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The Monotonicity of the Linear Complementarity Problem*

Ximing Fang[1] and Minhai Huang[1,†]

Abstract The monotonicity of the linear complementarity problem (LCP) is discussed in this paper. Both the monotone property about the single element of the solution and the monotone property of the whole solution are presented. In order to illustrate the results, some corresponding numerical experiments are provided.

Keywords Linear complementarity problem, Solution, Monotonicity.

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1. Introduction

The linear complementarity problem is to find a vector $z \in R^n$ such that

$$z^{\mathrm{T}}(Mz+q)=0, \quad z \geq 0, \quad Mz+q \geq 0, \tag{1.1}$$

where $M=(m_{ij}) \in R^{n\times n}$ and $q \in R^n$. Problem (1) is usually denoted by LCP(M,q), which has many applications such as the elastic contact problems, the free boundary problems, the linear and quadratic programming problems and the market equilibrium problems (see [1,3,7,10,15,21,26,27] and the references therein).

The theory research and the numerical algorithms for the LCP(M,q) have been studied in recent decades. The theory research includes the existence and uniqueness of the solution, the stability and sensitivity of the solution, the relationship between the LCP(M,q) and other problems, etc., (see [6,7,10,14,18,20,21,23,24]). It is well-known that the LCP(M,q) has a unique solution for any $q \in R^n$, if and only if the system matrix M is a P-matrix. The positive definite matrix and the H_+-matrix are two types of P-matrices, both of which have been studied by many authors (see [1, 9, 10, 14, 19]). For the stability and sensitivity of the solution, Mathias, Pang, Cottle and other researchers discussed the error problem and the perturbation problem of the solution, and many interesting results have been obtained, including the Lipschitzian continuous property of the solution (see [4,5,7,8,12,16,17,23,25]). For the numerical algorithms of the LCP(M,q), all kinds of solving methods have been presented, including the direct methods and the iteration methods such as

[†]the corresponding author.
Email address: fangxm504@163.com (X. Fang), hmh9520@sina.com (M. Huang)
[1]Department of Mathematics and Statistics, Zhaoqing University, Guangdong 526000, China
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the Lemke method, the projected method, and the modulus-based matrix splitting iteration method, etc.. Most of the solving methods are very efficient, and for the detailed materials, readers can refer to [1, 13, 15, 27] and the references therein.

Although there are many theories for the LCP(M, q), there are few theoretical studies on the monotony. In this paper, we will study the monotonicity problem of the solution. We will present that the solution possesses the monotone decreasing property for an arbitrary single variable, when matrix M is a P-matrix and the whole solution is monotone decreasing under some conditions when the system matrix M is an M-matrix. Besides, some conclusions related to the solution and the corresponding experiments will be provided.

The outline of this paper is as follows. We briefly introduce some definitions, then present the main conclusions in Section 2. The numerical experiments are shown and discussed in Section 3. We end this paper by some concluding remarks in Section 4.

2. Preliminaries and main results

First, we review several definitions as follows.

Definition 2.1. (Murty [19], Gale and Nikaido [11]) M is said to be a P-matrix, if all its principal minors are positive.

Definition 2.2. (Ostrowski [22], Berman and Plemmons [2]) A matrix $M \in R^{n\times n}$ is called an M-matrix, if

$$M^{-1} \geq 0 \quad \text{and} \quad m_{ij} \leq 0 (i \neq j) \quad for \quad i, j = 1, 2, ..., n. \tag{2.1}$$

Definition 2.3. (Murty [20]) Let $M \in R^{n\times n}$ be a matrix, the complementarity set of column vectors of M is a set $\{A_{.j}, j = 1, 2, ..., n\}$ such that $A_{.j}$ is either $I_{.j}$ or $-M_{.j}$, for each $j = 1, 2, ..., n$.

Definition 2.4. (Murty [20]) Let $M \in R^{n\times n}$ be a matrix and $\{A_{.j}, j = 1, 2, ..., n\}$ be any complementarity set of column vectors of M, a complementarity cone of M is the set

$$pos\{A_{.j}, j = 1, 2, ..., n\} = \{\sum_{j=1}^{n} \beta_j A_{.j}, \beta_j \geq 0, j = 1, 2, ..., n\}. \tag{2.2}$$

In the following, we give the main conclusions of this paper.

