Journal of Nonlinear Modeling and Analysis
Volume 3, Number 4, December 2021, 647–661
http://jnma.ca; http://jnma.ijournal.cn
DOI:10.12150/jnma.2021.647

Solvability and Stability for Singular Fractional (p, q)-difference Equation*

Zhongyun Qin[1] and Shurong Sun[1,†]

Abstract In this paper, we initiate the solvability and stability for a class of singular fractional (p, q)-difference equations. First, we obtain an existence theorem of solution for the fractional (p, q)-difference equation. Then, by using a fractional (p, q)-Gronwall inequality, some stability criteria of solution are established, which also implies the uniqueness of solution.

Keywords Fractional (p, q)-difference equation, Existence of solution, Stability, (p, q)-Gronwall inequality, Uniqueness.

MSC(2010) 34A12, 39A30, 39A70.

1. Introduction

The q-difference is a kind of discrete calculus. In the early 20th century, the q-difference was first systematically studied by Jackson [9]. After the 1970s, the theory of q-difference has been extensively studied. Mitlagel Leffler proposed the theory of fractional q-difference operators, and later the related theories such as q-Laplace and Fourier Transform, q-Sturm-Liouville theory, q-Taylor expansion, q-Bernstein polynomial and so on attracted a great deal of attention [2,7,10]. In recent years, q-difference has been more and more frequently used in natural science and engineering. It plays an important role in mathematical physical models, dynamical systems, quantum physics and economics. For more details, the reader may refer to [8, 12, 15].

Motivated by these applications of q-calculus which is also called quantum calculus, many researchers have developed the theory of quantum calculus based on two-parameter p and q. In 1991, Chakrabarti and Jagannathan first investigated the (p, q)-calculus in quantum algebras [5]. For some results on the study of (p, q)-calculus, we refer to [11, 13, 14, 17]. The (p, q)-calculus is used efficiently in many fields such as physical sciences, hypergeometric series, lie group, special functions, approximation theory, Bezier curves and surfaces and etc.

The problem of fractional calculus in discrete settings has become an active research area [1,3,6]. Agarwal [1] and Al-Salam [3] introduced fractional q-difference calculus, while Diaz and Osler [6] studied fractional difference calculus. Recently, Brikshavana and Sitthiwirattham havd studied the fractional Hahn calculus [4]. In

†the corresponding author.
Email address: sshrong@163.com (S. Sun), Qinzhongyun1@163.com (Z. Qin)
[1]School of Mathematical Sciences, University of Jinan, Jinan, Shandong 250022, China
*The authors were supported by Natural Science Foundation of China (No. 62073153) and Shandong Provincial Natural Science Foundation (No. ZR2020MA016).

2020, Soontharanonl and Sitthiwirattham introduced the fractional (p,q)-calculus [18]. In the meantime, they studied the existence of a fractional (p,q)-difference equation. Compared with q-difference equations, (p,q)-difference equations have two quantization parameters p and q, which are genuinely independent. They have wider applicability in concrete mathematical models of quantum mechanics and fluid mechanics [5].

Although many interesting results related to discrete analogues of some topics of continuous fractional calculus have been studied, the theory of discrete fractional calculus remains much less developed than that of continuous fractional calculus. In particular, there are few papers about fractional (p,q)-calculus. As far as we know, the stability of fractional (p,q)-difference equation has not been studied, even for regular fractional (p,q)-difference equation. Up to now, no research has existed about solvability for Caputo type fractional (p,q)-difference equations. The gap mentioned is the motivation for this research.

In this paper, we consider the solvability and stability of the fractional (p,q)-difference equation:

$$\begin{cases} {}^cD^{\alpha}_{p,q}x(t) = f\left(t, x(t)\right), \ t > 0, \\ x(0) = x_0, \end{cases} \tag{1.1}$$

where $0 < \alpha < 1$, $0 < q < p \leq 1$, and ${}^cD^{\alpha}_{p,q}$ is Caputo type fractional (p,q)-difference operator. In this paper, we first prove that the fractional (p,q)-difference equation has at least one solution if $t^{\alpha}f(t,x)$ is continuous on variables t and x by Ascoli-Arzela's lemma. Furthermore, we establish a fractional (p,q)-Gronwall inequality. By the fractional (p,q)-Gronwall inequality, we obtain a stability criterion.

This paper is structured as follows: In Section 2, we present necessary definitions, properties and lemmas. In Section 3 and Section 4, some results on the existence of solution and stability are obtained. An example is given in Section 5. Finally, we end the paper with a conclusion.

2. Preliminaries

In this section, we present basic definitions, notations, and lemmas that will be used in this paper. Let $0 < q < p \leq 1$. We introduce the notation [18]:

$$[k]_{p,q} := \begin{cases} \dfrac{p^k - q^k}{p - q} = p^{k-1}[k]_{\frac{q}{p}}, & k \in \mathbb{N}, \\ 1, & k = 0, \end{cases}$$

and the (p,q)-analogue factorial is defined as:

$$[k]_{p,q}! := \begin{cases} [k]_{p,q}[k-1]_{p,q} \cdots [1]_{p,q} = \displaystyle\prod_{i=1}^{k} \frac{p^i - q^i}{p - q}, & k \in \mathbb{N}, \\ 1, & k = 0. \end{cases}$$

The (p,q)-analogue of the power function $(a-b)^{(n)}_{p,q}$ with $n \in N_0 := \{0,1,2,\ldots\}$ is given by

$$(a-b)^{(0)}_{p,q} := 1, \qquad (a-b)^{(n)}_{p,q} := \prod_{k=0}^{n-1}(ap^k - bq^k), \qquad a, b \in \mathbb{R}.$$

If $\alpha \in \mathbb{R}$, the general form is given by

$$(a-b)_q^{(\alpha)} := a^\alpha \prod_{i=0}^{\infty} \Big[\frac{1-(\frac{b}{a})q^i}{1-(\frac{b}{a})q^{\alpha+i}}\Big], \quad a \neq 0.$$

Furthermore, for $\alpha \in \mathbb{R}$, the fractional (p,q)-analogue of the power function is given by

$$(a-b)_{p,q}^{(\alpha)} = p^{\binom{\alpha}{2}}(a-b)_{\frac{q}{p}}^{(\alpha)} = a^\alpha \prod_{i=0}^{\infty} \frac{1}{p^\alpha}\Big[\frac{1-(\frac{b}{a})(\frac{q}{p})^i}{1-(\frac{b}{a})(\frac{q}{p})^{\alpha+i}}\Big], \quad a \neq 0,$$

where $\binom{\alpha}{2} = \frac{\alpha(\alpha-1)}{2}$, and $(a-b)_{p,q}^{(\alpha)} = a^\alpha(1-\frac{b}{a})_{p,q}^{(\alpha)}$. For $0 < q < p \leq 1$, the (p,q)-gamma and (p,q)-beta functions are defined by

$$\Gamma_{p,q}(x) = \begin{cases} \dfrac{(p-q)_{p,q}^{(x-1)}}{(p-q)^{x-1}} = \dfrac{(1-\frac{q}{p})_{p,q}^{(x-1)}}{(1-\frac{q}{p})^{x-1}}, x \in \mathbb{R} \setminus \{0,-1,-2,\cdots\}, \\ [x-1]_{p,q}!, \qquad\qquad\qquad\qquad x \in \mathbb{N}, \end{cases}$$

$$B_{p,q}(x,y) := \int_0^1 t^{x-1}(1-qt)_{p,q}^{\underline{y-1}} d_{p,q}t = p^{\frac{1}{2}(y-1)(2x+y-2)} \frac{\Gamma_{p,q}(x)\Gamma_{p,q}(y)}{\Gamma_{p,q}(x+y)},$$

respectively.

