Journal of Nonlinear Modeling and Analysis
Volume 4, Number 3, September 2022, 475–487
http://jnma.ca; http://jnma.ijournal.cn
DOI:10.12150/jnma.2022.475

A Study on Oscillatory and Asymptotic Nature of Impulsive Neutral Differential Equations of Order Three*

Abdur Raheem[1,†], Asma Afreen[1] and Areefa Khatoon[1]

Abstract In this paper, we consider a class of third-order neutral impulsive differential equations. An equivalent class of neutral differential equations is obtained by using a suitable substitution. Some new oscillation results are proved. Moreover, we discuss the asymptotic behavior of the solution. The results presented here are illustrated via examples.

Keywords Neutral differential equations, Impulsive conditions, Oscillation criteria, Asymptotic behavior.

MSC(2010) 34K11, 34K40, 34G10, 34G45, 35R12.

1. Introduction

Here, we establish the oscillation results for the following model of neutral impulsive differential equation of order three:

$$\begin{cases} \Big(s_1(t)\upsilon^{(2)}(t)\Big)^{(1)} + s_2(t)u^{(1)}(\eta_2(t)) = 0, \quad t \neq t_p, \\ u^{(r)}(t_p) - u^{(r)}(t_p^-) = d_p u^{(r)}(t_p^-), \quad r = 0,1,2, \\ \qquad\qquad\qquad\qquad\qquad\qquad p = 1,2,3,\dots, \end{cases} \tag{1.1}$$

where $\upsilon(t) = u(t)+\alpha u(\eta_1(t))$, $\eta_1(t) \leq t$, $\eta_2(t) \leq t$, $t > t_0$, $\alpha > 0$, and $\upsilon^{(r)}(t)$ denotes the derivative of order r with respect to t.

It is well-known that the motions on the earth are not always uniform, as various kinds of resistance appear during the motions. For example, suppose high-intensity forces act for a short duration of time. In that case, motions caused by these forces are called impulsive motions, and the differential equations describing these motions are called impulsive differential equations.

The differential equations with impulsive effect can be used to simulate those discontinuous processes in which impulses occur. Therefore, it has become an important tool to handle the real process of mathematical models and phenomena

†the corresponding author.
Email address: araheem.iitk3239@gmail.com (A. Raheem), areefakhatoon@gmail.com (A. Khatoon), afreen.asma52@gmail.com (A. Afreen)
[1]Department of Mathematics, Aligarh Muslim University, Aligarh–202002, India
*The authors were supported by University Grant Commission (UGC), India (Grant Nos. MANF F.82-27/2019 (SA-III)/4453, F.82-27/2019 (SA-III)/191620066959).

such as optimal control, electric circuit, biotechnology, population dynamics, fractals, neural network, viscoelasticity and chemical technology. For more details on impulsive differential equations, please refer to [15]. One of the main advantages of the impulses can be seen in [23], as they provided the model in which the mass point might oscillate in the presence of impulsive effect and in the absence of impulsive effect, the mass point did not oscillate. For more work on impulsive effect, please refer to [7,14,19].

In 1989, some researchers started to investigate the oscillatory nature of differential equations with impulses, and they were at the initial stage of development. Later on, the authors of [7,9–11,18,23] extended the study of oscillation to parabolic and hyperbolic impulsive partial differential equations. A hybrid evolution system with impulsive conditions has been studied by Sadhasivam and Deepa [21]. In the last few decades, many researchers [2,4,13,20,22,25,29] have applied the Riccati technique to the study of the oscillatory behavior for various types of second-order differential equations. Some works on the oscillatory and asymptotic behavior of the solutions to higher-order impulsive differential equations have been carried out in [8,16,27]. Li [16] investigated the oscillatory and asymptotic behavior of the solutions to a higher-order delay differential equation with impulses by using comparison results with an associated non-impulsive delay differential equation.

Basic definitions and results on oscillation for neutral type differential equations can be found in [5,15]. Due to the wide applicability of neutral differential equations in various fields of science and engineering, there is a great interest in obtaining new oscillation criteria for a different types of differential equations (see, for instance, [2–4,6,12,13,17,22,25,26,29]). We have often seen that even non-impulsive neutral delay differential equations may have solutions of oscillatory nature due to some additional control. An improved sufficient condition for the oscillation and asymptotic stability has been obtained in this paper [26]. Guan and Shen [12] examined the oscillation criteria of a first-order impulsive differential equation with variable delays. Oscillation theorems for third-order delay differential equations were discussed by Tiryaki and Aktas [24]. Arul and Shobha [2] generalized neutral differential equations of order two and presented some new oscillation criteria by using Riccati transformation under some conditions.

Recently, Zhang and Li [28] have studied the oscillatory behaviour of the solutions to second-order impulsive neutral dynamic system with positive and negative coefficients. Moreover, in [22], the authors have presented some new necessary and sufficient conditions for the oscillation of a class of second-order neutral delay impulsive differential equations.

