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Hopf Bifurcation Analysis of a Class of Abstract Delay Differential Equation*

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Abstract The dynamics of a class of abstract delay differential equations are investigated. We prove that a sequence of Hopf bifurcations occur at the origin equilibrium as the delay increases. By using the theory of normal form and centre manifold, the direction of Hopf bifurcations and the stability of the bifurcating periodic solutions is derived. Then, the existence of the global Hopf bifurcation of the system is discussed by applying the global Hopf bifurcation theorem of general functional differential equation.

Keywords Hopf bifurcation, Delay, Stability, Normal form, Periodic solution.

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1. Introduction

Since the last century, people have successively proposed a large number of delay differential equations problems in many fields of natural and social sciences such as galaxy evolution [9], optics [3, 20], nuclear physics [1, 31], chemical circulation systems, neural networks [33], population dynamics [28], ecosystems [24], infectious diseases [2, 17, 30], etc. For example, population growth model

$$N'(t) = K\left(1 - \frac{N(t-\tau)}{p}\right)N(t) \tag{1.1}$$

is a nonlinear delay differential equation. Neutral delay differential equation proposed in the study of energy loss in power networks

$$\dot{x}(t) = A\dot{x}(t-\tau) + Bx(t) - Cx(t-\tau) \tag{1.2}$$

is also a very typical example. The proposition of these problems has aroused increasing attention on the study of differential equations with delay. Similar arguments can be found in [6, 10, 11].

Before the 1930s, the research content of functional differential equations was limited to the special properties of some special types of equations. Volterra [25, 26]

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used the relationship between the functional differential range and some physical systems to define the energy function to observe the asymptotic behavior of system in a short time. This was a milestone in the development of the theoretical system of functional differential equations. With the in-depth study of these problems, the theory of functional differential equations has been continuously improved, and many monographs on the theory of functional differential equations have appeared. Bellman and Danskin [3], Bellman and Cooke [18] put forward the stability theorem of the linear difference differential equation of constant system, Krasovskii [12] also gave. Hale [7] explained the theory of functional differential equations more comprehensively from the aspects of stability, boundedness, periodic solutions, vibration and asymptotic properties, and almost periodic solutions. Further, Hale and Lunel published the Introduction to Functional Differential Equations in 1993, which made a good summary of the research on finite delay functional differential equations.

Delay differential equation is an important branch of differential equation. It is a kind of differential equation whose derivative function of time depends on the value of the solution at the past time point [22]. It is used to describe the motion phenomenon related to the state of motion and historical time. Specifically, the differential equation describing the development process of a specific system stated over time of the objective world is called a differential dynamic system. If the development of the system state depends not only on the current state, but also on the state of the system at certain moments or time segments in the past, this type of dynamic system is called a time-delay differential dynamic system [5].

Generally speaking, delay differential systems have more complex dynamic properties than corresponding ordinary differential systems. This is because the time lag can change the stability of the equilibrium point of the system and lead to the occurrence of Hopf branching and chaos. Therefore, it is a very meaningful subject to study the influence of time delay on the dynamics of the system. In fact, there is a wide literature on the dynamic systems with time delay, we refer the readers to [8, 16, 19, 21, 23, 32] respectively and references therein.

A two-agent opinion dynamical system with processing delay

$$\begin{cases} \dot{x}_1(t) = \frac{1}{2}\alpha a_{12}\left(x_2(t-\tau) - x_1(t-\tau)\right) \\ \dot{x}_2(t) = \frac{1}{2}\alpha a_{21}\left(x_1(t-\tau) - x_2(t-\tau)\right) \end{cases} \tag{1.3}$$

is discussed in [29]. The author transforms the dynamic problem into a kind of delay differential equation

$$\dot{x}(t) = \alpha p(x(t-\tau)), \tag{1.4}$$

and analyze the asymptotic stability of its origin. Wei [15] studied the dynamic properties of a scalar delay differential equation

$$\dot{x}(t) = -\gamma x(t) + \beta f(x(t-\tau)). \tag{1.5}$$

Equation (1.5) proved that the Hopf bifurcation sequence occurs with the increase of time delay at the equilibrium point. Furthermore, the results of the existence of the global Hopf bifurcation are studied, and the global existence of multiple periodic solutions is established.

Inspired by the above, this paper combines the system equations in [29] and [15], and considers the following abstract differential equations with time delay:

$$\begin{cases} \dot{x}_1(t) = \alpha f_1\left(x_2(t-\tau)\right), \\ \dot{x}_2(t) = -\gamma x_2(t) + \beta f_2\left(x_1(t-\tau)\right), \end{cases} \tag{1.6}$$

where τ is a parameter.

This paper is divided into four sections: In Section 2, we investigate the equilibrium and the occurrence of Hopf bifurcations. In Section 3, we derive sufficient conditions for the stability and the direction of bifurcating periodic solutions. A global Hopf bifurcation is established in Section 4.