Definition 2.1. ([16]) Let $0 < q < p \leq 1$. The (p,q)-derivative of the function f is defined as

$$D_{p,q}f(t) := \frac{f(pt)-f(qt)}{(p-q)t}, \ t \neq 0,$$

and $(D_{p,q}f)(0) = \lim_{t\to 0}(D_{p,q}f)(t)$, provided that f is differentiable at 0. Meanwhile, the high order (p,q)-derivative $D_{p,q}^n f(t)$ is defined by

$$D_{p,q}^n f(t) = \begin{cases} f(t), n = 0, \\ D_{p,q}D_{p,q}^{n-1}f(t), n \in \mathbb{N}^+. \end{cases}$$

Definition 2.2. ([16]) Let $0 < q < p \leq 1$, f be an arbitrary function, and x be a real number. The (p,q)-integral of the function f is defined as

$$\int_0^x f(t)d_{p,q}t = (p-q)x\sum_{k=0}^{\infty} \frac{q^k}{p^{k+1}} f\left(\frac{q^k}{p^{k+1}}x\right), \tag{2.1}$$

provided that the series of right side in (2.1) converges. In this case, f is called (p,q)-integrable on $[0,x]$, and denote

$$I_{p,q}f(x) = \int_0^x f(t)d_{p,q}t.$$

Definition 2.3. ([16]) Let $0 < q < p \leq 1$, f be an arbitrary function, a and b be two real numbers. Then, we define

$$\int_a^b f(t)d_{p,q}t = \int_0^b f(t)d_{p,q}t - \int_0^a f(t)d_{p,q}t.$$

Lemma 2.1. *Let $0 < q < p \leq 1$, a and b be two real numbers. Then, the following formulas hold that:*

(a) $\int_a^a f(t)d_{p,q}t = 0$;

(b) $\int_a^b \alpha f(t)d_{p,q}t = \alpha \int_a^b f(t)d_{p,q}t,\ \alpha \in \mathbb{R}$;

(c) $\int_a^b f(t)d_{p,q}t = -\int_b^a f(t)d_{p,q}t$;

(d) $\int_a^b f(t)d_{p,q}t = \int_c^b f(t)d_{p,q}t + \int_a^c f(t)d_{p,q}t,\ c \in \mathbb{R},\ a < c < b$;

(e) $\int_a^b [f(t) + g(t)]d_{p,q}t = \int_a^b f(t)d_{p,q}t + \int_a^b g(t)d_{p,q}t$.

By (2.1), the above lemma can be easily proved. Therefore, we omit it.

Definition 2.4. ([18]) Let $\alpha > 0,\ 0 < q < p \leq 1$, and f be an arbitrary function on $[0, \infty)$. The fractional (p, q)-integral is defined by

$$I_{p,q}^{\alpha} f(t) = \frac{1}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} \int_0^t (t - qs)_{p,q}^{(\alpha-1)} f\left(\frac{s}{p^{\alpha-1}}\right) d_{p,q}s,$$

and $I_{p,q}^{0} f(t) = f(t)$.

Definition 2.5. ([18]) Let $\alpha > 0,\ 0 < q < p \leq 1$, and $f : (0, \infty) \to \mathbb{R}$ be an arbitrary function. For any $t \in (0, \infty)$, the fractional (p, q)-difference operator of Caputo type of order α is defined by

$$\begin{aligned} {}^{c}D_{p,q}^{\alpha} f(t) &:= I_{p,q}^{N-\alpha} D_{p,q}^{N} f(t) \\ &= \frac{1}{p^{\binom{N-\alpha}{2}}\Gamma_{p,q}(N - \alpha)} \int_0^t (t - qs)_{p,q}^{(N-\alpha-1)} D_{p,q}^{N} f\left(\frac{s}{p^{N-\alpha-1}}\right) d_{p,q}s, \end{aligned}$$

and ${}^{c}D_{p,q}^{0} f(t) = f(t)$, where $N - 1 < \alpha \leq N,\ N \in \mathbb{N}$.

Lemma 2.2. *([18]) For $\alpha > 0$, $0 < q < p \leq 1$, and $f : (0, \infty) \to \mathbb{R}$, we get*

$$D_{p,q}^{\alpha} I_{p,q}^{\alpha} f(t) = f(t).$$

Lemma 2.3. *([18]) For $\alpha,\ \beta > 0$, and $0 < q < p \leq 1$, (p, q)-integral and (p, q)-difference operators have the following properties:*

(a) $I_{p,q}^{\alpha}[I_{p,q}^{\beta} f(x)] = I_{p,q}^{\beta}[I_{p,q}^{\alpha} f(x)] = I_{p,q}^{\alpha+\beta} f(x)$,

(b) $D_{p,q} I_{p,q} f(x) = f(x)$, *and* $I_{p,q} D_{p,q} f(x) = f(x) - f(0)$.

Lemma 2.4. *(Variable substitution) Let $0 < q < p \leq 1$ and $0 < \alpha < 1$. Then, we have*

$$\int_0^t (t - qs)_{p,q}^{(\alpha-1)} d_{p,q}s = t^{\alpha} \int_0^1 (1 - q\tau)_{p,q}^{(\alpha-1)} d_{p,q}\tau, \tag{2.2}$$

and

$$\int_0^t (t - qs)_{p,q}^{(\alpha-1)} s^{-\alpha} d_{p,q}s = \int_0^1 (1 - q\tau)_{p,q}^{(\alpha-1)} \tau^{-\alpha} d_{p,q}\tau. \tag{2.3}$$

Proof. According to Definition 2.2, we have

$$\begin{aligned} \int_0^t (t - qs)_{p,q}^{(\alpha-1)} d_{p,q}s &= (p - q)t \sum_{k=0}^{\infty} \frac{q^k}{p^{k+1}} \left(t - q\frac{q^k}{p^{k+1}}t\right)_{p,q}^{(\alpha-1)} \\ &= (p - q)t \sum_{k=0}^{\infty} \frac{q^k}{p^{k+1}} t^{\alpha-1} \left(1 - q\frac{q^k}{p^{k+1}}\right)_{p,q}^{(\alpha-1)} \\ &= t^{\alpha}(p - q) \sum_{k=0}^{\infty} \frac{q^k}{p^{k+1}} \left(1 - q\frac{q^k}{p^{k+1}}\right)_{p,q}^{(\alpha-1)}, \end{aligned}$$

and

$$\int_0^1 (1-q\tau)_{p,q}^{(\alpha-1)} d_{p,q}\tau = (p-q)\sum_{k=0}^{\infty} \frac{q^k}{p^{k+1}} \left(1-q\frac{q^k}{p^{k+1}}\right)_{p,q}^{(\alpha-1)}.$$