Motivated by all the above works, we obtain some new oscillation results of impulsive neutral differential equations of order three by converting them to the non-impulsive neutral differential equations. Also, there are only a few papers that deal with this technique.

The rest of this paper is organized as follows. Section 2 contains some basic lemmas and assumptions, which are required for the next sections. In Section 3, some new-type oscillation results are obtained for problem (1.1) by using Riccati transformation. In the last section, the results are illustrated by examples.

2. Preliminaries and assumptions

Throughout the paper, we consider the following assumptions:

(C1) The functions $s_r : (t_0, \infty) \to \mathbb{R}^+$, $r = 1, 2$ are continuous, and there exists $G > 0$ such that $s_1(t) \leq G$.

(C2) $\eta_r : (t_0, \infty) \to \mathbb{R}$, $r = 1, 2$ are continuous functions with the following conditions:

(i) $\eta_1(t) \leq t$, $\eta_2(t) \leq t$, $\eta_1(\eta_2(t)) = \eta_2(\eta_1(t))$;

(ii) $\eta_1^{(1)}(t) = \eta_2^{(1)}(t) = 1$;

(iii) $\lim_{t\to\infty} \eta_1(t) = \infty$, $\lim_{t\to\infty} \eta_2(t) = \infty$.

Lemma 2.1. $\zeta(t) = \prod_{t_0<t_p\leq t} (1 + d_p)^{-1} u(t)$ *satisfies*

$$\Big(s_1(t)V^{(2)}(t)\Big)^{(1)} + s_2(t) \prod_{\eta_2(t)<t_p\leq t} (1 + d_p)^{-1} \zeta^{(1)}(\eta_2(t)) = 0, \tag{2.1}$$

where

$$V(t) = \zeta(t) + \alpha \prod_{\eta_1(t)<t_p\leq t} (1 + d_p)^{-1} \zeta(\eta_1(t)), \tag{2.2}$$

if $u(t)$ *satisfies (1.1) on the interval* (t_0, ∞).

Proof. Let $\zeta(t) = \prod_{t_0<t_p\leq t} (1 + d_p)^{-1} u(t)$ satisfy (2.1). Then, we will show that $u(t)$ satisfies (1.1) on the interval (t_0, ∞).
Obviously,

$$u(t) = \prod_{t_0<t_p\leq t} (1 + d_p)\zeta(t) \text{ and } \upsilon(t) = \prod_{t_0<t_p\leq t} (1 + d_p)V(t).$$

Thus,

$$\Big(s_1(t)\upsilon^{(2)}(t)\Big)^{(1)} = \prod_{t_0<t_p\leq t} (1 + d_p)\Big(s_1(t)V^{(2)}(t)\Big)^{(1)}.$$

Using (2.1), we get

$$\begin{aligned}\Big(s_1(t)\upsilon^{(2)}(t)\Big)^{(1)} &= \prod_{t_0<t_p\leq t} (1 + d_p)\Big(- s_2(t) \prod_{\eta_2(t)<t_p\leq t} (1 + d_p)^{-1} \zeta^{(1)}(\eta_2(t))\Big) \\ &= -s_2(t) \prod_{t_0<t_p\leq \eta_2(t)} (1 + d_p)\zeta^{(1)}(\eta_2(t)).\end{aligned}$$

Therefore, for $t \neq t_p$, we have

$$\Big(s_1(t)\upsilon^{(2)}(t)\Big)^{(1)} + s_2(t)u^{(1)}(\eta_2(t)) = 0.$$

Also, we obtain

$$u^{(r)}(t) = \prod_{t_0<t_p\leq t} (1 + d_p)\zeta^{(r)}(t), \quad r = 0, 1, 2.$$

$$\implies u^{(r)}(t_p) = (1+d_p)u^{(r)}(t_p^-).$$

This shows that $u(t)$ satisfies (1.1).

Conversely, assume that $u(t) = \prod_{t_0<t_p\le t}(1+d_p)\zeta(t)$ satisfies (1.1). Then, we will show that $\zeta(t)$ satisfies (2.1) on (t_0,∞).

As $V(t) = \prod_{t_0<t_p\le t}(1+d_p)^{-1}\upsilon(t)$, we have

$$\Big(s_1(t)V^{(2)}(t)\Big)^{(1)} = \prod_{t_0<t_p\le t}(1+d_p)^{-1}\Big(s_1(t)\upsilon^{(2)}(t)\Big)^{(1)}.$$

Using (1.1), we obtain

$$\begin{aligned}\Big(s_1(t)V^{(2)}(t)\Big)^{(1)} &= -s_2(t)\prod_{t_0<t_p\le t}(1+d_p)^{-1}u^{(1)}(\eta_2(t))\\ &= -s_2(t)\prod_{\eta_2(t)<t_p\le t}(1+d_p)^{-1}\zeta^{(1)}(\eta_2(t)).\end{aligned}$$

On the other hand, we can easily show that $\zeta^{(r)}(t_p^-) = \zeta^{(r)}(t_p)$. This shows that $\zeta(t)$ satisfies (2.1). □