2. Local existence of periodic solutions

In this section, we study the stability of the equilibrium and the existence of local Hopf bifurcations. Firstly, we make the following assumptions on (1.6).

(H_1) $f_i \in C^3$, $i = 1, 2$. $f_i(0) = 0$, and $f_i(x) \neq 0$ for $x \in N$ and $x \neq 0$.

(H_2) $\alpha, \beta, \gamma > 0$ and $\tau \geq 0$.

Under the assumption (H_1), origin is a fixed point to equation (1.6). Linearizing the equation around origin gives

$$\begin{cases} \dot{x}_1(t) = \alpha f_1'(0) x_2(t-\tau), \\ \dot{x}_2(t) = -\gamma x_2(t) + \beta f_2'(0) x_1(t-\tau). \end{cases} \tag{2.1}$$

In other words,

$$\begin{cases} \dot{x}_1(t) = a_1 x_2(t-\tau), \\ \dot{x}_2(t) = -\gamma x_2(t) + a_2 x_1(t-\tau), \end{cases} \tag{2.2}$$

where $a_1 = \alpha f_1'(0)$, $a_2 = \beta f_2'(0)$.

The characteristic equation associated with (2.2) is

$$\begin{bmatrix} \lambda & -a_1 e^{-\lambda\tau} \\ -a_2 e^{-\lambda\tau} & \lambda + \gamma \end{bmatrix} = 0. \tag{2.3}$$

That is,

$$\lambda(\lambda + \gamma) - a_1 a_2 e^{-2\lambda\tau} = 0. \tag{2.4}$$

For $\tau = 0$, we have

$$\lambda(\lambda + \gamma) - a_1 a_2 = 0. \tag{2.5}$$

Equation (2.5) has two roots

$$\lambda_{1,2} = \frac{1}{2}\left[-\gamma \pm \left(\gamma^2 + 4a_1a_2\right)^{\frac{1}{2}}\right].$$

Thus, equation (2.5) has strictly negative real part, if and only if

$$a_1 a_2 < -\frac{\gamma^2}{4}.$$

Lemma 2.1. *Assume that*

$$(H_3)\, a_1a_2 < -\frac{\gamma^2}{4}$$

hold, all the roots of equation (2.5) has negative real part, which is the system (1.6) is asymptotically stable.

For $\tau > 0$, $i\omega(\omega > 0)$ is a root, if and only if

$$i\omega(i\omega+\gamma) - a_1a_2e^{-2\lambda\tau} = i\omega(i\omega+\gamma) - a_1a_2(\cos 2\omega\tau - i\sin 2\omega\tau) = 0.$$

Separating the real and imaginary parts, we get

$$-\omega^2 = a_1a_2\cos 2\omega\tau,\ \omega\gamma = -a_1a_2\sin 2\omega\tau, \tag{2.6}$$

which leads to

$$\omega^4 + \omega^2\gamma^2 = a_1^2a_2^2. \tag{2.7}$$

Let $u = \omega^2$, then equation (2.7) becomes

$$u^2 + u\gamma^2 - a_1^2a_2^2 = 0.$$

Obviously, this equation has two real roots

$$u_{1,2} = \frac{-\gamma^2 \pm \sqrt{\gamma^4 + 4a_1^2a_2^2}}{2}.$$

Meanwhile, $u_1 > 0$, $u_2 < 0$. Therefore, (2.7) has a real root

$$\omega_0 = \sqrt{\frac{-\gamma^2 + \sqrt{\gamma^4 + 4a_1^2a_2^2}}{2}}.$$

Furthermore,

$$\left|\frac{-\omega^2}{a_1a_2}\right| < 1.$$

Let

$$\tau_k = \frac{1}{2\omega_0}\left(\arccos\frac{-\omega_0^2}{a_1a_2} + 2k\pi\right), \tag{2.8}$$

where $k = 0, 1, 2, \cdots$. Denote $F(\lambda,\tau) = \lambda^2 + \gamma\lambda - a_1a_2e^{-2\lambda\tau}$. If equation (2.4) has not a pair of simple imaginary roots $\pm i\omega$, we have

$$\left.\frac{\partial F}{\partial \lambda}\right|_{\tau=r_k} = \left(2\lambda + \gamma + 2\tau a_1a_2e^{-2\lambda\tau}\right)\Big|_{\tau=\tau_k} = 0.$$

Combining with (2.4),

$$\begin{cases} 2i\omega_0 + \gamma + 2\tau_k a_1a_2e^{-2i\omega_0\tau_k} = 0, \\ -\omega_0^2 + i\omega_0\gamma - a_1a_2e^{-2i\omega_0\tau_k} = 0. \end{cases}$$

Hence,

$$(2+\gamma)\omega_0 = 0.$$

This is contradictory. Therefore, equation (2.4) has a pair of simple imaginary roots $\pm i\omega$ when $\tau = \tau_k$. Let $\lambda_k(\tau) = \alpha_k(\tau) + i\omega_k(\tau)$ denote a root of (2.4) near $\tau = \tau_k$, satisfying $\alpha_k(\tau_k) = 0$ and $\omega_k(\tau_k) = \omega_0$. Then, we have some results as follows.