Hence, the equality (2.2) holds. Furthermore, according to Definition 2.2, we have

$$\begin{aligned}\int_0^t (t-qs)_{p,q}^{(\alpha-1)} s^{-\alpha} d_{p,q}s &= (p-q)t\sum_{k=0}^{\infty} \frac{q^k}{p^{k+1}} \left(t-q\frac{q^k}{p^{k+1}}t\right)_{p,q}^{(\alpha-1)} (\frac{q^k}{p^{k+1}}t)^{-\alpha} \\ &= (p-q)t\sum_{k=0}^{\infty} \frac{q^k}{p^{k+1}} t^{\alpha-1} \left(1-q\frac{q^k}{p^{k+1}}\right)_{p,q}^{(\alpha-1)} t^{-\alpha}(\frac{q^k}{p^{k+1}})^{-\alpha} \\ &= (p-q)\sum_{k=0}^{\infty} \frac{q^k}{p^{k+1}} \left(1-q\frac{q^k}{p^{k+1}}\right)_{p,q}^{(\alpha-1)} (\frac{q^k}{p^{k+1}})^{-\alpha},\end{aligned}$$

and

$$\int_0^1 (1-q\tau)_{p,q}^{(\alpha-1)} \tau^{-\alpha} d_{p,q}\tau = (p-q)\sum_{k=0}^{\infty} \frac{q^k}{p^{k+1}} \left(1-q\frac{q^k}{p^{k+1}}\right)_{p,q}^{(\alpha-1)} (\frac{q^k}{p^{k+1}})^{-\alpha}.$$

Hence, the equality (2.3) holds. This completes the proof.

3. The solvability of fractional (p, q)-difference equation

We consider the following Caputo type fractional (p, q)-difference equation with initial value:

$$\begin{cases} {}^cD_{p,q}^{\alpha} x(t) = f\big(t, x(t)\big), \ t > 0, \\ x(0) = x_0, \end{cases} \tag{3.1}$$

where $0 < \alpha < 1$, $0 < q < p \le 1$, ${}^cD_{p,q}^{\alpha}$ is Caputo type fractional (p, q)-difference operator, and $t^\alpha f(t, x)$ is continuous on $[0, \infty) \times (-\infty, \infty)$. If a continuous function $x(t)$ satisfies the (3.1), $x(t)$ is called a solution of (3.1).

Lemma 3.1. *Let $0 < q < p \le 1$ and $0 < \alpha < 1$. Then, $x(t)$ is a solution of (3.1), if and only if it is the solution of the following integral equation*

$$x(t) = x_0 + \frac{1}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} \int_0^t (t-qs)_{p,q}^{(\alpha-1)} f\Big(\frac{s}{p^{\alpha-1}}, x\Big(\frac{s}{p^{\alpha-1}}\Big)\Big) d_{p,q}s, \ t \in (0, \infty), \ x(0) = x_0. \tag{3.2}$$

Proof. Assume $x(t)$ is a solution of (3.1). Taking operator $I_{p,q}^{\alpha}$ on both sides of (3.1), and by Definition 2.5, we have

$$I_{p,q}^{\alpha} I_{p,q}^{1-\alpha} D_{p,q} x(t) = I_{p,q}^{\alpha} f\big(t, x(t)\big), \ t \in (0, \infty).$$

By Lemma 2.3 and Definition 2.4, we get

$$I_{p,q} D_{p,q} x(t) = \frac{1}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} \int_0^t (t-qs)_{p,q}^{(\alpha-1)} f\Big(\frac{s}{p^{\alpha-1}}, x\Big(\frac{s}{p^{\alpha-1}}\Big)\Big) d_{p,q}s.$$

Furthermore, from Lemma 2.3, we have

$$x(t)-x(0)=x(t)-x_0=\frac{1}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)}\int_0^t (t-qs)_{p,q}^{(\alpha-1)} f\Big(\frac{s}{p^{\alpha-1}}, x\Big(\frac{s}{p^{\alpha-1}}\Big)\Big) d_{p,q}s.$$

Next, we prove the sufficiency of this lemma. Assuming $x(t)$ satisfies (3.2), from Lemma 2.3, Definitions 2.4 and 2.5, we have

$$I_{p,q}^{\alpha}\,{}^{c}D_{p,q}^{\alpha}x(t)=I_{p,q}D_{p,q}x(t)=I_{p,q}^{\alpha}f\big(t,x(t)\big),\ t\in(0,\infty).$$

Taking operator $D_{p,q}^{\alpha}$ on both sides of above formula, and by Lemma 2.2, we can derive

$${}^{c}D_{p,q}^{\alpha}x(t)=f\big(t,x(t)\big).$$

This completes the proof.

Lemma 3.2. *Let $0<q<p\le 1$. Then, we can get*

$$\left(1-\left(\frac{q}{p}\right)^{k+1}\right)_{p,q}^{(\alpha-1)} \le \frac{p^{\binom{\alpha-1}{2}}}{1-(\frac{q}{p})^{\alpha}}, \forall k\in\mathbb{N}^+, \alpha>0. \tag{3.3}$$

Proof. By the definition of fractional (p,q)-power function, we have

$$\begin{aligned}\Big(1-\Big(\frac{q}{p}\Big)^{k+1}\Big)_{p,q}^{(\alpha-1)} &= p^{\binom{\alpha-1}{2}}\Big(1-\Big(\frac{q}{p}\Big)^{k+1}\Big)_{\frac{q}{p}}^{(\alpha-1)}\\ &= p^{\binom{\alpha-1}{2}}\prod_{i=0}^{\infty}\Big[\frac{1-(\frac{q}{p})^{k+1}(\frac{q}{p})^{i}}{1-(\frac{q}{p})^{k+1}(\frac{q}{p})^{\alpha-1+i}}\Big]\\ &= p^{\binom{\alpha-1}{2}}\lim_{N\to\infty}\prod_{i=0}^{N}\Big[\frac{1-(\frac{q}{p})^{i+k+1}}{1-(\frac{q}{p})^{i+k+\alpha}}\Big]\\ &= p^{\binom{\alpha-1}{2}}\lim_{N\to\infty}S_N.\end{aligned}$$

Since

$$\begin{aligned}S_N=\prod_{i=0}^{N}\frac{1-(\frac{q}{p})^{i+k+1}}{1-(\frac{q}{p})^{i+k+\alpha}} &= \frac{1}{1-(\frac{q}{p})^{k+\alpha}}\frac{1-(\frac{q}{p})^{k+1}}{1-(\frac{q}{p})^{k+\alpha+1}}\cdots\frac{1-(\frac{q}{p})^{k+N}}{1-(\frac{q}{p})^{k+\alpha+N}}\frac{1-(\frac{q}{p})^{k+N+1}}{1}\\ &\le \frac{1-(\frac{q}{p})^{k+N+1}}{1-(\frac{q}{p})^{k+\alpha}}\le\frac{1}{1-(\frac{q}{p})^{\alpha}},\end{aligned}$$

the inequality (3.3) holds. This completes the proof.

