t_k)^{\alpha_k}}{\Gamma_{q_k}(\alpha_k+1)},\\
B_1=&2a+3b+6b_1(t_k-t_{k-1})+\frac{4b_1(t_k-t_{k-1})^2}{1+q_{k-1}}+\frac{2b_1(t_k-t_{k-1})^3}{\Gamma_{q_{k-1}}(4)}+\frac{b_1(t-t_k)^{\alpha_k}}{\Gamma_{q_k}(\alpha_k+1)}.
\end{aligned}
$$

(iii) A maps bounded sets into equicontinuous sets of $PC(J,\mathbb{R})$.

Let $\tau_2,\tau_1\in J_k\in(t_k,t_{k+1}]$ for some $k\in\{0,1,2,\cdots,m\}$ and B_R be bound set of $PC(J,\mathbb{R})$ as before. Then, for $x\in B_R$, we have

$$
\begin{aligned}
&|Ax(\tau_2)-Ax(\tau_1)|\\
&\le\frac{(\tau_2-t_k)^{\alpha_k-1}-(\tau_1-t_k)^{\alpha_k-1}}{\Gamma_{q_k}(\alpha_k)}\Bigg[|\beta|+\Bigg[\sum_{0<t_k<t}\varphi_k(x(t_k))+{}_{t_{k-1}}I_{q_{k-1}}f(t_k,x(t_k))\\
&-{}_{t_k}I_{q_k}f(t_k,x(t_k))\Bigg]\Bigg]+\frac{(\tau_2-t_k)^{\alpha_k-2}-(\tau_1-t_k)^{\alpha_k-1}}{\Gamma_{q_k}(\alpha_k-1)}\Bigg[|\gamma|+|\beta|t_{k-1}\\
&+\sum_{0<t_k<t}\Bigg[\varphi_k^*(x(t_k))-{}_{t_{k-1}}I^2_{q_{k-1}}f(t_k,x(t_k))-{}_{t_k}I^2_{q_k}f(t_k,x(t_k)\Bigg]\\
&+\sum_{0<t_k<t}(t_k-t_{k-1})_{t_{k-1}}I_{q_{k-1}}f(t_k,x(t_k))+\sum_{0<t_k<t}\sum_{0<t_j<t_k}\big[{}_{t_{j-1}}I_{q_{j-1}}f(t_j,x(t_j))\big]\\
&+\sum_{0<t_k<t}\varphi_j(x(t_j))\Bigg]+\frac{(\tau_2-t_k)^{\alpha_k-3}-(\tau_1-t_k)^{\alpha_k-1}}{\Gamma_{q_k}(\alpha_k-2)}\Bigg[|\gamma t_{k-1}|+|\beta t^2_{k-1}|\\
&+(t_k-t_{k-1})+\sum_{0<t_k<t}\sum_{0<t_j<t_k}(t_k-t_{k-1})\big[{}_{tj-1}I^2_{q_{j-1}}f(t_j,x(t_j))-{}_{t_j}I^2_{q_j}f(t_j,x(t_j))\\
&+|\gamma|+|\beta t_{k-1}|+\varphi^*_{k-1}x(t_{k-1})\big]+\sum_{0<t_k<t}\varphi_k^*(x(t_k))\\
&+\sum_{0<t_k<t}\sum_{0<t_j<t_k}\big[{}_{tj-1}I^3_{q_{j-1}}f(t_j,x(t_j))-{}_{tj}I^3_{q_j}f(t_j,x(t_j))\big]\Bigg]\\
&+\frac{(\tau_2-t_k)^{\alpha_k}-(\tau_1-t_k)^{\alpha_k}}{\Gamma_{q_k}(\alpha_k+1)}f(t,x(t))\to 0(\tau_2\to\tau_1).
\end{aligned}
$$

As a consequence of the Arzela-Asoli theorem, we can conclude that $A:PC(J,\mathbb{R})\to PC(J,\mathbb{R})$ is completely continuous. The proof has been completed.

