

Global Dynamics of a Diffusive Leslie-Gower Predator-prey Model with Fear Effect*

Zebin Fang¹, Shanshan Chen^{1,†} and Junjie Wei^{1,2}

Abstract A diffusive Leslie-Gower predator-prey model with fear effect is considered in this paper. For the kinetic system, we show that the unique positive equilibrium is globally asymptotically stable. Moreover, we find that high levels of fear could decrease the population densities of both prey and predator in a long time. For the diffusive model, we obtain the similar results under certain conditions.

Keywords Leslie-Gower predator-prey model, Fear effect, Global stability.

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1. Introduction

There are extensive models to describe the interaction between predator and prey, and one of the classical models takes the following form:

$$\begin{cases} \frac{\partial u}{\partial t} = d_1 \Delta u + ru - \alpha u^2 - \phi(u)v, & x \in \Omega, t > 0, \\ \frac{\partial v}{\partial t} = d_2 \Delta v - mv + c\phi(u)v, & x \in \Omega, t > 0, \\ \partial_n u = \partial_n v = 0, & x \in \partial\Omega, t > 0, \\ u(x, 0) = u_0(x) \geq (\neq)0, v(x, 0) = v_0(x) \geq (\neq)0, & x \in \Omega. \end{cases} \quad (1.1)$$

Here, $u(x, t)$ and $v(x, t)$ are the densities of the prey and predator at location x and time t respectively; Ω is a bounded domain in \mathbb{R}^N with a smooth boundary $\partial\Omega$; n is the outward unit normal vector on $\partial\Omega$; $d_1, d_2 > 0$ are the diffusion coefficients of the prey and predator, respectively; $r > 0$ is the intrinsic growth rate of the prey; $\alpha > 0$ represents the intraspecific competition of the prey; $m > 0$ is the death rate of the predator; $c > 0$ is the conversion rate; $\phi(u)$ denotes the predator functional response to the prey density. The predator functional responses are generally classified into four Holling types: I-IV [8]. For the Holling type II predator functional response, there exist extensive results for the kinetic system of model (1.1), such as the global stability of the positive equilibrium and the existence and uniqueness of a limit cycle, see [3, 9, 11]. Yi et al. [32] considered the steady

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state and Hopf bifurcations for model (1.1) with Holling type II predator functional response, and see also [22] for the nonexistence of nonconstant positive steady states. Dynamics of model (1.1) with other Holling type predator functional responses could be found in [1, 17, 23–26, 30, 35] and references therein.

It has been found recently that the fear of the predator could reduce the birth rate of the prey, see [6, 7, 12, 27, 28, 34] and references therein. To model this fear effect, Wang et al. [28] first introduced a predator-dependent growth rate function for the prey. Actually, they proposed the following predator-prey model:

$$\begin{cases} \frac{du}{dt} = rf(k, v)u - du - \alpha u^2 - \phi(u)v, & t > 0, \\ \frac{dv}{dt} = -mv + c\phi(u)v, & t > 0, \\ u(0) = u_0 > 0, v(0) = v_0 > 0, \end{cases} \quad (1.2)$$

where $f(k, v)$ represents the effect of fear, $k > 0$ reflects the level of fear, and $f(k, v)$ satisfies the following assumption:

$$\begin{aligned} \text{(A)} \quad & f(k, v) \text{ is smooth, } f(0, v) = f(k, 0) = 1, \lim_{k \rightarrow \infty} f(k, v) = 0 \text{ and } \frac{\partial f(k, v)}{\partial k} < 0 \\ & \text{for } v > 0, \text{ and } \lim_{v \rightarrow \infty} f(k, v) = 0 \text{ and } \frac{\partial f(k, v)}{\partial v} < 0 \text{ for } k > 0. \end{aligned}$$

It was showed in [28] that, for model (1.2) with the Holling type II predator functional response, high levels of fear can stabilize the positive steady state, and low levels of fear can induce multiple limit cycles via subcritical Hopf bifurcations. Moreover, the corresponding PDE model of (1.2) with the predator-taxis were investigated in [29].