Lemma 3.3. *Let $0<\alpha<1$, $0<q<p\le 1$. Assume that function family $\left\{t^{\alpha}f_v\left(\frac{t}{p^{\alpha}}\right)\right\}$ is equicontinuous on interval $[0,\infty)$ and*

$$x_v(t)=x_0+\frac{1}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)}\int_0^t (t-qs)_{p,q}^{(\alpha-1)} f_v\Big(\frac{s}{p^{\alpha-1}}\Big)d_{p,q}s,\ t\in(0,\infty). \tag{3.4}$$

Then, the function family $\{x_v(t)\}$ is also equicontinuous on $[0,\infty)$.

Proof. For any $t_1,\ t_2 \in [0,\infty)$, by (3.4), Definition 2.2 and Lemma 3.2, we have

$$\begin{aligned}
&|x_v(t_1) - x_v(t_2)| \\
=& \left| \frac{1}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} \int_0^{t_1} (t_1 - qs)_{p,q}^{(\alpha-1)} f_v\left(\frac{s}{p^{\alpha-1}}\right) d_{p,q}s \right. \\
&\left. - \frac{1}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} \int_0^{t_2} (t_2 - qs)_{p,q}^{(\alpha-1)} f_v\left(\frac{s}{p^{\alpha-1}}\right) d_{p,q}s \right| \\
\le& \frac{p-q}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} \sum_{k=0}^{\infty} \frac{q^k}{p^{k+1}} \left(1 - \left(\frac{q}{p}\right)^{k+1}\right)_{p,q}^{(\alpha-1)} \left| t_1^\alpha f_v\left(\frac{q^k}{p^{k+\alpha}} t_1\right) - t_2^\alpha f_v\left(\frac{q^k}{p^{k+\alpha}} t_2\right) \right| \\
\le& \frac{p-q}{\Gamma_{p,q}(\alpha) p^{\binom{\alpha}{2}}} \frac{1}{1-(\frac{q}{p})^\alpha} \sum_{k=0}^{\infty} \frac{q^k}{p^{k+1}} \left(\frac{p}{q}\right)^{k\alpha} \left| \left(\frac{q^k}{p^k} t_1\right)^\alpha f_v\left(\frac{q^k t_1}{p^k} \frac{1}{p^\alpha}\right) \right. \\
&\left. - \left(\frac{q^k}{p^k} t_2\right)^\alpha f_v\left(\frac{q^k t_2}{p^k} \frac{1}{p^\alpha}\right) \right|.
\end{aligned} \tag{3.5}$$

Let $g_v(t) = t^\alpha f_v(\frac{t}{p^\alpha})$. Then, we obtain

$$\begin{aligned}
&|x_v(t_1) - x_v(t_2)| \\
&\le \frac{p-q}{\Gamma_{p,q}(\alpha) p^{\binom{\alpha}{2}}} \frac{1}{1-(\frac{q}{p})^\alpha} \sum_{k=0}^{\infty} \frac{q^k}{p^{k+1}} \left(\frac{p}{q}\right)^{k\alpha} \left| g_v\left(\frac{q^k}{p^k} t_1\right) - g_v\left(\frac{q^k}{p^k} t_2\right) \right|.
\end{aligned}$$

Since $\{g_v(t)\}$ is equicontinuous on $[0,\infty)$, $0 < (\frac{q}{p})^k < 1$, and $\sum\limits_{k=0}^{\infty} \frac{q^k}{p^{k+1}} \left(\frac{p}{q}\right)^{k\alpha} = \frac{1}{q\left[1-\left(\frac{q}{p}\right)^{1-\alpha}\right]}$, by the equicontinuous definition, the function family $\{x_v(t)\}$ is also equicontinuous. This completes the proof.

Now, we present the solvability of fractional (p,q)-difference initial value problem (3.1).

Theorem 3.1. *Assume that for $0 \le \beta < \alpha < 1$, any $b > 0$, function $t^\alpha f(t,x)$ is continuous on domain $R_0 = \{(t,x) : 0 \le t \le b, |x - x_0| \le d\}$. If $|t^\alpha f(t,x)| \le \frac{d}{\Gamma_{p,q}(1-\alpha)}$ when $(t,x) \in R_0$. Then, the fractional (p,q)-difference equation* (3.1) *has at least a continuous solution $x(t)$ for $t \in [0,\infty)$.*

Proof. Let $0 < \delta < 1$. For $0 < v \le \delta$, we define the function $x_v\left(\frac{t}{p^{\alpha-1}}\right) = x_0$ on $[-\delta, 0]$ and

$$x_v(t) = x_0 + \frac{1}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} \int_0^t (t-qs)_{p,q}^{(\alpha-1)} f\left(\frac{s}{p^{\alpha-1}}, x_v\left(\frac{s-v}{p^{\alpha-1}}\right)\right) d_{p,q}s,\ x_v(0) = x_0, \tag{3.6}$$

on $[0,\gamma_1]$, where $\gamma_1 = \min\{b, v\}$. Here are two cases to prove it.

If $\gamma_1 = b$, from (3.6), we obtain

$$|x_v(t) - x_0| = \frac{p^{\alpha(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} \left| \int_0^t (t-qs)_{p,q}^{(\alpha-1)} s^{-\alpha} \cdot (\frac{s}{p^{\alpha-1}})^\alpha f\left(\frac{s}{p^{\alpha-1}}, x_v\left(\frac{s-v}{p^{\alpha-1}}\right)\right) d_{p,q}s \right|.$$