Lemma 2.2.

$$\left.\frac{d\,\mathrm{Re}\,\lambda(\tau)}{d\tau}\right|_{\tau=\tau_k} > 0.$$

Proof. Differentiating both sides of (2.4) with respect to τ, we obtain

$$2\lambda\frac{d\lambda}{d\tau} + \gamma\frac{d\lambda}{d\tau} - a_1a_2e^{-2\lambda\tau}\left(-2\lambda - 2\tau\frac{d\lambda}{d\tau}\right) = 0.$$

Therefore,

$$\frac{d\lambda}{d\tau} = \frac{-2a_1a_2\lambda}{(2\lambda+\gamma)e^{2\lambda\tau} + 2a_1a_2\tau},$$

and hence

$$\begin{aligned}\left.\frac{d\lambda}{d\tau}\right|_{\tau=\tau_k} &= \frac{-2a_1a_2i\omega_0}{(2i\omega_0+\gamma)\,e^{2i\omega_0\tau_k} + 2a_1a_2\tau_k}\\ &= \frac{-2a_1a_2i\omega_0}{(2i\omega_0+\gamma)\left(\cos 2\omega_0\tau_k + i\sin 2\omega_0\tau_k\right) + 2a_1a_2\tau_k}\\ &= \frac{-2a_1a_2i\omega_0}{(2i\omega_0+\gamma)\left(\frac{-\omega_0^2}{a_1a_2} - \frac{\omega_0\gamma}{a_1a_2}i\right) + 2a_1a_2\tau_k}\\ &= \frac{2a_1^2a_2^2i\omega_0}{(2i\omega_0+\gamma)\left(\omega_0^2+\omega_0\gamma i\right) - 2a_1^2a_2^2\tau_k}\\ &= \frac{1}{M}\left\{2a_1^2a_2^2\omega_0^2\left(\gamma\omega_0^2 + 2a_1^2a_2^2\tau_k\right) - 2ia_1^2a_2^2\omega_0\left(2\omega_0^3+\omega_0\gamma^2\right)\right\},\end{aligned} \tag{2.9}$$

where

$$M = \left(\omega_0^2\gamma + 2a_1^2a_2^2\tau_k\right)^2 + \left(\omega_0\gamma^2 + 2\omega_0^3\right)^2.$$

Hence,

$$\left.\frac{d\,\mathrm{Re}\,\lambda(\tau)}{d\tau}\right|_{\tau=\tau_k} = \frac{1}{M}\left\{2a_1^2a_2^2\omega_0^2\left(\gamma^2\omega_0 + 2\omega_0^3\right)\right\} > 0,$$

which implies our claim. □

Lemma 2.3. *For the system (1.6), assume that $(H_1)-(H_3)$ hold, we have*

(i) equation (2.4) has a pair of simple imaginary roots $\pm i\omega$ when $\tau = \tau_k$, where $k = 0, 1, 2, \ldots$.

(ii) for $\tau \in [0, \tau_0)$, all roots of equation (2.4) have negative real parts, for $\tau = \tau_0$, all roots still have negative real parts except $\pm i\omega_0$.

(iii) for $\tau \in (\tau_k, \tau_{k+1}]$, equation (2.4) has $2(k+1)$ roots with positive real parts.

Proof. Equation (2.4) has simple imaginary roots $\pm i\omega_0$, if and only if $\tau = \tau_k$, the conclusion on the number of eigenvalues with positive real parts can be arrived at according to Lemma 2.2, Dieudonné [7], Ruan and Wei [22]. The details are omitted. □

Summarizing the above discussion directly draws the conclusion on the stability of origin as follows.

Theorem 2.1. *For system (1.6), assume that $(H_1)-(H_3)$ hold, we get*

(i) $x = 0$ is asymptotically stable for $\tau \in [0, \tau_0)$.

(ii) $x = 0$ is unstable for $\tau > \tau_0$.

(iii) the system undergoes a Hopf bifurcation at the origin when $\tau = \tau_k$, for $k = 0, 1, 2, \ldots$.

3. Stability and direction of the Hopf bifurcation

In the previous section, we obtained conditions for the Hopf bifurcation to occur when $\tau = \tau_k$, $k = 0, 1, 2, ...$ This subsection will investigate the direction of the Hopf bifurcation and the stability of the bifurcating solution when τ passes τ_0, employing the center manifold theory and techniques from Hassard et al. [15] and Wei [29].