Theorem 3.2. *Assume that (H_5) and (H_6) hold. Suppose that $B_1<1$ holds further, then the boundary value problem (1.2) has at least one solution.*

Proof. In view of Lemma 3.1, it is easy to know that A is completely continuous. It is clear that $x(t) \in PC(J, \mathbb{R})$ is a solution of boundary value problem (1.2), if and only if x is a fixed point of A. We need to show that the set

$$E = \{x(t) \in PC(J, \mathbb{R}) : x(t) = \lambda Ax(t), 0 \le \lambda \le 1\}$$

is bounded, which is independent of λ. Let $x(t) \in E$, then $x(t) = \lambda Ax(t)$ for some $0 \le \lambda \le 1$.

By (H_5) and (H_6), for each $t \in (t_k, t_{k+1}]$, according to (ii) of Lemma 3.1, we have

$$|(x(t))| = |\lambda A(x(t))| \le |A(x(t))| \le A_1 + B_1\|x\|.$$

Consequently,

$$\|x\| \le \frac{A}{1 - B_1} := M_1.$$

This show that the set E is bounded. By Theorem 2.3, we can draw the conclusion that the boundary value problem (1.2) has at least one solution, by which we complete the proof.

In order to get the main result with Schaefer's fixed point theorem well, we replace $(H_5)'$ with (H_5).

$(H_5)'$ there exists $L > 0$ such that $|f(t, x(t))| \le L(1+|x(t)|), \forall t \in (t_k, t_{k+1}], x(t) \in \mathbb{R}$.

Theorem 3.3. *Assume that $(H_5)'$ and (H_6) hold, then the boundary value problem (1.2) has at least one solution.*

Proof. In view of Lemma 3.1, it is easy to know that A is completely continuous. It is clear that $x(t) \in PC(J, \mathbb{R})$ is a solution of boundary value problem (1.2), if and only if $x(t)$ is a fixed point of A. We need to show that the set

$$E = \{x(t) \in PC(J, \mathbb{R}) : x(t) = \lambda Ax(t), 0 \le \lambda \le 1\}$$

is bounded, which is independent of λ. Let $x(t) \in E$, then $x(t) = \lambda Ax(t)$ for some $0 \le \lambda \le 1$. By $(H_5)'$ and (H_6), for each $t \in (t_k, t_{k+1}]$, according to (ii) of Lemma 3.1, we have

$$\begin{aligned}
&|(Ax)(t)| \\
&\le \frac{(t-t_k)^{\alpha_k-1}}{\Gamma_{q_k}(\alpha_k)}\left[|\beta| + a + 2{}_{t_k}I_{q_k}L(1+|x(t_k)|)\right] \\
&+ \frac{(t-t_k)^{\alpha_k-2}}{\Gamma_{q_k}(\alpha_k-1)}\Bigg[|\gamma| + |\beta| + b + 2{}_{t_k}I_{q_k}^2 L(1+|x(t_k)|) + (t_k - t_{k-1}){}_{t_k}I_{q_k}L(1+|x(t_k)|) \\
&+{}_{t_k}I_{q_k}L(1+|x(t_k)|) + a\Bigg] + \frac{(t-t_k)^{\alpha_k-3}}{\Gamma_{q_k}(\alpha_k-2)}\Bigg[|\gamma| + |\beta| \\
&+ (t_k - t_{k-1})\left[2{}_{t_k}I_{q_k}^2 L(1+|x(t_k)|) + |\gamma| + |\beta| + 2b\right] + 2{}_{t_k}I_{q_k}^3 L(1+|x(t_k)|) \\
&+ (t_k - t_{k-1})^2\left[|\beta| + a + 2{}_{t_k}I_{q_k}L(1+|x(t_k)|)\right] +{}_{t_k}I_{q_k}^{\alpha_k} f(t, x(t))\Bigg]
\end{aligned}$$

