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Transversal Heteroclinic Bifurcation in Hybrid Systems with Application to Linked Rocking Blocks*

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Abstract In this paper, we study heteroclinic bifurcation and the appearance of chaos in time-perturbed piecewise smooth hybrid systems with discontinuities on finitely many switching manifolds. The unperturbed system has a heteroclinic orbit connecting hyperbolic saddles of the unperturbed system that crosses every switching manifold transversally, possibly multiple times. By applying a functional analytical method, we obtain a set of Melnikov functions whose zeros correspond to the occurrence of chaos of the system. As an application, we present an example of quasiperiodically excited piecewise smooth system with impacts formed by two linked rocking blocks.

Keywords Melnikov method, Hybrid system, Heteroclinic bifurcation, Chaos, Linked rocking blocks.

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1. Introduction

It is very important and interesting in the theory of dynamical systems to investigate the occurrence of chaos. Common routes to chaos for smooth systems include period-doubling, intermittency, torus bifurcation and homoclinic bifurcation [26,42]. In particular, for a periodically excited smooth system with a homoclinic orbit, the perturbed stable and unstable manifolds intersect transversally under some conditions, which implies the existence of Smale horseshoe chaos via Smale-Birkhoff Homoclinic Theorem. The Melnikov method is a powerful analytical tool that can be used to determine whether transversal homoclinic intersection occurs [19,25,26,40].

In recent years, with the development of science and technology, there have been lots of works devoted to the study of bifurcation phenomena in piecewise smooth (PWS) dynamical systems [9, 16, 19, 38, 39, 46]. This is because many problems from real applications in fields such as mechanics, electrical engineering and control theory are modelled by PWS systems. For such systems, a typical route to chaos is through discontinuity-induced bifurcations, such as grazing, sliding, border-collision and chattering [2, 9, 13, 16, 19, 20, 32, 33, 39, 46].

In [15, 43], Chow, Rand and Shaw studied homoclinic bifurcations for a class of periodically excited linear inverted pendulum. Their numerical results suggest

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that homoclinic bifurcation can also lead to chaotic motions for PWS systems. In the last decades, many efforts have been made on extending the Melnikov method established for smooth systems to PWS systems. To mention only a few of them, see [2, 3, 14, 18–20, 22, 31, 36, 45]. It is assumed in these works that the unperturbed homoclinic or periodic orbit intersects the switching manifold transversally. In [1, 4, 5, 7, 19], Battelli, Fečkan, Awrejcewicz et al. extended the Melnikov method to sliding homoclinic bifurcation of general n-dimensional PWS systems. Grazing homoclinic bifurcation for a periodically excited nonlinear inverted pendulum was also studied in [17]. Calamai, Franca and Pospíšil [12, 21] investigated homoclinic bifurcations in PWS systems with the critical point lies on the switching manifold. In particular, they proved that in this case, the existence of a transversal homoclinic point does not imply chaos. In [4–8, 19], by using a functional analytical method, Battelli and Fečkan proved rigorously that if a certain Melnikov function has a simple zero, then under some recurrent conditions, a time-perturbed PWS system with transversal or sliding homoclinic orbit behaves chaotically in the sense that the system has a Smale-like horseshoe.

In 1988, Bertozzi extended the Smale-Birkhoff Homoclinic Theorem and the Melnikov method, so they are applicable to heteroclinic bifurcations for smooth systems [10]. It is natural to ask if the transversal intersections of the perturbed stable and unstable manifolds of a heteroclinic orbit of PWS systems result in chaotic motions. Unfortunately, the Heteroclinic Theorem of Berttozzi [10] requires the corresponding Poincaré map to be differentiable. Thus, it cannot be applied to PWS systems, because this condition is not satisfied by most of the PWS systems. Nevertheless, the study of heteroclinic bifurcations in time-perturbed PWS systems has attracted increasing attention. Heteroclinic bifurcations for models of periodically excited slender rigid blocks were studied in the works of Bruhn and Koch [11], Hogan [28], Lenci and Rega in [34]. In [23], Granados, Hogan and Seara presented the Melnikov method for heteroclinic and subharmonic bifurcations in a periodically excited piecewise planar Hamiltonian system with two zones. The Melnikov method for heteroclinic bifurcations of a planar PWS system with impacts and of a general planar PWS system with finitely many zones were developed in [35] and [44] respectively. Although it is not rigorously proved, numerical simulations on concrete examples given in these works suggest that chaotic behavior can be resulted from heteroclinic bifurcations in PWS systems.

Recently, by applying the aforementioned functional analytic method developed by Battelli and Fečkan in [4–8, 19], Li and Du [37] studied the appearance of chaos in time-perturbed n-dimensional PWS systems with heteroclinic orbit. They derived a set of Melnikov type functions whose zeros correspond to the occurrence of chaos of the system. To reduce the complexity, they assumed that the switching manifolds are supersurfaces intersecting at a connected $(n-2)$-dimensional submanifold, the unperturbed system has a hyperbolic saddle in each subregion and a heteroclinic orbit connecting those saddles that crosses every switching manifold transversally exactly once. However, in real applications, discontinuities of a PWS system may occur on more complicated sets and impacts may occur, when the flow of the system reaches the switching manifolds. Thus, it is necessary to extend the results obtained in [37] to systems with other types of switching manifolds and other types of PWS systems, for example, systems with impacts considered in [23, 35].

The aim of this paper is to extend the results of [37] to more general systems, namely n-dimensional time-perturbed PWS hybrid systems. We assume that the unperturbed system has an orbit connecting hyperbolic saddles of the unperturbed system that crosses every switching manifold transversally. Comparing with the work of [37], in this paper, we do not require that the switching manifolds intersect each other, and we allow impacts occur when the flow of the system reaches the switching manifold. Furthermore, the heteroclinic orbit may exit and enter a subregion multiple times. Consequently, the heteroclinic orbit may connect multiple hyperbolic saddles in the same subregion. We obtain a set of Melnikov type functions, and show that their zeros correspond to the occurrence of chaos of the system. Obviously, the system considered in this paper is more general, and the results obtained here can be applied to more general situations such as the ones studied in [23, 35].

As an application, in this paper, we study heteroclinic bifurcation and chaos for quasiperiodically excited system consisting of two slender rocking blocks coupled by a light spring. The slender rocking block model is an important PWS system that can be widely used in earthquake engineering and robotics. The single block model was first proposed by Housner in [30] and its dynamics have been extensively investigated in the past. See, for example, [11, 23, 27, 28, 30]. In [24], Granados, Hogan and Seara considered a periodically excited mechanical system consisting of two slender rocking blocks coupled by a light spring. For simplicity, assume that both blocks are identical. Under the assumption that on impact with the rigid base, neither block loses energy, the Arnold diffusion of the system was studied in [24]. As pointed out in [24], such a phenomenon can be seen as one possible mechanism for block overturning. As in [24], we assume that both blocks are identical. However, we assume that on impact with the rigid base, and both blocks lose energy. We further assume that the system can be periodically or quasiperiodically excited, which is more realistic, because many systems arising from real application are externally excited by more than one frequencies. We show that the unperturbed system has a transversal heteroclinic orbit. Then, we compute the corresponding Melnikov functions and prove that the heteroclinic bifurcation of this four-dimensional system results in chaos under certain conditions.

Our paper is organized as follows: In Section 2, we present some basic assumptions and the main result of the paper, namely Theorem 2.1. In Section 3, we describe how to prove Theorem 2.1. In Section 4, we discuss chaotic behavior in time-perturbed hybrid systems whose unperturbed system has a transversal heteroclinic orbit. In Section 5, we present an example of quasiperiodically excited piecewise smooth system with impacts formed by two linked rocking blocks. Some concluding remarks are given in Section 6.

2. Preliminaries

First, we introduce some notations. Let k, l be positive integers. For two column vectors $v_1, v_2 \in \mathbb{R}^k$, $\langle v_1, v_2\rangle$ and $|v_1|$ are defined by $\langle v_1, v_2\rangle = v_1^T v_2$ and $|v_1| = \sqrt{\langle v_1, v_1\rangle}$ respectively. For a $k \times k$ real matrix A, $\|A\|$ is defined by $\|A\| = \max_{|x|=1} |Ax|$. The identity matrix of proper size is denoted by I. For a given linear map $L : \mathbb{R}^k \mapsto \mathbb{R}^l$, its range and kernel are denoted by $\mathcal{R}L$ and $\mathcal{N}L$ respectively. To simplify notations, without causing confusions, as in [8], we use the notation $\|\cdot\|$ for any norm on a Banach space $\mathcal{X}$ instead of notation like $\|\cdot\|_{\mathcal{X}}$. The boundary and closure of a set $E \subset \mathbb{R}^k$ are denoted by ∂E and $\bar{E}$ respectively.

The gradient of a smooth scalar function $h : \mathbb{R}^k \mapsto \mathbb{R}$ is denoted by ∇h, and the divergence and the Jacobian matrix of a smooth map $G : \mathbb{R}^k \mapsto \mathbb{R}^k$ are denoted by div$G$ and DG respectively. Let $\ell^\infty(\mathbb{R}^k)$ be the Banach space:

$$\ell^\infty(\mathbb{R}^k) = \left\{ \{x_j\}_{j\in\mathbb{Z}} : \ x_j \in \mathbb{R}^k \text{ for } j \in \mathbb{Z}, \sup_{j\in\mathbb{Z}} |x_j| < \infty \right\}$$

with the norm $\|\{x_j\}_{j\in\mathbb{Z}}\| = \sup_{j\in\mathbb{Z}} |x_j|$. Let

$$\ell_1^\infty(\mathbb{R}) = \left\{ \{\alpha_j\}_{j\in\mathbb{Z}} \in \ell^\infty(\mathbb{R}) : \ \sup_{j\in\mathbb{Z}} |\alpha_j - \alpha_{j-1}| < 1 \right\}.$$

Then, $\ell_1^\infty(\mathbb{R})$ is an open nonempty subset of $\ell^\infty(\mathbb{R})$.