Theorem 2.1. *Suppose $M \in R^{n\times n}$ is a P-matrix, and $\hat{q}, \tilde{q} \in R^n$ satisfy $\hat{q}, \tilde{q} \in pos\{A_{.j}, j = 1, 2, ..., n\}$, which is a complementarity cone of M. If the solutions of (1) with $q = \hat{q}, \tilde{q}$ are denoted by $\hat{z}, \tilde{z}$ respectively, then*

(i) $\lambda\hat{q} + \mu\tilde{q} \in pos\{A_{.j}, j = 1, 2, ..., n\}, \lambda \geq 0, \mu \geq 0$*;*

(ii) $\lambda\hat{z} + \mu\tilde{z}$ *is the solution of (1), when* $q = \lambda\hat{q} + \mu\tilde{q}$.

Proof. The conclusion (i) can be easily proved based on the definition of the complementarity cone of M. We only prove (ii) in the following.

It is well-known that the LCP(M, q) has a unique solution for any $q \in R^n$, if the matrix M is a P-matrix, and the LCP(M, q) has a unique solution for a $q \in R^n$, if and only if q belongs to some complementarity cone of M. From the condition, we know that both the solution $\hat{z}$ and $\tilde{z}$ are unique. Since $\hat{q}$ and $\tilde{q}$ belong to the same complementarity cone of M, if we denote the nonnegative vectors $M\hat{z} + \hat{q}$, $M\tilde{z} + \tilde{q}$ by $\hat{\omega}$, $\tilde{\omega}$ respectively, then $\hat{z}$ and $\tilde{z}$ have the same nonnegative elements' positions corresponding to the same positions where both $\hat{\omega}$ and $\tilde{\omega}$ are zeros. Meanwhile, $\hat{\omega}$ and $\tilde{\omega}$ have the same nonnegative elements' positions corresponding to the same positions where both $\hat{z}$ and $\tilde{z}$ are zeros. Therefore, there are four complementarity relationships: $\hat{z}^{\mathrm{T}}\hat{\omega} = 0$, $\tilde{z}^{\mathrm{T}}\tilde{\omega} = 0$, $\hat{z}^{\mathrm{T}}\tilde{\omega} = 0$ and $\tilde{z}^{\mathrm{T}}\hat{\omega} = 0$. That is,

$$\begin{pmatrix} I & -M \end{pmatrix} \begin{pmatrix} \hat{\omega} \\ \hat{z} \end{pmatrix} = \hat{q}, \begin{pmatrix} I & -M \end{pmatrix} \begin{pmatrix} \tilde{\omega} \\ \tilde{z} \end{pmatrix} = \tilde{q}, \hat{z}^{\mathrm{T}}\tilde{\omega} = 0, \tilde{z}^{\mathrm{T}}\hat{\omega} = 0.$$

Thus, for arbitrary nonnegative real numbers $\lambda \geq 0, \mu \geq 0$, we have

$$\begin{pmatrix} I & -M \end{pmatrix} \begin{pmatrix} \lambda\hat{\omega} + \mu\tilde{\omega} \\ \lambda\hat{z} + \mu\tilde{z} \end{pmatrix} = \lambda\hat{q} + \mu\tilde{q}$$

with

$$(\lambda\hat{z} + \mu\tilde{z})^{\mathrm{T}}(\lambda\hat{\omega} + \mu\tilde{\omega}) = (\lambda\hat{z} + \mu\tilde{z})^{\mathrm{T}}[M(\lambda\hat{z} + \mu\tilde{z}) + (\lambda\hat{q} + \mu\tilde{q})] = 0,$$
$$\lambda\hat{z} + \mu\tilde{z} \geq 0,\ M(\lambda\hat{z} + \mu\tilde{z}) + (\lambda\hat{q} + \mu\tilde{q}) \geq 0.$$

Therefore, $\lambda\hat{z} + \mu\tilde{z}$ is the solution of (1) with $q = \lambda\hat{q} + \mu\tilde{q}$, and it is unique. Then, the conclusion (ii) is proved. □

We have the following monotonicity conclusion of the solution for the arbitrary single variable of q.