Notice that by Definition 2.2,

$$\begin{aligned}
&\left|\int_0^t (t-qs)_{p,q}^{(\alpha-1)} s^{-\alpha}\cdot(\frac{s}{p^{\alpha-1}})^{\alpha} f\left(\frac{s}{p^{\alpha-1}}, x_v\left(\frac{s}{p^{\alpha-1}}\right)\right) d_{p,q}s\right| \\
&= (p-q)\left|\sum_{k=0}^{\infty}\frac{q^k}{p^{k+1}}\left(1-q\frac{q^k}{p^{k+1}}\right)_{p,q}^{(\alpha-1)}\left(\frac{q^k}{p^{k+1}}\right)^{-\alpha}\left(\frac{1}{p^{\alpha-1}}\frac{q^k}{p^{k+1}}t\right)^{\alpha}\right. \\
&\quad \left.\times f\left(\frac{1}{p^{\alpha-1}}\frac{q^k}{p^{k+1}}t, x_v\left(\frac{1}{p^{\alpha-1}}\frac{q^k}{p^{k+1}}t\right)\right)\right| \\
&\le (p-q)\sum_{k=0}^{\infty}\left|\frac{q^k}{p^{k+1}}\left(1-q\frac{q^k}{p^{k+1}}\right)_{p,q}^{(\alpha-1)}\left(\frac{q^k}{p^{k+1}}\right)^{-\alpha}\left(\frac{1}{p^{\alpha-1}}\frac{q^k}{p^{k+1}}t\right)^{\alpha}\right. \\
&\quad \left.\times f\left(\frac{1}{p^{\alpha-1}}\frac{q^k}{p^{k+1}}t, x_v\left(\frac{1}{p^{\alpha-1}}\frac{q^k}{p^{k+1}}t\right)\right)\right|
\end{aligned}$$

Since for any $t\in[0,\infty)$, $t^{\alpha}f(t,x)$ is continuous, then

$$(\frac{1}{p^{\alpha-1}}\frac{q^k}{p^{k+1}}t)^{\alpha} f\left(\frac{1}{p^{\alpha-1}}\frac{q^k}{p^{k+1}}t, x_v\left(\frac{1}{p^{\alpha-1}}\frac{q^k}{p^{k+1}}t\right)\right)$$

is bounded. Besides, by D'Alembert's test, we get

$$\sum_{k=0}^{\infty}\left|\frac{q^k}{p^{k+1}}\left(1-q\frac{q^k}{p^{k+1}}\right)_{p,q}^{(\alpha-1)}(\frac{q^k}{p^{k+1}})^{-\alpha}\right|$$

is convergent. Hence,

$$\sum_{k=0}^{\infty}\left|\frac{q^k}{p^{k+1}}\left(1-q\frac{q^k}{p^{k+1}}\right)_{p,q}^{(\alpha-1)}(\frac{q^k}{p^{k+1}})^{-\alpha}(\frac{1}{p^{\alpha-1}}\frac{q^k}{p^{k+1}}t)^{\alpha} f\left(\frac{1}{p^{\alpha-1}}\frac{q^k}{p^{k+1}}, x_v\left(\frac{1}{p^{\alpha-1}}\frac{q^k}{p^{k+1}}\right)\right)\right|$$

is also convergent. Thus, we have

$$\begin{aligned}
&\left|\int_0^t (t-qs)_{p,q}^{(\alpha-1)} s^{-\alpha}\cdot(\frac{s}{p^{\alpha-1}})^{\alpha} f\left(\frac{s}{p^{\alpha-1}}, x_v\Big(\frac{s-v}{p^{\alpha-1}}\Big)\right) d_{p,q}s\right| \\
&\le (p-q)\sum_{k=0}^{\infty}\frac{q^k}{p^{k+1}}\left|\left(1-q\frac{q^k}{p^{k+1}}\right)_{p,q}^{(\alpha-1)}(\frac{q^k}{p^{k+1}})^{-\alpha}(\frac{1}{p^{\alpha-1}}\frac{q^k}{p^{k+1}}t)^{\alpha}\right. \\
&\quad \left.\times f\left(\frac{1}{p^{\alpha-1}}\frac{q^k}{p^{k+1}}t, x_v\left(\frac{1}{p^{\alpha-1}}\left(\frac{q^k}{p^{k+1}}t-v\right)\right)\right)\right| \\
&= \int_0^t\left|(t-qs)_{p,q}^{(\alpha-1)} s^{-\alpha}\cdot(\frac{s}{p^{\alpha-1}})^{\alpha} f\left(\frac{s}{p^{\alpha-1}}, x_v\Big(\frac{s-v}{p^{\alpha-1}}\Big)\right)\right| d_{p,q}s.
\end{aligned} \tag{3.7}$$

Then, by (3.7), $|t^{\alpha}f(t,x)| \le \frac{d}{\Gamma_{p,q}(1-\alpha)}$ when $(t,x)\in R_0$, Lemma 2.4, and the definition of (p,q)-beta function, we can get

$$\begin{aligned}|x_v(t)-x_0| &\le \frac{p^{\alpha(\alpha-1)}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)}\int_0^t \Bigg|(t-qs)_{p,q}^{(\alpha-1)}s^{-\alpha}\cdot(\frac{s}{p^{\alpha-1}})^{\alpha} \\ &\quad\times f\left(\frac{s}{p^{\alpha-1}}, x_v\Big(\frac{s-v}{p^{\alpha-1}}\Big)\right)\Bigg| d_{p,q}s \\ &\le \frac{p^{\alpha(\alpha-1)}\cdot d}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)\Gamma_{p,q}(1-\alpha)}\int_0^t (t-qs)_{p,q}^{(\alpha-1)}s^{-\alpha}d_{p,q}s \\ &= \frac{p^{\alpha(\alpha-1)}\cdot d}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)\Gamma_{p,q}(1-\alpha)}\int_0^1 (1-q\tau)_{p,q}^{(\alpha-1)}\tau^{-\alpha}d_{p,q}\tau \\ &\le \frac{p^{\alpha(\alpha-1)}\cdot d}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)\Gamma_{p,q}(1-\alpha)}B_{p,q}(1-\alpha,\alpha)=d,\ t\in[0,\gamma_1].\end{aligned} \tag{3.8}$$

Hence, the function family $\{x_v(t)\}$ is uniformly bounded on $[0,\gamma_1]$. Noting that $x_v(\frac{s-v}{p^{\alpha-1}})=x_0$ is equicontinuous for $s\in[0,\gamma_1]$ and $t^{\alpha}f(t,x)$ is uniformly continuous on R_0, by Lemma 3.3, we conclude that function family $\{x_v(t)\}$ is equicontinuous on $[0,\gamma_1]$. By Ascoli-Arzela's lemma, there exists a sequence $\{v_n\}$ such that $\lim\limits_{v_n\to 0} x_{v_n}(t)=x(t)$ uniformly on $[0,\gamma_1]$. Since $t^{\alpha}f(t,x)$ is uniformly continuous, we have that $t^{\alpha}f\big(\frac{t}{p^{\alpha-1}}, x_{v_n}(\frac{t-v_n}{p^{\alpha-1}})\big)$ converges to $t^{\alpha}f\big(\frac{t}{p^{\alpha-1}}, x(\frac{t}{p^{\alpha-1}})\big)$ as $v_n\to 0$, and by virtue of $f\big(\frac{t}{p^{\alpha-1}}, x_{v_n}(\frac{t-v_n}{p^{\alpha-1}})\big)=t^{-\alpha}t^{\alpha}f\left(\frac{t}{p^{\alpha-1}}, x_{v_n}\left(\frac{t-v_n}{p^{\alpha-1}}\right)\right)$, taking $v=v_n$ in (3.6) and letting $v_n\to 0$, we can obtain that $x(t)$ is a solution of equation (3.1) on $[0,b]$.