Lemma 2.2 ([16]). *A non-zero solution $u(t)$ of (1.1) is oscillatory on (t_0,∞), if the corresponding solution $\zeta(t) = \prod_{t_0<t_p\le t}(1+d_p)^{-1}u(t)$ of (2.1) is oscillatory on (t_0,∞). Moreover, $\lim_{t\to\infty} u^{(r)}(t)=0$, if $\lim_{t\to\infty}\zeta^{(r)}(t)=0, \quad r=0,1,2.$*

3. Main results

Theorem 3.1. *Suppose one of the following three conditions:*

$$\int_{t_0}^{\infty}\nu^2 s_2(\nu)\Upsilon(\nu)d\nu = -\infty, \tag{3.1}$$

$$\int_{t_0}^{t}J(\nu)\Upsilon(\nu)d\nu = \infty \tag{3.2}$$

and

$$\int_{t_0}^{t}\left(S_1(\nu)J(\nu)\Upsilon(\nu) - \frac{s_1(\nu)+\alpha s_1(\eta_1(\nu))}{4S_1(\nu)s_1^2(\nu)}\right)d\nu = \infty, \tag{3.3}$$

where

$$J(t) = \min\{s_2(t), s_2(\eta_1(t))\},\ \Upsilon(t) = \prod_{\eta_2(t)<t_p\le t}(1+d_p)^{-1},\ S_1(t) = \int_t^{\infty}\frac{d\nu}{s_1(\nu)}$$

holds. Then, all solutions $u(t)$ of (1.1) such that $u(t)u^{(1)}(t)>0$ or $u(t)u^{(1)}(t)<0$, for $t\ge\varrho$ are oscillatory.

Proof. On contrary, assume that $\zeta(t) \neq 0$ be a non-oscillatory solution of (2.1). Further, if we assume that $\zeta(t) > 0$ for $t \geq \varrho$, then the following three cases arise.

Case 1. If there exists $\varrho \geq t_0$ such that $\zeta(t) > 0$, $\zeta^{(1)}(t) < 0$, $\zeta^{(2)}(t) > 0$, or $V(t) > 0$, $V^{(1)}(t) < 0$, $V^{(2)}(t) > 0$, for $t \geq \varrho$. Since $\zeta^{(2)}(\eta_2(t)) > 0$, $\zeta^{(1)}(\eta_2(t))$ is increasing for $t \geq \varrho$. Therefore, for $t \geq \varrho$, from (2.1), we get

$$\begin{aligned}\Big(s_1(t)V^{(2)}(t)\Big)^{(1)} &= -s_2(t) \prod_{\eta_2(t)<t_p\leq t} (1+d_p)^{-1}\zeta^{(1)}(\eta_2(t))\\ &\leq -s_2(t) \prod_{\eta_2(t)<t_p\leq t} (1+d_p)^{-1}\zeta^{(1)}(\eta_2(\varrho))\\ &\leq -Ms_2(t)\Upsilon(t),\end{aligned}$$

where $M = \zeta^{(1)}(\eta_2(\varrho)) < 0$. Integrating ϱ and t after multiplying by ν^2, we get

$$\int_\varrho^t \nu^2\Big(s_1(\nu)V^{(2)}(\nu)\Big)^{(1)} d\nu \leq -M\int_\varrho^t \nu^2 s_2(\nu)\Upsilon(\nu)d\nu. \tag{3.4}$$

On the other hand,

$$\begin{aligned}\int_\varrho^t \nu^2\Big(s_1(\nu)V^{(2)}(\nu)\Big)^{(1)} d\nu &= \int_\varrho^t \nu^2 d\big(s_1(\nu)V^{(2)}(\nu)\big)\\ &= \Big[\nu^2 s_1(\nu)V^{(2)}(\nu)\Big]_\varrho^t - \int_\varrho^t 2\nu s_1(\nu)V^{(2)}(\nu)d\nu\\ &\geq \Big[\nu^2 s_1(\nu)V^{(2)}(\nu)\Big]_\varrho^t - 2G\int_\varrho^t \nu V^{(2)}(\nu)d\nu\\ &= \Big[\nu^2 s_1(\nu)V^{(2)}(\nu)\Big]_\varrho^t - 2G\Big[\nu V^{(1)}(\nu)\Big]_\varrho^t\\ &\quad +2G\int_\varrho^t V^{(1)}(\nu)d\nu\\ &= t^2 s_1(t)V^{(2)}(t) - \varrho^2 s_1(\varrho)V^{(2)}(\varrho)\\ &\quad -2G\big[tV^{(1)}(t) - \varrho V^{(1)}(\varrho)\big] + 2G\big[V(t) - V(\varrho)\big]\\ &\geq -\varrho^2 s_1(\varrho)V^{(2)}(\varrho) + 2G\varrho V^{(1)}(\varrho) - 2GV(\varrho).\end{aligned}$$

From (3.4), we have

$$-\int_\varrho^t \nu^2 s_2(\nu)\Upsilon(\nu)d\nu \leq \frac{1}{M}\big\{\varrho^2 s_1(\varrho)V^{(2)}(\varrho) - 2G\varrho V^{(1)}(\varrho) + 2GV(\varrho)\big\} < \infty.$$

On letting $t \to \infty$, we get a contradiction to condition (3.1).