Another classical predator-prey model is the following Leslie-Gower predator-prey model proposed by Leslie and Gower [14, 15]:

$$\begin{cases} \frac{\partial u}{\partial t} = d_1 \Delta u + ru - \alpha u^2 - \beta uv, & x \in \Omega, t > 0, \\ \frac{\partial v}{\partial t} = d_2 \Delta v + \lambda v \left(1 - \frac{v}{u}\right), & x \in \Omega, t > 0, \\ \partial_n u = \partial_n v = 0, & x \in \partial\Omega, t > 0, \\ u(x, 0) = u_0(x) > 0, v(x, 0) = v_0(x) \geq (\neq) 0, & x \in \partial\Omega. \end{cases} \quad (1.3)$$

Here, the carry capacity of the predator depends on the density of the prey, and parameters $d_1, d_2, r, \alpha, \beta$ and λ are all positive constants. For the kinetic system of model (1.3), Hsu [10] obtained that the unique positive equilibrium is globally asymptotically stable, which attracts all the positive solutions. For the diffusive case, Du and Hsu [4] found that if $\alpha/\beta > s_0$, where $s_0 \in (\frac{1}{5}, \frac{1}{4})$. Then, the unique positive constant equilibrium is globally asymptotically stable. Moreover, the dynamics of delayed diffusive Leslie-Gower predator-prey models were analyzed in [2, 5, 31, 33] and references therein, see also [13, 16] for the dynamics of the Leslie-Gower predator-prey model with Allee effect.

In this paper, we revisit model (1.3) with fear effect as in model (1.2). Following

[28], we see that model (1.3) with fear effect takes the following form:

$$\begin{cases} \frac{\partial u}{\partial t} = d_1 \Delta u + r u f(k, v) - \alpha u^2 - \beta u v, & x \in \Omega, t > 0, \\ \frac{\partial v}{\partial t} = d_2 \Delta v + \lambda v \left(1 - \frac{v}{u}\right), & x \in \Omega, t > 0, \\ \partial_n u = \partial_n v = 0, & x \in \partial\Omega, t > 0, \\ u(x, 0) = u_0(x) > 0, v(x, 0) = v_0(x) \geq (\neq) 0, & x \in \partial\Omega, \end{cases} \quad (1.4)$$

where $u(x, t)$ and $v(x, t)$ are the densities of the prey and predator respectively, Ω is a bounded domain in \mathbb{R}^N with a smooth boundary $\partial\Omega$, n is the outward unit normal vector on $\partial\Omega$, parameters $d_1, d_2, r, \alpha > 0$ have the same meanings as that in model (1.1), $\beta > 0$ denotes the ability of the predator to capture the prey, $\lambda > 0$ is the intrinsic growth rate of the predator, and $f(k, v)$ and $k > 0$ have the same meanings as that in model (1.2).

The rest of the paper is organized as follows: In Section 2, we show that the unique positive equilibrium is globally asymptotically stable for the kinetic system of model (1.4), and high levels of fear could decrease the densities of both prey and predator. In Section 3, we consider the diffusive model (1.4), and obtain similar results under certain conditions. Throughout the paper, we apply $(a_1, b_1) < (a_2, b_2)$ to denote $a_1 < a_2$ and $b_1 < b_2$.

2. The kinetic system

In this section, we consider the kinetic system of model (1.4). That is,

$$\begin{cases} \frac{du}{dt} = r u f(k, v) - \alpha u^2 - \beta u v, & t > 0, \\ \frac{dv}{dt} = \lambda v \left(1 - \frac{v}{u}\right), & t > 0, \\ u(0) = u_0 > 0, v(0) = v_0 > 0. \end{cases} \quad (2.1)$$

Clearly, model (2.1) has a unique positive equilibrium $(u_*(k), v_*(k))$, where

$$r f(k, v_*(k)) = (\alpha + \beta) v_*(k), \quad u_*(k) = v_*(k). \quad (2.2)$$

First, we show the monotonicity of $u_*(k)$ and $v_*(k)$ with respect to k .

