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Positive Periodic Solutions for a Single-species Model with Delay Weak Kernel and Cycle Mortality*

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Abstract In this paper, by using the Krasnoselskii's fixed-point theorem, we study the existence of positive periodic solutions of the following single-species model with delay weak kernel and cycle mortality:

$$x'(t) = rx(t)\Big[1 - \frac{1}{K}\int_{-\infty}^{t} \alpha e^{-\alpha(t-s)} x(s)ds\Big] - a(t)x(t),$$

and get the necessary conditions for the existence of positive periodic solutions. Finally, an example and numerical simulation are used to illustrate the validity of our results.

Keywords Positive periodic solutions, Single-species model, Delay, Cycle mortality.

MSC(2010) 34C25, 34C60, 92D25.

1. Introduction

As we all know, the application of delay differential equations in population dynamics can be traced back to the 1920s, and in the past 100 years, the theory of population dynamics has achieved significant development. For example, see [2, 3, 9, 10]. In 1980, Gurney et al. [8] studied the delayed Nicholson's blowflies equation

$$N'(t) = PN(t-\tau)e^{-\alpha N(t-\tau)} - \delta N(t), \tag{1.1}$$

where $N(t)$ represents the population of mature adults at time t, $\frac{1}{\alpha}$ denotes the population size at which the complete population reproduces at its maximum rate, P denotes the maximum possible per capita egg production rate, $\tau > 0$ is a delay term and $\delta > 0$ is the mortality rate.

Consider the different practical conditions, model (1.1) is generalized to more general models. In 2008, Li et al. [11] used the Krasnoselskii's fixed-point theorem to prove the existence of the positive periodic solution of the following generalized

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Nicholson's blowflies model:

$$x'(t) = -\delta(t)x(t) + \sum_{i=1}^{m} p_i(t)x(t-\tau_i(t))e^{-q_i(t)x(t-\tau_i(t))}, \quad t \geq 0.$$

In [1], by using the Schauder's fixed-point theorem, they study the existence of positive periodic solutions of Nicholson's blowflies differential equation with iterative harvest function:

$$N'(t) = p(t)N(t-\tau)e^{-\gamma(t)N(t-\tau)} - a(t)N(t) - qN(t-\tau)E(t, N(t), ..., N^{[n]}(t)),$$

where E denotes the harvesting effort, defined as the intensity of the human activities to harvest the flies and $q \geq 0$ is the so-called the catchability coefficient, which express the fraction of the population that is removed by one unit of harvesting effort.

In 1934, Volterra [14] proposed a more accurate model based on the Logistic model:

$$\frac{dN(t)}{dt} = N(t)\Big[1 - \frac{1}{K}\int_{-\infty}^{t} G(t-s)N(s)ds\Big],$$

where $G(t)$ called the delay kernel, is a weighting factor which says how much emphasis should be given to the size of the population at earlier times to determine the present effect on resource availability. The delay kernel is usually normalized so that $\int_0^\infty G(u)du = 1$. Two special cases including

$$G(u) = \alpha e^{-\alpha u},$$

$$G(u) = \alpha^2 u e^{-\alpha u}$$

are called weak delay kernel and strong delay kernel.

Based on the discussions above, in this paper, we consider a single-species model with delay weak kernel and death term as follows:

$$x'(t) = rx(t)\Big[1 - \frac{1}{K}\int_{-\infty}^{t} \alpha e^{-\alpha(t-s)}x(s)ds\Big] - a(t)x(t), \tag{1.2}$$

where $r > 0$ is the intrinsic rate of growth for population, $x(t)$ is the size of the population at time t, $K > 0$ is the population's carrying capacity and $a(t)$ is the death rate and we assume that (H) holds in this article:

(H) $a : \mathbb{R} \to (0, 1)$ is T-periodic continuous coefficients satisfying $a(t) = a(t+T)$ for $t \in \mathbb{R}$.

In recent years, the existence of positive T-periodic solutions of periodic ecological models with delays have been studied by many authors. For examole, see [4–6, 12, 13, 15, 16] and references therein. However, it is worth mentioning that there is no conclusion about the existence of positive T-periodic solutions of equation (1.2). Therefore, our results are completely new.

The paper is organized as follows: In Section 2, we give a simple analysis of equation (1.2) and some definitions and lemmas are given. In Section 3, we use the Krasnoselskii's fixed-point theorem to obtain the positive periodic solution of the model (1.2) under some given conditions. In Section 4, an example is given to illustrate our results obtained in the previous section.