Case 2. If there exists $\varrho' > t_0$ such that $\zeta(t) > 0$, $\zeta^{(1)}(t) > 0$, $\zeta^{(2)}(t) > 0$, or $V(t) > 0$, $V^{(1)}(t) > 0$, $V^{(2)}(t) > 0$, for $t \geq \varrho'$. From (2.1), we have

$$\begin{aligned}&\Big(s_1(t)V^{(2)}(t)\Big)^{(1)} + \alpha\Big(s_1(\eta_1(t))V^{(2)}(\eta_1(t))\Big)^{(1)} + J(t)\Upsilon(t)\\ &\times\Big[\zeta^{(1)}(\eta_2(t)) + \alpha \prod_{\eta_1(\eta_2(t))<t_p\leq\eta_2(t)} (1+d_p)^{-1}\zeta^{(1)}(\eta_1(\eta_2(t)))\Big] \leq 0.\end{aligned}$$

Using (2.2), for $t \geq \varrho'$, we obtain

$$\Big(s_1(t)V^{(2)}(t)\Big)^{(1)} + \alpha\Big(s_1(\eta_1(t))V^{(2)}(\eta_1(t))\Big)^{(1)} + J(t)\Upsilon(t)V^{(1)}(\eta_2(t)) \leq 0. \quad (3.5)$$

Using the fact $V^{(1)}(\eta_2(t)) \geq c > 0$ and integrating from ϱ' to t, we get

$$\int_{\varrho'}^{t} J(\nu)\Upsilon(\nu)d\nu < \infty,$$

which is a contradiction to condition (3.2).

Case 3. If there exists $\varrho'' \geq t_0$ such that $\zeta(t) > 0$, $\zeta^{(1)}(t) > 0$, $\zeta^{(2)}(t) < 0$, or $V(t) > 0$, $V^{(1)}(t) > 0$, $V^{(2)}(t) < 0$, for $t \geq \varrho''$.
Define

$$\chi(t) = \frac{s_1(t)V^{(2)}(t)}{V^{(1)}(t)}, \quad t \geq \varrho''. \quad (3.6)$$

Since $s_1(\nu)V^{(2)}(\nu)$ is decreasing, for $\nu \geq t$, we have

$$s_1(\nu)V^{(2)}(\nu) \leq s_1(t)V^{(2)}(t).$$
$$\implies V^{(2)}(\nu) \leq s_1(t)\frac{V^{(2)}(t)}{s_1(\nu)}.$$

Integrating from t to μ, we get

$$V^{(1)}(\mu) - V^{(1)}(t) \leq s_1(t)V^{(2)}(t)\int_t^{\mu} \frac{d\nu}{s_1(\nu)}, \quad \mu \geq t \geq \varrho''.$$

Taking limit as $\mu \to \infty$, we get

$$\frac{s_1(t)V^{(2)}(t)}{V^{(1)}(t)}S_1(t) > -1, \quad t \geq \varrho''.$$
$$\implies -1 < \chi(t)S_1(t) < 0, \quad t \geq \varrho''. \quad (3.7)$$

On the other hand, we define

$$\kappa(t) = \frac{s_1(\eta_1(t))V^{(2)}(\eta_1(t))}{V^{(1)}(t)}, \quad t \geq \varrho''. \quad (3.8)$$

Since $s_1(t)V^{(2)}(t)$ is decreasing, we have $s_1(t)V^{(2)}(t) \leq s_1(\eta_1(t))V^{(2)}(\eta_1(t))$. Then, $\chi(t) \leq \kappa(t)$. From (3.7), we have

$$-1 < \kappa(t)S_1(t) < 0, \quad t \geq \varrho''. \quad (3.9)$$

Differentiating (3.6), we get

$$\chi^{(1)}(t) = \frac{\Big(s_1(t)V^{(2)}(t)\Big)^{(1)}}{V^{(1)}(t)} - \frac{s_1(t)V^{(2)}(t)}{(V^{(1)}(t))^2}V^{(2)}(t)$$
$$= \frac{\Big(s_1(t)V^{(2)}(t)\Big)^{(1)}}{V^{(1)}(t)} - \frac{\chi^2(t)}{s_1(t)}, \quad t \geq \varrho''. \quad (3.10)$$

Differentiating (3.8), we get

$$\kappa^{(1)}(t) \leq \frac{\Big(s_1(\eta_1(t))V^{(2)}(\eta_1(t))\Big)^{(1)}}{V^{(1)}(t)} - \frac{Y^2(t)}{s_1(\eta_1(t))}, \quad t \geq \varrho''. \tag{3.11}$$