Let $y(t) = x(\tau t)$. Then, (1.6) becomes

$$\begin{cases} \dot{y}_1(t) = \alpha\tau f_1\left(y_2(t-1)\right), \\ \dot{y}_2(t) = -\gamma\tau y_2(t) + \beta\tau f_2\left(y_1(t-1)\right). \end{cases} \tag{3.1}$$

Linearizing the equation around origin gives

$$\begin{cases} \dot{y}_1(t) = \alpha\tau f_1'(0) y_2(t-1), \\ \dot{y}_2(t) = -\gamma\tau y_2(t) + \beta\tau f_2'(0) y_1(t-1). \end{cases} \tag{3.2}$$

We have

$$\begin{cases} \dot{y}_1(t) = a_1\tau y_2(t-1), \\ \dot{y}_2(t) = -\gamma\tau y_2(t) + a_2\tau y_1(t-1), \end{cases} \tag{3.3}$$

where $a_1 = \alpha f_1'(0)$, $a_2 = \beta f_2'(0)$. Correspondingly, the characteristic (2.4) becomes

$$z(z+\gamma\tau) - a_1 a_2 \tau^2 e^{-2z} = 0 \tag{3.4}$$

with $z = \lambda\tau$ for $\tau \neq 0$. From the conclusion of Lemma 2.3 we know that, all roots of equation (3.3) except $\pm i\tau_0\omega_0$ have negative real parts. Furthermore, by Lemma 2.2, the root of (3.3)

$$z(\tau) = \tau\alpha(\tau) + i\tau\omega(\tau),$$

with $\alpha(\tau_0) = 0$ and $\omega(\tau_0) = \omega_0$ satisfies

$$\left.\frac{d\,\mathrm{Re}\,\tau\lambda(\tau)}{d\tau}\right|_{\tau=\tau_k} > 0.$$

Set $\tau = \tau_0 + \mu$, $\mu \in \mathbb{R}$. Then, $\mu = 0$ is a Hopf bifurcation value for (2.3). Rewrite (2.3) as

$$\dot{y}(t) = (\tau_0 + \mu)\left[By(t) + Cy(t-1)\right], \tag{3.5}$$

where $y(t) = (y_1(t), y_2(t))^T$, $B = \begin{bmatrix} 0 & 0 \\ 0 & -\gamma \end{bmatrix}$, $C = \begin{bmatrix} 0 & a_1 \\ a_2 & 0 \end{bmatrix}$.

For $\phi = (\phi_1, \phi_2)^T \in C\left([-1, 0], \mathbb{R}^2\right)$, let

$$L_\mu(\phi) = (\tau_0 + \mu)\left[B\phi(0) + C\phi(-1)\right] \tag{3.6}$$

and

$$F(\mu, \phi) = (\tau_0 + \mu)\left[c_{11}\phi^2(-1) + d_{11}\phi^3(-1)\right] + \cdots, \tag{3.7}$$

where $c_{11} = \frac{1}{2}\begin{bmatrix} 0 & \alpha f_1''(0) \\ \beta f_2''(0) & 0 \end{bmatrix}$, $d_{11} = \frac{1}{3!}\begin{bmatrix} 0 & \alpha f_1'''(0) \\ \beta f_2'''(0) & 0 \end{bmatrix}$.

By the Riesz representation theorem, there exists a function $\eta(\theta, \mu)$ of the bounded variation for $\theta \in [-1, 0]$ such that

$$L_\mu(\phi) = \int_{-1}^{0} d\eta\,(\theta, \mu)\,\phi(\theta), \quad \forall \phi \in C\left([-1, 0], \mathbb{R}^2\right). \tag{3.8}$$

In fact, we can take

$$\eta\,(\theta, \mu) = (\tau_0 + \mu)B\delta(\theta) + (\tau_0 + \mu)C\delta(\theta + 1), \tag{3.9}$$

where $\delta(x)$ denotes the Dirac Function

$$\delta(x) = \begin{cases} 1, x = 0, \\ 0, x \neq 0. \end{cases}$$

For $\phi \in C\left([-1, 0], \mathbb{R}^2\right)$, we define

$$A(\mu)\phi = \begin{cases} \frac{d\phi(\theta)}{d\theta}, & \theta \in [-1, 0), \\ \int_{-1}^{0} d\eta(\xi, \mu)\phi(\theta), & \theta = 0, \end{cases}$$

and

$$N(\mu)\phi = \begin{cases} 0, & \theta \in [-1, 0), \\ F(\mu, \phi), & \theta = 0. \end{cases}$$

Then, (3.5) can be rewritten as

$$\dot{y}_t = A(\mu)y_t + N(\mu)y_t, \tag{3.10}$$

where $y_t(\theta) = y(t + \theta)$ for $\theta \in [-1, 0]$. For $\psi \in C\left([0, 1], \mathbb{R}^2\right)$, define

$$A^*\psi = \begin{cases} \frac{d\psi(s)}{ds}, & s \in (0, 1], \\ \int_{-1}^{0} d\eta(t, 0)\psi(-t), & s = 0. \end{cases}$$