2. Auxiliary lemmas and preparations

For the sake of convenience, we would like to introduce some notations, definitions, lemmas and assumptions which are used in what follows in this section.

Definition 2.1 ([7]). Let $\mathbb{M}$ be a real Banach space. A nonempty, closed and convex set $\mathbb{Q} \subset \mathbb{M}$ is a cone if it satisfies the following two conditions:
(i) $x \in \mathbb{Q}$, $\lambda \geq 0$ imply $\lambda x \in \mathbb{Q}$;
(ii) $x \in \mathbb{Q}$, $-x \in \mathbb{Q}$ imply $x = \theta$, where θ is the zero element of $\mathbb{Q}$.

Definition 2.2 ([7]). An operator $P : \mathbb{M} \to \mathbb{M}$ is completely continuous, if it is continuous and maps bounded sets into relatively compact set.

The following is the well-known Kresnoselskii's fixed-point theorem in a cone.

Lemma 2.1 ([7]). *Let $\mathbb{M}$ be a Banach space, and let $\mathbb{Q} \subset \mathbb{M}$ be a cone. Assume that Ω_1 and Ω_2 are open subset of $\mathbb{M}$ with $\theta \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$, and let*

$$P : \mathbb{Q} \cap (\overline{\Omega}_2 \setminus \Omega_1) \to \mathbb{Q}$$

be a completely continuous operator such that
(i) $\|Pu\| \leq \|u\|$, $u \in \mathbb{Q} \cap \partial\Omega_1$, and $\|Pu\| \geq \|u\|$, $u \in \mathbb{Q} \cap \partial\Omega_2$; or
(ii) $\|Pu\| \geq \|u\|$, $u \in \mathbb{Q} \cap \partial\Omega_1$, and $\|Pu\| \leq \|u\|$, $u \in \mathbb{Q} \cap \partial\Omega_2$.
Then, P has a fixed point in $\mathbb{Q} \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

Here, we introduce some notations used throughout this paper. Let

$$\Gamma_1 = \frac{e^{-\int_0^T a(s)ds}}{e^{\int_0^T a(s)ds} - 1}, \quad \Gamma_2 = \frac{e^{\int_0^T a(s)ds}}{e^{\int_0^T a(s)ds} - 1}, \quad \text{and} \quad k = \frac{\Gamma_1}{\Gamma_2}.$$

For $T > 0$, let

$$\mathbb{M} = \{x \in C(\mathbb{R}, \mathbb{R}) : x(x + T) = x(t) \ \text{ for } \ t \in \mathbb{R}\}$$

be the Banach space of T-periodic continuous functions equipped with the norm

$$\|x\| = \max_{t \in \mathbb{R}} |x(t)| = \max_{t \in [0,T]} |x(t)|.$$

Define a subset in $\mathbb{M}$ by

$$\mathbb{Q} = \{x \in \mathbb{M} : x(t) \geq k\|x\|, \quad t \in [0, T]\}.$$

It is easy to see that $\mathbb{Q}$ is a cone in $\mathbb{M}$.
For such a solution x of (1.2), there are $\xi \in [0, T]$ and $\eta \in [0, T]$ such that

$$L_1 = x(\xi) = \min_{t \in [0,T]} x(t),$$

$$L_2 = x(\eta) = \max_{t \in [0,T]} x(t).$$

Lemma 2.2. *Let $a(t) \neq 0$, and $x \in \mathbb{Q}$. Then, x is a solution of (1.2), if and only if*

$$x(t) = \int_t^{t+T} rx(u)\Big[1 - \frac{1}{K}\int_{-\infty}^{u} \alpha e^{-\alpha(u-s)} x(s)ds\Big] G(t,u)du, \tag{2.1}$$

where

$$G(t,u) = \frac{e^{\int_t^u a(s)ds}}{e^{\int_0^T a(s)ds} - 1}.$$

Proof. Let $x \in \mathbb{Q}$ be a solution of equation (1.2). We have

$$(x'(t) + a(t)x(t))e^{\int_0^t a(s)ds} = rx(t)\Big[1 - \frac{1}{K}\int_{-\infty}^{t} \alpha e^{-\alpha(t-s)} x(s)ds\Big] e^{\int_0^t a(s)ds},$$

which is equivalent to

$$\frac{d}{dt}(x(t)e^{\int_0^t a(s)ds}) = rx(t)\Big[1 - \frac{1}{K}\int_{-\infty}^{t} \alpha e^{-\alpha(t-s)} x(s)ds\Big] e^{\int_0^t a(s)ds}.$$