Theorem 2.2. *Suppose $M \in R^{n\times n}$ is a P-matrix, $q_1, ..., q_{i-1}, q_{i+1}, ..., q_n$ are fixed, and denote the ith element of the solution z of (1.1) by z_i. Then, z_i is a monotonic decreasing function of q_i, for $i = 1, 2, ..., n$.*

Proof. Suppose

$$\begin{pmatrix} \mathfrak{q}_1 \\ \mathfrak{q}_2 \end{pmatrix} = \begin{pmatrix} q(1:i-1) \\ q(i+1:n) \end{pmatrix}$$

and $\beta > 0$ is an arbitrary real number. Suppose z is the solution of (1) with $\omega = Mz + q$, when

$$q = (\mathfrak{q}_1; q_i; \mathfrak{q}_2),$$

and $\tilde{z}$ is the solution of (1) with $\tilde{\omega} = M\tilde{z} + \tilde{q}$, when

$$\tilde{q} = (\mathfrak{q}_1; q_i + \beta; \mathfrak{q}_2).$$

Then,

(i) if $\tilde{z}_i = 0$, it is obvious that $z_i \geq \tilde{z}_i$;

(ii) if $\tilde{z}_i > 0$, we will prove $z_i \geq \tilde{z}_i$.

Suppose not, then $z_i < \tilde{z}_i$. Under this condition, we have $\tilde{\omega}_i = 0$ by the complementarity relationship. Meanwhile, we also have $\omega_i = 0$. Otherwise, if $\omega_i > 0$, then $z_i = 0$. Thus, from

$$\begin{pmatrix} \omega(1:i-1) \\ \omega_i + \beta \\ \omega(i+1:n) \end{pmatrix} = M \begin{pmatrix} z(1:i-1) \\ 0 \\ z(i+1:n) \end{pmatrix} + \tilde{q},$$

we know that z is another solution of (1), when $q = \tilde{q}$, which is a contradiction.

Hence, based on $z_i < \tilde{z}_i$, $\tilde{\omega}_i = 0$ and $\omega_i = 0$, if we set

$$q = \begin{pmatrix} \mathfrak{q}_1 + M(1:i-1,i)z_i \\ q_i + \beta + m_{ii}z_i \\ \mathfrak{q}_2 + M(i+1:n,i)z_i \end{pmatrix},$$

in (1). Then, (1) has two different solutions

$$z = \begin{pmatrix} z(1:i-1) \\ 0 \\ z(i+1:n) \end{pmatrix} \quad \text{and} \quad z = \begin{pmatrix} \tilde{z}(1:i-1) \\ \tilde{z}_i - z_i \\ \tilde{z}(i+1:n) \end{pmatrix},$$

which contradicts with the uniqueness of the solution of (1) for any $q \in R^n$. Therefore, if $\tilde{z}_i > 0$,

$$z_i \geq \tilde{z}_i.$$

Combining (i) and (ii), the conclusion is proved. □

Remark 2.1. Theorem 2.2 shows the relationship between q_i and z_i, for each $i = 1, 2, ..., n$. In general, this monotone decreasing property cannot be extended to the whole solution. We give a example as follows.

Let

$$M = \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}, \hat{q} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \bar{q} = \begin{pmatrix} -1 \\ -2 \end{pmatrix}.$$

Then, the solutions of LCP$(M, \hat{q})$ and LCP$(M, \bar{q})$ are

$$\hat{z} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \bar{z} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

respectively. We can find that the inequality $\hat{z}_2 < \bar{z}_2$ holds from Theorem 2.2, since $\hat{q}_2 > \bar{q}_2$ and $\hat{q}_1 = \bar{q}_1$. However, the inequality $\hat{z} \leq \bar{z}$ does not hold.

In the following, we consider the monotonicity of the whole solution z to the LCP(M, q). First, we present a conclusion related to the complementarity cone as follows.

Theorem 2.3. *Suppose $M \in R^{n\times n}$ is an M-matrix, and $q = (q_1, q_2, ..., q_n)^{\mathrm{T}} \in R^n$. Denote $S_N = \{i|q_i < 0\} = \{i_1, i_2, ..., i_s\}$, $S_P = \{j|q_j \geq 0\} = \{j_1, j_2, ..., j_t\}$, and $\mathfrak{q}_1 = (q_{i_1}, q_{i_2}, ..., q_{i_s})^{\mathrm{T}}$. Denote the principal submatrix of M by $M_{(S_N,S_N)}$, and $v = M^{-1}_{(S_N,S_N)}(-\mathfrak{q}_1)$. If $q_i + M_{(i,S_N)}v \geq 0$ for $i \in S_P$, then the following relations hold:*

(i) *if $i \in S_N$, then $-M_{.i} \in \{A_{.j}, j = 1, 2, ..., n\}$;*

(ii) *if $i \in S_P$, then $I_i \in \{A_{.j}, j = 1, 2, ..., n\}$;*

(iii) *if $i \in S_N$, then $z_i \geq 0$, and z_i comes from the elements of v in accordance with the natural order;*

(iv) *if $i \in S_P$, then $z_i = 0$.*

Here, $\{A_{.j}, j = 1, 2, ..., n\}$ is the complementarity set of M, z is the solution of the LCP(M, q), and $M_{(i,S_N)}$ is the part of the ith row of M with the column elements' indices coming from S_N.