Let $\Omega \subseteq \mathbb{R}^n$ ($n \geq 2$) be an open disc in $\mathbb{R}^n$ and $m \geq 2$ be an integer, $\mathcal{J} = \{1, 2, \cdots, m\}$. Assume that Ω is split into m disjoint open regions Ω_1, Ω_2, $\cdots$, Ω_m by the discontinuity sets $\mathcal{C}_{ij} = \bar\Omega_i \cap \bar\Omega_j$ for $1 \leq i < j \leq m$, where each $\mathcal{C}_{ij}$ is either the empty set, when $\bar\Omega_i \cap \bar\Omega_j = \emptyset$, or a hypersurface given by $\mathcal{C}_{ij} = \{x \in \bar\Omega : h_{ij}(x) = 0\}$ with $h_{ij} \in C^2(\bar\Omega, \mathbb{R})$, when $\bar\Omega_i \cap \bar\Omega_j \neq \emptyset$.

For $i \in \mathcal{J}$, let $f_i \in C_b^2(\bar\Omega_i, \mathbb{R}^n)$, $g_i \in C_b^2(\bar\Omega_i \times \mathbb{R} \times \mathbb{R}, \mathbb{R}^n)$, namely f_i and g_i have uniformly bounded derivatives up to the second order on $\bar\Omega_i$ and $\bar\Omega_i \times \mathbb{R} \times \mathbb{R}$, respectively. Moreover, we assume that their second order derivatives are uniformly continuous.

Now, we consider the following PWS system defined on Ω:

$$\dot x = f_i(x) + \varepsilon g_i(x, t, \varepsilon), \quad x \in \bar\Omega_i, \quad i \in \mathcal{J}, \tag{2.1}$$

plus a set of reset maps

$$x \mapsto \mathcal{R}_{ij}(x, \varepsilon), \quad x \in \mathcal{C}_{ij}, \quad \text{for } 1 \leq i < j \leq m \text{ and } \mathcal{C}_{ij} \neq \emptyset, \tag{2.2}$$

where $\varepsilon \in \Lambda := (-\varepsilon_0, \varepsilon_0)$ for some $\varepsilon_0 > 0$, and for each $1 \leq i < j \leq m$ with $\mathcal{C}_{ij} \neq \emptyset$, $\mathcal{R}_{ij} \in C_b^2(\mathbb{R}^n \times \Lambda, \mathbb{R}^n)$, namely, $\mathcal{R}_{ij}$ has uniformly bounded derivatives up to the second order and its second order derivative is uniformly continuous. Furthermore, we assume that $\mathcal{R}_{ij}$ maps the set $\mathcal{C}_{ij}$ back to itself for $1 \leq i < j \leq m$. Obviously, the reset maps (2.2) are the generalizations of the impact laws defined on impact manifolds of an impact system.

In the sequel, we assume that $\mathcal{C}_{ij} = \mathcal{C}_{ji}$ and $\mathcal{R}_{ij} = \mathcal{R}_{ji}$ for $1 \leq i < j \leq m$. Let $D_1\mathcal{R}_{ij}(x, \varepsilon)$ and $D_2\mathcal{R}_{ij}(x, \varepsilon)$ be the derivatives of $\mathcal{R}_{ij}(x, \varepsilon)$ with respect to the variables x and ε respectively for $(x, \varepsilon) \in \mathbb{R}^n \times \Lambda$.

When $\varepsilon = 0$, the unperturbed system of (2.1-2.2) has the following form:

$$\begin{cases} \dot x = f_i(x), & x \in \bar\Omega_i, \quad i \in \mathcal{J}, \\ x \mapsto \mathcal{R}_{ij}(x, 0), & x \in \mathcal{C}_{ij}, \quad \text{for } 1 \leq i < j \leq m \text{ and } \mathcal{C}_{ij} \neq \emptyset. \end{cases} \tag{2.3}$$

Let the following assumptions hold:

(H1) System (2.3) has N ($N \geq 2$) hyperbolic saddles p_1, p_2, $\cdots$, p_N such that, there is a sequence i_1, i_2, $\cdots$, $i_N \in \mathcal{J}$ with $p_j \in \Omega_{i_j}$ and $\mathcal{C}_{i_j i_{j+1}} \neq \emptyset$ for $j \in \mathcal{J}_N := \{1, 2, \cdots, N\}$, where for the rest of this paper, we always assume that $i_0 = i_N$ and $i_{N+1} = i_1$.

(H2) System (2.3) has a heteroclinic cycle Γ which consists of $2N$ branches $\Gamma_j^s = \{\gamma_j^s(t):\ t\in[0,+\infty)\}\subset\Omega_{i_j}$, $\Gamma_j^u = \{\gamma_j^u(t):\ t\in(-\infty,0]\}\subset\Omega_{i_j}$ $(j\in\mathcal{J}_N)$ such that

$$\Gamma = \bigcup_{j=1}^{N}\left(\Gamma_j^u \bigcup \{p_j\} \bigcup \Gamma_j^s\right),$$

where for each $j\in\mathcal{J}_N$, $\gamma_j^{u,s}(t)$ are solutions of (2.3) in Ω_{i_j}, Γ_j^u and Γ_{j+1}^s intersect $\mathcal{C}_{i_j i_{j+1}}$ transversally at $\gamma_j^u(0)$ and $\gamma_{j+1}^s(0)\in\mathcal{C}_{i_j i_{j+1}}$ respectively with $\mathcal{R}_{i_j i_{j+1}}(\gamma_j^u(0),0)=\gamma_{j+1}^s(0)\in\mathcal{C}_{i_j i_{j+1}}$. Furthermore,

$$\lim_{t\to+\infty}\gamma_j^s(t) = \lim_{t\to-\infty}\gamma_j^u(t) = p_j.$$

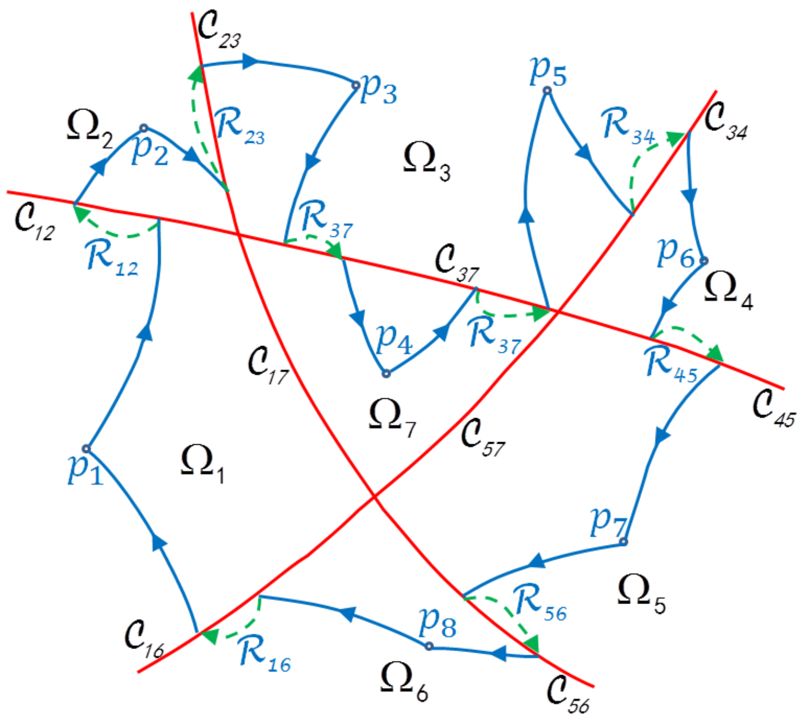

Figure 1. A heteroclinic cycle Γ of system (2.3) with $n=2$, $N=8$ and $m=7$

A heteroclinic cycle Γ of system (2.3) with $n=2$, $N=8$ and $m=7$ is shown in Figure 1. As it can be seen from Figure 1 that, in this case, we have $i_1=1$, $i_2=2$, $i_3=3$, $i_4=7$, $i_5=3$, $i_6=4$, $i_7=5$, $i_8=6$. Thus, the sequence i_1, i_2, $\cdots$, $i_N\in\mathcal{J}$ is not necessarily strictly increasing. Consequently, each zone may contain multiple saddles, the cycle Γ may cross a switching manifold transversally multiple times. Referring to Figure 1, Ω_3 contains two saddles p_3 and p_5, Γ crosses $\mathcal{C}_{37}$ transversally twice. Here, we take $\gamma_j^u(0)$ and $\gamma_{j+1}^s(0)$ as the same point due to the reset map $\mathcal{R}_{i_j i_{j+1}}$ for $j\in\mathcal{J}_N$. Hence, the system considered in this paper is more general than the one considered in [37]. Since (2.3) is autonomous without loss of generality and for the sake of simplicity, we assume that for each of $j\in\mathcal{J}_N$, the time at which $\gamma_j^{u,s}(t)$ reaches the corresponding switching manifold is $t = 0$ in (H2).