Using (3.10) and (3.11), we get

$$\begin{aligned}\chi^{(1)}(t) + \alpha\kappa^{(1)}(t) \leq & \frac{\Big(s_1(t)V^{(2)}(t)\Big)^{(1)}}{V^{(1)}(t)} + \alpha\frac{\Big(s_1(\eta_1(t))V^{(2)}(\eta_1(t))\Big)^{(1)}}{V^{(1)}(t)} \\ & -\frac{\chi^2(t)}{s_1(t)} - \alpha\frac{\kappa^2(t)}{s_1(\eta_1(t))}.\end{aligned}$$

Using (3.5), we get

$$\chi^{(1)}(t) + \alpha\kappa^{(1)}(t) \leq -J(t)\Upsilon(t) - \frac{\chi^2(t)}{s_1(t)} - \alpha\frac{\kappa^2(t)}{s_1(\eta_1(t))}.$$

Integrating the above inequality from ϱ'' to t after multiplying by $S_1(t)$, we get

$$\begin{aligned}\int_{\varrho''}^{t} S_1(\nu)\Big[\chi^{(1)}(\nu) + \alpha\kappa^{(1)}(\nu)\Big]d\nu \leq & -\int_{\varrho''}^{t} S_1(\nu)J(\nu)\Upsilon(\nu)d\nu \\ & -\int_{\varrho''}^{t} S_1(\nu)\left(\frac{\chi^2(\nu)}{s_1(\nu)} + \alpha\frac{\kappa^2(\nu)}{s_1(\eta_1(\nu))}\right)d\nu.\end{aligned}$$

After some simplification, by using (3.7) and (3.9), we obtain

$$\begin{aligned}-(1+\alpha) + \int_{\varrho''}^{t}\left(\frac{S_1(\nu)\chi^2(\nu)}{s_1(\nu)} + \frac{\chi(\nu)}{s_1(\nu)}\right)d\nu - S_1(\varrho'')\big(\chi(\varrho'') + \alpha\kappa(\varrho'')\big) \\ +\alpha\int_{\varrho''}^{t}\left(\frac{S_1(\nu)\kappa^2(\nu)}{s_1(\eta_1(\nu))} + \frac{\kappa(\nu)}{s_1(\nu)}\right)d\nu < -\int_{\varrho''}^{t} S_1(\nu)J(\nu)\Upsilon(\nu)d\nu.\end{aligned}$$

Using the inequality $\lambda AB^{\lambda-1} - A^{\lambda} \leq (\lambda-1)B^{\lambda},\ \lambda \geq 1$, we obtain

$$\begin{aligned}\int_{\varrho''}^{t}\left(S_1(\nu)J(\nu)\Upsilon(\nu) - \frac{s_1(\nu) + \alpha s_1(\eta_1(\nu))}{4S_1(\nu)s_1^2(\nu)}\right)d\nu \\ \leq (1+\alpha) + S_1(\varrho'')\Big(\chi(\varrho'') + \alpha\kappa(\varrho'')\Big) < \infty,\end{aligned}$$

which is a contradiction to condition (3.3). Applying Lemma 2.2, the result follows. □

Theorem 3.2. *Let $u(t)$ be an eventually positive solution of (1.1) such that $u^{(1)}(t) > 0$, $u^{(2)}(t) < 0$, and there exists a constant $K > 0$ such that*

$$\prod_{t_0 < t_p \leq t}(1 + d_p)^{-1} \leq K.$$

Further, if

$$\int_t^{\infty} \frac{1}{s_1(\nu_1)} \left[\int_{\varrho}^{\nu_1} s_2(\nu) d\nu \right] d\nu_1 = \infty, \tag{3.12}$$

then $\lim_{t\to\infty} u^{(1)}(t) = 0$.

Proof. Since $u(t)$ has the property $u(t) > 0$, $u^{(1)}(t) > 0$, $u^{(2)}(t) < 0$, or $\zeta(t) > 0$, $\zeta^{(1)}(t) > 0$, $\zeta^{(2)}(t) < 0$, and consequently $V(t) > 0$, $V^{(1)}(t) > 0$, $V^{(2)}(t) < 0$, for $t \geq \varrho$. Let $\lim_{t\to\infty} V^{(1)}(t) = A \geq 0$. Then, we need to prove that $A = 0$. On contrary, assume $A > 0$. Since $V^{(1)}(t)$ is decreasing for $t \geq \varrho$, we have $A + \epsilon > V^{(1)}(t) > A$, for all $\epsilon > 0$. Choosing $0 < \epsilon < \frac{A(1-\alpha K)}{\alpha K}$, it is easy to see that

$$\begin{aligned}
\zeta^{(1)}(t) &= V^{(1)}(t) - \alpha \prod_{\eta_1(t) < t_p \leq t} (1 + d_p)^{-1} \zeta^{(1)}(\eta_1(t)) \\
&\geq A - \alpha \prod_{\eta_1(t) < t_p \leq t} (1 + d_p)^{-1} V^{(1)}(\eta_1(t)) \\
&\geq A - \alpha \prod_{\eta_1(t) < t_p \leq t} (1 + d_p)^{-1} (A + \epsilon) \\
&\geq A - \alpha K(A + \epsilon) = P(A + \epsilon) > PV^{(1)}(t),
\end{aligned} \tag{3.13}$$

where $P = \frac{A - \alpha K(A+\epsilon)}{A+\epsilon} > 0$.

