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The First Integrals and Related Properties of a Class of Quintic Systems with a Uniform Isochronous Center*

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Abstract In this paper, we give all the first integrals of the quintic systems which have a uniform isochronous center, and use them to determine the qualitative behavior of the periodic solutions of their equivalent non-autonomous systems. Meanwhile, we clearly describe the local phase portraits of singularity at infinity.

Keywords First integrals, Equivalence system, Uniform isochronous center, Infinite singularity, Local phase portraits.

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1. Introduction

Consider a planar polynomial differential system

$$\begin{cases} x' = ax + by + \sum_{k=2}^{\infty} P_k(x,y), \\ y' = cx + dy + \sum_{k=2}^{\infty} Q_k(x,y), \end{cases} \tag{1.1}$$

where $P_k(x,y)$ and $Q_k(x,y)$ are real homogeneous polynomials in x and y of degree k, which is an integer and greater than or equal to two.

Determining a singular point of a planar polynomial differential system to be a center is called Poincaré center-focus problem, which has been exhaustively studied in the last century, and it is closely related to the Hilbert 16th problem. Nevertheless, in spite of all efforts, there is no general method to solve this problem. Up to now, only for quadratic systems and some special systems the center conditions have been obtained [1, 2, 9, 12, 16, 18, 21, 22, 31]. Poincaré and Lyapunov [13] have provided such criterion: The origin point of (1.1) (with $a + d = 0$, $ad - bc > 0$) is a center, if and only if it possesses a nonconstant real analytic first integral in a neighbourhood of the origin (or there is a nonzero analytic integrating factor in a neighbourhood of the origin). However, in general, it is very difficult to find the integrating factor or the first integral. As increasing difficulties are encountered in the process of finding the central condition, some scholars have been trying to look

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for the conditions under which system (1.1) possesses a special center. The composition center has been discussed by Alwash and Lloyd [5,7] (see, for instance, [5,7]). Discussions of isochronous centres [4,6,8] have recently been provided by Algaba [3], Conti [14] and Villarini [25] as well as by Christopher and Devlin [11], these works refer to a number of articles on the topic, and have mentioned systems such as (1.1).

From [19], we get that if a planar differential polynomial system $x' = X(x,y), y' = Y(x,y)$ of degree n has a center at the origin of coordinates, then this center is an uniform isochronous, if and only if we carry out a linear change of variables and a scaling of time, it can be written into the form

$$\begin{cases} x' = -y + xP(x,y), \\ y' = x + yP(x,y), \end{cases} \tag{1.2}$$

where $P = \sum_{k=1}^{n-1} P_k(x,y)$, and $P_k(x,y)$ is a homogeneous polynomial in x and y of degree k.

System (1.2) is called rigid system [1,5]. For the rigid system (1.2) with $P = P_1 + P_m$ or $P = P_2 + P_{2m}$, m is an arbitrary positive integer. In [1] and [29,30], the authors used different methods to obtain the center conditions and point out that this center is also a composition center and uniform isochronous center.

In [8, 13, 19, 20], the authors have discussed the center conditions and phase portraits of system (1.2) with $P(x,y)$ as a quadratic or a cubic polynomial. In this paper, we shall be primarily concerned with the quintic system, which has the form

$$\begin{cases} x' = -y + x(P_2(x,y) + P_4(x,y)), \\ y' = x + y(P_2(x,y) + P_4(x,y)), \end{cases} \tag{1.3}$$

where $P_k = \sum_{i+j=k} p_{ij} x^i y^j$, $(k = 2,4)$, p_{ij} are real constants. First, we will give the first integrals of this system, when it has a center at origin point. Secondly, we shall establish some time-varying systems, which are equivalent (with the coinciding reflecting function [23]) to (1.3), and we use this autonomous system to determine the qualitative behavior of the periodic solutions of their equivalent non-autonomous systems. Finally, we will describe the local phase portraits of singularity at infinity.

2. The first integral

In this section, we will apply the method of Darboux [10, 15, 17] to sufficiently discover many algebraic integrals of system (1.3) and construct its first integrals .

Let $f(x,y) \in C[x,y]$, $f(x,y)$ not be identically zero. The algebraic curve $f(x,y) = 0$ is an invariant algebraic curve ($f(x,y)$ called algebraic integral) of the polynomial system

$$x' = X(x,y),\ y' = Y(x,y), \tag{2.1}$$

if for some polynomial $h(x,y) \in C[x,y]$, we have

$$f_x(x,y)X(x,y) + f_y(x,y)Y(x,y) = h(x,y)f(x,y).$$

The polynomial $h(x,y)$ is called the cofactor of the invariant algebraic curve $f(x,y) = 0$.

If $H(t,x,y)$ is a nonconstant analytic function and such that

$$H_t + H_x X(x,y) + H_y Y(x,y) \equiv 0,$$

then $H(t,x,y)$ is called the first integral of (2.1).