Proof. From the definition S_N and S_P, we know $s + t = n$. Since M is an M-matrix, $M_{(S_N,S_N)}$ is also an M-matrix. Therefore, the vector $v = M^{-1}_{(S_N,S_N)}(-\mathfrak{q}_1)$ is a positive vector. If we denote the indices of the vector v by $i_1, i_2, ..., i_s$ corresponding to the elements of set S_N, and set $z = (z_1, z_2, ..., z_n)^{\mathrm{T}}$, where $z_i = 0$, if $i \in S_P$ and $z_i = z_{i_k} = v_{i_k}$, if $i \in S_N$, and set $\omega = (\omega_1, \omega_2, ..., \omega_n)^{\mathrm{T}}$, where $\omega_i = 0$, if $i \in S_N$ and $\omega_i = q_i + M_{(i,S_N)}v$, if $i \in S_P$. Then, we have $z \geq 0, \omega \geq 0$ with

$$\begin{pmatrix} I & -M \end{pmatrix} \begin{pmatrix} \omega \\ z \end{pmatrix} = q, \quad z^{\mathrm{T}}\omega = 0.$$

Thus, z is the solution of the LCP(M, q). From the constructions of z, ω, and the unique property of solution of the LCP(M, q) with an M-matrix, we know $q \in pos\{A_{.j}, j = 1, 2, ..., n\}$, which satisfies $-M_{.i} \in \{A_{.j}, j = 1, 2, ..., n\}, i \in S_N$ and $I_i \in \{A_{.j}, j = 1, 2, ..., n\}, i \in S_P$, $z_i \geq 0$, if $i \in S_N$ and $z_i = 0$, if $i \in S_P$. Therefore, the conclusion is established. □

From Theorem 2.3 and the proof, we can obtain the following conclusion easily.

Corollary 2.1. *Suppose $M \in R^{n\times n}$ is an M-matrix, and $q = (q_1, q_2, ..., q_n)^{\mathrm{T}} \in R^n$ satisfy $q_i \neq 0, i = 1, 2, ..., n$. Denote $S_N = \{i|q_i < 0\} = \{i_1, i_2, ..., i_s\}$, $S_P = \{j|q_j > 0\} = \{j_1, j_2, ..., j_t\}$, and $\mathfrak{q}_1 = (q_{i_1}, q_{i_2}, ..., q_{i_s})^{\mathrm{T}}$. Denote the principal submatrix of M by $M_{(S_N,S_N)}$ and $v = M^{-1}_{(S_N,S_N)}(-\mathfrak{q}_1)$. If $q_i + M_{(i,S_N)}v \geq 0$ for $i \in S_P$, then the following relations hold:*

(i) *if $i \in S_N$, then $-M_{.i} \in \{A_{.j}, j = 1, 2, ..., n\}$;*

(ii) *if $i \in S_P$, then $I_i \in \{A_{.j}, j = 1, 2, ..., n\}$;*

(iii) *if $i \in S_N$, then $z_i > 0$, and z_i comes from the elements of v in accordance with the natural order;*

(iv) *if $i \in S_P$, then $z_i = 0$.*

Here, $\{A_{.j}, j = 1, 2, ..., n\}$ is the complementarity set of M, z is the solution of the LCP(M, q), and $M_{(i,S_N)}$ is the part of the ith row of M with the column elements' indices coming from S_N.

From Theorem 2.3 and Corollary 2.1, we not only deduce the complementarity cone of M which q belongs to, but also deduce the zero elements' positions and the positive elements' positions of the solution. In addition, we can obtain a conclusion for the invariance of the solution as follows.