The integration from t to $t+T$ gives

$$\begin{aligned} &x(t+T)e^{\int_0^{t+T} a(s)ds} - x(t)e^{\int_0^t a(s)ds} \\ &= \int_t^{t+T} rx(u)\Big[1 - \frac{1}{K}\int_{-\infty}^{u} \alpha e^{-\alpha(u-s)} x(s)ds\Big] e^{\int_0^u a(s)ds}du. \end{aligned}$$

By the periodic properties, we obtain

$$\begin{aligned} &x(t)e^{\int_0^t a(s)ds}[e^{\int_t^{t+T} a(s)ds} - 1] \\ &= \int_t^{t+T} rx(u)\Big[1 - \frac{1}{K}\int_{-\infty}^{u} \alpha e^{-\alpha(u-s)} x(s)ds\Big] e^{\int_0^u a(s)ds}du. \end{aligned}$$

Thus,

$$x(t) = \int_t^{t+T} rx(u)\Big[1 - \frac{1}{K}\int_{-\infty}^{u} \alpha e^{-\alpha(u-s)} x(s)ds\Big] \frac{e^{\int_t^u a(s)ds}}{e^{\int_0^T a(s)ds} - 1}du.$$

The proof is complete. □

Remark 2.1. The function G satisfies the following property:

$$\Gamma_1 = \frac{e^{-\int_0^T a(s)ds}}{e^{\int_0^T a(s)ds} - 1} \leq G(t,u) \leq \frac{e^{\int_0^T a(s)ds}}{e^{\int_0^T a(s)ds} - 1} = \Gamma_2.$$

Define an operator $P : \mathbb{M} \to \mathbb{M}$

$$(Px)(t) = \int_t^{t+T} rx(u)\Big[1 - \frac{1}{K}\int_{-\infty}^{u} \alpha e^{-\alpha(u-s)} x(s)ds\Big] G(t,u)du.$$

For $\forall x \in \mathbb{Q},\ t \in [0,T]$,

$$x(t) \leq \Gamma_2 \int_t^{t+T} rx(u)\Big[1 - \frac{1}{K}\int_{-\infty}^{u} \alpha e^{-\alpha(u-s)} x(s)ds\Big]du,$$

and

$$x(t) \geq \Gamma_1 \int_t^{t+T} rx(u)\Big[1 - \frac{1}{K}\int_{-\infty}^{u} \alpha e^{-\alpha(u-s)} x(s)ds\Big]du.$$

Thus,

$$(Px)(t) \geq \frac{\Gamma_1}{\Gamma_2}\|Px\| = k\|Px\|.$$

Hence,

$$P\mathbb{Q} \subset \mathbb{Q}.$$

Lemma 2.3. *Let (H) holds. Then, the operator $P : \mathbb{Q} \to \mathbb{Q}$ is completely continuous.*

Proof. We need to verify the following two points:
(i) P is continuous;
(ii) P maps any bounded subset of $\mathbb{Q}$ into a relatively compact subset of $\mathbb{Q}$.
Point(i): Let $\varphi, \psi \in \mathbb{Q}$. From (2.1), we get

$$\begin{aligned}
&\Big|(P\varphi)(t) - (P\psi)(t)\Big| \\
&= \Big|\int_t^{t+T} r\varphi(u)\Big[1 - \frac{1}{K}\int_{-\infty}^{u} \alpha e^{-\alpha(u-s)}\varphi(s)ds\Big]G(t,u)du \\
&\quad - \int_t^{t+T} r\psi(u)\Big[1 - \frac{1}{K}\int_{-\infty}^{u} \alpha e^{-\alpha(u-s)}\psi(s)ds\Big]G(t,u)du\Big| \\
&\leq \Big|\int_t^{t+T} rG(t,u)(\varphi(u) - \psi(u))du\Big| \\
&\quad + \Big|\int_t^{t+T} \frac{r}{K}G(t,u)\Big[\int_{-\infty}^{u} \alpha e^{-\alpha(u-s)}(\varphi(u)\varphi(s) - \psi(u)\psi(s))ds\Big]du\Big| \\
&\leq r\Gamma_2 T\|\varphi - \psi\| + 2\frac{r}{K}\Gamma_2 T L_2\|\varphi - \psi\| \\
&\leq r\Gamma_2 T(1 + \frac{2L_2}{K})\|\varphi - \psi\|.
\end{aligned}$$