For technical purposes, for each $i \in \mathcal{J}$ and $j \in \mathcal{J}_N$, we extend the domains for $\gamma_j^{u,s}$ so that they are defined on $\mathbb{R}$ and extend the domains of f_i and g_i such that $f_i \in C_b^2(\mathbb{R}^n, \mathbb{R}^n)$, $g_i \in C_b^2(\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}, \mathbb{R}^n)$ and

$$\begin{aligned} \sup\left\{|f_i(x)| : \ x \in \mathbb{R}^n\right\} &\le 2\sup\left\{|f_i(x)| : x \in \bar{\Omega}_i\right\}, \\ \sup\left\{|g_i(x,t,\varepsilon)| : \ (x,t,\varepsilon) \in \mathbb{R}^{n+2}\right\} &\le 2\sup\left\{|g_i(x,t,\varepsilon)| : (x,t,\varepsilon) \in \bar{\Omega}_i \times \mathbb{R} \times \Lambda\right\}. \end{aligned}$$

In the sequel, we use the same notations for the extend functions as the original ones and assume that up to the second order all the derivatives of $\mathcal{R}_{ij}$ and the extended f_i and g_i are bounded, uniformly continuous. Those assumptions for the extend functions have no effect on the results obtained in this paper, because only the values of f_i and g_i for x in a compact neighborhood of Γ are needed.

Since for each $j \in \mathcal{J}_N$, p_j is a hyperbolic saddle of (2.3) in Ω_{i_j}, the linear variational system

$$\dot{x} = Df_{i_j}(\gamma_j^s(t))x, \quad t \ge -1 \tag{2.4}$$

has an exponential dichotomy on $[-1, +\infty)$ with projection $P_{i_j}^s$, constant K and exponent $\rho > 0$ such that

$$\begin{aligned} \|X_{i_j}^s(t)P_{i_j}^s(X_{i_j}^s(s))^{-1}\| &\le Ke^{-\rho(t-s)} \quad \text{for } -1 \le s \le t, \\ \|X_{i_j}^s(t)(I - P_{i_j}^s)(X_{i_j}^s(s))^{-1}\| &\le Ke^{\rho(t-s)} \quad \text{for } -1 \le t \le s, \end{aligned}$$

where $X_{i_j}^s(t)$ is the fundamental matrix solution of (2.4) with $X_{i_j}^s(0) = I$ for $t \in [-1, +\infty)$. Similarly, for each $j \in \mathcal{J}_N$, the linear variational system

$$\dot{x} = Df_{i_j}(\gamma_j^u(t))x, \quad t \le 1 \tag{2.5}$$

has an exponential dichotomy on $(-\infty, 1]$ with projection $P_{i_j}^u$, constant K and exponent $\rho > 0$ such that

$$\begin{aligned} \|X_{i_j}^u(t)P_{i_j}^u(X_{i_j}^u(s))^{-1}\| &\le Ke^{-\rho(t-s)} \quad \text{for } s \le t \le 1, \\ \|X_{i_j}^u(t)(I - P_{i_j}^u)(X_{i_j}^u(s))^{-1}\| &\le Ke^{\rho(t-s)} \quad \text{for } t \le s \le 1, \end{aligned}$$

where $X_{i_j}^u(t)$ is the fundamental matrix solution of (2.5) with $X_{i_j}^u(0) = I$ for $t \in (-\infty, 1]$. Here, without loss of generality, we assume that K and ρ of the dichotomies on $(-\infty, 1]$ and $[-1, +\infty)$ for all $j \in \mathcal{J}_N$ are the same.

For $j \in \mathcal{J}_N$, let $R_{i_j i_{j+1}}^u : \mathbb{R}^n \mapsto \mathbb{R}^n$ be the projection onto $\mathcal{N}\nabla h_{i_j i_{j+1}}(\gamma_j^u(0))$ along $\dot{\gamma}_j^u(0)$ and $R_{i_j i_{j+1}}^s : \mathbb{R}^n \mapsto \mathbb{R}^n$ be the projection onto $\mathcal{N}\nabla h_{i_j i_{j+1}}(\gamma_{j+1}^s(0))$ along $\dot{\gamma}_{j+1}^s(0)$ and given by

$$\begin{aligned} R_{i_j i_{j+1}}^s w &= w - \frac{\langle \nabla h_{i_j i_{j+1}}(\gamma_{j+1}^s(0)), w\rangle}{\langle \nabla h_{i_j i_{j+1}}(\gamma_{j+1}^s(0)), \dot{\gamma}_{j+1}^s(0)\rangle}\dot{\gamma}_{j+1}^s(0), \\ R_{i_j i_{j+1}}^u w &= w - \frac{\langle \nabla h_{i_j i_{j+1}}(\gamma_j^u(0)), w\rangle}{\langle \nabla h_{i_j i_{j+1}}(\gamma_j^u(0)), \dot{\gamma}_j^u(0)\rangle}\dot{\gamma}_j^u(0) \end{aligned}$$

for $w \in \mathbb{R}^n$. Let

$$S_j^u = \mathcal{N}P_{i_j}^u \cap \mathcal{N}\nabla h_{i_j i_{j+1}}(\gamma_j^u(0)),$$

$$S_j^s = \mathcal{R}P_{i_j}^s \cap \mathcal{N}\nabla h_{i_{j-1}i_j}(\gamma_j^s(0)),$$
$$\tilde{S}_j^u = D_1\mathcal{R}_{i_ji_{j+1}}(\gamma_j^u(0),0)S_j^u.$$

Since $\dim S_j^u = \dim \mathcal{N}P_{i_j}^u - 1$, $\dim S_j^s = \dim \mathcal{R}P_{i_j}^s - 1$, we have $\dim\left(\mathcal{R}P_{i_{j+1}}^s + \tilde{S}_j^u\right) \leq n-1$. We assume that

(H3) For each $j \in \mathcal{J}_N$, $\mathcal{R}P_{i_{j+1}}^s + \tilde{S}_j^u$ has codimension one in $\mathbb{R}^n$.

Similar to that of [4,5,7,8,19], condition (H3) is a kind of nondegeneracy condition on Γ. By (H3), for each $j \in \mathcal{J}_N$, there is a unitary vector $\psi_{i_j} \in (\mathcal{R}P_{i_{j+1}}^s + \tilde{S}_j^u)^\perp$ such that

$$\mathbb{R}^n = (\mathcal{R}P_{i_{j+1}}^s + \tilde{S}_j^u) \oplus \mathrm{span}(\psi_{i_j}).$$

Let $\psi_{i_1}, \cdots, \psi_{i_N}$ be fixed in the sequel.

Let P_{i_j} be the stable projection of the dichotomy of the linear system $\dot{x} = Df_{i_j}(p_j)x$ for $j \in \mathcal{J}_N$. Let

$$\tilde{P}_{i_j}^s(t) = X_{i_j}^s(t)P_{i_j}^s(X_{i_j}^s(t))^{-1}, \quad \tilde{P}_{i_j}^u(t) = X_{i_j}^u(-t)P_{i_j}^u(X_{i_j}^u(-t))^{-1}.$$

Then, by a result in [41], we have

$$\lim_{t\to\infty} \|\tilde{P}_{i_j}^{u,s}(t) - P_{i_j}\| = 0, \quad j \in \mathcal{J}_N.$$

Hence, there is a $T \gg 1$ such that for any $t_1, t_2 \geq T$, we have

$$\mathcal{N}\tilde{P}_{i_{j+1}}^s(t_1) \oplus \mathcal{R}\tilde{P}_{i_j}^u(t_2) = \mathbb{R}^n, \quad j \in \mathcal{J}_N. \tag{2.6}$$

Again, here we assume that the values T for all $j \in \mathcal{J}_N$ are the same, and let T be fixed in the sequel.

For each $j \in \mathcal{J}_N$, define

$$\Psi_j(t) = \begin{cases} ((X_{i_{j+1}}^s(t))^{-1})^T(I - (P_{i_{j+1}}^s)^T)(R_{i_ji_{j+1}}^s)^T\psi_{i_j}, & t \geq 0, \\ ((X_{i_j}^u(t))^{-1})^T(P_{i_j}^u)^T(R_{i_ji_{j+1}}^u)^T(D_1\mathcal{R}_{i_ji_{j+1}}(\gamma_j^u(0),0))^T\psi_{i_j}, & t \leq 0. \end{cases} \tag{2.7}$$

For $\alpha \in \mathbb{R}$, let

$$\mathcal{M}_j^u(\alpha) = \int_{-\infty}^0 \Psi_j^T(\tau)g_{i_j}(\gamma_j^u(\tau), \tau+\alpha, 0)d\tau, \tag{2.8}$$

$$\mathcal{M}_j^s(\alpha) = \int_0^{+\infty} \Psi_j^T(\tau)g_{i_{j+1}}(\gamma_{j+1}^s(\tau), \tau+\alpha, 0)d\tau. \tag{2.9}$$

We define the Melnikov functions by

$$\mathcal{M}_j(\alpha) = \psi_{i_j}^T D_2\mathcal{R}_{i_ji_{j+1}}(\gamma_j^u(0),0) + \mathcal{M}_j^u(\alpha) + \mathcal{M}_j^s(\alpha), \tag{2.10}$$

Clearly, under our assumptions, $\mathcal{M}_j^u(\alpha)$, $\mathcal{M}_j^s(\alpha)$ and $\mathcal{M}_j(\alpha)$ are all C^2 functions.