$$\leq \frac{T'}{\Gamma'}\Big[5|\beta| + 3(a+b+|\gamma|) + (3+T)L(1+|x(t_k)|)t_k + 2T^2 t_k L(1+|x(t_k)|)$$
$$+ (2+2T)\frac{(t_k - t_{k-1})^2}{1+q_{k-1}}L(1+|x(t_k)|) + 2\frac{(t_k-t_{k-1})^3}{1+q_{k-1}^2}L(1+|x(t_k)|)\Big]$$
$$+{}_{t_k}I_{q_k}^{\alpha_k} f(t,x(t))$$
$$= A' + \frac{1}{\Gamma_{q_k}(\alpha_k)}\int_{t_k}^{t} {}_{t_k}(t - {}_{t_k}\phi_{q_k}(t))_{q_k}^{\alpha_k-1} L(1+|x(s)|)_{t_k} d_{q_k}s$$
$$= A' + \frac{L}{\Gamma_{q_k}(\alpha_k)}\int_{t_k}^{t} {}_{t_k}(t - {}_{t_k}\phi_{q_k}(t))_{q_k}^{\alpha_k-1}{}_{t_k} d_{q_k}s$$
$$+ \frac{L}{\Gamma_{q_k}(\alpha_k)}\int_{t_k}^{t} {}_{t_k}(t - {}_{t_k}\phi_{q_k}(t))_{q_k}^{\alpha_k-1}|x(s)|_{t_k} d_{q_k}s$$
$$= A' + \frac{L}{\Gamma_{q_k}(\alpha_k)}(t-t_k)^{\alpha_k}B_q(\alpha_k,1) + \frac{L}{\Gamma_{q_k}(\alpha_k)}\int_{t_k}^{t} {}_{t_k}(t - {}_{t_k}\phi_{q_k}(t))_{q_k}^{\alpha_k-1}|x(s)|_{t_k} d_{q_k}s$$
$$= A'' + \frac{L}{\Gamma_{q_k}(\alpha_k)}\int_{t_k}^{t} {}_{t_k}(t - {}_{t_k}\phi_{q_k}(t))_{q_k}^{\alpha_k-1}|x(s)|_{t_k} d_{q_k}s,$$

where

$$A' = \frac{T'}{\Gamma'}\Big[5|\beta| + 3(a+b+|\gamma|) + (3+T)L(1+|x(t_k)|)t_k + 2T^2 t_k L(1+|x(t_k)|)$$
$$+ (2+2T)\frac{(t_k - t_{k-1})^2}{1+q_{k-1}}L(1+|x(t_k)|) + 2\frac{(t_k-t_{k-1})^3}{1+q_{k-1}^2}L(1+|x(t_k)|)\Big],$$
$$A'' = A' + \frac{L}{\Gamma_{q_k}(\alpha_k)}(t-t_k)^{\alpha_k}B_q(\alpha_k,1).$$

We need to show that the set

$$E = \{x(t) \in PC(J,\mathbb{R}) : x(t) = \lambda Ax(t), 0 \leq \lambda \leq 1\}$$

is bounded, which is independent of λ. Let $x(t) \in E$, then $x(t) = \lambda Ax(t)$ for some $0 \leq \lambda \leq 1$.

By $(H_5)'$ and (H_6), for each $t \in (t_k, t_{k+1}]$, according to (ii) of Lemma 3.1, we have

$$|x(t)| = |\lambda A(x(t))| \leq |A(x(t))| \leq A'' + \frac{L}{\Gamma_{q_k}(\alpha_k)}\int_{t_k}^{t} {}_{t_k}(t - {}_{t_k}\phi_{q_k}(t))_{q_k}^{\alpha_k-1}|x(s)|_{t_k} d_{q_k}s.$$

According to q-Gronwall inequality, we deduce

$$|x(t)| \leq A'' + \int_{t_k}^{t}\sum_{n=1}^{\infty}\frac{\frac{L}{(\Gamma_{q_k}(\alpha_k)}\Gamma_{q_k}(\alpha_k))^n}{\Gamma_{q_k}(n\alpha_k)}(t -_{t_k} \phi_{q_k}(t))^{\alpha_k-1}A'' d_{q_k}s := M'$$

This show that the set E is bounded. As a consequence of Schaefer's fixed point theorem, we deduce that A has at least one fixed point, which means that the problem (1.2) has at least one solution. The proof is complete.