Therefore, from (2.1), we have

$$\begin{aligned}
\Big(s_1(t)V^{(2)}(t)\Big)^{(1)} &= -s_2(t) \prod_{\eta_2(t) < t_p \leq t} (1 + d_p)^{-1} \zeta^{(1)}(\eta_2(t)) \\
&\leq -s_2(t) KPV^{(1)}(t) \\
&\leq -s_2(t) KPA.
\end{aligned}$$

Integrating from ϱ to t, we get

$$\begin{aligned}
s_1(t)V^{(2)}(t) &\leq -KPA \int_{\varrho}^{t} s_2(\nu) d\nu. \\
\implies V^{(2)}(t) &\leq -\frac{KPA}{s_1(t)} \int_{\varrho}^{t} s_2(\nu) d\nu.
\end{aligned}$$

Again, integrating from t to ∞, we get

$$-V^{(1)}(t) \leq -KPA \int_t^{\infty} \frac{1}{s_1(\nu_1)} \left[\int_{\varrho}^{\nu_1} s_2(\nu) d\nu \right] d\nu_1.$$

$$\implies V^{(1)}(t) \geq KPA \int_t^{\infty} \frac{1}{s_1(\nu_1)} \left[\int_{\varrho}^{\nu_1} s_2(\nu) d\nu \right] d\nu_1,$$

which contradicts condition (3.12). Hence, $\lim_{t\to\infty} V^{(1)}(t) = 0$.

Since $\zeta^{(1)}(t) \leq V^{(1)}(t)$, we have

$$\lim_{t\to\infty} \zeta^{(1)}(t) = 0.$$

Therefore, by applying Lemma 2.2, we get

$$\lim_{t\to\infty} u^{(1)}(t) = 0.$$

□

4. Applications

In this section, we provide some examples to interpret the main results.

Example 4.1. Consider the given system

$$\begin{cases} \Big(t^{-3}\big(u(t)+\alpha u(t-\pi)\big)^{(2)}\Big)^{(1)} + t^{-4}u^{(1)}(t-2\pi) = 0, \quad t \neq t_p, \\ u^{(r)}(t_p) - u^{(r)}(t_p^-) = \frac{1}{p}u^{(r)}(t_p^-), \quad r = 0,1,2 \quad p = 1,2,3,\dots. \end{cases} \tag{4.1}$$

Here, $s_1(t) = t^{-3}$, $s_2(t) = t^{-4}$, $\eta_1(t) = t-\pi$, $\eta_2(t) = t-2\pi$,
$d_p = \frac{1}{p}$, $t_p = p\pi$, $\Upsilon(t) = \prod_{t-2\pi<t_p\leq t}\left(\frac{p}{p+1}\right)$.

We see that

$$\begin{aligned} \int_{t_0}^{\infty} \nu^2 s_2(\nu)\Upsilon(\nu)d\nu &= \int_{t_0}^{\infty} \nu^{-2} \prod_{\nu-2\pi<t_p\leq\nu}\Big(\frac{p}{p+1}\Big)d\nu \\ &= \int_{t_0}^{t_1} \nu^{-2} \prod_{\nu-2\pi<t_p\leq\nu}\Big(\frac{p}{p+1}\Big)d\nu + \int_{t_1}^{t_2} \nu^{-2} \prod_{\nu-2\pi<t_p\leq\nu}\Big(\frac{p}{p+1}\Big)d\nu \\ &\quad + \int_{t_2}^{t_3} \nu^{-2} \prod_{\nu-2\pi<t_p\leq\nu}\Big(\frac{p}{p+1}\Big)d\nu + \cdots \\ &= \frac{1}{2}t_0 - \pi\left[\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots\right] \\ &= -\infty. \end{aligned}$$

Thus, condition (3.1) is satisfied. If $u(t) > 0$, $u^{(1)}(t) < 0$, $u^{(2)}(t) > 0$, for $t \geq \varrho$, then by using Case (1) of Theorem 3.1, every non-zero solution $u(t)$ of system (4.1) such that $u(t)u^{(1)}(t) > 0$ or $u(t)u^{(1)}(t) < 0$ is oscillatory.

Example 4.2. Consider the following system

$$\begin{cases} \Big(t^{-\frac{1}{2}}\big(u(t)+\alpha u(t-\pi)\big)^{(2)}\Big)^{(1)} + tu^{(1)}(t-2\pi) = 0, \quad t \neq t_p, \\ u^{(r)}(t_p) - u^{(r)}(t_p^-) = \frac{1}{p}u^{(r)}(t_p^-), \quad r = 0,1,2, \quad p = 1,2,3,\dots. \end{cases} \tag{4.2}$$