Therefore, the operator P is continuous.
Point(ii): For $\forall x \in \mathbb{Q},\ t \in [0,T]$,

$$\begin{aligned}
\Big|(Px)(t)\Big| = (Px)(t) &= \int_t^{t+T} rx(u)\Big[1 - \frac{1}{K}\int_{-\infty}^{u} \alpha e^{-\alpha(u-s)} x(s)ds\Big]G(t,u)du \\
&\leq\ r\Gamma_2 T L_2(1 - \frac{L_1}{K}).
\end{aligned} \tag{2.2}$$

Hence, $\{Px : x \in \mathbb{Q}\}$ is a family of uniformly bounded. Now, we show that the operator P is equicontinuous. For $\forall x \in \mathbb{Q},\ t_1, t_2 \in [0, T]$ with $t_1 < t_2$, we have

$$\begin{aligned}
&\Big|(Px)(t_1) - (Px)(t_2)\Big| \\
&= \Big| \int_{t_2}^{t_2+T} rx(u)\Big[1 - \frac{1}{K}\int_{-\infty}^{u} \alpha e^{-\alpha(u-s)} x(s)ds\Big] G(t_2, u)du \\
&\quad - \int_{t_1}^{t_1+T} rx(u)\Big[1 - \frac{1}{K}\int_{-\infty}^{u} \alpha e^{-\alpha(u-s)} x(s)ds\Big] G(t_1, u)du\Big| \\
&\le \int_{t_2}^{t_1} rx(u)\Big[1 - \frac{1}{K}\int_{-\infty}^{u} \alpha e^{-\alpha(u-s)} x(s)ds\Big] G(t_2, u)du \\
&\quad + \int_{t_1}^{t_1+T} rx(u)\Big[1 - \frac{1}{K}\int_{-\infty}^{u} \alpha e^{-\alpha(u-s)} x(s)ds\Big] \Big|G(t_2, u) - G(t_1, u)\Big| du \\
&\quad + \int_{t_1+T}^{t_2+T} rx(u)\Big[1 - \frac{1}{K}\int_{-\infty}^{u} \alpha e^{-\alpha(u-s)} x(s)ds\Big] G(t_2, u)du. \qquad (2.3)
\end{aligned}$$

We write

$$\Big|G(t_2, u) - G(t_1, u)\Big| = \frac{1}{e^{\int_0^T a(s)ds} - 1}\Big|e^{\int_{t_2}^{u} a(s)ds} - e^{\int_{t_1}^{u} a(s)ds}\Big|.$$

We have

$$\int_{t_1}^{t_1+T} \Big|e^{\int_{t_2}^{u} a(s)ds} - e^{\int_{t_1}^{u} a(s)ds}\Big| du = \int_{t_1}^{t_1+T} e^{\int_{t_2}^{u} a(s)ds}\Big|1 - e^{\int_{t_1}^{t_2} a(s)ds}\Big| du.$$

This immediately implies that

$$\int_{t_1}^{t_1+T} \Big|e^{\int_{t_2}^{u} a(s)ds} - e^{\int_{t_1}^{u} a(s)ds}\Big| du \le T\|a_0\||t_2 - t_1|e^{\int_0^T a(s)ds}.$$

Consequently,

$$\begin{aligned}
\int_{t_1}^{t_1+T} \Big|G(t_2, u) - G(t_1, u)\Big| du &\le T\|a_0\||t_2 - t_1|\frac{e^{\int_0^T a(s)ds}}{e^{\int_0^T a(s)ds} - 1} \\
&= T\Gamma_2\|a_0\||t_2 - t_1|. \qquad (2.4)
\end{aligned}$$

It follows from (2.3) and (2.4) that

$$\begin{aligned}
\Big|(Px)(t_1) - (Px)(t_2)\Big| &\le 2r\Gamma_2 L_2(1 - \frac{L_1}{K})|t_2 - t_1| + r\Gamma_2 T L_2(1 - \frac{L_1}{K})\|a_0\||t_2 - t_1| \\
&\le r\Gamma_2 L_2(1 - \frac{L_1}{K})(2 + T\|a_0\|)|t_2 - t_1|.
\end{aligned}$$

Therefore, the operator P is equicontinuous. By the Arzela-Ascoli Theorem, we know that the operator P is completely continuous. The proof is complete. □