Consider a planar PWS system as a special case, i.e., $n=2$. For each $j \in \mathcal{J}_N$, define

$$\beta_{i_j}^u(t) = \exp\left(-\int_0^t \mathrm{div} f_{i_j}(\gamma_j^u(s))ds\right),$$

$$\beta^s_{i_j}(t) = \exp\bigg(-\int_0^t \mathrm{div} f_{i_j}(\gamma^s_j(s))ds\bigg),$$
$$F^u_{i_j}(t,\alpha) = f_{i_j}(\gamma^u_j(t)) \wedge g_{i_j}(\gamma^u_j(t), t+\alpha, 0),$$
$$F^s_{i_j}(t,\alpha) = f_{i_j}(\gamma^s_j(t)) \wedge g_{i_j}(\gamma^s_j(t), t+\alpha, 0).$$

Then, similar to the method in [37], we have the following explicit expressions for the Melnikov functions $\mathcal{M}_j(\alpha)$ for $j \in \mathcal{J}_N$:

$$\begin{aligned}\mathcal{M}_j(\alpha) = \Pi_{i_{j+1}}\bigg\{&((f_{i_{j+1}}(\gamma^s_{j+1}(0)))^{\perp})^T D_2\mathcal{R}_{i_j i_{j+1}}(\gamma^u_j(0),0)\\ &+\frac{\langle(\nabla h_{i_j i_{j+1}}(\gamma^u_j(0)))^{\perp}, (D_1\mathcal{R}_{i_j i_{j+1}}(\gamma^u_j(0),0))^T(f_{i_{j+1}}(\gamma^s_{j+1}(0)))^{\perp}\rangle}{\langle\nabla h_{i_j i_{j+1}}(\gamma^u_j(0)), \dot{\gamma}^u_j(0)\rangle}\\ &\cdot\int_{-\infty}^0 F^u_{i_j}(\tau,\alpha)\beta^u_{i_j}(\tau)d\tau + \int_0^{+\infty} F^s_{i_{j+1}}(\tau,\alpha)\beta^s_{i_{j+1}}(\tau)d\tau\bigg\},\end{aligned}$$

where $\Pi_{i_j} = 1/|f_{i_j}(\gamma^s_j(0))|$.

In this paper, we aim to investigate the chaotic behaviors of system (2.1) near the heteroclinic orbit Γ. For this purpose, we need to look for the solution defined on $\mathbb{R}$ of (2.1) that belongs to a small neighborhood of Γ. For this, we have the following theorem, which is also the main result of this paper:

Theorem 2.1. *Assume that for each $i \in \mathcal{J}$, f_i and g_i are functions with uniformly bounded derivatives upto the second order on $\bar{\Omega}_i$ and $\bar{\Omega}_i \times \mathbb{R}^2$ respectively and their second order derivatives are uniformly continuous. Suppose that the assumptions $(H1-H3)$ hold. Then, for given constant $c_0 > 0$, there exist constants $\delta_0 > 0$ and $c_1 > 0$, such that for any δ with $0 < \delta < \delta_0$, there is a $\bar{\varepsilon}_\delta > 0$ such that for any ε with $0 < |\varepsilon| < \bar{\varepsilon}_\delta$, for any increasing sequence $\mathcal{T} = \{T_k\}_{k\in\mathbb{Z}}$ with $T_{k+1} - T_k > 1 - 2\rho^{-1}\ln|\varepsilon|$ for any $k \in \mathbb{Z}$ such that, for $j \in \mathcal{J}_N$, $k \in \mathbb{Z}$ and for some $\varpi_0 = \{\alpha^0_k\}_{k\in\mathbb{Z}} \in \ell^\infty_1(\mathbb{R})$ with*

$$\mathcal{M}_j\left(T_{2Nk+2j} + \alpha^0_{Nk+j}\right) = 0 \text{ and } \inf_{k\in\mathbb{Z}} |\mathcal{M}'_j\left(T_{2Nk+2j} + \alpha^0_{Nk+j}\right)| > c_0, \quad (2.11)$$

there exists a unique sequence $\{\alpha_k(\mathcal{T},\varepsilon)\}_{k\in\mathbb{Z}} := \{\alpha_k\}_{k\in\mathbb{Z}} \in \ell^\infty_1(\mathbb{R})$ with $|\alpha_k(\mathcal{T},\varepsilon) - \alpha^0_k| < c_1|\varepsilon|$ for any $k \in \mathbb{Z}$, and a unique solution $x(t,\mathcal{T},\varepsilon)$ of system (2.1) such that for any $j \in \mathcal{J}_N$ and $k \in \mathbb{Z}$,

(1) $x(t,\mathcal{T},\varepsilon) \in \Omega_{i_j}$ for $t \in (T_{2Nk+2j-2}+\alpha_{Nk+j-1}, T_{2Nk+2j}+\alpha_{Nk+j})$, $x(t\pm,\mathcal{T},\varepsilon) \in \mathcal{C}_{i_j i_{j+1}}$ with $x(t+,\mathcal{T},\varepsilon) = \mathcal{R}_{i_j i_{j+1}}(x(t-,\mathcal{T},\varepsilon),\varepsilon)$ for $t = T_{2Nk+2j} + \alpha_{Nk+j}$;

(2) let $J^s_{jk} = [T_{2Nk+2j-2}+\alpha_{Nk+j-1}, T_{2Nk+2j-1}+\alpha_{Nk+j-1}]$, $J^u_{jk} = [T_{2Nk+2j-1}+\alpha_{Nk+j-1}, T_{2Nk+2j} + \alpha_{Nk+j}]$, then we have

$$\sup_{t\in J^s_{jk}} \left|x(t,\mathcal{T},\varepsilon) - \gamma^s_j(t - T_{2Nk+2j-2} - \alpha_{Nk+j-1})\right| < \delta,$$
$$\sup_{t\in J^u_{jk}} \left|x(t,\mathcal{T},\varepsilon) - \gamma^u_j(t - T_{2Nk+2j} - \alpha_{Nk+j})\right| < \delta.$$

3. Proof of Theorem 2.1

In this section, we describe how to prove Theorem 2.1. For this purpose, we first discuss the orbits close to the heteroclinic orbit Γ.

Let $\{T_k\}_{k\in\mathbb{Z}}$ be an increasing sequence with $T_{k+1}-T_k \geq T+1$, where T is given in Section 2. For $j \in \mathcal{J}_N$, define

$$\begin{aligned}
P^s_{i_j,k} &= X^s_{i_j}(T_{2Nk+2j-1}-T_{2Nk+2j-2}+1)P^s_{i_j}(X^s_{i_j}(T_{2Nk+2j-1}-T_{2Nk+2j-2}+1))^{-1}\\
&= \tilde{P}^s_{i_j}(T_{2Nk+2j-1}-T_{2Nk+2j-2}+1),\\
P^u_{i_j,k} &= X^u_{i_j}(T_{2Nk+2j-1}-T_{2Nk+2j}-1)P^u_{i_j}(X^u_{i_j}(T_{2Nk+2j-1}-T_{2Nk+2j}-1))^{-1}\\
&= \tilde{P}^u_{i_j}(T_{2Nk+2j-1}-T_{2Nk+2j}-1).
\end{aligned}$$

Let $\delta > 0$ be a positive number such that for any $t \geq 0$ and $j \in \mathcal{J}_N$, the closed balls $\bar{B}(\gamma^s_j(t),\delta)$ and $\bar{B}(\gamma^u_j(-t),\delta)$ are subsets of Ω. Let

$$\begin{aligned}
N_{i_j} &= \sup\left\{|g_{i_j}(x,t,\varepsilon)| : (x,t,\varepsilon)\in\mathbb{R}^{n+2}\right\},\\
N'_{i_j} &= \sup\left\{\left|\frac{\partial g_{i_j}}{\partial x}(x,t,\varepsilon)\right| : (x,t,\varepsilon)\in\mathbb{R}^{n+2}\right\},\\
\Delta^s_{i_j}(\delta) &= \sup_{|x|\leq\delta}\sup_{t\geq 0}\left|Df_{i_j}(x+\gamma^s_j(t)) - Df_{i_j}(\gamma^s_j(t))\right|,\\
\Delta^u_{i_j}(\delta) &= \sup_{|x|\leq\delta}\sup_{t\leq 0}\left|Df_{i_j}(x+\gamma^u_j(t)) - Df_{i_j}(\gamma^u_j(t))\right|.
\end{aligned}$$

By the results given in [4,5,7,8,19], we have the following two Propositions, which show how to construct solutions near $\gamma^u_j(t)$ on $[T_{2Nk+2j-1}+\alpha-1, T_{2Nk+2j}+\alpha]$ and solutions near $\gamma^s_j(t)$ on $[T_{2Nk+2j-2}+\alpha, T_{2Nk+2j-1}+\alpha+1]$ for $j\in\mathcal{J}_N$, $k\in\mathbb{Z}$ and $\alpha\in\mathbb{R}$:

Proposition 3.1. Suppose that the assumptions $(H1)$ and $(H2)$ hold and $j \in \mathcal{J}_N$, $k\in\mathbb{Z}$, $\alpha\in\mathbb{R}$. Let $\delta>0$ and $(\xi^u_{i_j},\varphi^u_{i_j},\alpha,\varepsilon)\in\mathcal{N}P^u_{i_j}\times\mathcal{R}P^u_{i_j,k}\times\mathbb{R}^2$ be such that

$$2K(|\xi^u_{i_j}|+|\varphi^u_{i_j}|+2\rho^{-1}N_{i_j}|\varepsilon|)\leq\delta,\quad 4K\rho^{-1}(\Delta^u_{i_j}(\delta)+N'_{i_j}|\varepsilon|)<1.$$