Here, $s_1(t) = \dfrac{1}{\sqrt{t}}$, $s_2(t) = t$, $\eta_1(t) = t-\pi$, $\eta_2(t) = t-2\pi$,
$d_p = \frac{1}{p}$, $t_p = p\pi$, $J(t) = \min\{t, t-\pi\} = t-\pi$,

$$\Upsilon(t) = \prod_{\eta_2(t)<t_p\le t}(1+d_p)^{-1} = \prod_{t-2\pi<t_p\le t}\Big(\tfrac{p}{p+1}\Big), \quad S_1(t) = \int_t^\infty \sqrt{\nu}d\nu = \infty.$$

We see that

$$\begin{aligned}\int_{t_0}^\infty J(\nu)\Upsilon(\nu)d\nu &= \int_{t_0}^\infty (\nu-\pi)\prod_{\nu-2\pi<t_p\le\nu}\Big(\frac{p}{p+1}\Big)d\nu\\ &= \frac{1}{2}\int_{t_0}^{t_1}(\nu-\pi)d\nu + \frac{1}{3}\int_{t_1}^{t_2}(\nu-\pi)d\nu + \frac{1}{4}\int_{t_2}^{t_3}(\nu-\pi)d\nu + \cdots\\ &= \frac{1}{2}\Big[\frac{(t_1-\pi)^2}{2} - \frac{(t_0-\pi)^2}{2}\Big] + \frac{1}{3}\Big[\frac{(t_2-\pi)^2}{2} - \frac{(t_1-\pi)^2}{2}\Big]\\ &\quad + \frac{1}{4}\Big[\frac{(t_3-\pi)^2}{2} - \frac{(t_2-\pi)^2}{2}\Big]\\ &= \infty.\end{aligned}$$

Thus, condition (3.2) is satisfied. If $u(t) > 0$, $u^{(1)}(t) > 0$, $u^{(2)}(t) > 0$, for $t \ge \varrho$, then by using Case (2) of Theorem 3.1, every non-zero solution $u(t)$ of system (4.1) such that $u(t)u^{(1)}(t) > 0$ or $u(t)u^{(1)}(t) < 0$ is oscillatory.
Further, we can easily check that

$$\int_{t_0}^\infty \Big(S_1(\nu)J(\nu)\Upsilon(\nu) - \frac{s_1(\nu)+\alpha s_1(\eta_1(\nu))}{4S_1(\nu)s_1^2(\nu)}\Big)d\nu = \infty.$$

Thus, condition (3.3) is satisfied. If $u(t) > 0$, $u^{(1)}(t) > 0$, $u^{(2)}(t) < 0$, for $t \ge \varrho$, then by using Case (3) of Theorem 3.1, every non-zero solution $u(t)$ of system (4.1) such that $u(t)u^{(1)}(t) > 0$ or $u(t)u^{(1)}(t) < 0$ is oscillatory.

Example 4.3. Consider the given system

$$\begin{cases}\Big(t^{-1}\big(u(t)+\alpha u(t-\pi)\big)^{(2)}\Big)^{(1)} + t^{-4}u^{(1)}(t-2\pi) = 0, \quad t \ne t_p,\\ u^{(r)}(t_p) - u^{(r)}(t_p^-) = \Big(\frac{1+2p}{p^2}\Big)u^{(r)}(t_p^-), \quad r = 0,1,2 \quad p = 1,2,3,\dots.\end{cases} \tag{4.3}$$

Here, $s_1(t) = t^{-1}$, $s_2(t) = t^{-4}$, $\eta_1(t) = t-\pi$, $\eta_2(t) = t-2\pi$,
$d_p = \dfrac{1+2p}{p^2}$, $t_p = p\pi$.
Obviously,

$$\prod_{t_0<t_p\le t}\Big(\frac{p}{p+1}\Big)^2 \le K, \quad \text{for some} \quad K > 0.$$

We see that

$$\int_t^\infty \frac{1}{s_1(\nu_1)}\Big[\int_\varrho^{\nu_1} s_2(\nu)d\nu\Big]d\nu_1 = \infty.$$

Thus, condition (3.12) is satisfied. If $u(t) > 0$, $u^{(1)}(t) > 0$, $u^{(2)}(t) > 0$, for $t \ge \varrho$, then by using Theorem 3.2, we get $\lim_{t\to\infty} u^{(1)}(t) = 0$.

5. Conclusion

In this work, we have established some sufficient conditions for the oscillation of solutions for a class of neutral impulsive differential equations of order three, and discussed the asymptotic behavior of solutions. The results are demonstrated by examples.

Acknowledgements

The authors would like to thank the reviewer and editors for their valuable suggestions.

References

[1] S. H. Abdallah, *Oscillatory and non-oscillatory behaviour of second-order neutral delay differential equations*, Applied Mathematics and Computation, 2003, 135(2–3), 333–344.

[2] R. Arul and V. S. Shobha, *Oscillation of Second Order Nonlinear Neutral Differential Equations with Mixed Neutral Term*, Journal of Applied Mathematics and Physics, 2015, 3(9), 1080–1089.

[3] B. Baculíková and J. Dzurina, *Oscillation of third-order neutral differential equations*, Mathematical and Computer Modelling, 2010, 52(1–2), 215–226.