Lemma 2.4. *Let (H) holds. Then, there are positive constants A and B such that for $x \in \mathbb{Q}$,*

$$\|Px\| \le B \qquad \text{and} \qquad \|Px\| \ge A.$$

Proof. From (2.2), for $\forall x \in \mathbb{Q}, t \in [0, T]$,

$$\begin{aligned}\Big|(Px)(t)\Big| = (Px)(t) &= \int_t^{t+T} rx(u)\Big[1 - \frac{1}{K}\int_{-\infty}^{u} \alpha e^{-\alpha(u-s)}x(s)ds\Big]G(t,u)du \\ &\leq\ r\Gamma_2 T L_2(1 - \frac{L_1}{K}) = B.\end{aligned}$$

Then,

$$\|Px\| \leq B.$$

For $\forall x \in \mathbb{Q}, t \in [0, T]$,

$$\begin{aligned}\Big|(Px)(t)\Big| = (Px)(t) &= \int_t^{t+T} rx(u)\Big[1 - \frac{1}{K}\int_{-\infty}^{u} \alpha e^{-\alpha(u-s)}x(s)ds\Big]G(t,u)du \\ &\geq\ r\Gamma_1 T L_1(1 - \frac{L_2}{K}) = A > 0.\end{aligned}$$

For $\forall x \in \mathbb{Q}$,

$$\|Px\| \geq A.$$

Thus, the proof of Lemma 2.4 is complete. □

3. Main results

Theorem 3.1. *Suppose $a(t) \neq 0$. If (H) holds, then equation (1.2) has at least one positive T-periodic solution x in $\mathbb{Q}$.*

Proof. Let

$$\Omega_1 = \{x \in \mathbb{M} : \|x\| < A\},$$

and

$$\Omega_2 = \{x \in \mathbb{M} : \|x\| < B\}.$$

Obviously, Ω_1 and Ω_2 are open bounded subsets in $\mathbb{M}$, and $\theta \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$. From Lemma 2.3, $P : \mathbb{Q} \cap (\overline{\Omega}_2 \setminus \Omega_1) \to \mathbb{Q}$ is completely continuous.

If $x \in \mathbb{Q} \cap \partial\Omega_2$, which implies that $\|x\| = B$, and from Lemma 2.4, $\|Px\| \leq B$. Hence, $\|Px\| \leq \|x\|$ for $x \in \mathbb{Q} \cap \partial\Omega_2$.

If $x \in \mathbb{Q} \cap \partial\Omega_1$, which implies that $\|x\| = A$, and from Lemma 2.4, $\|Px\| \geq A$. Hence, $\|Px\| \geq \|x\|$ for $x \in \mathbb{Q} \cap \partial\Omega_1$.

From Lemma 2.1, the operator P has at least one fixed point lying in $\mathbb{Q} \cap (\overline{\Omega}_2 \setminus \Omega_1)$. For example, equation (1.2) has at least one positive T-periodic solution. Theorem 3.1 is proved. □

Theorem 3.2. *Assume that (H) holds, and that*

$$r \leq a(t) \quad for \quad t \in [0, T]. \tag{3.1}$$

Then, every positive solution of Equation (1.2) tends to zero as $t \to \infty$.

Proof. Let $x(t)$ be any positive solution of equation (1.2). Equation (1.2) changes into

$$(x(t)e^{\int_0^t a(s)ds})' = rx(t)\Big[1 - \frac{1}{K}\int_{-\infty}^{t} \alpha e^{-\alpha(t-s)}x(s)ds\Big]e^{\int_0^t a(s)ds}.$$

Integrating the above from $t_0 > 0$ to $t > t_0$, we have

$$x(t) = x(t_0)e^{-\int_{t_0}^t a(s)ds} + \int_{t_0}^{t} rx(u)\Big[1 - \frac{1}{K}\int_{-\infty}^{u} \alpha e^{-\alpha(u-s)}x(s)ds\Big]e^{\int_t^u a(s)ds}du.$$

From (3.1),

$$x(t) \le x(t_0)e^{-\int_{t_0}^t a(s)ds} + (1 - \frac{L_1}{K})\int_{t_0}^{t} x(u)a(u)e^{\int_t^u a(s)ds}du.$$

Let $\zeta = \limsup\limits_{t\to\infty} x(t)$, then $0 \le \zeta < \infty$. Below we prove that $\zeta = 0$. We divide it into three cases.