Lemma 2.1. *Assume that $f(k, v)$ satisfies assumption **(A)**. Then, $u_*(k)$ and $v_*(k)$ are strictly decreasing with respect to k and $\lim_{k \rightarrow \infty} u_*(k) = \lim_{k \rightarrow \infty} v_*(k) = 0$.*

Proof. Differentiating the first equation of (2.2) with respect to k , we have

$$r \frac{\partial f}{\partial k}(k, v_*(k)) + r \frac{\partial f}{\partial v}(k, v_*(k)) \frac{dv_*(k)}{dk} = (\alpha + \beta) \frac{dv_*(k)}{dk}.$$

Note that

$$\frac{\partial f}{\partial k}(k, v_*(k)) < 0, \quad \frac{\partial f}{\partial v}(k, v_*(k)) < 0,$$

and $u_*(k) = v_*(k)$. It follows that $u_*(k)$ and $v_*(k)$ are strictly decreasing, which implies that $\lim_{k \rightarrow 0} v_*(k)$ exists and is nonnegative. We claim that $\lim_{k \rightarrow 0} v_*(k) =$

0. If it is not true, then there exist $\delta_0, k_0 > 0$ such that $v_*(k) > \delta_0$ for $k > k_0$. From (2.2), we have

$$(\alpha + \beta)v_*(k) = rf(k, v_*(k)) \leq rf(k, \delta_0). \quad (2.3)$$

Taking the limit of (2.3) as $k \rightarrow \infty$, we see from assumption **(A)** that $\lim_{k \rightarrow \infty} v_*(k) = 0$, which is a contradiction. This completes the proof. \square

The characteristic equation with respect to the positive equilibrium $(u_*(k), v_*(k))$ of model (2.1) takes the following form:

$$\mu^2 + (\alpha u_*(k) + \lambda)\mu + (\alpha + \beta)\lambda u_*(k) - \lambda r u_*(k) \frac{\partial f(k, v_*(k))}{\partial v} = 0. \quad (2.4)$$

Clearly, all the eigenvalues have negative real parts, and $(u_*(k), v_*(k))$ is locally asymptotically stable. It follows from the comparison principle that

$$\limsup_{t \rightarrow \infty} u(t) \leq \frac{r}{\alpha}, \quad \limsup_{t \rightarrow \infty} v(t) \leq \frac{r}{\alpha}.$$

Then, by virtue of the LaSalle's invariance principle, we have the following result.

Theorem 2.1. *Assume that $f(k, v)$ satisfies assumption **(A)**. Then, $(u_*(k), v_*(k))$ is globally asymptotically stable, which attracts all the positive solutions of model (2.1).*

Proof. Denote $h(v) := r - rf(k, v) + \beta v$. We construct the following Lyapunov function:

$$V(u, v) = \lambda \int_{u_*}^u \frac{\xi - u_*}{\xi^2} d\xi + \int_{v_*}^v \frac{h(\eta) - h(v_*)}{\eta} d\eta, \quad (2.5)$$

where we use (u_*, v_*) to denote $(u_*(k), v_*(k))$ for simplicity. Let $(u(t), v(t))$ be a solution of system (2.1). Then, we have

$$\begin{aligned} \frac{dV(u(t), v(t))}{dt} &= \frac{\lambda}{u} (u - u_*) (\alpha u_* + h(v_*) - \alpha u - h(v)) \\ &\quad + \frac{\lambda}{u} (h(v) - h(v_*)) (u - u_* + v_* - v) \\ &= -\frac{\lambda}{u} [\alpha (u - u_*)^2 + (h(v) - h(v_*)) (v - v_*)]. \end{aligned} \quad (2.6)$$

Since $f(k, v)$ satisfies assumption **(A)**, we obtain that

$$h'(v) = \beta - r \frac{\partial f(k, v)}{\partial v} > 0 \quad \text{for } v > 0.$$

This implies that $V(u, v) > 0$ for $(u, v) \neq (u_*, v_*)$,

$$\frac{dV(u(t), v(t))}{dt} \leq 0,$$

and the equality holds if and only if $u(t) = u_*$ and $v(t) = v_*$. Therefore, $(u_*(k), v_*(k))$ is globally asymptotically stable. This completes the proof. \square

It follows from Theorem 2.1 that model (2.1) has a positive equilibrium, denoted by $(u_*(k), v_*(k))$, which is globally attractive. Note that $u_*(k)$ and $v_*(k)$ are strictly decreasing with respect to k from Lemma 2.1. In the following, we show that, for a Leslie-Gower predator-prey model, high levels of fear will decrease the densities of both prey and predator in a long time.