4. Examples

Consider the following impulsive fractional q_k-difference initial value problem

$$\begin{cases} {}_{t_k}D^{\frac{k^2-2k+5}{2k^3+k+2}}_{\frac{k^3-3k+7}{2k^4+k+8}}x(t) = \frac{e^{-\cos^2 t|x(t)|}}{20(t+1)^2(1+|x(t)|)}, \ t \in [0, \frac{1}{2}], \ t \neq t_k, \\ \widetilde{\Delta}\, x(t_k) = \frac{|x(t_k)|}{4(k+2)+|x(t_k)|}, \ k = 1, 2, \ t_k = \frac{k}{5}, \\ \Delta^* x(t_k) = \frac{|x(t_k)|}{6(k+3)+|x(t_k)|}, \ k = 1, 2, \ t_k = \frac{k}{5}, \\ \Delta^{**} x(t_k) = \frac{|x(t_k)|}{5(k+4)+|x(t_k)|}, \ k = 1, 2, \ t_k = \frac{k}{5}, \\ x(0) = 0, \ {}_0D^{\frac{3}{2}}_{\frac{7}{8}}x(0) = \frac{3}{4}, \ {}_0D^{\frac{1}{2}}_{\frac{7}{8}}x(0) = \frac{2}{3}. \end{cases} \tag{4.1}$$

Here,

$$\alpha_k = \frac{k^2-2k+5}{2k^3+k+2}, \ q_k = \frac{k^3-3k+7}{2k^4+k+8}, k = 0, 1, 2, m = 2, T = \frac{1}{2}, \beta = \frac{3}{4}, \gamma = \frac{2}{3},$$

$$f(t, x(t)) = \frac{e^{-\cos^2 t|x(t)|}}{20(t+1)^2(1+|x(t)|)}, \varphi_k(x(t_k)) = \frac{|x(t_k)|}{4(k+2)+|x(t_k)|},$$

$$\varphi_k^*(x(t_k)) = \frac{|x(t_k)|}{6(k+3)+|x(t_k)|}, \varphi_k^{**}(x(t_k)) = \frac{|x(t_k)|}{5(k+4)+|x(t_k)|}.$$

Since

$$|f(t, x) - f(t, y)| \leq \frac{1}{20}\|x - y\|,$$

$$|\varphi_k(x) - \varphi_k(y)| \leq \frac{1}{12}\|x - y\|,$$

$$|\varphi_k^*(x) - \varphi_k^*(y)| \leq \frac{1}{24}\|x - y\|,$$

$$|\varphi_k^{**}(x) - \varphi_k^{**}(y)| \leq \frac{1}{25}\|x - y\|,$$

we choose $\gamma = \frac{13}{4}$, and we have that $(H_1), (H_2), (H_3)$ and (H_4) are satisfied with $L = \frac{1}{20}, \ M = \frac{1}{12}, M^* = \frac{1}{24}, \ M^{**} = \frac{1}{25}$.