For $\phi \in C\left([-1, 0], \mathbb{R}^2\right)$ and $\psi \in C\left([0, 1], \mathbb{R}^2\right)$, define a bilinear form

$$\langle \psi(s), \phi(\theta) \rangle = \bar{\psi}(0)\phi(0) - \int_{-1}^{0}\int_{s-0}^{\theta} \bar{\psi}(\xi - \theta)d\eta(\theta)\phi(\xi)d\xi, \tag{3.11}$$

where $\eta(\theta) = \eta(\theta, 0)$. We know that A^* and $A = A(0)$ are adjoint operators, so $\pm i\tau_0\omega_0$ are also eigenvalues of A^*. It is obtained that $q(\theta) = (1, \varepsilon)^T e^{i\tau_0\omega_0\theta}$ is an eigenvector of A corresponding to the eigenvalue $i\tau_0\omega_0$ and $q^*(s) = D(1, \varepsilon^*)e^{i\tau_0\omega_0 s}$ is an eigenvector of A^* with respect to $-i\tau_0\omega_0$. Furthermore,

$$\langle q^*, q \rangle = 1,\ \langle q^*, \overline{q} \rangle = 0,$$

where

$$\varepsilon = \frac{i\omega_0}{\alpha f_1'(0)} e^{i\omega_0\tau_0},\ \varepsilon^* = \frac{1}{\beta f_2'(0)} e^{i\omega_0\tau_0}$$

and

$$D = 1 + \varepsilon\bar{\varepsilon}^* + \alpha\tau_0 f_1'(0)\varepsilon e^{-i\omega_0\tau_0} + \beta\tau_0 f_2'(0)\bar{\varepsilon}^* e^{-i\omega_0\tau_0}.$$

With the help of these preliminaries, we immediately give some useful data in what follows, which can lead to significant properties of the Hopf bifurcation. Following the algorithms given by Hassardal [15] and using a computation process similar to that given by Wei and Li [?], we first compute the center manifold C_0 at $\mu = 0$. Let y_t be the solution of (3.5) when $\mu = 0$. Define

$$z(t) = \langle q^*, y_t \rangle\,,\ W(t,\theta) = y_t(\theta) - 2\,\mathrm{Re}\{z(t)q(\theta)\}.$$

On the center manifold C_0 we have

$$W(t,\theta) = W(z(t), \bar{z}(t), \theta),$$

where

$$W(z(t), \bar{z}(t), \theta) = W_{20}\frac{z^2}{2} + W_{11}z\bar{z} + W_{02}\frac{\bar{z}^2}{2} + \cdots,$$

z and $\bar{z}$ are local coordinates for the center manifold C_0 in the direction of q^* and $\bar{q}^*$. Note that W is real if y_t is real. We consider only real solutions. For a solution $y_t \in C_0$ of (3.5), since $\mu = 0$, we have

$$\begin{aligned}\dot{z}(t) =& \mathrm{i}\omega_0\tau_0 z + q^*(\theta)f(W + 2\,\mathrm{Re}\{z(t)q(\theta)\})\\ =& \mathrm{i}\omega_0\tau_0 z + \bar{q}^*(0)f(W(z,\bar{z},0) + 2\,\mathrm{Re}\{z(t)q(0)\})\\ =& \mathrm{i}\omega_0\tau_0 z + \bar{q}^*(0)f_0(z,\bar{z}).\end{aligned}$$

We rewrite the equation as

$$\dot{z}(t) = i\omega_0\tau_0 z + g(z,\bar{z}) \tag{3.12}$$

with

$$g(z,\bar{z}) = \bar{q}^*(0)f_0(z,\bar{z}) = g_{20}\frac{z^2}{2} + g_{11}z\bar{z} + g_{02}\frac{\bar{z}^2}{2} + g_{21}\frac{z^2\bar{z}}{2} + \cdots. \tag{3.13}$$

By (3.10) and (3.12), we have

$$\begin{aligned}W' =& \begin{cases} AW - 2\,\mathrm{Re}\left\{\bar{q}^*(0)fq(\theta)\right\}, & -1 \le \theta < 0\\ AW - 2\,\mathrm{Re}\left\{\bar{q}^*(0)fq(\theta)\right\} + f, \ \theta = 0 \end{cases}\\ =& AW + H(z,\bar{z},\theta),\end{aligned}$$

where

$$H(z,\bar{z},\theta) = H_{20}(\theta)\frac{z^2}{2} + H_{11}z\bar{z} + H_{02}(\theta)\frac{\bar{z}^2}{2} + \cdots. \tag{3.14}$$

Expanding the above series and comparing the corresponding coefficients, we obtain

$$(A - 2i\omega_0\tau_0)W_{20}(\theta) = -H_{20}(\theta), \tag{3.15}$$

$$AW_{11}(\theta) = -H_{11}(\theta). \tag{3.16}$$

Note that

$$y_t = W(t,\theta) + zq(\theta) + \bar{z}\bar{q}(\theta),\ q(\theta) = (1,\varepsilon)^T e^{i\tau_0\omega_0\theta}.$$