Theorem 2.4. *Suppose $M \in R^{n\times n}$ is an M-matrix, and $\hat{q} = (\hat{q}_1, \hat{q}_2, ..., \hat{q}_n)^{\mathrm{T}} \in R^n$. Denote $S_N = \{i|\hat{q}_i < 0\} = \{i_1, i_2, ..., i_s\}$, $S_P = \{j|\hat{q}_j \geq 0\} = \{j_1, j_2, ..., j_t\}$, and $\hat{\mathfrak{q}}_1 = (\hat{q}_{i_1}, \hat{q}_{i_2}, ..., \hat{q}_{i_s})^{\mathrm{T}}$. Denote the principal submatrix of M by $M_{(S_N,S_N)}$, and $\hat{v} = M^{-1}_{(S_N,S_N)}(-\hat{\mathfrak{q}}_1)$. If $\hat{q}_i + M_{(i,S_N)}\hat{v} \geq 0$, for $i \in S_P$. Then, for any nonnegative vector $\beta \in R^n$ with the elements satisfying: $\beta_i = 0$, if $\hat{q}_i < 0$, $\beta_i \geq 0$, if $\hat{q}_i \geq 0$, for $i = 1, 2, ..., n$, $\tilde{q} = \hat{q} + \beta$ and $\hat{q}$ belong to the same complementrity cone of M and $\tilde{z} = \hat{z}$, where $\tilde{z}$ is the solution of the LCP(M, q), when $q = \tilde{q}$.*

Proof. From the character of β, we know that $\hat{q}$ and $\tilde{q}$ have the same negative elements with the same positions, and only at the nonnegative elements' positions, the elements of $\tilde{q}$ maybe become larger. Therefore, for the LCP(M, q), when $q = \tilde{q}$, we have $\tilde{v} = M^{-1}_{(S_N,S_N)}(-\tilde{\mathfrak{q}}_1) = M^{-1}_{(S_N,S_N)}(-\hat{\mathfrak{q}}_1) = \hat{v}$ is a nonnegative vector. Thus, if $\hat{q}_i + M_{(i,S_N)}\hat{v} \geq 0$ for $i \in S_P$, then for $\tilde{q}$, we also have $\tilde{q}_i + M_{(i,S_N)}\tilde{v} \geq \hat{q}_i + M_{(i,S_N)}\hat{v} \geq 0$ for $i \in S_P$. Hence, from Theorem 2.3, (i)-(iv) hold. Thus, $\tilde{q}$ and $\hat{q}$ belong to the same complementarity cone of M. The solution $\tilde{z}$ of (1), when $q = \tilde{q}$ satisfies $\tilde{z} = \hat{z}$ from $\tilde{v} = \hat{v}$. Then, the conclusion is proved. □

Remark 2.2. Theorem 2.4 means that the nonnegative elements of the vector q are increased, but the solution of the LCP(M, q) keeps the same. The following conclusion shows the monotonicity of solution of the LCP(M, q) with an M-matrix under a certain condition.

Theorem 2.5. *Suppose $M \in R^{n\times n}$ is an M-matrix, and $\hat{q} = (\hat{q}_1, \hat{q}_2, ..., \hat{q}_n)^{\mathrm{T}}, \tilde{q} = (\tilde{q}_1, \tilde{q}_2, ..., \tilde{q}_n)^{\mathrm{T}} \in R^n$ satisfy $\hat{q} \leq \tilde{q}$, and possess the same negative elements' indices set $S_N = \{i_1, i_2, ..., i_s\}$ and the nonnegative elements' indices set $S_P = \{j_1, j_2, ..., j_t\}$. Denote $\hat{\mathfrak{q}}_1 = (\hat{q}_{i_1}, \hat{q}_{i_2}, ..., \hat{q}_{i_s})^{\mathrm{T}}$, $\tilde{\mathfrak{q}}_1 = (\tilde{q}_{i_1}, \tilde{q}_{i_2}, ..., \tilde{q}_{i_s})^{\mathrm{T}}$, and denote the principal submatrix of M by $M_{(S_N,S_N)}$ and $\hat{v} = M^{-1}_{(S_N,S_N)}(-\hat{\mathfrak{q}}_1)$, $\tilde{v} = M^{-1}_{(S_N,S_N)}(-\tilde{\mathfrak{q}}_1)$. If $\hat{q}_i + M_{(i,S_N)}\hat{v} \geq 0$ and $\tilde{q}_i + M_{(i,S_N)}\tilde{v} \geq 0$ for $i \in S_P$, then the solution $\hat{z}$ of LCP($M, \hat{q}$) and the solution $\tilde{z}$ of LCP($M, \tilde{q}$) satisfy $\hat{z} \geq \tilde{z}$.*