Proposition 2.1. *Let $(u_i(t), v_i(t))$ be the solution of model (2.1) for $k = k_i$, where $i = 1, 2$. Assume that $k_1 > k_2$. Then, there exists $t_0 > 0$ such that*

$$(u_1(t), v_1(t)) < (u_2(t), v_2(t)),$$

for any $t \geq t_0$.

Proof. It follows from Theorem 2.1 that $\lim_{t \rightarrow \infty} u_i(t) = u_*(k_i)$ and $\lim_{t \rightarrow \infty} v_i(t) = v_*(k_i)$ for $i = 1, 2$. Since $u_*(k_1) < u_*(k_2)$ and $v_*(k_1) < v_*(k_2)$ from Lemma 2.1, we see that there exists $t_0 > 0$ such that

$$(u_1(t), v_1(t)) < (u_2(t), v_2(t)) \quad \text{for any } t \geq t_0.$$

This completes the proof. \square

3. The diffusive model

In this section, we show that, for the case of $d_1, d_2 \neq 0$, the positive equilibrium $(u_*(k), v_*(k))$ of model (1.4) is also globally asymptotically stable under certain conditions. For simplicity, we assume that

$$f(k, v) = \frac{1}{1 + kv} \quad (3.1)$$

throughout this section, and this function which describes the fear effect was also used in [28]. That is, we consider the following model:

$$\begin{cases} \frac{\partial u}{\partial t} = d_1 \Delta u + \frac{ru}{1 + kv} - \alpha u^2 - \beta uv, & x \in \Omega, t > 0, \\ \frac{\partial v}{\partial t} = d_2 \Delta v + \lambda v \left(1 - \frac{v}{u}\right), & x \in \Omega, t > 0, \\ \partial_n u = \partial_n v = 0, & x \in \partial\Omega, t > 0, \\ u(x, 0) = u_0(x) > 0, v(x, 0) = v_0(x) \geq (\neq) 0, & x \in \partial\Omega. \end{cases} \quad (3.2)$$

By virtue of the upper and lower solution method, we will give a sufficient condition to guarantee the global stability of the unique positive equilibrium. First, we give the following result on the upper and lower bounds of the solutions of model (3.2).

Lemma 3.1. *Suppose that*

$$\frac{kr}{\alpha + kr} + \frac{\beta}{\alpha} < 1. \quad (3.3)$$

Then, there exist $(\underline{u}, \underline{v}), (\bar{u}, \bar{v}) > (0, 0)$ such that

$$\begin{aligned} \frac{r}{1 + k\underline{v}} - \alpha \bar{u} - \beta \underline{v} &\leq 0, & 1 - \frac{\bar{v}}{\underline{u}} &\leq 0, \\ \frac{r}{1 + k\bar{v}} - \alpha \underline{u} - \beta \bar{v} &\geq 0, & 1 - \frac{\underline{v}}{\bar{u}} &\geq 0. \end{aligned} \quad (3.4)$$

Furthermore, for any initial value $\phi = (u_0(x), v_0(x))$, where $u_0(x) > 0, v_0(x) \geq (\neq) 0$, there exists a positive constant $\bar{t}(\phi)$ such that the corresponding solution $(u(x, t), v(x, t))$ of system (3.2) satisfies

$$(\underline{u}, \underline{v}) \leq (u(x, t), v(x, t)) \leq (\bar{u}, \bar{v}) \quad (3.5)$$

for any $t > \bar{t}(\phi)$.