We get

$$\begin{aligned} y(t-1) =& z\mathrm{e}^{-\mathrm{i}\omega_0\tau_0} + \bar{z}\mathrm{e}^{\mathrm{i}\omega_0\tau_0} \\ &+ W_{20}(-1)\frac{z^2}{2} + W_{11}(-1)z\bar{z} + W_{02}(-1)\frac{\bar{z}^2}{2} + \cdots . \end{aligned}$$

This relation and (3.13) imply

$$\begin{aligned} g(z,\bar{z}) =& \bar{q}^*(0)f_0(z,\bar{z}) \\ =& \tau_0\bar{D}\left[f_1(0,y_t) + \bar{\varepsilon}^* f_2(0,y_t)\right] \\ =& \tau_0\left[\bar{D}(\tau_0+\mu)\alpha f_1''(0)\left[\varepsilon^2 e^{-2i\omega_0\tau_0}\frac{z^2}{2} + \varepsilon\bar{\varepsilon}z\bar{z} + \bar{\varepsilon}^2 e^{2i\omega_0\tau_0}\frac{\bar{z}^2}{2}\right.\right. \\ &+ \left.\left(2\varepsilon e^{-i\omega_0\tau_0}W_{11}^{(2)}(-1) + \bar{\varepsilon}e^{i\omega_0\tau_0}W_{20}^{(2)}(-1)\right)\frac{z^2\bar{z}}{2}\right] \\ &+ (\tau_0+\mu)\alpha f_1''(0)\varepsilon^2\bar{\varepsilon}e^{-i\omega_0\tau_0}\frac{z^2\bar{z}}{2} \\ &+ (\tau_0+\mu)\beta f_2''(0)\left[e^{-2i\omega_0\tau_0}\frac{z^2}{2} + z\bar{z} + e^{2i\omega_0\tau_0}\frac{\bar{z}^2}{2}\right. \\ &+ \left.\left(2e^{-i\omega_0\tau_0}W_{11}^{(1)}(-1) + e^{i\omega_0\tau_0}W_{20}^{(1)}(-1)\right)\frac{z^2\bar{z}}{2}\right] \\ &+ \left.(\tau_0+\mu)\beta f_2'''(0)e^{-i\omega_0\tau_0}\frac{z^2\bar{z}}{2}\right]. \end{aligned}$$

Comparing coefficients with (3.13) and using $\bar{q}^*(0) = \bar{D}$, we have

$$\begin{aligned} g_{20} &= \bar{D}\tau_0\left[(\tau_0+\mu)\left(\alpha f_1''(0)\varepsilon^2 + \beta f_2''(0)\bar{\varepsilon}^*\right)\right]e^{-2i\tau_0\omega_0}, \\ g_{11} &= \bar{D}\tau_0\left[(\tau_0+\mu)\left(\alpha f_1''(0)\varepsilon\bar{\varepsilon} + \beta f_2''(0)\bar{\varepsilon}^*\right)\right], \\ g_{02} &= \bar{D}\tau_0\left[(\tau_0+\mu)\left(\alpha f_1''(0)\bar{\varepsilon}^2 + \beta f_2''(0)\bar{\varepsilon}^*\right)\right]e^{2i\tau_0\omega_0}, \\ g_{21} &= \bar{D}\tau_0\left[(\tau_0+\mu)\left[2\alpha f_1''(0)\varepsilon W_{11}^{(2)}(-1) + \alpha f_1''(0)\varepsilon^2\bar{\varepsilon}\right.\right. \\ &\quad \left. +2\beta f_2''(0)W_{11}^{(1)}(-1) + \beta f_2'''(0)\bar{\varepsilon}^*\right]e^{-i\tau_0\omega_0} \\ &\quad \left. + (\tau_0+\mu)\left[\alpha f_1''(0)\bar{\varepsilon}W_{20}^{(2)}(-1) + \beta f_2''(0)W_{20}^{(1)}(-1)\right]e^{i\tau_0\omega_0}\right]. \end{aligned} \tag{3.17}$$

Since for $\theta \in [-1,0)$,

$$H(z,\bar{z},\theta) = -\bar{q}^*(0)f_0q(\theta) - \bar{q}^*(0)\bar{f}_0\bar{q}(\theta) = -g(z,\bar{z})q(\theta) - \bar{g}(z,\bar{z})\bar{q}(\theta).$$

Comparing coefficients with (3.14), we get

$$H_{11}(\theta) = -g_{11}q(\theta) - \bar{g}_{11}\bar{q}(\theta),\ H_{20}(\theta) = -g_{20}q(\theta) - \bar{g}_{02}\bar{q}(\theta).$$

Substituting these relations into (3.15), we can derive the following equation

$$\dot{W}_{20}(\theta) = 2i\omega_0\tau_0W_{20}(\theta) + g_{20}q(\theta) + g_{02}\bar{q}(\theta).$$

Solving for $W_{20}(0)$, we obtain

$$W_{20}(\theta) = \frac{ig_{20}}{\omega_0\tau_0}q(0)e^{i\omega_0\tau_0\theta} + \frac{i\bar{g}_{02}}{3\omega_0\tau_0}\bar{q}(0)e^{-i\omega_0\tau_0\theta} + E_1e^{2i\omega_0\tau_0\theta}. \tag{3.18}$$