Suppose that $f_i(x,y)(i=1,2,..m)$ are the independent algebraic integrals of (2.1) and the cofactor of $f_i(x,y)$ is $h_i(x,y)$. If there is a set of constants $k_i(i=1,2,...,m)$ that are not all zero, such that $\sum_{i=}^{m} k_i h_i(x,y) \equiv 0$, then the function $H = f_1^{k_1}(x,y) f_2^{k_2}(x,y) ... f_m^{k_m}(x,y)$ is the first integral of system (2.1).

For system (1.3), in [30], the center conditions are given in a particularly succinct form and more elegantly and economically expressed as follows.

Lemma 2.1 ([30]). Suppose that $P_2(x,y) \neq 0$. The origin point of system (1.3) is a center, if and only if

$$\int_0^{2\pi} P_2(\cos\theta, \sin\theta) d\theta = 0, \int_0^{2\pi} (\int_0^{\theta} P_2(\cos\tau, \sin\tau) d\tau)^k P_4(\cos\theta, \sin\theta) d\theta = 0, (k=0,1,2),$$

i.e.,

$$p_{20} + p_{02} = 0; \tag{2.2}$$

$$p_{22} + 3(p_{40} + p_{04}) = 0; \tag{2.3}$$

$$p_{11}(p_{04} - p_{40}) + p_{20}(p_{31} + p_{13}) = 0; \tag{2.4}$$

$$(p_{11}^2 - 4p_{20}^2)(p_{40} + p_{04}) - p_{11} p_{20}(p_{31} - p_{13}) = 0. \tag{2.5}$$

Theorem 2.1. Suppose that $P_2 \cdot P_4 \neq 0$ and the conditions of Lemma 2.1 are satisfied. Then, system (1.3) can be brought to the form

$$\begin{cases} x' = -y + x^2 y(1 + Ax^2 + By^2), \\ y' = x + xy^2(1 + Ax^2 + By^2), \end{cases} \tag{2.6}$$

where A and B are constants and $A^2 + B^2 \neq 0$. Furthermore, the first integral of system (2.6) is described in what follows.

Case 1. $B - A = \frac{1}{4}$,

$$H_1 = \frac{x^2 + y^2}{1 + 2Ax^2 + 2By^2} e^{\frac{1}{1+2Ax^2+2By^2}};$$

Case 2. $B - A > \frac{1}{4}$,

$$H_2 = \frac{(x^2+y^2)^2}{B - A + Ax^2 + By^2 + (Ax^2 + By^2)^2} e^{\frac{-2}{\sqrt{\sigma_1}} arctan \frac{1+2Ax^2+2By^2}{\sqrt{\sigma_1}}};$$

Case 3. $B - A < \frac{1}{4}$ and $B - A \neq 0$,

$$H_3 = \frac{(x^2+y^2)^2}{B - A + Ax^2 + By^2 + (Ax^2 + By^2)^2} (\frac{2Ax^2 + 2By^2 + 1 + \sqrt{-\sigma_1}}{2Ax^2 + 2By^2 + 1 - \sqrt{-\sigma_1}})^{\frac{1}{\sqrt{-\sigma_1}}};$$

Case 4. $B = A$,

$$H_4 = \frac{x^2 + y^2}{1 + Ax^2 + Ay^2} e^{\frac{1+y^2}{A(x^2+y^2)}},$$

where $\sigma_1 = 4(B - A) - 1$.

Proof. As $P_2 \neq 0$, $p_{20}^2 + p_{11}^2 \neq 0$.

Case (a). $p_{20} = 0, p_{11} \neq 0$. By conditions (2.2)–(2.5), it follow that $p_{22} = p_{40} = p_{04} = 0$, and system (1.3) becomes

$$\begin{cases} x' = -y + x^2y(p_{11} + p_{31}x^2 + p_{13}y^2), \\ y' = x + y^2x(p_{11} + p_{31}x^2 + p_{13}y^2). \end{cases}$$

Case (b). $p_{20} \neq 0, p_{11} = 0$. Applying conditions (2.2)–(2.5), we get

$$p_{20} = p_{02} = 0,\ p_{22} = 0,\ p_{40} + p_{04} = 0,\ p_{31} = p_{13} = 0.$$

Applying the transformation: $u = x - y$, $v = x + y$, system (1.3) changes into

$$\begin{cases} u' = -v + u^2v(p_{20} + \frac{1}{4}(2p_{40} - p_{31})u^2 + \frac{1}{4}(2p_{40} + p_{31})v^2), \\ v' = u + v^2u(p_{20} + \frac{1}{4}(2p_{40} - p_{31})u^2 + \frac{1}{4}(2p_{40} + p_{31})v^2). \end{cases}$$