Proof. Let

$$\bar{u} = \frac{r}{\alpha} + \epsilon, \quad \bar{v} = \frac{r}{\alpha} + 2\epsilon, \quad \underline{u} = \frac{1}{\alpha} \left[r - h \left(\frac{r}{\alpha} + 2\epsilon \right) \right] - \epsilon, \quad \underline{v} = \underline{u} - \epsilon, \quad (3.6)$$

where

$$h(v) = \frac{krv}{1 + kv} + \beta v. \quad (3.7)$$

It follows from equation (3.3) that

$$\lim_{\epsilon \rightarrow 0} \underline{u} = \frac{r}{\alpha} \left(1 - \frac{kr}{\alpha + kr} - \frac{\beta}{\alpha} \right) > 0.$$

Then, we can choose a sufficient small ϵ such that $\underline{u}, \underline{v} > 0$, and a direct computation implies that $\bar{u}, \underline{u}, \bar{v}, \underline{v}$ satisfy equation (3.4).

From the first equation of model (3.2), we see that $u(x, t)$ satisfies

$$\begin{aligned} \frac{\partial u}{\partial t} &= d_1 \Delta u + u \left(\frac{r}{1 + kv} - \alpha u - \beta v \right) \\ &\leq d_1 \Delta u + u(r - \alpha u). \end{aligned}$$

It follows from the comparison principle that, for any initial value ϕ , there exists $t_1(\phi) > 0$ such that $u(x, t) \leq \bar{u}$ for any $t > t_1(\phi)$ and $x \in \bar{\Omega}$. Consequently, $v(x, t)$ satisfies

$$\frac{\partial v}{\partial t} = d_2 \Delta v + \lambda v \left(1 - \frac{v}{u} \right) \leq d_2 \Delta v + \lambda v \left(1 - \frac{v}{\bar{u}} \right)$$

for $t > t_1(\phi)$. Similarly, we obtain that, for any initial value ϕ , there exists $t_2(\phi) > t_1(\phi)$ such that $v(x, t) \leq \bar{v}$ for any $t > t_2(\phi)$ and $x \in \bar{\Omega}$. Then, $u(x, t)$ satisfies

$$\begin{aligned} \frac{\partial u}{\partial t} &= d_1 \Delta u + u(r - h(v) - \alpha u) \\ &\geq d_1 \Delta u + u(r - h(\bar{v}) - \alpha u) \end{aligned}$$

for $t > t_2(\phi)$, where $h(v)$ is defined as in equation (3.7). This implies that, for any initial value ϕ , there exists $t_3(\phi) > t_2(\phi)$ such that $u(x, t) \geq \underline{u}$ for any $t > t_3(\phi)$ and $x \in \bar{\Omega}$. Consequently, we have

$$\frac{\partial v}{\partial t} \geq d_2 \Delta v + \lambda v \left(1 - \frac{v}{\underline{u}} \right), \quad (3.8)$$

for $t > t_3(\phi)$. Then, there exists $\bar{t}(\phi) > t_3(\phi)$ such that $v(x, t) \geq \underline{v}$ for any $t > \bar{t}(\phi)$ and $x \in \bar{\Omega}$. This completes the proof. \square

Then, by virtue of the upper and lower solution method [18–21], we show that the unique positive equilibrium is globally attractive under certain conditions.

Theorem 3.1. *Assume that equation (3.3) is satisfied. Then, for any initial value $\phi = (u_0(x), v_0(x))$, where $u_0(x) > 0$, $v_0(x) \geq (\neq) 0$, the corresponding solution of system (3.2) converges uniformly to the positive constant equilibrium $(u_*(k), v_*(k))$ as $t \rightarrow \infty$.*

Proof. It follows from lemma 3.1 that (\bar{u}, \bar{v}) and $(\underline{u}, \underline{v})$ defined in equation (3.6) is a pair of coupled upper and lower solution of system (3.2) for a sufficiently small

ϵ . A direct computation implies that there exists $K > 0$ such that, for any $(\underline{u}, \underline{v}) \leq (u_1, v_1), (u_2, v_2) \leq (\bar{u}, \bar{v})$,

$$\begin{aligned} & \left| u_1 \left(\frac{r}{1 + kv_1} - \alpha u_1 - \beta v_1 \right) - u_2 \left(\frac{r}{1 + kv_2} - \alpha u_2 - \beta v_2 \right) \right| \\ & \leq K(|u_1 - u_2| + |v_1 - v_2|), \\ & \left| \lambda v_1 \left(1 - \frac{v_1}{u_1} \right) - \lambda v_2 \left(1 - \frac{v_2}{u_2} \right) \right| \\ & \leq K(|u_1 - u_2| + |v_1 - v_2|). \end{aligned}$$