[4] B. Baculíková and J. Dzurina, *Oscillation theorems for second-order non linear neutral differential equations*, Computers & Mathematics with Applications, 2011, 62(12), 4472–4478.

[5] D. D. Bainov and D. P. Mishev, *Oscillation Theory for Neutral Differential Equations with Delay*, IOP Publishing Ltd, New York, 1991.

[6] G. E. Chatzarakis and S. R. Grace, *Oscillation of 2nd-order Nonlinear Noncanonical Difference Equations with Deviating Argument*, Journal of Nonlinear Modeling and Analysis, 2021, 3(4), 495–504.

[7] L. Feng, Y. Sun and Z. Han, *Philos-type oscillation criteria for impulsive fractional differential equations*, Journal of Applied Mathematics and Computing, 2020, 62(1–2), 361–376.

[8] X. Fu and X. Li, *Oscillation of higher order impulsive differential equations of mixed type with constant argument at fixed time*, Mathematical and Computer Modelling, 2008, 48(5–6), 776–786.

[9] X. Fu, X. Liu and S. Sivalogannathan, *Oscillation criteria for impulsive parabolic systems*, Applicable Analysis, 2001, 79(1–2), 239–255.

[10] X. Fu, X. Liu and S. Sivaloganathan, *Oscillation criteria for impulsive parabolic differential equations with delay*, Journal of Mathematical Analysis and Applications, 2002, 268(2), 647–664.

[11] X. Fu and L. Zhang, *Forced oscillation for impulsive hyperbolic boundary value problems with delay*, Applied Mathematics and Computation, 2004, 158(3), 761–780.

[12] K. Guan and J. Shen, *Oscillation criteria for a first-order impulsive neutral differential equation of Euler form*, Computers & Mathematics with Applications, 2009, 58(4), 670–677.

[13] Z. Han, T. Li, S. Sun and W. Chen, *On the oscillation of second order neutral delay differential equation*, Advances in Difference Equations, 2010, Article ID 289340, 8 pages.

[14] F. Jiang, *Existence of Periodic Solutions in Impulsive Differential Equations*, Journal of Nonlinear Modeling and Analysis, 2021, 3(1), 53–70.

[15] V. Lakshmikantham, D. D. Bainov and P. S. Simeonov, *Theory of Impulsive Differential Equations*, World Scientific, Singapore, 1989.

[16] X. Li, *Oscillation properties of higher order impulsive delay differential equations*, International Journal of Differential Equations, 2007, 2(2), 209–219.

[17] X. Lina and X. H. Tangb, *Oscillation of solutions of neutral differential equations with a superlinear neutral term*, Applied Mathematics Letters, 2007, 20(9), 1016–1022.

[18] J. Luo, *Oscillation of hyperbolic partial differential equations with impulses*, Applied Mathematics and Computation, 2002, 133(2–3), 309–318.

[19] A. Raheem and Md. Maqbul, *Oscillation criteria for impulsive partial fractional differential equations*, Computers & Mathematics with Applications, 2017, 73(8), 1781–1788.

[20] S. Ruan, *Oscillations of Second Order Neutral Differential Equations*, Canadian Mathematical Bulletin, 1993, 36(4), 485–496.

[21] V. Sadhasivam and M. Deepa, *Oscillation criteria for fractional impulsive hybrid partial differential equations*, Problemy Analiza. Issues of Analysis, 2019, 8(26), 2, 73–91.

[22] S. S. Santra, A. Ghosh, O. Bazighifan, K. M. Khedher and T. A. Nofal, *Second-order impulsive differential systems with mixed and several delays*, Advances in Difference Equations, 2021, 318, 12 pages.

[23] J. Sugie and K. Ishihara, *Philos-type oscillation criteria for linear differential equations with impulsive effects*, Journal of Mathematical Analysis and Applications, 2019, 470(2), 911–930.

[24] A. Tiryaki and M. F. Aktas, *Oscillation criteria of a certain class of third order nonlinear delay differential equations with damping*, Journal of Mathematical Analysis and Applications, 2007, 325(1), 54–68.

[25] R. Xu and F. Meng, *New Kamenev-type oscillation criteria for second order neutral nonlinear differential equations*, Applied Mathematics and Computation, 2007, 188(2), 1364–1370.

[26] J. Yan, *Oscillation of nonlinear delay impulsive differential equations and inequalities*, Journal of Mathematical Analysis and Applications, 2002, 265(2), 332–342.

[27] C. Zhang and W. Feng, *Oscillation for higher order nonlinear ordinary differential equations with impulses*, Electronic Journal of Differential Equations, 2006, 18, 12 pages.

[28] S. Zhang and Q. Li, *Oscillatory Properties for Second-Order Impulsive Neutral Dynamic Equations with Positive and Negative Coefficients on Time Scales*, Journal of Mathematics, 2021, Article ID 3980250, 7 pages.

[29] R. Zhuang and W. Li, *Interval oscillation criteria for second order neutral nonlinear differential equations*, Applied Mathematics and Computation, 2004, 157(1), 39–51.