Similarly,

$$W_{11}(\theta) = -\frac{ig_{11}}{\omega_0\tau_0}q(0)e^{i\omega_0\tau_0\theta} + \frac{i\bar{g}_{11}}{\omega_0\tau_0}\bar{q}(0)e^{-i\omega_0\tau_0\theta} + E_2, \tag{3.19}$$

where

$$E_1 = \left(\alpha f_1''(0) + \beta f_2''(0)\bar{\varepsilon}^*\varepsilon^2\right) e^{-2i\tau_0\omega_0}\left[2i\omega_0 + \gamma - \left(\alpha f_1'(0) + \beta f_2'(0)\right) e^{-2i\tau_0\omega_0}\right]^{-1},$$

$$E_2 = \left(\alpha f_1''(0) + \beta f_2''(0)\varepsilon\bar{\varepsilon}\right)\left[\gamma - \left(\alpha f_1'(0) + \beta f_2'(0)\right)\right]^{-1}.$$

Definitely, each g_{ij} can be definitely computed out according to equation (1.6) with all required parameters clear. Finally, we can compute the following quantities:

$$c_1(0) = \frac{i}{2\tau_0\omega_0}\left(g_{11}g_{20} - 2\left|g_{11}\right|^2 - \frac{\left|g_{02}\right|^2}{3}\right) + \frac{g_{21}}{2},$$

$$\mu_2 = -\frac{\operatorname{Re}\left(c_1(0)\right)}{\operatorname{Re}\left(\lambda'\left(\tau_0\right)\right)},\ \beta_2 = 2\operatorname{Re}\left(c_1(0)\right),\ T_2 = -\frac{\operatorname{Im}\left(c_1(0)\right) + \mu_2\operatorname{Im}\left(\lambda'\left(\tau_0\right)\right)}{\omega_0\tau_0}.$$

It is well-known that μ_2 determines the direction of the Hopf bifurcation: if $\mu_2 > 0(\mu_2 < 0)$, then the Hopf bifurcation is supercritical (subcritical) and the bifurcating periodic solutions exist for $\tau > \tau_0(\tau < \tau_0)$; β_2 determines the stability of bifurcating periodic solutions: the bifurcating periodic solutions are orbitally asymptotically stable (unstable) if $\beta_2 < 0(\beta_2 > 0)$, and T_2 determines the period of the bifurcating periodic solutions: the period increases (decreases) if $T_2 > 0(T_2 < 0)$. From the discussion in Section 2, we know that $\operatorname{Re}\left(\lambda'\left(\tau_0\right)\right)$. Therefore, we have the following result.

Theorem 3.1. *The direction of the Hopf bifurcation of the system (1.6) at the origin, when $\tau = \tau_0$ is supercritical (subcritical) and the bifurcating periodic solutions, are orbitally asymptotically stable (unstable) if* $\operatorname{Re}\left(c_1(0)\right) < 0(> 0)$.

4. Global existence of periodic solutions

In this section, we study the global continuation of periodic solutions bifurcating from the point $(0, \tau_k)$, $k = 0, 1, 2, \cdots$ for equation (1.6) by using a global Hopf bifurcation theorem given by Wu [**?**]. Firstly, set

$$\dot{x}(t) = F(z_t, \tau, p), \tag{4.1}$$

where $z = (x_1, x_2)^T$, $z_t(\theta) = z(t+\theta) \in C\left([-\tau, 0], \mathbb{R}^2\right)$. For the sake of convenience, we introduce some notation:

$$X = C\left([-\tau, 0], \mathbb{R}\right),$$

$$\Sigma = C\ell\{(z, \tau, \mathrm{p}) : x \text{ is a poriodic solution of (1.3)}\} \subset X \times \mathbb{R} \times \mathbb{R}_+,$$

$$N = \{(\hat{z}, \tau, p) | F(\hat{z}, \tau, p) = 0\}.$$

Let $C(0, \tau_k, \frac{2\pi}{\omega_0})$ denote the connect component of $(0, \tau_k, \frac{2\pi}{\omega_0})$ in Σ, where τ_k and ω_0 are defined in (2.8).

(H_4) There exists $L > 0$ such that

$$|f_i(x)| \leq L, \forall x \in \mathbb{R},$$

where $i = 1, 2$.