Then, we define two iteration sequences $(\bar{u}^{(m)}, \bar{v}^{(m)})$ and $(\underline{v}^{(m)}, \underline{u}^{(m)})$ in the following. For $m \geq 1$,

$$\begin{aligned} \bar{u}^{(m)} &= \bar{u}^{(m-1)} + \frac{\bar{u}^{(m-1)}}{K} \left[\frac{r}{1 + k\bar{v}^{(m-1)}} - \alpha \bar{u}^{(m-1)} - \beta \bar{v}^{(m-1)} \right], \\ \underline{u}^{(m)} &= \underline{u}^{(m-1)} + \frac{\underline{u}^{(m-1)}}{K} \left[\frac{r}{1 + k\underline{v}^{(m-1)}} - \alpha \underline{u}^{(m-1)} - \beta \underline{v}^{(m-1)} \right], \\ \bar{v}^{(m)} &= \bar{v}^{(m-1)} + \frac{\lambda \bar{v}^{(m-1)}}{K} \left(1 - \frac{\bar{v}^{(m-1)}}{\bar{u}^{(m-1)}} \right), \\ \underline{v}^{(m)} &= \underline{v}^{(m-1)} + \frac{\lambda \underline{v}^{(m-1)}}{K} \left(1 - \frac{\underline{v}^{(m-1)}}{\underline{u}^{(m-1)}} \right), \end{aligned}$$

and $(\bar{u}^{(0)}, \bar{v}^{(0)}) = (\bar{u}, \bar{v})$ and $(\underline{u}^{(0)}, \underline{v}^{(0)}) = (\underline{u}, \underline{v})$. It follows from [21] that, for $m \geq 1$,

$$(\underline{u}, \underline{v}) \leq (\underline{u}^{(m)}, \underline{v}^{(m)}) \leq (\underline{u}^{(m+1)}, \underline{v}^{(m+1)}) \leq (\bar{u}^{(m+1)}, \bar{v}^{(m+1)}) \leq (\bar{u}^{(m)}, \bar{v}^{(m)}) \leq (\bar{u}, \bar{v}).$$

Consequently, there exist (\tilde{u}, \tilde{v}) and (\check{u}, \check{v}) , where $(\underline{u}, \underline{v}) \leq (\check{u}, \check{v}) \leq (\tilde{u}, \tilde{v}) \leq (\bar{u}, \bar{v})$, such that $\lim_{m \rightarrow \infty} \bar{u}^{(m)} = \tilde{u}$, $\lim_{m \rightarrow \infty} \bar{v}^{(m)} = \tilde{v}$, $\lim_{m \rightarrow \infty} \underline{u}^{(m)} = \check{u}$, $\lim_{m \rightarrow \infty} \underline{v}^{(m)} = \check{v}$ and

$$\begin{aligned} \frac{r}{1 + k\tilde{v}} - \alpha \tilde{u} - \beta \tilde{v} &= 0, \quad 1 - \frac{\tilde{v}}{\tilde{u}} = 0, \\ \frac{r}{1 + k\check{v}} - \alpha \check{u} - \beta \check{v} &= 0, \quad 1 - \frac{\check{v}}{\check{u}} = 0. \end{aligned} \tag{3.9}$$

The above construction of $\{\bar{u}^{(m)}\}_{m=0}^{\infty}, \{\underline{u}^{(m)}\}_{m=0}^{\infty}, \{\bar{v}^{(m)}\}_{m=0}^{\infty}, \{\underline{v}^{(m)}\}_{m=0}^{\infty}$ and their corresponding properties are standard for the upper and lower solution method [21], and we only include here for the sake of completeness. It follows from equation (3.9) that

$$\begin{aligned} \frac{r}{1 + k\tilde{v}} - \alpha \tilde{v} - \beta \tilde{v} &= 0, \\ \frac{r}{1 + k\check{v}} - \alpha \check{v} - \beta \check{v} &= 0. \end{aligned} \tag{3.10}$$