Then, equation $\dot{x}=f_{i_j}(x)+\varepsilon g_{i_j}(x,t,\varepsilon)$ has a unique bounded solution $x^u_{i_j,k}(t) := x^u_{i_j,k}(t,\xi^u_{i_j},\varphi^u_{i_j},\alpha,\varepsilon)$ on $[T_{2Nk+2j-1}+\alpha-1, T_{2Nk+2j}+\alpha]$, which is C^2 in $(\xi^u_{i_j},\varphi^u_{i_j},\alpha,\varepsilon)$ and satisfies:

$$|x^u_{i_j,k}(t+T_{2Nk+2j}+\alpha,\xi^u_{i_j},\varphi^u_{i_j},\alpha,\varepsilon)-\gamma^u_j(t)|\leq 2K(|\xi^u_{i_j}|+|\varphi^u_{i_j}|+2\rho^{-1}N_{i_j}|\varepsilon|)\leq\delta$$

for any $t\in[T_{2Nk+2j-1}-T_{2Nk+2j}-1,0]$ together with

$$\begin{aligned}
&P^u_{i_j}x^u_{i_j,k}(T_{2Nk+2j}+\alpha,\xi^u_{i_j},\varphi^u_{i_j},\alpha,\varepsilon)=\xi^u_{i_j},\\
&P^u_{i_j,k}x^u_{i_j,k}(T_{2Nk+2j-1}+\alpha-1,\xi^u_{i_j},\varphi^u_{i_j},\alpha,\varepsilon)=\varphi^u_{i_j}.
\end{aligned}$$

Moreover, $x^u_{i_j,k}(t+\alpha,\xi^u_{i_j},\varphi^u_{i_j},\alpha,\varepsilon)$ and its derivatives with respect to $(\xi^u_{i_j},\varphi^u_{i_j},\alpha,\varepsilon)$ are also bounded in $[T_{2Nk+2j-1}-1,T_{2Nk+2j}]$ uniformly with respect to $(\xi^u_{i_j},\varphi^u_{i_j},\alpha,\varepsilon)$ and $k\in\mathbb{Z}$, uniformly continuous in $(\xi^u_{i_j},\varphi^u_{i_j},\alpha,\varepsilon)$ uniformly with respect to $t\in[T_{2Nk+2j-1}-1,T_{2Nk+2j}]$ and $k\in\mathbb{Z}$, and satisfy:

$$\begin{aligned}
&\frac{\partial x^u_{i_j,k}}{\partial\xi^u_{i_j}}(t+\alpha,0,0,\alpha,0)=X^u_{i_j}(t-T_{2Nk+2j})(I-P^u_{i_j}),\\
&\frac{\partial x^u_{i_j,k}}{\partial\varphi^u_{i_j}}(t+\alpha,0,0,\alpha,0)=\Upsilon^{u,1}_{kj}(t,T_{2Nk+2j-1}-1),
\end{aligned}$$

$$\begin{aligned}\frac{\partial x^u_{i_j,k}}{\partial \varepsilon}(t+\alpha,0,0,\alpha,0) = &\int_{T_{2Nk+2j-1}-1}^{t} \Upsilon^{u,1}_{kj}(t,\tau) g^u_{jk}(\tau,\alpha)d\tau \\ &- \int_{t}^{T_{2Nk+2j}} \Upsilon^{u,2}_{kj}(t,\tau) g^u_{jk}(\tau,\alpha)d\tau,\end{aligned}$$

where

$$\begin{aligned} g^u_{jk}(\tau,\alpha) &= g_{i_j}(\gamma^u_j(\tau - T_{2Nk+2j}),\tau+\alpha,0), \\ \Upsilon^{u,1}_{kj}(t,\tau) &= X^u_{i_j}(t-T_{2Nk+2j})P^u_{i_j}(X^u_{i_j}(\tau - T_{2Nk+2j}))^{-1}, \\ \Upsilon^{u,2}_{kj}(t,\tau) &= X^u_{i_j}(t-T_{2Nk+2j})(I-P^u_{i_j})(X^u_{i_j}(\tau - T_{2Nk+2j}))^{-1}. \end{aligned}$$

Proposition 3.2. Suppose that the assumptions $(H1)$ and $(H2)$ hold and $j \in \mathcal{J}_N$, $k \in \mathbb{Z}$, $\alpha \in \mathbb{R}$. Let $\delta > 0$ and $(\xi^s_{i_j}, \varphi^s_{i_j}, \alpha, \varepsilon) \in \mathcal{R}P^s_{i_j} \times \mathcal{N}P^s_{i_j,k} \times \mathbb{R}^2$ be such that

$$2K(|\xi^s_{i_j}| + |\varphi^s_{i_j}| + 2\rho^{-1}N_{i_j}|\varepsilon|) \le \delta, \quad 4K\rho^{-1}(\Delta^s_{i_j}(\delta) + N'_{i_j}|\varepsilon|) < 1.$$

Then, equation $\dot{x} = f_{i_j}(x) + \varepsilon g_{i_j}(x,t,\varepsilon)$ has a unique bounded solution $x^s_{i_j,k}(t) := x^s_{i_j,k}(t, \xi^s_{i_j}, \varphi^s_{i_j}, \alpha, \varepsilon)$ on $[T_{2Nk+2j-2}+\alpha, T_{2Nk+2j-1}+\alpha+1]$, which is C^2 in $(\xi^s_{i_j}, \varphi^s_{i_j}, \alpha, \varepsilon)$ and satisfies:

$$|x^s_{i_j,k}(T_{2Nk+2j-2}+\alpha, \xi^s_{i_j}, \varphi^s_{i_j}, \alpha, \varepsilon) - \gamma^s_j(t)| \le 2K(|\xi^s_{i_j}| + |\varphi^s_{i_j}| + 2\rho^{-1}N_{i_j}|\varepsilon|) \le \delta$$

for any $t \in [0, T_{2Nk+2j-1} - T_{2Nk+2j-2} + 1]$ together with

$$\begin{aligned} P^s_{i_j} x^s_{i_j,k}(T_{2Nk+2j-2}+\alpha, \xi^s_{i_j}, \varphi^s_{i_j}, \alpha, \varepsilon) &= \xi^s_{i_j}, \\ P^s_{i_j,k} x^s_{i_j,k}(T_{2Nk+2j-1}+\alpha+1, \xi^s_{i_j}, \varphi^s_{i_j}, \alpha, \varepsilon) &= \varphi^s_{i_j}. \end{aligned}$$

Moreover, $x^s_{i_j,k}(t+\alpha, \xi^s_{i_j}, \varphi^s_{i_j}, \alpha, \varepsilon)$ and its derivatives with respect to $(\xi^s_{i_j}, \varphi^s_{i_j}, \alpha, \varepsilon)$ are also bounded in $[T_{2Nk+2j-2}, T_{2Nk+2j-1}+1]$ uniformly with respect to $(\xi^s_{i_j}, \varphi^s_{i_j}, \alpha, \varepsilon)$ and $k \in \mathbb{Z}$, uniformly continuous in $(\xi^s_{i_j}, \varphi^s_{i_j}, \alpha, \varepsilon)$ uniformly with respect to $t \in [T_{2Nk+2j-2}, T_{2Nk+2j-1}+1]$ and $k \in \mathbb{Z}$, and satisfy:

$$\begin{aligned} \frac{\partial x^s_{i_j,k}}{\partial \xi^s_{i_j}}(t+\alpha,0,0,\alpha,0) &= X^s_{i_j}(t-T_{2Nk+2j-2})P^s_{i_j}, \\ \frac{\partial x^s_{i_j,k}}{\partial \varphi^s_{i_j}}(t+\alpha,0,0,\alpha,0) &= \Upsilon^{s,2}_{kj}(t, T_{2Nk+2j-1}+1), \\ \frac{\partial x^s_{i_j,k}}{\partial \varepsilon}(t+\alpha,0,0,\alpha,0) &= \int_{T_{2Nk+2j-2}}^{t} \Upsilon^{s,1}_{kj}(t,\tau) g^s_{jk}(\tau,\alpha)d\tau \\ &\quad - \int_{t}^{T_{2Nk+2j-1}+1} \Upsilon^{s,2}_{kj}(t,\tau) g^s_{jk}(\tau,\alpha)d\tau, \end{aligned}$$

where

$$\begin{aligned} g^s_{jk}(\tau,\alpha) &= g_{i_j}(\gamma^s_j(\tau - T_{2Nk+2j-2}),\tau+\alpha,0), \\ \Upsilon^{s,1}_{kj}(t,\tau) &= X^s_{i_j}(t-T_{2Nk+2j-2})P^s_{i_j}(X^s_{i_j}(\tau - T_{2Nk+2j-2}))^{-1}, \\ \Upsilon^{s,2}_{kj}(t,\tau) &= X^s_{i_j}(t-T_{2Nk+2j-2})(I-P^s_{i_j})(X^s_{i_j}(\tau - T_{2Nk+2j-2}))^{-1}. \end{aligned}$$