Case (c). $p_{20} \cdot p_{11} \neq 0$. Using conditions (2.2)–(2.5) and applying the transformation: $u = x - \delta y$, $v = \delta x + y$, where $\delta = \frac{-p_{11} + \sqrt{p_{11}^2 + 4p_{20}^2}}{2p_{20}}$, system (1.3) becomes

$$\begin{cases} u' = -v + \frac{p_{20}}{\delta}u^2v(1 + \check{A}u^2 + \check{B}v^2), \\ v' = u + \frac{p_{20}}{\delta}uv^2(1 + \check{A}u^2 + \check{B}v^2), \end{cases}$$

where

$$\check{A} = \frac{1}{(1+\delta^2)^2}(p_{40} - p_{04} + \frac{\delta}{p_{20}}(p_{11}(p_{40} + p_{04}) - p_{20}p_{31})),$$
$$\check{B} = \frac{1}{(1+\delta^2)^2}(p_{40} - p_{04} - \frac{\delta}{p_{20}}(p_{11}(p_{40} + p_{04}) + p_{20}p_{13})).$$

In summary, under condition (2.2)–(2.5), system (1.3) can be transformed into the form

$$\begin{cases} u' = -v + u^2v(A_0 + A_1u^2 + A_2v^2), \\ v' = u + uv^2(A_0 + A_1u^2 + A_2v^2) \end{cases} \tag{2.7}$$

with $A_0 \neq 0$. Without losing generality, we can assume that $A_0 > 0$, as when $A_0 < 0$, by using $t \to -t$, it ensures that $A_0 > 0$. By applying the transformation $x = \sqrt{A_0}u$, $y = \sqrt{A_0}v$, system (2.7) can be brought to the form of (2.6).

In order to integrate system (2.6), we use the method of Darboux [10, 15] to sufficiently discover many algebraic integrals of system (2.6).

Case 1. $B - A = \frac{1}{4}$. It is not difficult to verify that system (2.6) has algebraic integrals:

$$f_1 = x^2 + y^2,\ f_2 = 1 + 2Ax^2 + 2By^2,\ f_3 = e^{\frac{1}{1+2Ax^2+2By^2}},$$

and their co-factors are respectively

$$h_1 = 2xy(1 + Ax^2 + By^2),\ h_2 = xy(1 + 2Ax^2 + 2By^2),\ h_3 = -xy,$$

and taking $k_1 = 1, k_2 = -1, k_3 = 1$, which ensure that $k_1h_1 + k_2h_2 + k_3h_3 \equiv 0$. Therefore, it follows that system (2.6) possesses the first integral H_1.

Case 2. $B - A > \frac{1}{4}$. System (2.6) has algebraic integrals:

$$f_1 = x^2+y^2,\ f_2 = B-A+Ax^2+By^2+(Ax^2+By^2)^2,\ f_3 = e^{arctan\frac{1}{\sqrt{\sigma_1}}(1+2Ax^2+2By^2)}$$

and their co-factors are respectively

$$h_1 = 2xy(1 + Ax^2 + By^2),\ h_2 = 2xy(1 + 2Ax^2 + 2By^2),\ h_3 = \sqrt{\sigma_1}xy,$$

and taking $k_1 = 2, k_2 = -1, k_3 = -\frac{2}{\sqrt{\sigma_1}}$, we obtain $k_1h_1 + k_2h_2 + k_3h_3 \equiv 0$. Thus, the function H_2 is the first integral of system (2.6).

Case 3. $B - A < \frac{1}{4}$, $B - A \neq 0$. The system (2.6) has algebraic integrals:

$$f_1 = x^2 + y^2,\ f_2 = 1 - \sqrt{-\sigma_1} + 2Ax^2 + 2By^2,\ f_3 = 1 + \sqrt{-\sigma_1} + 2Ax^2 + 2By^2,$$

and their co-factors are respectively

$$h_1 = 2xy(1+Ax^2+By^2),\ h_2 = xy(1+\sqrt{-\sigma_1}+2Ax^2+2By^2),\ h_3 = xy(1-\sqrt{-\sigma_1}+2Ax^2+2By^2),$$

and taking $k_1 = 2, k_2 = -1 - \frac{1}{\sqrt{-\sigma_1}}, k_3 = -1 + \frac{1}{\sqrt{-\sigma_1}}$, which follow that $k_1h_1 + k_2h_2 + k_3h_3 \equiv 0$. Thus, system (2.6) has the first integral H_3.

Case 4. $B = A$. System (2.6) has algebraic integrals:

$$f_1 = x^2 + y^2,\ f_2 = 1 + Ax^2 + Ay^2,\ f_3 = e^{\frac{1+y^2}{x^2+y^2}},$$

and their co-factors are respectively

$$h_1 = 2xy(1 + Ax^2 + Ay^2),\ h_2 = 2Axy(x^2 + y^2),\ h_3 = -2Axy,$$

and taking $k_1 = 1, k_2 = -1, k_3 = \frac{1}{A}$, we have $k_1h_1 + k_2h_2 + k_3h_3 \equiv 0$. Therefore, the function H_4 is the first integral of system (2.6). □

Remark 1. Obviously, $H = \arctan\frac{y}{x} - t = c$ is the first integral of system (1.3). Thus, by Theorem 2.1, all the first integrals of (2.6) are known.

Similar to what have been mentioned above, we can get the following conclusions.