We find that $\widetilde{T} = 0.01205530547, \ \widetilde{\Gamma} = 0.875$, and

$$\begin{aligned} \psi_1 = \frac{\widetilde{T}}{\widetilde{\Gamma}}\Big\{&(M + 2M^*)m + Lt_m + \sum_{j=1}^{m} \frac{(t_j - t_{j-1})^3}{\Gamma_{q_{j-1}}(4)} L + \sum_{j=1}^{m} (t_j - t_{j-1})^2 M(m-1) \\ &+ \sum_{j=1}^{m} (t_j - t_{j-1})^3 L + \sum_{j=1}^{m} L(t_j - t_{j-1})(1 - t_{j-1}) - \sum_{j=1}^{m} \frac{(t_j - t_{j-1})^2}{1 + q_{j-1}} L + \sum_{j=1}^{m} (j-1) \\ &M + \sum_{j=1}^{m} \frac{(t_j - t_{j-1})^3}{1 + q_{j-1}} L + \sum_{j=1}^{m} (t_j - t_{j-1})(j-1)M^* + L\Big\} \\ &\approx 0.1317463686. \end{aligned}$$

Therefore, according to Theorem 3.1, the initial value problem (4.1) has a unique solution on J. This means that our hypothesis is reasonable, that is, the uniqueness of the solution to the equation.

Example 4.2. Consider the following impulsive fractional q_k-difference equations

$$\begin{cases} {}_{t_k}D^{\frac{k^2-2k+5}{2k^3+k+2}}_{\frac{k^3-3k+7}{2k^4+k+8}}x(t) = \frac{x(t)\sin x^2(t)}{16(1+e^{t^2})(e^t+|x(t)|)},\ t \in [0, \frac{1}{2}],\ t \neq t_k, \\ \widetilde{\Delta}\, x(t_k) = \frac{|x(t_k)|}{4(k+2)},\ k = 1, 2,\ t_k = \frac{k}{5}, \\ \Delta^* x(t_k) = \frac{|x(t_k)|}{6(k+3))|},\ k = 1, 2,\ t_k = \frac{k}{5}, \\ \Delta^{**} x(t_k) = \frac{|x(t_k)|}{5(k+4))|},\ k = 1, 2,\ t_k = \frac{k}{5}, \\ x(0) = 0,\ {}_0D^{\frac{3}{2}}_{\frac{7}{8}}x(0) = \frac{3}{4},\ {}_0D^{\frac{1}{2}}_{\frac{7}{8}}x(0) = \frac{2}{3}. \end{cases} \tag{4.2}$$

Here, $\alpha_k = \frac{k^2-2k+5}{2k^3+k+2}$, $q_k = \frac{k^3-3k+7}{2k^4+k+8}, k = 0, 1, 2, m = 2, T = \frac{1}{2}, \beta = \frac{3}{4}, \gamma = \frac{2}{3}, f(t, x(t)) = \frac{x(t)\sin x^2(t)}{16(1+e^{t^2})(e^t+|x(t)|)}, \varphi_k(x(t_k)) = \frac{|x(t_k)|}{4(k+2)}, \varphi^*_k(x(t_k)) = \frac{|x(t_k)|}{6(k+3)}, \varphi^{**}_k(x(t_k)) = \frac{|x(t_k)|}{5(k+4)}$.

Obviously, we have $|f(t, x(t))| \leq \frac{1}{64}(1 + |x|), L = \frac{1}{64}, a = \frac{1}{12}, b = \frac{1}{18}, M' = 0.087$. Thus, all the assumptions in Theorem 3.3 are satisfied, which means that (4.2) has at least one solution.

In equation (4.2), if we set $f(t, x) = \frac{e^{-t}}{10} + \ln(1 + |t|)x$, where $a(t) = \frac{e^{-t}}{10}, b(t) = \ln(1+|t|)$, we can obtain $a = \frac{1}{10}, b = \ln 2$, and by calculation, we get $B_1 = 0.12 < 1$. That is to say, Theorem 3.2 is satisfied. However, if we set $f(t, x) = e^{-t} + \sin t|x|$, where $a(t) = e^{-t}, b(t) = \sin t$, by calculation, we get $a = b = 1, B_1 = 1.167 > 1$. That is to say, $B_1 < 1$ is not necessary for the function $f(t, x)$ to be true.

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