Now, let

$$\ell_n^\infty = \Big\{ \big\{ \big(\varphi_{i_1,k}^u, \varphi_{i_1,k}^s, \cdots, \varphi_{i_N,k}^u, \varphi_{i_N,k}^s, \xi_{i_1,k}^u, \xi_{i_1,k}^s, \cdots, \xi_{i_N,k}^u, \xi_{i_N,k}^s \big) \big\}_{k\in\mathbb{Z}} \in \ell^\infty \left(\mathbb{R}^{4nN}\right) : \\ \varphi_{i_j,k}^u \in \mathcal{R}P_{i_j,k}^u, \varphi_{i_j,k}^s \in \mathcal{N}P_{i_j,k}^s, \xi_{i_j,k}^u \in \mathcal{N}P_{i_j}^u, \xi_{i_j,k}^s \in \mathcal{R}P_{i_j}^s, \forall j \in \mathcal{J}_N, k \in \mathbb{Z} \Big\}$$

be the Banach space with the norm:

$$\|\theta\| = \sup_{k\in\mathbb{Z}} \max_{j\in\mathcal{J}_N} \left\{ |\varphi_{i_j,k}^u|, |\varphi_{i_j,k}^s|, |\xi_{i_j,k}^u|, |\xi_{i_j,k}^s| \right\}$$

for $\theta = \left\{ \left(\varphi_{i_1,k}^u, \varphi_{i_1,k}^s, \cdots, \varphi_{i_N,k}^u, \varphi_{i_N,k}^s, \xi_{i_1,k}^u, \xi_{i_1,k}^s, \cdots, \xi_{i_N,k}^u, \xi_{i_N,k}^s \right) \right\}_{k\in\mathbb{Z}} \in \ell_n^\infty$. Let δ_0 be the largest positive number such that for all $j \in \mathcal{J}_N$, the following is satisfied:

$$4K\rho^{-1} \left[\Delta_{i_j}^{u,s}(\delta_0) + \frac{N'_{i_j}\rho}{4KN_{i_j}}\delta_0 \right] \le 1.$$

Let $\delta \in (0, \delta_0)$ and $\varepsilon_\delta = \min_{j\in\mathcal{J}_N} \left\{ \frac{\rho\delta}{8KN_{i_j}} \right\}$. For any $\varepsilon \in (-\varepsilon_\delta, \varepsilon_\delta)$, define

$$\ell_{\delta,\varepsilon}^\infty = \Big\{ \big\{ \big(\varphi_{i_1,k}^u, \varphi_{i_1,k}^s, \cdots, \varphi_{i_N,k}^u, \varphi_{i_N,k}^s, \xi_{i_1,k}^u, \xi_{i_1,k}^s, \cdots, \xi_{i_N,k}^u, \xi_{i_N,k}^s \big) \big\}_{k\in\mathbb{Z}} \in \ell_n^\infty : \\ 2K(|\xi_{i_j,k}^{u,s}| + |\varphi_{i_j,k}^{u,s}| + 2\rho^{-1}N_{i_j}|\varepsilon|) \le \delta, j \in \mathcal{J}_N, k \in \mathbb{Z} \Big\},$$
$$\ell_\delta^\infty = \left\{ (\theta, \varpi, \varepsilon) \in \ell_{\delta,\varepsilon}^\infty \times \ell_1^\infty(\mathbb{R}) \times (-\varepsilon_\delta, \varepsilon_\delta) \right\}.$$

Then, $\ell_{\delta,\varepsilon}^\infty$ and ℓ_δ^∞ are open nonempty subsets of ℓ_n^∞ and $\ell_n^\infty \times \ell^\infty(\mathbb{R}) \times \mathbb{R}$ respectively. We assume that $\theta = \left\{ \left(\varphi_{i_1,k}^u, \varphi_{i_1,k}^s, \cdots, \varphi_{i_N,k}^u, \varphi_{i_N,k}^s, \xi_{i_1,k}^u, \xi_{i_1,k}^s, \cdots, \xi_{i_N,k}^u, \xi_{i_N,k}^s \right) \right\}_{k\in\mathbb{Z}}$, $\varpi = \{\alpha_k\}_{k\in\mathbb{Z}}$ for $(\theta, \varpi, \varepsilon) \in \ell_\delta^\infty$ in the sequel.

Take an increasing sequence $\mathcal{T} = \{T_k\}_{k\in\mathbb{Z}}$ with $T_{k+1} - T_k > T + 1$, where T is as above. By Proposition 3.1 and Proposition 3.2, for $(\theta, \varpi, \varepsilon) \in \ell_\delta^\infty$ and for each $j \in \mathcal{J}_N$, we have solutions $x_{i_j,k}^u(t) = x_{i_j,k}^u(t, \xi_{i_j,k}^u, \varphi_{i_j,k}^u, \alpha_{Nk+j}, \varepsilon)$, $x_{i_j,k}^s(t) = x_{i_j,k}^s(t, \xi_{i_j,k}^s, \varphi_{i_j,k}^s, \alpha_{Nk+j-1}, \varepsilon)$ of $\dot{x} = f_{i_j}(x) + \varepsilon g_{i_j}(x, t, \varepsilon)$ defined on $[T_{2Nk+2j-1} + \alpha_{Nk+j} - 1, T_{2Nk+2j} + \alpha_{Nk+j}]$ and $[T_{2Nk+2j-2} + \alpha_{Nk+j-1}, T_{2Nk+2j-1} + \alpha_{Nk+j-1} + 1]$ respectively. Since for any $k \in \mathbb{Z}$, $T_{k+1} - T_k > T + 1 > 1$ and $|\alpha_{k+1} - \alpha_k| < 1$, we have

$$T_{2Nk+2j-1} + \alpha_{Nk+j} - 1 < T_{2Nk+2j-1} + \alpha_{Nk+j-1} < T_{2Nk+2j} + \alpha_{Nk+j}.$$

Thus, both $x_{i_j,k}^u(t)$ and $x_{i_j,k}^s(t)$ are defined at the time $t = T_{2Nk+2j-1} + \alpha_{Nk+j-1}$. Hence, we can consider the following infinite set of equations for $(\theta, \varpi, \varepsilon) \in \ell_\delta^\infty$:

$$\mathcal{G}_\mathcal{T}(\theta, \varpi, \varepsilon) = 0, \tag{3.1}$$

where $\mathcal{G}_\mathcal{T} : \ell_\delta^\infty \mapsto \ell^\infty(\mathbb{R}^{2nN+2N})$ is given by

$$\mathcal{G}_\mathcal{T}(\theta, \varpi, \varepsilon) =$$

$$\left\{\begin{pmatrix} x^s_{i_j,k}(T_{2Nk+2j-1}+\alpha_{Nk+j-1},\xi^s_{i_j,k},\varphi^s_{i_j,k},\alpha_{Nk+j-1},\varepsilon) \\ \quad -x^u_{i_j,k}(T_{2Nk+2j-1}+\alpha_{Nk+j-1},\xi^u_{i_j,k},\varphi^u_{i_j,k},\alpha_{Nk+j},\varepsilon) \\ x^s_{i_{j+1},k}(T_{2Nk+2j}+\alpha_{Nk+j},\xi^s_{i_{j+1},k},\varphi^s_{i_{j+1},k},\alpha_{Nk+j},\varepsilon) \\ \quad -\mathcal{R}_{i_ji_{j+1}}(x^u_{i_j,k}(T_{2Nk+2j}+\alpha_{Nk+j},\xi^u_{i_j,k},\varphi^u_{i_j,k},\alpha_{Nk+j},\varepsilon),\varepsilon) \\ h_{i_{j-1}i_j}(x^s_{i_j,k}(T_{2Nk+2j-2}+\alpha_{Nk+j-1},\xi^s_{i_j,k},\varphi^s_{i_j,k},\alpha_{Nk+j-1},\varepsilon)) \\ h_{i_ji_{j+1}}(x^u_{i_j,k}(T_{2Nk+2j}+\alpha_{Nk+j},\xi^u_{i_j,k},\varphi^u_{i_j,k},\alpha_{Nk+j},\varepsilon)) \end{pmatrix}_{j\in\mathcal{J}_N}\right\}_{k\in\mathbb{Z}}.$$

Clearly, a solution $(\theta,\varpi,\varepsilon)$ of equation (3.1) implies the existence of solutions $x^u_{i_j,k}(t)$ and $x^s_{i_j,k}(t)$ as described above with

$$x^u_{i_j,k}(T_{2Nk+2j-1}+\alpha_{Nk+j-1}) = x^s_{i_j,k}(T_{2Nk+2j-1}+\alpha_{Nk+j-1})$$

for each $j\in\mathcal{J}_N$, $k\in\mathbb{Z}$. From which we can construct a solution $x(t,\theta,\varpi,\varepsilon)$ of system (2.1) near the heteroclinic orbit Γ by setting

$$x(t,\theta,\varpi,\varepsilon)=\begin{cases} x^s_{i_j,k}(t),\ t\in[T_{2Nk+2j-2}+\alpha_{Nk+j-1},T_{2Nk+2j-1}+\alpha_{Nk+j-1}), \\ x^u_{i_j,k}(t),\ t\in[T_{2Nk+2j-1}+\alpha_{Nk+j-1},T_{2Nk+2j}+\alpha_{Nk+j}) \end{cases}$$

for $j\in\mathcal{J}_N$, $k\in\mathbb{Z}$. Clearly, we have $x(t+,\theta,\varpi,\varepsilon)=\mathcal{R}_{i_ji_{j+1}}(x(t-,\theta,\varpi,\varepsilon),\varepsilon)$ for $t=T_{2Nk+2j}+\alpha_{Nk+j}$.

The rest of the proof of Theorem 2.1 is similar to the proof of Theorem 2.1 given in [37]. We omit it here for the sake of brevity.

4. Chaotic behavior

In this section, we discuss chaotic behaviors of system (2.1). Let the following assumption hold so that the conditions of Theorem 2.1 are satisfied:

(H4) For any ε with $0<|\varepsilon|<\bar{\varepsilon}_\delta$, there is an increasing sequence $\mathcal{T}=\{T_k\}_{k\in\mathbb{Z}}$ with $T_{k+1}-T_k>1-2\rho^{-1}\ln|\varepsilon|$ for any $k\in\mathbb{Z}$ and a $\varpi_0=\{\alpha^0_k\}_{k\in\mathbb{Z}}\in\ell^\infty_1(\mathbb{R})$ with $\|\varpi_0\|<\frac{1}{2}$ such that (2.11) holds for each $j\in\mathcal{J}_N$.