Theorem 2.2. If $P_4(x, y) \equiv 0$, $P_2 \neq 0$, then the origin point of system (1.3) is a center, if and only if

$$p_{20} + p_{02} = 0,$$

and it can be brought to the form

$$\begin{cases} x' = -y + x^2y, \\ y' = x + xy^2, \end{cases} \tag{2.8}$$

and its first integral is

$$H_5 = \frac{x^2 + y^2}{1 + y^2}.$$

Theorem 2.3. If $P_2(x, y) \equiv 0$, $P_4 \neq 0$, then the origin point of system

$$\begin{cases} x' = -y + xP_4(x, y), \\ y' = x + yP_4(x, y) \end{cases} \tag{2.9}$$

is a center, if and only if

$$p_{22} + 3(p_{40} + p_{04}) = 0,$$

and this system has the first integral:

$$H_6 = \frac{(x^2 + y^2)^2}{1 + 4xy(p_{40}x^2 - p_{04}y^2) + (p_{13} + p_{31})y^4 + 2p_{31}x^2y^2}.$$

3. Equivalence system

In this section, we will establish some time varying differential systems, which are equivalent to the above autonomous differential systems (2.6), (2.8) and (2.9), by which to determine the qualitative behaviors of the periodic solutions of their equivalent time varying systems.

Now, we present some results and concepts necessary to our study.

Consider the differential system

$$x' = X(t, x), \tag{3.1}$$

which has a continuously differentiable right-hand side and general solution $\phi(t; t_0, x_0)$.

Definition 3.1 ([23]). For system (3.1), $F(t, x) := \phi(-t; t, x)$ is called its reflecting function.

By [23], we see that a differentiable function $F(t, x)$ is a reflecting function of system (3.1), if and only if it is a solution of the Cauchy problem

$$F_t + F_x X(t, x) + X(-t, F) = 0, F(0, x) = x.$$

If system (3.1) is 2ω-periodic with respect to t, $F(t, x)$ is its reflecting function, then $T(x) := F(-\omega, x) = \phi(\omega; -\omega, x)$ is the Poincaré mapping of (3.1) over the period $[-\omega, \omega]$. Thus, the solution $x = \phi(t; -\omega, x_0)$ of (3.1) defined on $[-\omega, \omega]$ is 2ω-periodic, if and only if x_0 is a fixed point of $T(x)$.

Definition 3.2 ([23]). If the reflecting functions of two differential systems are coincident in their common domain, then these systems are said to be equivalent.

All the equivalent 2ω-periodic systems have a common Poincaré mapping over the period $[-\omega, \omega]$, and the qualitative behavior of the periodic solutions of these equivalent systems are the same. By this, we can study the qualitative behavior of the solutions of a complicated system by using a simple differential system. Unfortunately, in general, it is very difficult to find out the reflecting function of (3.1). How to judge whether the two systems are equivalent when we do not know their reflecting function? This is a very important and interesting problem. In [23, 24], Mironenko has studied the problem and obtained some valuable and interesting conclusions.

In [24], the author provided a useful criterion for the equivalence of two equations. Afterwards, this conclusion has been generalized by [31].

Lemma 3.1 ([24, 31]). If the vector function $\Delta_i(t, x)$ is a solution of the equation

$$\Delta_t(t, x) + \Delta_x X(t, x) - X_x(t, x)\Delta(t, x) = 0, \tag{3.2}$$

then system (3.1) is equivalent to system

$$x' = X(t,x) + \sum_{i=1}^{m} \alpha_i(t, H(t,x,y))\Delta_i(t,x), \tag{3.3}$$

where $\alpha_i(t, H(t,x,y))$ is an arbitrary differentiable scalar function and $\alpha_i(t, H(t,x,y)) + \alpha_i(-t, H(t,x,y)) = 0$, $H(t,x,y)$ is a first integral of (3.1).

In addition, if systems (3.1) and (3.3) are 2ω-periodic with respect to t, then the qualitative behavior of their 2ω-periodic solutions are the same.

Remark 2. If $X(t,0) = 0$, the zero solution $x = 0$ (i.e., origin point) of (3.1) is called a center, if there exists an open neighborhood U of $x = 0$, for every $x_0 \in U$ the solution $x = \phi(t; t_0, x_0)$ through x_0 is periodic.

Theorem 3.1. System (2.6) is equivalent to system

$$\begin{cases} x' = -y + x^2y(1 + Ax^2 + By^2) + x\alpha(t,H)(B - A + Ax^2 + By^2 + (Ax^2 + By^2)^2), \\ y' = x + xy^2(1 + Ax^2 + By^2) + y\alpha(t,H)(B - A + Ax^2 + By^2 + (Ax^2 + By^2)^2), \end{cases} \tag{3.4}$$

where $\alpha(t, H(x,y))$ is an arbitrary differentiable scalar function and such that $\alpha(t,H) + \alpha(-t,H) = 0$, H is the first integral of system (2.6), i.e., $H = H_i (i = 1,2,3)$, corresponding to $B - A = \frac{1}{4}$, $B - A > 1/4$, $B - A < 1/4$ respectively.