Proof. Since $\hat{q}, \tilde{q}$ have the same negative elements' indices set S_N, and $\hat{q}_i + M_{(i,S_N)}\hat{v} \geq 0$, $\tilde{q}_i + M_{(i,S_N)}\tilde{v} \geq 0$ for $i \in S_P$, combining with Theorem 2.3, we know that the solutions $\hat{z}$ and $\tilde{z}$ have the same zero elements' indices set S_P, and the same nonnegative elements' indices set S_N. Therefore, we only need to prove that there is the monotonic decreasing relationship on the nonnegative elements' indices set S_N.

From the proof of Theorem 2.3, we know that the nonnegative elements of the solutions $\hat{z}, \tilde{z}$ come from the positive vector $\hat{v}, \tilde{v}$ respectively, and the elements selecting order corresponds to the set S_N. For $M^{-1}_{(S_N,S_N)} \geq 0$, $\hat{\mathfrak{q}}_1 \leq \tilde{\mathfrak{q}}_1 < 0$, we have

$$\hat{v} = M^{-1}_{(S_N,S_N)}(-\hat{\mathfrak{q}}_1) \geq M^{-1}_{(S_N,S_N)}(-\tilde{\mathfrak{q}}_1) = \tilde{v}.$$

Thus, $\hat{z} \geq \tilde{z}$ holds, and the conclusion is proved. □

Remark 2.3. Theorems 2.2-2.5 discuss the solution z of the LCP(M, q), and the difference is that Theorem 2.2 is for the single variable of the solution when the system matrix M is a P-matrix, and the other three theorems are for the whole solution, when the system matrix M is an M-matrix.

3. Numerical examples

In this section, we show experiments to illustrate the presented results, that is, Theorems 2.1-2.5. The first two examples are low-order cases, and the third example is a high-order case.

Example 3.1. We set the system matrix M in the LCP(M, q) to be

$$M = \begin{pmatrix} 1 & 0 & -1 & 1 \\ -1 & 2 & 0 & -1 \\ 0 & -2 & 3 & 1 \\ -1 & 2 & -3 & 4 \end{pmatrix}$$

and consider q for two cases. That is,

$$\tilde{q} = (-1, 3, -3, 6)^{\mathrm{T}}, \hat{q} = (1, 1, -6, 8)^{\mathrm{T}}.$$

Then, we know that M is a P-matrix, and the solutions are

$$\tilde{z} = (1, 0, 2, 0)^{\mathrm{T}}, \hat{z} = (2, 0, 1, 0)^{\mathrm{T}}$$

for the LCP$(M, \tilde{q})$ and the LCP$(M, \hat{q})$ respectively. Meanwhile, we have $\tilde{q}, \hat{q} \in pos\{-M_{\cdot 1}, I_2, -M_{\cdot 3}, I_4\}$. Then, from Theorem 2.1, we know that $\lambda\tilde{z} + \mu\hat{z}$ is the unique solution of the LCP$(M, \bar{q})$ with $\bar{q} = \lambda\tilde{q} + \mu\hat{q}$ and $\lambda \geq 0,\ \mu \geq 0$.

Example 3.2. We set M in the LCP(M, q) to be

$$M = \begin{pmatrix} 1 & 0 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 3 \end{pmatrix}$$

and consider q for three cases. That is,

$$\hat{q} = \begin{pmatrix} -1 \\ 3 \\ -1 \end{pmatrix}, \tilde{q} = \begin{pmatrix} -1 \\ 3+\beta \\ -1 \end{pmatrix}, \bar{q} = \begin{pmatrix} -1 \\ 4 \\ -\frac{1}{2} \end{pmatrix}$$

respectively. It is easy to know that the matrix M is an M-matrix, and for $\hat{q}$, $\tilde{q}$ and $\bar{q}$, we have the same sets $S_P = \{2\}$ and $S_N = \{1, 3\}$, when $\beta \geq 0$.