Let $\mathcal{E}$ be the set of bi-infinite sequences of elements of $S:=\{0,1\}$. Then, $(\mathcal{E},d)$ is a totally disconnected compact metric space with the distance $d(e,e')$ of $e=\{\cdots e_{-l}\cdots e_{-1}.e_0\cdots e_l\cdots\}$, $e'=\{\cdots e'_{-l}\cdots e'_{-l}.e'_0\cdots e'_l\cdots\}\in\mathcal{E}$ being given by

$$d(e,e')=\sum_{l=-\infty}^{\infty}\frac{|e_l-e'_l|}{2^{|l|}},$$

where for any $l\in\mathbb{Z}$, $e_l,e'_l\in S$. The *Bernoulli shift* $\sigma:\mathcal{E}\to\mathcal{E}$ is defined as $\sigma(e)=\{\cdots e_{-l}\cdots e_{-1}e_0.e_1\cdots e_l\cdots\}$ for $e=\{\cdots e_{-l}\cdots e_{-1}.e_0\cdots e_l\cdots\}\in\mathcal{E}$. Then, σ is a homeomorphism having (1) a countable infinity of periodic orbits of all possible periods, (2) an uncountable infinity of nonperiodic orbits and (3) a dense orbit [19, p. 18].

Let $\mathcal{T}$ and ϖ_0 be as in (H4). For any $e=\{\cdots e_{-l}\cdots e_{-1}.e_0\cdots e_l\cdots\}\in\mathcal{E}$, let $\{n^e_k\}_k$ be a fixed increasing sequence of integers such that $e_l=1$, if and only if

$l = n_k^e$. Define $\mathcal{T}^e = \{T_l^e\}_{l\in\mathbb{Z}}$ and $\varpi_0^e = \{\alpha_l^{0e}\}_{l\in\mathbb{Z}}$ as

$$T_l^e = \begin{cases} T_{2Nn_k^e}, & \text{if } l = 2Nk, \\ T_{2Nn_k^e+1}, & \text{if } l = 2Nk+1, \\ \quad\vdots & \\ T_{2Nn_k^e+2N-1}, & \text{if } l = 2Nk+2N-1, \end{cases}$$
$$\alpha_l^{0e} = \alpha_{n_l^e}^0.$$

Similar to that of [37], we have the following result:

Theorem 4.1. *Assume that for each $i \in \mathcal{J}$, f_i and g_i are functions with uniformly bounded derivatives upto the second order on $\bar{\Omega}_i$ and $\bar{\Omega}_i \times \mathbb{R}^2$ respectively and their second order derivatives are uniformly continuous. Suppose that the assumptions $(H1-H4)$ hold. Let $\mathcal{T}$, ϖ_0 be as in $(H4)$ and $\mathcal{T}^e$, ϖ_0^e be defined above. Then, for any sufficiently small $|\varepsilon| \neq 0$ and for any $e \in \mathcal{E}$, there is a unique sequence $\{\alpha_l^e(\mathcal{T}^e, \varepsilon)\}_{l\in\mathbb{Z}} := \{\alpha_l^e\}_{l\in\mathbb{Z}} \in \ell_1^\infty(\mathbb{R})$ with $|\alpha_l^e - \alpha_l^{0e}| < c_1|\varepsilon|$, for any $l \in \mathbb{Z}$ and a unique solution $x(t, \mathcal{T}, e, \varepsilon)$ of system (2.1), depending only on e and $\mathcal{T}$, such that:*

(1) *If for both $l = n_k^e$ and $\bar{l} = n_{k+1}^e$, $e_l = e_{\bar{l}} = 1$, then for any $j \in \mathcal{J}_N$,*

$$\sup_{t\in J_{jk}^{e,s}} \left|x(t, \mathcal{T}, e, \varepsilon) - \gamma_j^s(t - T_{2Nk+2j-2}^e - \alpha_{Nk+j-1}^e)\right| < \delta,$$
$$\sup_{t\in J_{jk}^{e,u}} \left|x(t, \mathcal{T}, e, \varepsilon) - \gamma_j^u(t - T_{2Nk+2j}^e - \alpha_{Nk+j}^e)\right| < \delta,$$

where $J_{jk}^{e,s} = [T_{2Nk+2j-2}^e + \alpha_{Nk+j-1}^e, T_{2Nk+2j-1}^e + \alpha_{Nk+j-1}^e]$, $J_{jk}^{e,u} = [T_{2Nk+2j-1}^e + \alpha_{Nk+j-1}^e, T_{2Nk+2j}^e + \alpha_{Nk+j}^e]$.

(2) *If there is an integer k_- such that $e_{n_{k_-}^e} = 1$ and $e_l = 0$ for any $l < n_{k_-}^e$, then*

$$\sup_{t\in J_{1k_-}^{e,u}} \left|x(t, \mathcal{T}^e, \varepsilon) - \gamma_1^u(t - T_{2Nk_-+2}^e - \alpha_{Nk_-+1}^e)\right| < \delta,$$

where $J_{1k_-}^{e,u} = (-\infty, T_{2Nk_-+2}^e + \alpha_{Nk_-+1}^e]$.

(3) *If there is an integer k_+ such that $e_{n_{k_+}^e} = 1$ and $e_l = 0$ for any $l > n_{k_+}^e$, then*

$$\sup_{t\in J_{1(k_++1)}^{e,s}} \left|x(t, \mathcal{T}, e, \varepsilon) - \gamma_1^s(t - T_{2N(k_++1)}^e - \alpha_{N(k_++1)}^e)\right| < \delta.$$

where $J_{1(k_++1)}^{e,s} = [T_{2N(k_++1)}^e + \alpha_{N(k_++1)}^e, +\infty)$.

(4) *If $e = 0$, then for each $j \in \mathcal{J}_N$ and sufficiently small $|\varepsilon|$, there is a unique bounded solution defined on $t \in \mathbb{R}$, denoted by $x_j(t, \mathcal{T}, 0, \varepsilon)$, of system (2.1), such that $x_j(t, \mathcal{T}, 0, \varepsilon) \in \Omega_{i_j}$ for all $t \in \mathbb{R}$ and*

$$\sup_{t\in\mathbb{R}} |x_j(t, \mathcal{T}, 0, \varepsilon) - p_j| < \delta.$$

Furthermore, for $l \in \mathbb{Z}$, $x(t, \mathcal{T}^{(l+1)}, \sigma(e), \varepsilon) = x(t, \mathcal{T}^{(l)}, e, \varepsilon)$ for any $t \in \mathbb{R}$ and $e \in \mathcal{E}$, where $\mathcal{T}^{(l)} = \{T_{k+2Nl}\}_{k\in\mathbb{Z}}$.

The proof of Theorem 4.1 is similar to the arguments given before Theorem 5.1 of [37]. Thus, we omit it for brevity.

For $l \in \mathbb{Z}$, let $S_l = \{x(T_{2l}, \mathcal{T}^{(l)}, e, \varepsilon) : e \in \mathcal{E}\} \subset \mathbb{R}^n$. Define the map $F_l : S_l \mapsto S_{l+1}$, so that for $\xi \in S_l$, $F_l(\xi)$ is the value at time T_{2l+1} of the solution $x(t)$ of (2.1) such that $x(T_{2l}) = \xi$, i.e., $F_l(\xi) = x(T_{2l+1})$. Let $\Phi_l : \mathcal{E} \mapsto S_l$ be defined as $\Phi_l(e) = x(T_{2l}, \mathcal{T}^{(l)}, e, \varepsilon)$. Then, for each $l \in \mathbb{Z}$, S_l is compact in $\mathbb{R}^n$, F_l and Φ_l are homeomorphisms. Moreover, we have the following result:

Theorem 4.2. *Under the assumptions of Theorem 4.1, for any sufficiently small $|\varepsilon| > 0$ and for any $l \in \mathbb{Z}$, the commute diagram*

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{\ \sigma\ } & \mathcal{E} \\ {\scriptstyle \Phi_l}\downarrow & & \downarrow{\scriptstyle \Phi_{l+1}} \\ S_l & \xrightarrow{\ F_l\ } & S_{l+1} \end{array}$$

holds. Furthermore, $\Phi_l : \mathcal{E} \mapsto S_l$ is a homeomorphism for any $l \in \mathbb{Z}$.

Now, we assume that for all $i \in \mathcal{J}$, $g_i(x, t, \varepsilon)$ are almost periodic or periodic in t uniformly in (x, ε). It is easy to see that Theorems 6.1 and 6.2 given in [37] are still true for system (2.1). Here, we omit them for brevity.

5. Application to linked rocking blocks

In this section, we consider heteroclinic bifurcation and chaos for a quasiperiodically excited system consisting of two slender rocking blocks coupled by a light spring, as is depicted in Figure 2. In [24], Granados, Hogan and Seara studied the Arnold diffusion of this model, when it is periodically excited. They assumed that on impact with the rigid base, and neither block loses energy. They pointed out that the Arnold diffusion can be seen as one possible mechanism for block overturning.