Furthermore, if $\alpha(t + 2\pi, H) = \alpha(t,H)$, then the origin point of system (3.4) is a center, too.

Proof. It is not difficult to verify that the function

$$\Delta = (x(B-A+Ax^2+By^2+(Ax^2+By^2)^2),\ y(B-A+Ax^2+By^2+(Ax^2+By^2)^2))^T$$

is a solution of the equation (3.2) with $X = (-y + x^2y(1 + Ax^2 + By^2),\ x + y^2x(1 + Ax^2 + By^2))^T$. By Lemma 3.1, we see that the system (2.6) is equivalent to (3.4). In view of Lemma 2.1, the origin point of (2.6) is a center. Thus, the zero solution $x = 0, y = 0$ of system (3.4) is a center, too. □

Theorem 3.2. If $A \neq 0$, then the differential system

$$\begin{cases} x' = -y + x^2y(1 + Ax^2 + Ay^2), \\ y' = x + xy^2(1 + Ax^2 + Ay^2). \end{cases} \tag{3.5}$$

is equivalent to system

$$\begin{cases} x' = -y + x^2y(1 + Ax^2 + Ay^2) + \alpha_1(t,H_4)\Delta_{11} + x\alpha_2(t,H_4)(x^2 + y^2)(1 + Ax^2 + Ay^2), \\ y' = x + xy^2(1 + Ax^2 + Ay^2) + \alpha(t,H_4)\Delta_{21} + y\alpha_2(t,H_4)(x^2 + y^2)(1 + Ax^2 + Ay^2), \end{cases} \tag{3.6}$$

where

$$\Delta_{11} = c_1y + x(1 + A(x^2 + y^2))\varphi,$$
$$\Delta_{21} = -c_1x + y(1 + A(x^2 + y^2))\varphi,$$
$$\varphi = \beta_1(t)x^2 + \beta_2(t)xy + \beta_3(t)y^2,$$

$\beta_1 = c_2 + c_3 \cos 2t + c_4 \sin 2t$, $\beta_2 = -c_1 + 2c_3 \sin 2t - 2c_4 \cos 2t$, $\beta_3 = c_2 - c_3 \cos 2t - c_4 \sin 2t$, where $c_i (i = 1, 2, ..., 4)$ are arbitrary constants, $\alpha_i(t, H_4)) \, (i = 1, 2)$ are arbitrary differentiable scalar functions and such that $\alpha_i(t, H_4) + \alpha_i(-t, H_4) = 0 \, (i = 1, 2)$, H_4 is the same as it in Theorem 2.1.

Furthermore, if $\alpha_i(t + 2\pi, H_4) = \alpha_i(t, H_4) \, (i = 1, 2)$, then the origin point of system (3.6) is a center, too.

Proof. By checking, we see that the functions $\Delta_1 = (\Delta_{11}, \Delta_{21})^T$, $\Delta_2 = (x(x^2 + y^2)(1 + Ax^2 + Ay^2), \, y(x^2 + y^2)(1 + Ax^2 + Ay^2))^T$ are the solutions of the equation (3.2) with $X = (-y + x^2y(1 + Ax^2 + Ay^2), \, x + y^2x(1 + Ax^2 + Ay^2))^T$. Further, by using Lemma 2.1 and Lemma 3.1, it follows that system (3.5) and system (3.6) are equivalent, and the origin point of system (3.6) is a center, too. □

Similar to the above, we can get the following conclusions.

Theorem 3.3. System (2.8) is equivalent to system

$$\begin{cases} x' = -y + x^2y + \alpha_1 x(1 + y^2) + \alpha_2 x(x^2 - 1) + \alpha_3 x(x^2 + y^2) + \alpha_4(c_1 y + x\varphi), \\ y' = x + xy^2 + \alpha_1 y(1 + y^2) + \alpha_2 y(x^2 - 1) + \alpha_3 y(x^2 + y^2) + \alpha_4(-c_1 x + y\varphi), \end{cases} \tag{3.7}$$

where c_1 is an arbitrary constant, $\alpha_i = \alpha_i(t, H_5) \, (i = 1, 2, 3, 4)$ are arbitrary differentiable scalar functions and such that $\alpha_i(t, H_5) + \alpha_i(-t, H_5) = 0 \, (i = 1, 2, 3, 4)$, H_5 is the same as it in Theorem 2.2 and φ is the same as it in Theorem 3.2.

Furthermore, if $\alpha_i(t + 2\pi, H_5) = \alpha_i(t, H_5) \, (i = 1, 2, 3, 4)$, then the origin point of system (3.7) is a center, too.