If $\gamma_1<b$ (or $\gamma_1=v$), we can employ (3.6) to extend $x_v(t)$ to interval $[-\delta,\gamma_2]$ where $\gamma_2=\min\{b,2v\}$. Similar to the argument of (3.8), we can get

$$|x_v(t)-x_0| \le \frac{p^{\alpha(\alpha-1)}\cdot d}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)\Gamma_{p,q}(1-\alpha)}B_{p,q}(1-\alpha,\alpha)=d,\ t\in[0,\gamma_2]. \tag{3.9}$$

Hence, the function family $\{x_v(t)\}$ is uniformly bounded on $[0,\gamma_2]$. Besides, noting that $\frac{s-v}{p^{\alpha-1}}\in[-\delta,0]$ when $s\in[0,\gamma_2]$, the above argument had shown that $x_v(\frac{s-v}{p^{\alpha-1}})$ is equicontinuous on $[0,\gamma_2]$. By Lemma 3.3 and (3.9), we conclude that function family $\{x_v(t)\}$ is equicontinuous and uniformly bounded on $[0,\gamma_2]$ again. Continuing this process, we extend $x_v(t)$ over $[0,b]$ such that $\{x_v(t)\}$ is equicontinuous and $|x_v(t)-x_0|\le d$. Finally, as the same derivation process in first case, we get that (3.1) at least has one solution on $[0,b]$. By the arbitrariness of b, we deduce that (3.1) has at least a continuous solution $x(t)$ on $[0,\infty)$. This completes the proof.

4. The stability of fractional (p,q)-difference equation

In this section, the result of stability for the (p,q)-difference equation (3.1) is given by using a (p,q)-Gronwall inequality which also implies the uniqueness of solution. Now, an integral operator is defined as follows:

$$Jx(t):=\lambda I_{p,q}^{\alpha}x(t)=\frac{\lambda}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)}\int_0^t (t-qs)_{p,q}^{(\alpha-1)}x\left(\frac{s}{p^{\alpha-1}}\right)d_{p,q}s,\ t\in[0,\infty), \tag{4.1}$$

and $J^n x(t) = JJ^{n-1}x(t)$.

Lemma 4.1. *Let $\lambda \geq 0$. Then, the following inequality holds*

$$|J^k x(t)| \leq \frac{\lambda^k t^{\alpha k}}{\Gamma_{p,q}(k\alpha+1)} \sup_{0\leq s\leq t} |x(s)|, \ k \in \mathbb{N}^+, \ t \in [0,\infty). \tag{4.2}$$

Proof. Now, we use mathematical induction to prove. For $k = 1$, from (4.1), we have

$$|Jx(t)| = \frac{\lambda}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} \left| \int_0^t (t-qs)_{p,q}^{(\alpha-1)} x\left(\frac{s}{p^{\alpha-1}}\right) d_{p,q}s \right|.$$

From Definition 2.2, we have

$$\begin{aligned}\left| \int_0^t (t-qs)_{p,q}^{(\alpha-1)} x(\frac{s}{p^{\alpha-1}}) d_{p,q}s \right| &= (p-q)t \left| \sum_{k=0}^{\infty} \frac{q^k}{p^{k+1}} \left(t - q\frac{q^k}{p^{k+1}}t\right)_{p,q}^{(\alpha-1)} x(\frac{q^k}{p^{k+\alpha}}t) \right| \\ &\leq (p-q)t^{\alpha} \sum_{k=0}^{\infty} \left| \frac{q^k}{p^{k+1}} \left(1 - q\frac{q^k}{p^{k+1}}\right)_{p,q}^{(\alpha-1)} x(\frac{q^k}{p^{k+\alpha}}t) \right|.\end{aligned}$$

Since $x(t)$ is a continuous function on $t \in [0,\infty)$, then for any fixed t, we can deduce that $x(\frac{q^k}{p^{k+\alpha}}t)$ is bounded. Besides, since

$$0 < \left(1 - q\frac{q^k}{p^{k+1}}\right)_{p,q}^{(\alpha-1)} < \frac{1}{1-(\frac{q}{p})^{k+1}} \leq \frac{1}{1-\frac{q}{p}},$$

and $\sum\limits_{k=0}^{\infty} \left| \frac{q^k}{p^{k+1}} \right|$ is convergent, then $\sum\limits_{k=0}^{\infty} \left| \frac{q^k}{p^{k+1}} \left(1 - q\frac{q^k}{p^{k+1}}\right)_{p,q}^{(\alpha-1)} x(\frac{q^k}{p^{k+\alpha}}t) \right|$ is also convergent. Hence,

$$\begin{aligned}\left| \int_0^t (t-qs)_{p,q}^{(\alpha-1)} x(\frac{s}{p^{\alpha-1}}) d_{p,q}s \right| &\leq (p-q)t^{\alpha} \sum_{k=0}^{\infty} \left| \frac{q^k}{p^{k+1}} \left(1 - q\frac{q^k}{p^{k+1}}\right)_{p,q}^{(\alpha-1)} x(\frac{q^k}{p^{k+\alpha}}t) \right| \\ &= \int_0^t \left| (t-qs)_{p,q}^{(\alpha-1)} x(\frac{s}{p^{\alpha-1}}) \right| d_{p,q}s.\end{aligned} \tag{4.3}$$

By (4.3), Lemma 2.4 and the definition of (p,q)-beta function, we have

$$\begin{aligned}
|Jx(t)| &\leq \frac{\lambda}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} \int_0^t \left|(t-qs)_{p,q}^{(\alpha-1)} x\left(\frac{s}{p^{\alpha-1}}\right)\right| d_{p,q}s \\
&\leq \frac{\lambda}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} \sup_{0\leq s\leq t}\left|x\left(\frac{s}{p^{\alpha-1}}\right)\right| \int_0^t (t-qs)_{p,q}^{(\alpha-1)} d_{p,q}s \\
&= \frac{\lambda}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} t^{\alpha} \sup_{0\leq s\leq t}\left|x\left(\frac{s}{p^{\alpha-1}}\right)\right| \int_0^1 (1-q\tau)_{p,q}^{(\alpha-1)} d_{p,q}\tau \\
&= \frac{\lambda}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} t^{\alpha} B_{p,q}(1,\alpha) \sup_{0\leq s\leq t}\left|x\left(\frac{s}{p^{\alpha-1}}\right)\right| \\
&\leq \frac{\lambda}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} t^{\alpha} B_{p,q}(1,\alpha) \sup_{0\leq s\leq t}|x(s)| \\
&= \frac{\lambda p^{\frac{1}{2}(\alpha-1)\alpha}}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha+1)} t^{\alpha} \sup_{0\leq s\leq t}|x(s)| = \frac{\lambda}{\Gamma_{p,q}(\alpha+1)} t^{\alpha} \sup_{0\leq s\leq t}|x(s)|.
\end{aligned}$$