Case 1. When $x'(t) > 0$, choose $t_0 > 0$ such that $x'(t) > 0$ for $t > t_0$. Then, $0 < x(t_0) < x(t)$ for $t > t_0$. From (1.2),

$$\begin{aligned} 0 &< rx(t)\Big[1 - \frac{1}{K}\int_{-\infty}^{t} \alpha e^{-\alpha(t-s)}x(s)ds\Big] - a(t)x(t) \\ &< \Big[r(1 - \frac{L_1}{K}) - a(t)\Big]x(t) < 0. \end{aligned}$$

This contradiction shows that Case 1 is impossible.

Case 2. When $x'(t) < 0$. Choose $t_0 > 0$ such that $x'(t) < 0$ for $t > t_0$. Then, $\zeta < x(t) < x(t_0)$ for $t > t_0$. From (3.1), we have

$$\begin{aligned} x(t) &\le x(t_0)e^{-\int_{t_0}^t a(s)ds} + (1 - \frac{L_1}{K})\int_{t_0}^{t} x(u)a(u)e^{\int_t^u a(s)ds}du \\ &\le x(t_0)e^{-\int_{t_0}^t a(s)ds} + x(t_0)(1 - \frac{L_1}{K})\Big[1 - e^{-\int_{t_0}^t a(s)ds}\Big]. \end{aligned} \tag{3.2}$$

Let $t \to \infty$ in (3.2), we obtain

$$\zeta \le x(t_0)(1 - \frac{L_1}{K}).$$

Again, let $t_0 \to \infty$ in the above, we have that $\zeta \le \zeta(1 - \frac{L_1}{K})$, which implies that $\zeta = 0$.

Case 3. When $x'(t)$ is oscillatory, in this case, there is t_n with $t_n \to \infty$ as $n \to \infty$ such that

$$x'(t_n) = 0 \quad \text{for} n = 1, 2, \cdots, \quad \text{and} \quad \lim_{n\to\infty} x(t_n) = \zeta.$$

From (1.2), we have

$$a(t_n)x(t_n) = rx(t_n)\Big[1 - \frac{1}{K}\int_{-\infty}^{t_n} \alpha e^{-\alpha(t_n-s)}x(s)ds\Big]$$

$$\leq rx(t_n)(1-\frac{L_1}{K}).$$

Transform the above formula, we have

$$0 \leq \Big[r(1-\frac{L_1}{K})-a(t_n)\Big]x(t_n).$$

Let $n \to \infty$ in the above, we have that $\zeta = 0$. The proof is complete. □

From Theorem 3.2, we have the following results immediately.

Corollary 3.1. *Let (H) and (3.1) hold. Then, (1.2) has no positive T-periodic solution.*

Corollary 3.2. *Let (H) holds, and let $r > a(t)$. Then, (1.2) has at least one positive T-periodic solution.*

4. Example

In this section, we give an example to illustrate the correctness of our main results.

Example 4.1. Let $r = \frac{1}{2}$, $T = 35$ days, $L_1 = 10$, $L_2 = 30$, $K = 100$, $\alpha = 1$ and $a(t) = \frac{1}{200}\sin^2\frac{2\pi}{35}t$.

Proof. There are

$$\Gamma_1 = \frac{e^{-\int_0^T a(s)ds}}{e^{\int_0^T a(s)ds}-1} \simeq 0.85802,$$

$$\Gamma_2 = \frac{e^{\int_0^T a(s)ds}}{e^{\int_0^T a(s)ds}-1} \simeq 1.0221.$$

Thus,

$$\mathbb{Q} = \{x \in \mathbb{M} : x(t) \geq 0.83947\|x\|, \quad t \in [0,35]\}.$$

For $\forall x \in \mathbb{Q}$, we consider the following equation:

$$x'(t) = \frac{1}{2}x(t)\Big[1-\frac{1}{100}\int_{-\infty}^{t} e^{-(t-s)}x(s)ds\Big] - \frac{1}{200}(\sin^2\frac{2\pi}{35}t)x(t).$$

We have

$$A = r\Gamma_1 TL_1(1-\frac{L_2}{K}) = 105.11,$$

$$B = r\Gamma_2 TL_2(1-\frac{L_1}{K}) = 428.94.$$

Hence,

$$\|Px\| \geq 105.11 \qquad \text{and} \qquad \|Px\| \leq 428.94.$$

Since all conditions of Theorem 3.1 are satisfied, the considered equation has at least one positive periodic solution on $\mathbb{Q}$ and $105.11 \leq \|x\| \leq 428.94$. □

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