Theorem 3.4. Suppose that $3(p_{40} + p_{04}) + p_{22} = 0$. Then, system (2.9) is equivalent to system

$$\begin{cases} x' = -y + xP_4(x, y) + \alpha_1(t, H_6)\Delta_{11} + x\alpha_2(t, H_6)(x^2 + y^2)^2, \\ y' = x + yP_4(x, y) + \alpha_1(t, H_6)\Delta_{12} + y\alpha_2(t, H_6)(x^2 + y^2)^2, \end{cases} \tag{3.8}$$

where $\Delta_{11} = x\psi$, $\Delta_{12} = y\psi$, $\psi = 1 + 4xy(p_{40}x^2 - p_{04}y^2) + (p_{13} + p_{31})y^4 + 2p_{31}x^2y^2$, $\alpha_i(t, H_6)) \, (i = 1, 2)$ are arbitrary differentiable scalar functions and $\alpha_i(t, H_6) + \alpha_i(-t, H_6) = 0 \, (i = 1, 2)$, H_6 is the same as it in Theorem 2.3.

In addition, if $\alpha_i(t + 2\pi, H_6) = \alpha_i(t, H_6) \, (i = 1, 2)$, then the origin point of system (3.8) is a center, too.

Remark 3. By the above theorems, we see that if the origin point of a autonomous system is a center. Then, the origin point of all its equivalent non-autonomous systems is a center, too.

4. Infinite singular points

In this section, we will discuss the qualitative behavior of the infinite singular point of the above quintic system. First, we present a lemma necessary to our research.

Let $O(0, 0)$ be an isolated singular point of the system of the form

$$\begin{cases} x' = y + P(x, y), \\ y' = Q(x, y), \end{cases} \tag{4.1}$$

where $P(x,y)$ and $Q(x,y)$ are analytic in a neighborhood $U(\varrho, O)$ $(0 < \varrho \ll 1)$ and also $P(0,0) = Q(0,0) = P_x(0,0) = P_y(0,0) = Q_x(0,0) = Q_y(0,0) = 0$.

Lemma 4.1 ([17,26]). If there is a reversible topological transformation in $U(\varrho, O)$, by this, system (4.1) can be brought to the form

$$\begin{cases} x' = y, \\ y' = a_k x^k (1 + g_1(x)) + b_n x^n y (1 + g_2(x)) + y^2 G(x,y), \end{cases} \tag{4.2}$$

where $g_1(x), g_2(x), G(x,y)$ are analytic in $U(\varrho, O)$ and $g_1(0) = g_2(0) = G(0,0) = 0$, $k = 2m+1 (m \geq 1)$, $\lambda := b_n^2 + 4(m+1)a_{2m+1}$.

$\dot{1}$. If $a_{2m+1} > 0$, then $O(0,0)$ is a saddle;

$\dot{2}$. If $a_{2m+1} < 0$, $b_n = 0$, then $O(0,0)$ is a center of focus;

$\dot{3}$. If $a_{2m+1} < 0$, $b_n \neq 0$, $n > m$ or $m = n$ and $\lambda < 0$, then $O(0,0)$ is a center of focus;

$\dot{4}$. If $a_{2m+1} < 0$, $b_n \neq 0$, n is an even number, and $n < m$ or $n = m$ and also $\lambda \geq 0$, then $O(0,0)$ is a node;

$\dot{5}$. If $a_{2m+1} < 0$, $b_n \neq 0$, n is an odd number, and $n < m$ or $n = m$ and also $\lambda \geq 0$. Then, the phase portrait of (4.1) near the $O(0,0)$ consists of one hyperbolic and elliptic sector.

Let $D(0,0)$ be the infinite singular point of (2.6) on the X-axis, $E(0,0)$ be the infinite singular point on the Y-axis and $M(0,u_1)$, $N(0,u_2)$ $(u_{1,2} = \pm\sqrt{-\frac{A}{B}})$ be the infinite singular points on the U-axis.

Theorem 4.1. For system (2.6)

Case 1. $A \cdot B \neq 0$. When $A < 0$, $D(0,0)$ is a saddle, when $A > 0$, it is a center. When $B > 0$, $E(0,0)$ is a saddle, when $B < 0$, it is a center. When $A > 0$ and $B < 0$ the infinite singular points $M(0,u_1)$ and $N(0,u_2)$ are saddles. When $A < 0$ and $B > 0$ and also $B > A + \frac{1}{4}$, the infinite singular points $M(0,u_1)$ and $N(0,u_2)$ are centers. When $A < 0$ and $0 < B \leq \frac{1}{4} + A$, near $M(0,u_1)$ and $N(0,u_2)$, the phase portraits of (2.6) consists of one hyperbolic and elliptic sector.

Case 2. $A \neq 0, B = 0$. When $A < 0$, $D(0,0)$ is a saddle, when $A > 0$, it is a center. When $A > 0$ or $-\frac{1}{4} \leq A < 0$, $E(0,0)$ is a saddle, when $A < -\frac{1}{4}$, it is a center.

Case 3. $B \neq 0, A = 0$. When $B < 0$, $D(0,0)$ is a saddle, when $B > 0$, it is a center. When $B > 0$, $E(0,0)$ is a saddle, when $B < 0$, it is a center.

Case 4. $A = B = 0$. $D(0,0)$ is a center, $E(0,0)$ is a saddle.