Assume that inequality (4.2) holds for $k = n-1$. Then,

$$\begin{aligned}
\left|J^{n-1}\Big(\frac{t}{p^{\alpha-1}}\Big)\right| &\leq \frac{\lambda^{n-1}(\frac{t}{p^{\alpha-1}})^{\alpha(n-1)}}{\Gamma_{p,q}\big((n-1)\alpha+1\big)} \sup_{0\leq s\leq \frac{t}{p^{\alpha-1}}} |x(s)| \\
&\leq \frac{\lambda^{n-1}t^{\alpha(n-1)}}{p^{\alpha(\alpha-1)(n-1)}\Gamma_{p,q}\big((n-1)\alpha+1\big)} \sup_{0\leq s\leq t}|x(s)|.
\end{aligned} \tag{4.4}$$

When $k = n$, from (4.1), we obtain

$$\begin{aligned}
|J^n x(t)| = |JJ^{n-1}x(t)| &= \frac{\lambda}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} \left|\int_0^t (t-qs)_{p,q}^{(\alpha-1)} J^{n-1}x\Big(\frac{s}{p^{\alpha-1}}\Big) d_{p,q}s\right| \\
&\leq \frac{\lambda}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} \int_0^t \left|(t-qs)_{p,q}^{(\alpha-1)} J^{n-1}x\Big(\frac{s}{p^{\alpha-1}}\Big)\right| d_{p,q}s.
\end{aligned}$$

The above inequality can be easily proved by a similar method to (4.3). Therefore, we omit the process. Then, by (4.4), Lemma2.4 and the definition of (p,q)-beta function, we get

$$
\begin{aligned}
|J^n x(t)| &\le \frac{\lambda^n}{p^{\alpha(\alpha-1)(n-1)}p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)\Gamma_{p,q}\big((n-1)\alpha+1\big)} \sup_{0\le s\le t}|x(s)| \\
&\quad \times \int_0^t (t-qs)_{p,q}^{(\alpha-1)} s^{\alpha(n-1)} d_{p,q}s \\
&= \frac{\lambda^n t^{\alpha n}}{p^{\alpha(\alpha-1)(n-1)}p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)\Gamma_{p,q}\big((n-1)\alpha+1\big)} \sup_{0\le s\le t}|x(s)| \\
&\quad \times \int_0^1 (1-q\tau)_{p,q}^{(\alpha-1)} \tau^{\alpha(n-1)} d_{p,q}\tau \\
&\le \frac{\lambda^n t^{\alpha n}}{p^{\alpha(\alpha-1)(n-1)}p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)\Gamma_{p,q}\big((n-1)\alpha+1\big)} \sup_{0\le s\le t}|x(s)| \\
&\quad \times B_{p,q}\big((n-1)\alpha+1,\alpha\big) \\
&\le \frac{p^{\frac{1}{2}(\alpha-1)(2n\alpha-\alpha)}\lambda^n t^{\alpha n}}{p^{\alpha(\alpha-1)(n-1)}p^{\binom{\alpha}{2}}\Gamma_{p,q}(n\alpha+1)} \sup_{0\le s\le t}|x(s)| = \frac{\lambda^n t^{\alpha n}}{\Gamma_{p,q}(n\alpha+1)} \sup_{0\le s\le t}|x(s)|.
\end{aligned}
$$

The proof is completed.

Lemma 4.2. *(The (p,q)-Gronwall Inequality) Let $\beta(t)\ge 0$, and $\lambda \ge 0$. Assume that function $x:[0,\infty)\to\mathbb{R}^+$ is continuous and satisfies*

$$x(t) \le \beta(t) + \frac{\lambda}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} \int_0^t (t-qs)_{p,q}^{(\alpha-1)} x\left(\frac{s}{p^{\alpha-1}}\right) d_{p,q}s,\ t\in[0,\infty). \tag{4.5}$$

Then, the following inequality holds

$$x(t) \le E_{p,q}(\lambda,t)\sup_{0\le s\le t}\beta(s),\ t\in[0,\infty), \tag{4.6}$$

where $E_{p,q}(\lambda,t)=\sum\limits_{k=0}^{\infty}\frac{\lambda^k t^{\alpha k}}{\Gamma_{p,q}(k\alpha+1)}$.

Proof. For any $t\in[0,\infty)$, by (4.5) and recurrence, it yields

$$x(t)\le\beta(t)+Jx(t)\le\beta(t)+J\beta(t)+J^2x(t)\le\cdots\le\sum_{k=0}^{n-1}J^k\beta(t)+J^nx(t). \tag{4.7}$$

Hence, it follows from Lemma 4.1 that

$$x(t)\le\sum_{k=0}^{n-1}\frac{\lambda^k t^{\alpha k}}{\Gamma_{p,q}(k\alpha+1)}\sup_{0\le s\le t}\beta(s)+\frac{\lambda^n t^{\alpha n}}{\Gamma_{p,q}(n\alpha+1)}\sup_{0\le s\le t}|x(s)|. \tag{4.8}$$

Since $x(t)$ is continuous on $[0,\infty)$, $\frac{\lambda^n t^{\alpha n}}{\Gamma_{p,q}(n\alpha+1)}\sup_{0\le s\le t}|x(s)|\to 0$ when $n\to\infty$, taking $n\to\infty$ in (4.8), then we derive

$$x(t)\le E_{p,q}(\lambda,t)\sup_{0\le s\le t}\beta(s),$$

where $E_{p,q}(\lambda,t)=\sum\limits_{k=0}^{\infty}\frac{\lambda^k t^{\alpha k}}{\Gamma_{p,q}(k\alpha+1)}$. This completes the proof.

Now, we present the stability criterion of (p,q)-fractional difference equation (3.1).

Theorem 4.1. *Assume that function $f(t,x)$ satisfies the Lipschitz condition:*

$$|f(t,x) - f(t,x')| \leq \lambda |x - x'|, \tag{4.9}$$

where $t \in [0,\infty)$, and $x, x' \in \mathbb{R}$. Then, the solution of (p,q)-difference equation (3.1) *is stable with respect to the initial value. That is,*

$$|x_1(t) - x_2(t)| \leq E_{p,q}(\lambda, t)|x_1(0) - x_2(0)|, \ t \in [0,\infty), \tag{4.10}$$

where x_1 and x_2 are solutions of Equation (3.1) *with initial $x_1(0)$ and $x_2(0)$ respectively.*