Then,

$$\tilde{v} = \frac{1}{\alpha} \left(\frac{r}{1 + k\tilde{v}} - \beta \tilde{v} \right), \quad \check{v} = \frac{1}{\alpha} \left(\frac{r}{1 + k\check{v}} - \beta \check{v} \right). \tag{3.11}$$

Subtracting the first equation of (3.10) from the second equation, we have

$$\frac{kr(\tilde{v} - \check{v})}{(1 + k\tilde{v})(1 + k\check{v})} = (\alpha - \beta)(\tilde{v} - \check{v}). \quad (3.12)$$

We claim that $\tilde{v} = \check{v}$. If it is not true, then from equation (3.12), we have

$$(1 + k\tilde{v})(1 + k\check{v}) = \frac{kr}{\alpha - \beta}. \quad (3.13)$$

Plugging equation (3.11) into equation (3.13), we see that \tilde{v} and \check{v} are the roots of the following quadratic equation

$$\frac{k^2\beta}{\alpha}x^2 - \left(k - \frac{k\beta}{\alpha}\right)x + \frac{kr}{\alpha - \beta} - 1 - \frac{kr}{\alpha} = 0. \quad (3.14)$$

Note that k , r , α and β satisfy equation (3.3), which yields

$$\frac{kr}{\alpha - \beta} - 1 - \frac{kr}{\alpha} < 0.$$

Then, equation (3.14) cannot have two positive roots, which is a contradiction. Therefore, $\tilde{v} = \check{v}$, and consequently $\tilde{u} = \check{u}$. It follows from the upper and lower solution method [21] that the unique constant positive equilibrium $(u_*(k), v_*(k))$ is globally attractive, if equation (3.3) is satisfied. This completes the proof. \square

The characteristic equations with respect to the positive equilibrium $(u_*(k), v_*(k))$ of model (3.2) are as follows:

$$\begin{aligned} \mu^2 + [\alpha u_*(k) + \lambda + (d_1 + d_2)\gamma_i] \mu + (\alpha u_*(k) + d_1\gamma_i)(\lambda + d_2\gamma_i) \\ + \beta \lambda u_*(k) + \frac{\lambda r k u_*(k)}{(1 + k v_*(k))^2} = 0, \quad i = 0, 1, 2, \dots, \end{aligned} \quad (3.15)$$

where $\{\gamma_i\}_{i=1}^{\infty}$ satisfying

$$0 = \gamma_0 < \gamma_1 < \dots < \gamma_i < \gamma_{i+1} < \dots$$

are the eigenvalues of $-\Delta$. Clearly, all the roots of (3.15) have negative real parts, and consequently, $(u_*(k), v_*(k))$ is locally asymptotically stable. This, combined with Theorem 3.1, implies that $(u_*(k), v_*(k))$ is globally asymptotically stable.

Theorem 3.2. *Assume that equation (3.3) is satisfied. Then, $(u_*(k), v_*(k))$ is globally asymptotically stable, which attracts all the positive solutions of system (3.2).*

Similarly, we can also show that, under certain conditions, the high levels of fear will decrease the densities of both prey and predator for the diffusive case, and we omit the proof here.

Proposition 3.1. *Assume that $k_1 > k_2$, and k_i satisfies equation (3.3) for $i = 1, 2$. Let $(u_i(x, t), v_i(x, t))$ be the solution of model (3.2) for $k = k_i$, where $i = 1, 2$. Then, there exists $t_0 > 0$ such that*

$$(u_1(x, t), v_1(x, t)) < (u_2(x, t), v_2(x, t)),$$

for any $x \in \bar{\Omega}$ and $t \geq t_0$.