Proof. **Case 1.** $A \cdot B \neq 0$. By transforming $x = \frac{1}{z}$, $y = \frac{u}{z}$, $\frac{d\tau}{dt} = \frac{1}{z^3}$, system (2.6) becomes

$$\begin{cases} \frac{du}{d\tau} = z^3(1+u^2), \\ \frac{dz}{d\tau} = u(z^4 - z^2 - Bu^2 - A). \end{cases} \tag{4.3}$$

If $A \cdot B > 0$, system (4.3) has an unique singular point $D(0,0)$. If $A \cdot B < 0$, system (4.3) has three singular points $D(0,0)$ and $M(0,u_1)$ and $N(0,u_2)$.

Taking $z = x, y = u(z^4 - z^2 - Au^2 - A)$, system (4.3) turns into

$$\begin{cases} x' = y, \\ y' = -Ax^3(1 + \frac{1}{A}x^2 - \frac{1}{A}x^4) + \frac{2}{A}y^2 x(1 + o(x)) + \dots . \end{cases}$$

By Lemma 4.1, it implies that when $A<0$, $D(0,0)$ is a saddle, when $A>0$, $D(0,0)$ is a center or focus. On the other hand, similar to the Theorem 2.1, we can get the first integral of system (4.3) as follows: when $B=A+\frac{1}{4}$, $\tilde{H}_1=H_1(\frac{1}{z},\frac{u}{z})$; when $B>A+\frac{1}{4}$, $\tilde{H}_2=H_2(\frac{1}{z},\frac{u}{z})$; when $B<A+\frac{1}{4}$, $\tilde{H}_3=H_3(\frac{1}{z},\frac{u}{z})$, where $H_i(i=1,2,3)$ are the same as they are in Theorem 2.1. By their phase portraits (see Figure 1), when $A>0$, the singular point $D(0,0)$ being a center is manifest.

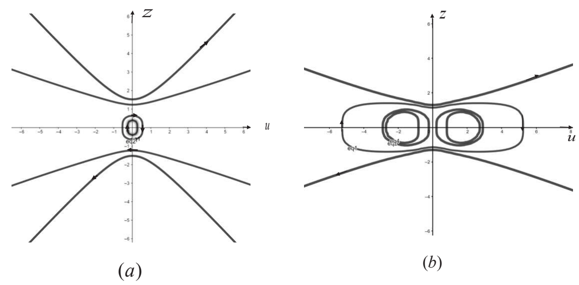

Figure 1.

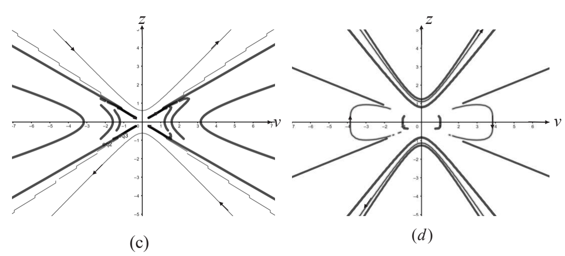

Figure 2.

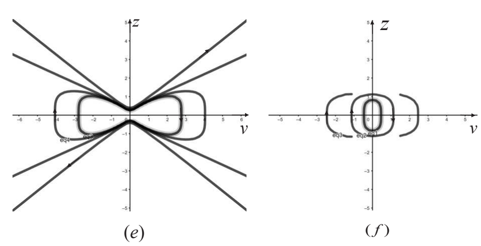

Figure 3.

Based on the similar analysis, we can get that when $B > 0$ the singular point $E(0,0)$ is a saddle, when $B < 0$, it is a center.

In the case $A \cdot B < 0$, taking $z = z, w = u - u_i$, then system (4.3) becomes

$$\begin{cases} w' = z^3(1 + (w + u_i)^2), \\ z' = (w + u_i)(z^4 - z^2 - Bw^2 - 2Bu_i w). \end{cases} \tag{4.4}$$

Putting $x = z$, $y = (w + u_i)(x^4 - x^2 - Bw^2 - 2Bu_i w)$, then (4.4) becomes

$$\begin{cases} x' = y, \\ y' = \frac{2A}{B}(B - A)x^3(1 + o(x)) - 2u_i xy(1 + o(x)) + \dots . \end{cases}$$

By Lemma 4.1, we see that when $A > 0$, $M(0, u_1)$, $N(0, u_2)$ are saddles. As $\lambda = 4u_i^2(1 - 4B + 4A)$, by this and Lemma 4.1, it implies that when $A < 0$ and $B > 0$ and also $B > A + \frac{1}{4}$, $M(0, u_1)$ and $N(0, u_2)$ are centers (see Figure 1(b)), when $A < 0$ and $0 < B \leq \frac{1}{4} + A$, near $M(0, u_1)$ and $N(0, -u_2)$, the phase portrait of (4.3) consists of one hyperbolic and elliptic sector.

Case 2. $A \neq 0, B = 0$. Similar to Case 1, by applying Lemma 4.1, we conclude that when $A < 0$, $D(0,0)$ is a saddle, when $A > 0$, $D(0,0)$ is a center (see Figure 3(f)).