Proof. Since x_1 and x_2 are solutions of Equation (3.1), by (3.2), we have

$$\begin{aligned}|x_1(t) - x_2(t)| \leq |x_1(0) - x_2(0)| &+ \frac{1}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} \left| \int_0^t (t - qs)_{p,q}^{(\alpha-1)} f\left(\frac{s}{p^{\alpha-1}}, x_1\left(\frac{s}{p^{\alpha-1}}\right)\right) \right. \\ &\left. - f\left(\frac{s}{p^{\alpha-1}}, x_2\left(\frac{s}{p^{\alpha-1}}\right)\right) d_{p,q}s \right| \\ \leq |x_1(0) - x_2(0)| &+ \frac{1}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} \int_0^t (t - qs)_{p,q}^{(\alpha-1)} \left| f\left(\frac{s}{p^{\alpha-1}}, x_1\left(\frac{s}{p^{\alpha-1}}\right)\right) \right. \\ &\left. - f\left(\frac{s}{p^{\alpha-1}}, x_2\left(\frac{s}{p^{\alpha-1}}\right)\right) \right| d_{p,q}s.\end{aligned}$$

Notice that the above inequality can be easily proved by a similar method to (4.3). Meanwhile, from (4.9), we derive

$$\begin{aligned}|x_1(t) - x_2(t)| &\leq |x_1(0) - x_2(0)| \\ &+ \frac{\lambda}{p^{\binom{\alpha}{2}}\Gamma_{p,q}(\alpha)} \int_0^t (t - qs)_{p,q}^{(\alpha-1)} \left| x_1\left(\frac{s}{p^{\alpha-1}}\right) - x_2\left(\frac{s}{p^{\alpha-1}}\right) \right| d_{p,q}s.\end{aligned}$$

Then, from Lemma 4.2, we can derive (4.10). Hence, the solution of (p,q)-difference equation (3.1) is stable with respect to the initial value. This completes the proof.

Remark 4.1. From (4.10), we can see the solution of (3.1) is unique with respect to given initial value if condition (4.9) holds.

5. Examples

Example 5.1. Consider the following Caputo type (p,q)-fractional initial value problem:

$$\begin{cases} {}^cD_{p,q}^{\frac{2}{3}} x(t) = t^{-\frac{1}{2}} x^2(t), \ t > 0 \\ x(0) = 0. \end{cases} \tag{5.1}$$

This corresponds to (3.1) with

$$\alpha = \frac{2}{3}, \ \beta = \frac{1}{2}, \ f(t,x) = t^{-\frac{1}{2}} x^2.$$

It is clear that $0 \leq \beta < \alpha < 1$, and for any $b > 0$, $t^{\frac{2}{3}} f(t,x)$ is continuous on domain $R_0 = \{(t,x) : 0 \leq t \leq b, |x| \leq d\}$, where $d = \frac{1}{b^{\frac{1}{6}}\Gamma_{p,q}(\frac{1}{3})}$. Then, we have

$$\left| t^{\frac{1}{6}} x^2 \right| \leq b^{\frac{1}{6}} d^2 = \frac{d}{\Gamma_{p,q}(\frac{1}{3})}, \ (t,x) \in R_0.$$

Therefore, by Theorem 3.1, (5.1) has at least a continuous solution $x(t)$ for $t \in [0, \infty)$.

6. Conclusion

In this paper, the solvability for a class of fractional singular (p, q)-difference equation is studied. In addition, by a fractional (p, q)-Gronwall inequality, we obtain the stability result, which also implies the uniqueness of solution. This paper investigates stability of fractional (p, q)-difference equation, and it generalizes the existence of solution and stability results of a fractional q-differential. These results can be further used in the fractional controlling or design of fractional controllers of discrete time. For example, discrete fractional network, chaos synchronization of discrete time and so on. We will continue to study the qualitative theory of fractional (p, q)-difference equations in our future work.

References

[1] R. Agarwal, *Certain fractional q-integrals and q-derivatives*, Mathematical Proceedings of the Cambridge Philosophical Society, 1969, 66(2), 365–370.

[2] B. Ahmad and J. Nieto, *Basic theory of nonlinear third-order q-difference equations and inclusions*, Mathematical Modelling and Analysis, 2013, 18(1), 122–135.

[3] W. Al-Salam, *Some Fractional q-Integrals and q-Derivatives*, Proceedings of the Edinburgh Mathematical Society, 1966, 15(2), 135–140.

[4] T. Brikshavana and T. Sitthiwirattham, *On fractional Hahn calculus*, Advances in Difference Equations, 2017, 354, 15 pages.
DOI: 10.1186/s13662-017-1412-y

[5] R. Chakrabarti and R. Jagannathan, *A (p, q)-oscillator realization of two-parameter quantum algebras*, Journal of Physics A: Mathematical and General, 1991, 24(13), 5683–5701.

[6] J. Diaz and T. Osler, *Differences of fractional order*, Mathematics of Computation, 1974, 28(125), 185–202.

[7] A. Dobrogowska, *The q-deformation of the Morse potential*, Applied Mathematics Letter, 2013, 26(1), 769–773.

[8] T. Ernst, *On several q-special matrices, including the q-Bernoulli and q-Euler matrices*, Linear Algebra and Its Applications, 2017, 542, 422–440.

[9] F. Jackson, *On q-difference equations*, American Journal of Mathematics, 1901, 32, 305–314.

[10] M. Jamei, F. Soleyman and W. Koepf, *Two Finite Sequences of Symmetric q-Orthogonal Polynomials Generated by Two q-Sturm-Liouville Problems*, Reports on Mathematical Physics, 2020, 85(1), 41–55.

[11] N. Kamsrisuk, C. Promsakon, S. Ntouyas and J. Tariboon, *Nonlocal boundary value problems for (p, q)-difference equations*, Differential Equations and Applications, 2018, 10, 183–195.

[12] H. Liu, *Application of the Variational Iteration Method to Strongly Nonlinear q-Difference Equations*, Journal of Applied Mathematics, 2012, Article ID 704138, 12 pages.
DOI: 10.1155/2012/704138

[13] M. Mursaleen, K. Ansari and A. Khan, *Some approximation results by (p, q)-analogue of Bernstein-Stancu operators*, Applied Mathematics and Computation, 2015, 264, 1–4.

[14] C. Promsakon, N. Kamsrisuk, S. Ntouyas and J. Tariboon, *On the Second-Order Quantum (p, q)-Difference Equations with Separated Boundary Conditions*, Advances in Mathematical Physics, 2018, Article ID 9089865.
DOI:10.1155/2018/9089865

[15] M. Riyasat, S. Khan and T. Nahid, *q-difference equations for the composite 2D q-Appell polynomials and their applications*, Cogent Mathematics, 2017, 4(1), Article ID 1376972, 23 pages.
DOI: 10.1080/23311835.2017.1376972

[16] P. Sadjang, *On the Fundamental Theorem of (p, q)-Calculus and Some (p, q)-Taylor Formulas*, Results in Mathematics, 2013, 73, 39.
DOI: 10.1007/s00025-018-0783-z

[17] V. Sahai and S. Yadav, *Representations of two parameter quantum algebras and (p, q)-special functions*, Journal of Mathematical Analysis and Applications, 2007, 335(1), 268–279.

[18] J. Soontharanon and T. Sitthiwirattham, *On fractional (p, q)-calculus*, Advances in Difference Equations, 2020, 354.
DOI: 10.1186/s13662-017-1412-y