Lemma 4.1. *Suppose that (H_4) is satisfied. Then, there exists constant k_0 such that when $\tau > \tau_{k_0}$, all periodic solutions to (1.6) are uniformly bounded.*

Proof. From the discussion in the second section, we can obtain

$$\frac{2\pi}{\omega_0} < \tau_1, \tag{4.2}$$

when $k > 1$. Moreover, by Lemma 2.2, we deduce that $(0, \tau_k, \frac{2\pi}{\omega_0})$ are isolated centres. Again by Lemma 2.2, we have

$$\begin{aligned}\gamma\left(0,\tau_k,\frac{2\pi}{\omega_0}\right) =& \deg_B\left(H^-\left(0,\tau_k,\frac{2\pi}{\omega_0}\right),\Omega_{\varepsilon\frac{2\pi}{\omega_0}}\right)\\ &-\deg_B\left(H^+\left(0,\tau_k,\frac{2\pi}{\omega_0}\right),\Omega_{\frac{2\pi}{\omega_0}}\right)\\ =& -1.\end{aligned} \tag{4.3}$$

By [?, Theorem 3.1, 3.2], we conclude that the connected component $C(0, \tau_k, \frac{2\pi}{\omega_0})$ through $(0, \tau_k, \frac{2\pi}{\omega_0})$ is unbounded. Let $r(t) = \sqrt{x_1^2(t) + x_2^2(t)}$, take the derivative with respect to both ends of $r(t)$ along the solution to the system (3.1),

$$\begin{aligned}\dot{r}(t) =& \frac{1}{r(t)}\left[x_1(t)\dot{x}_1(t) + x_2(t)\dot{x}_2(t)\right]\\ =& \frac{1}{r(t)}\left[x_1(t)\alpha f_1\left(x_2(t-\tau)\right) - \gamma x_2^2(t) + x_2(t)\beta f_2\left(x_1(t-\tau)\right)\right]\\ \le& \frac{1}{r(t)}\left[-\gamma x_2^2(t) + L\left(\alpha\left|x_1(t)\right| + \beta\left|x_2(t)\right|\right)\right]\end{aligned} \tag{4.4}$$

and

$$N \ge \max\{1, (\alpha+\beta)\gamma L\}.$$

If exists $t_0 > 0$, such that $r(t_0) = A \ge N$. Then, we have

$$\dot{r}\left(t_0\right) \le \frac{1}{A}\left[-\gamma A^2 + L(\alpha+\beta)A\right] = -\gamma A + (\alpha+\beta)L < 0.$$

Therefore, when $x(t) = (x_1(t), x_2(t))^T$ is periodic solutions of (1.6), we have that $r(t) < N$. Summarizing the above discussion, the conclusion follows. □

Lemma 4.2. *Assume that $(H_1) - (H_4)$ are satisfied. Then, problem (1.6) has no periodic solutions of period τ.*

Proof. Let $(x_1(t), x_2(t))$ is a $\tau-periodic$ solution of (1.6), then $x_1(t) = x_1(t-\tau)$ and $x_2(t) = x_2(t-\tau)$ are periodic solutions of the system of ordinary differential equations

$$\begin{cases}\dot{x_1}(t) = \alpha f_1\left(x_2(t)\right)) = P(x_1, x_2),\\ \dot{x_2}(t) = -\gamma x_2(t) + \beta f_2\left(x_1(t)\right)) = Q(x_1, x_2).\end{cases} \tag{4.5}$$

Let $(P(x_1, x_2), Q(x_1, x_2))$ denote the vector field of (4.4), then

$$\frac{\partial P\left(x_1,x_2\right)}{\partial x_1} + \frac{\partial Q\left(x_1,x_2\right)}{\partial x_2} = -\gamma < 0,$$

for all $(x_1, x_2) \in \mathbb{R}$. Thus, the classical Bendixson's negative criterion implies that (4.5) has no nonconstant periodic solutions. This completes the proof. From the discussion above, (1.6) has no periodic solutions of period $\frac{\pi}{n}\,(n \in N^+)$. By (4.2), $\frac{\tau}{k} < \frac{2\pi}{\omega_0} < \tau_1$. Further, by $(0, \tau, p) \in C\left(0, \tau, \frac{2\pi}{\omega_0}\right)$, when $\tau_k \to \tau_1$, $p \to \frac{2\pi}{\omega_0}$. $C\left(0, \tau, \frac{2\pi}{\omega_0}\right)$ is connected. Therefore, $p - periodic$ solutions to (1.6) are uniformly bounded. □

Theorem 4.1. *Assume that $(H_1) - (H_4)$ are satisfied, for each $\tau > \tau_k$, $k = 2, 3, \cdots$, equation (1.6) has at least $k-2$ periodic solutions, where τ_k is defined in (2.8).*

Proof. Lemma2.2 implies that $C\left(0, \tau, \frac{2\pi}{\omega_0}\right)$ is nonempty and unbounded, and by Lemma 4.1 and Lemma 4.2, the projection of $C\left(0, \tau, \frac{2\pi}{\omega_0}\right)$ onto $\tau - space$ is bounded below. Consequently, the projection of $C\left(0, \tau, \frac{2\pi}{\omega_0}\right)$ onto the $\tau - space$ includes $[\tau_k, \infty)$. This shows that for each $\tau > \tau_k (k > 1)$, equation (1.3) (or (4.1)) has $k-2$ periodic solutions. This completes the proof. □

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