(I) When $q = \hat{q} \leq \tilde{q}$, we have

$$M_{(S_N, S_N)} = \begin{pmatrix} 1 & -1 \\ -1 & 3 \end{pmatrix}, \hat{v} = M^{-1}_{(S_N, S_N)}(-\hat{\mathfrak{q}}_1) = \begin{pmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} -\hat{q}_1 \\ -\hat{q}_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix},$$

$$\hat{q}_2 + M_{(2, S_N)}\hat{v} = 3 + (-3) = 0.$$

From Theorem 2.3, we know that the solution $\hat{z}$ satisfies $\hat{z}_2 = 0$, $\hat{z}_1, \hat{z}_3 \geq 0$, and the solution is

$$\hat{z} = \begin{pmatrix} \hat{z}_1 \\ \hat{z}_2 \\ \hat{z}_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix},$$

with $\hat{q} \in pos\{-M_{.1}, I_2, -M_{.3}\}$. Meanwhile, we know that $\tilde{q} \in pos\{-M_{.1}, I_2, -M_{.3}\}$, and the solution $\tilde{z}$ of the LCP$(M, \tilde{q})$ satisfies $\tilde{z} = \hat{z}$ from Theorem 2.4.

(II) When $q = \bar{q} \geq \hat{q}$, from Theorem 2.3, similarly, we have

$$\bar{v} = M^{-1}_{(S_N,S_N)}(-\bar{\mathfrak{q}}_1) = \begin{pmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} -\bar{q}_1 \\ -\bar{q}_3 \end{pmatrix} = \begin{pmatrix} \frac{7}{4} \\ \frac{3}{4} \end{pmatrix}, \bar{q}_2 + M_{(2,S_N)}\bar{v} = 4 + (-\frac{5}{2}) = \frac{3}{2},$$

and the solution of the LCP$(M, \bar{q})$ is

$$\bar{z} = \begin{pmatrix} \bar{z}_1 \\ \bar{z}_2 \\ \bar{z}_3 \end{pmatrix} = \begin{pmatrix} \frac{7}{4} \\ 0 \\ \frac{3}{4} \end{pmatrix}.$$

We can find that $\bar{z} \leq \hat{z}$, which illustrates the monotonicity conclusion in Theorem 2.5.

Example 3.3. In this example, we consider a high-order case for Theorem 2.2. We set the matrix M to be

$$M_1 = \begin{pmatrix} 4 & -1 & & & \\ -1 & 4 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & 4 & -1 \\ & & & -1 & 4 \end{pmatrix} \text{ and } M_2 = \begin{pmatrix} 1 & -1 & & & \\ -1 & 2 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & n-1 & -1 \\ & & & -1 & n \end{pmatrix}$$

respectively, and set

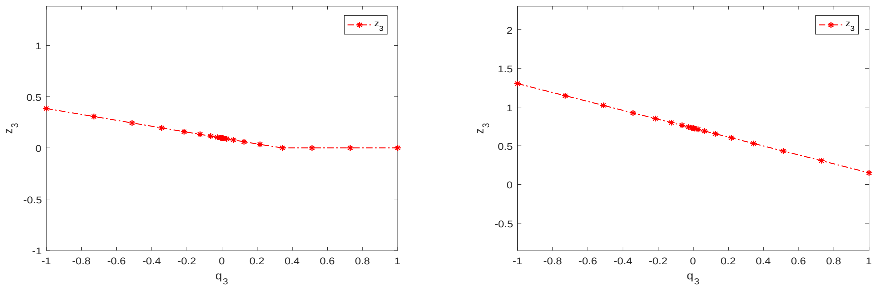
$$q = (-1, 0, x^3, -1, 0, 1, -1, 0, 1, \cdots)^{\mathrm{T}} \in R^n.$$

Then, both M_1 and M_2 are M-matrices, and both the LCP(M_1, q) and the LCP(M_2, q) have a unique solution for any $x \in R$. Specially, we set

$$x = -1 : \frac{1}{10} : 1$$

in our experiments, and solve the two linear complementarity problems by the block principal pivoting algorithm [9]. We set $n = 1000$, and consider the third element z_3 of the solution z. The numerical results are shown in Figure 1 as follows.

From (a) and (b) in Figure 1, we can see that z_3 is the decreasing function of q_3, which verifies the conclusion of Theorem 2.2.



(a) (b)

Figure 1. The monotonicity of z_3

4. Concluding remarks

In this paper, the monotonicity of solutions to the LCP(M, q) is considered, and the monotonicity about a single variable of the solution is presented. The distribution of zero elements and the monotonicity of the whole solution are also proposed, when the system matrix M is an M-matrix. The numerical experiments are illustrated to show the presented results.

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