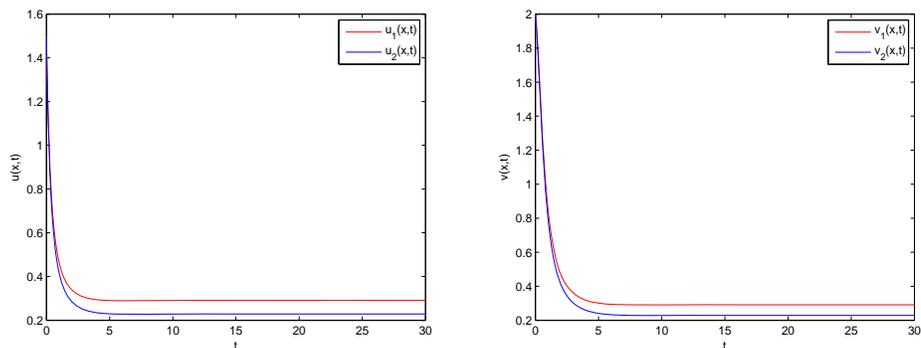


Figure 1. High levels of fear will decrease the densities of both prey and predator for the kinetic system. Here, $\alpha = 1$, $\beta = 0.5$, $\lambda = 1$, $r = 0.5$, the initial values $u(0) = 1.5$, and $v(0) = 2$, and $(u_1(x, t), v_1(x, t))$ is the corresponding solution with $k = 0.5$, and $(u_2(x, t), v_2(x, t))$ is the corresponding solution with $k = 2$.

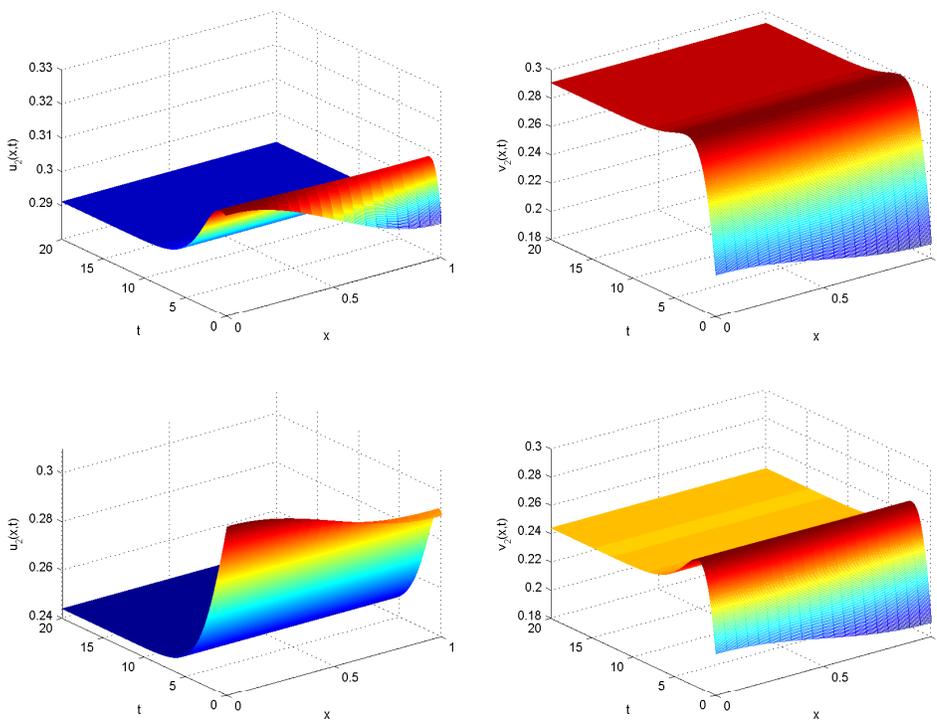


Figure 2. High levels of fear will decrease the densities of both prey and predator for the diffusive system. Here, $d_1 = 0.1$, $d_2 = 0.05$, $\alpha = 1$, $\beta = 0.5$, $\lambda = 1$, $r = 0.5$, $\Omega = (0, 1)$, and the initial values $u(x, t) = 0.3 + 0.01 \cos \pi x$, and $v(x, t) = 0.2 + 0.01 \cos \pi x$. (Upper) $k = 0.5$; (Lower) $k = 1.5$.

4. Numerical simulation: the effect of fear

It follows from Propositions 2.1 and 3.1 that high levels of fear will decrease the densities of both prey and predator. Moreover, in this section, we will numerically show this phenomenon, see Figure 1 for the kinetic system of model (1.4) and Figure 2 for the diffusive case. Here, we also choose $f(k, v) = \frac{1}{1 + kv}$ for simplicity.

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