By transforming $x = \frac{v}{z}$, $y = \frac{1}{z}$, $\frac{d\tau}{dt} = -\frac{1}{z^3}$, system (2.6) changes into

$$\begin{cases} \frac{dv}{d\tau} = z^3(1 + v^2), \\ \frac{dz}{d\tau} = v(z^4 + z^2 + Av^2). \end{cases} \tag{4.5}$$

Taking $v = r\cos\theta$, $z = r\sin\theta$, system (4.5) becomes

$$\begin{cases} r' = r^5 \cos\theta \sin^3\theta + r^3 R(\theta), \\ \theta' = r^2 U(\theta), \end{cases} \tag{4.6}$$

where $U(\theta) = (A - 2)\cos^2\theta + 3\cos^2\theta - 1$, $R(\theta) = \cos\theta\sin\theta(A\cos^2\theta + 2\sin^2\theta)$.

$\dot{1}$. If $\sigma_2 = 4A + 1 < 0$, then $U(\theta) < 0$, system (4.6) can be written as

$$\frac{dr}{d\theta} = \frac{1}{U(\theta)}(R(\theta)r + r^3\sin^3\theta\cos\theta), \tag{4.7}$$

its right side is continuously differentiable, and the coefficients satisfy the composition conditions [5, 7]. Thus, $r = 0$ is a composition center, i.e., $E(0,0)$ is a center (see Figure 3(e)).

$\dot{2}$. If $\sigma_2 = 4A+1 > 0, A < 0, U(\theta_i) = 0\,(i = 1, 2, 3, 4)$, $\theta_{1,2} = \arccos(\pm\sqrt{\frac{-3+\sqrt{\sigma_2}}{2(A-2)}})$, $\theta_{3,4} = \arccos(\pm\sqrt{\frac{-3-\sqrt{\sigma_2}}{2(A-2)}})$, and $R(\theta_1) > 0, R(\theta_2) < 0, R(\theta_3) > 0, R(\theta_4) < 0$, by this and applying the criterions of [27], there exists at least one trajectory tending to zero along the direction $\theta = \theta_1(\theta = \theta_3)$, when $t \longrightarrow -\infty$; there exists at least one trajectory tending to zero along the direction $\theta = \theta_2(\theta = \theta_4)$, when $t \longrightarrow +\infty$. Thus, $E(0,0)$ is a saddle (see Figure 2(c) and Figure 2(d)).

In the case of $A = -\frac{1}{4}$ or $A > 0$, similar to the above analysis, we conclude that $E(0,0)$ is a saddle, too.

Case 3. $A = 0, B \neq 0$. By transforming $x = \frac{1}{z}$, $y = \frac{u}{z}$, $\frac{d\tau}{dt} = \frac{1}{z^3}$, system (2.6) becomes

$$\begin{cases} \frac{du}{d\tau} = z^3(1+u^2), \\ \frac{dz}{d\tau} = u(z^4 - z^2 - Bu^2). \end{cases} \tag{4.8}$$

Using polar coordinates $u = r\cos\theta$, $z = r\sin\theta$ and transforming (4.8) into

$$\begin{cases} r' = r^3\cos\theta\sin\theta(-B\cos^2\theta + r^2\sin^2\theta), \\ \theta' = -r^2(B\cos^4\theta + \sin^2\theta). \end{cases}$$

i. If $B > 0$, $U(\theta) = B\cos^4\theta + \sin^2\theta > 0$, the $r = 0$ of equation

$$\frac{dr}{d\theta} = -\frac{r\cos\theta\sin\theta(r^2\sin^2\theta - B\cos^2\theta)}{B\cos^4\theta + \sin^2\theta}$$

is a composition center [5,7], i.e., $D(0,0)$ is a center.

2. If $B < 0$, $U(\theta_{1,2}) = 0$, $\theta_{1,2} = \arccos \pm(\sqrt{\frac{1-\sqrt{1-4B}}{2B}})$, by this and applying the criterions of [27], it follows that the $D(0,0)$ is a saddle.

Similar to Case 1, we obtain that when $B > 0$, $E(0,0)$ is a saddle, when $B < 0$, $E(0,0)$ is a center (see Figure 3(f)).

Case 4. $A = B = 0$. System (4.3) becomes

$$\begin{cases} \frac{du}{d\tau} = z(1+u^2), \\ \frac{dz}{d\tau} = u(z^2 - 1), \end{cases}$$

which has an analytic first integral

$$\frac{z^2-1}{1+u^2} = c,$$

and by this, we know that the singular point $D(0,0)$ is a center.

System (4.5) becomes

$$\begin{cases} \frac{dv}{d\tau} = z(1+v^2), \\ \frac{dz}{d\tau} = v(z^2 + 1), \end{cases}$$

and in polar coordinates, it can be written as the following

$$\begin{cases} \frac{dr}{d\tau} = r(2+r^2)\sin\theta\cos\theta, \\ \frac{d\theta}{d\tau} = \cos 2\theta. \end{cases}$$

Similar to Case 2, we see that the singular point $E(0,0)$ is a saddle. □

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