The blocks are rigid, of mass m_1 and m_2 and with semi-diagonal of length R_1 and R_2 respectively. The base is sufficiently flat, so that the i-th block rotates only about $O_i^{'}$ for $i = 1, 2$. Let α_1, α_2 be the angles formed by the lateral sides and the diagonals of the blocks. The state variables x_1 and x_3 are chosen so that $\alpha_1 x_1$ and $\alpha_2 x_3$ are the angles formed by the vertical and the lateral side of each block. When there is a rotation, x_1 (respectively x_3) is positive for rotation about O_1 (respectively O_2) and x_1 (respectively x_3) is negative for rotation about $O_1^{'}$ (respectively $O_2^{'}$). For slender blocks, $\alpha_i \ll 1$ for $i = 1, 2$.

As in [24], we assume that both blocks are identical, namely $m_1 = m_2$ and $\alpha_1 = \alpha_2$. However, we assume that on impact with the rigid base, and both blocks lose energy.

Let $x = (x_1, x_2, x_3, x_4)^T \in \mathbb{R}^4$, $h_a(x) = x_1$ and $h_b(x) = x_3$. Suppose that $\mathbb{R}^4$ is divided into four disjoint open regions Ω_i $(i = 1, \cdots, 4)$ by the super-surfaces $\mathcal{C}_{12}$, $\mathcal{C}_{23}$, $\mathcal{C}_{34}$ and $\mathcal{C}_{41}$, where $\mathcal{C}_{12} = \{x \in \mathbb{R}^4 : h_b(x) = 0, x_1 > 0\}$, $\mathcal{C}_{23} = \{x \in \mathbb{R}^4 : h_a(x) = 0, x_3 < 0\}$, $\mathcal{C}_{34} = \{x \in \mathbb{R}^4 : h_b(x) = 0, x_1 < 0\}$, $\mathcal{C}_{41} = \{x \in \mathbb{R}^4 : h_a(x) = 0, x_3 > 0\}$. Thus, $\Omega_1 = \{x \in \mathbb{R}^4 : x_1 > 0, x_3 > 0\}$, $\Omega_2 = \{x \in \mathbb{R}^4 : x_1 > 0, x_3 < 0\}$, $\Omega_3 = \{x \in \mathbb{R}^4 : x_1 < 0, x_3 < 0\}$, $\Omega_4 = \{x \in \mathbb{R}^4 : x_1 < 0, x_3 > 0\}$. According to [24], the dimensionless form of this coupled two slender rigid blocks can be modeled by

$$\dot{x} = f_i(x) + \varepsilon g_i(x, t), \quad x \in \bar{\Omega}_i, \quad i = 1, \cdots, 4, \tag{5.1}$$

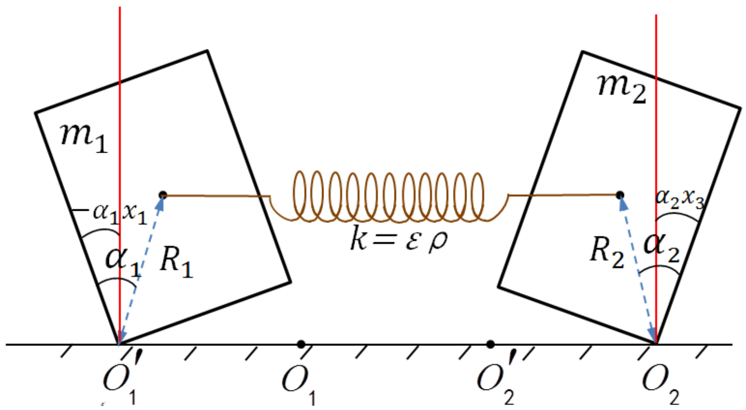

Figure 2. Two slender rocking blocks linked by a light spring

plus a set of reset maps

$$x \mapsto \mathcal{R}_{ij}(x,\varepsilon), \quad x \in \mathcal{C}_{ij}, \quad \text{for } (i,j) \in \{(1,2),(2,3),(3,4),(4,1)\}, \tag{5.2}$$

where for $i = 1, \cdots, 4$,

$$g_i(x,t) = \varrho(x_3 - x_1)(0,1,0,-1)^T - (\sigma_{i1}\sin\omega_1 t + \sigma_{i2}\sin\omega_2 t)(0,1,0,1)^T.$$

Moreover,

$$\begin{aligned}
f_1(x) &= (x_2, x_1 - 1, x_4, x_3 - 1)^T, \\
f_2(x) &= (x_2, x_1 - 1, x_4, x_3 + 1)^T, \\
f_3(x) &= (x_2, x_1 + 1, x_4, x_3 + 1)^T, \\
f_4(x) &= (x_2, x_1 + 1, x_4, x_3 - 1)^T, \\
\mathcal{R}_{12}(x,\varepsilon) &= (x_1, x_2, 0, (1-\varepsilon r)x_4)^T, \quad \text{for } x = (x_1, x_2, 0, x_4)^T \in \mathcal{C}_{12}, \\
\mathcal{R}_{23}(x,\varepsilon) &= (0, (1-\varepsilon r)x_2, x_3, x_4)^T, \quad \text{for } x = (0, x_2, x_3, x_4)^T \in \mathcal{C}_{23}.
\end{aligned}$$

$\mathcal{R}_{34}$ (respectively $\mathcal{R}_{41}$) has the same form as that for $\mathcal{R}_{12}$ (respectively $\mathcal{R}_{23}$) for $x \in \mathcal{C}_{34}$ (respectively $x \in \mathcal{C}_{41}$). Here, $\varepsilon\varrho > 0$ corresponds to the light spring constant, $1 - \varepsilon r \in (0,1]$ is the coefficient of restitution corresponding to the energy loses of the blocks on impact with the rigid base, σ_{i1}, σ_{i2} for $i = 1, \cdots, 4$ and ω_1, ω_2 are all positive constants.

When $\varepsilon = 0$, the unperturbed system of (5.1-5.2) has four hyperbolic saddles $p_1 = (1,0,1,0)^T \in \Omega_1$, $p_2 = (1,0,-1,0)^T \in \Omega_2$, $p_3 = (-1,0,-1,0)^T \in \Omega_3$ and $p_4 = (-1,0,1,0) \in \Omega_4$. Corresponding to p_1, the first (respectively second) block is at the unstable rest position such that its diagonal is perpendicular to O_1 (respectively O_2). Corresponding to p_2, the first (respectively second) block is at the unstable rest position such that its diagonal is perpendicular to O_1 (respectively O_2'). Corresponding to p_3, the first (respectively second) block is at the unstable rest position such that its diagonal is perpendicular to O_1' (respectively O_2'). Corresponding to p_4, the first (respectively second) block is at the unstable rest position such that its diagonal is perpendicular to O_1' (respectively O_2). The eigenvalues λ_{ij} for

$i = 1, \cdots, 4$ and $j = 1, \cdots, 4$ of the coefficient matrix of the linearized system of the unperturbed system of (5.1-5.2) at p_i are given by: $\lambda_{i1} = \lambda_{i2} = 1$, $\lambda_{i3} = \lambda_{i4} = -1$. The unperturbed system of (5.1-5.2) has a heteroclinic cycle Γ, which consists of eight branches $\Gamma_i^s := \{\gamma_i^s(t) : t \in [0, +\infty)\} \subset \bar{\Omega}_i$, $\Gamma_i^u := \{\gamma_i^u(t) : t \in (-\infty, 0]\} \subset \bar{\Omega}_i$ such that

$$\Gamma = \bigcup_{i=1}^{4} \left(\Gamma_i^s \bigcup \{p_i\} \bigcup \Gamma_i^u \right),$$

where $\gamma_i^{u,s}(t)$ are solutions of the unperturbed system of (5.1-5.2) given by

$$\begin{aligned}
\gamma_1^u(t) &= (1, 0, 1 - e^t, -e^t)^T, \quad t \in (-\infty, 0], \\
\gamma_1^s(t) &= (1 - e^{-t}, e^{-t}, 1, 0)^T, \quad t \in [0, +\infty), \\
\gamma_2^u(t) &= (1 - e^t, -e^t, -1, 0)^T, \quad t \in (-\infty, 0], \\
\gamma_2^s(t) &= (1, 0, -1 + e^{-t}, -e^{-t})^T, \quad t \in [0, +\infty), \\
\gamma_3^u(t) &= (-1, 0, -1 + e^t, e^t)^T, \quad t \in (-\infty, 0], \\
\gamma_3^s(t) &= (-1 + e^{-t}, -e^{-t}, -1, 0)^T, \quad t \in [0, +\infty), \\
\gamma_4^u(t) &= (-1 + e^t, e^t, 1, 0)^T, \quad t \in (-\infty, 0], \\
\gamma_4^s(t) &= (-1, 0, 1 - e^{-t}, e^{-t})^T, \quad t \in [0, +\infty),
\end{aligned}$$

with

$$\begin{aligned}
\dot{\gamma}_1^u(0) = (0, 0, -1, -1)^T, &\ \ \dot{\gamma}_1^s(0) = (1, -1, 0, 0)^T, \\
\dot{\gamma}_2^u(0) = (-1, -1, 0, 0)^T, &\ \ \dot{\gamma}_2^s(0) = (0, 0, -1, 1)^T, \\
\dot{\gamma}_3^u(0) = (0, 0, 1, 1)^T, &\ \ \dot{\gamma}_3^s(0) = (-1, 1, 0, 0)^T \\
\dot{\gamma}_4^u(0) = (1, 1, 0, 0)^T, &\ \ \dot{\gamma}_4^s(0) = (0, 0, 1, -1)^T.
\end{aligned}$$

Furthermore, we have $i_j = j$ for $j = 1, \cdots, 4$, $\gamma_1^u(0) = \gamma_2^s(0) = (1, 0, 0, -1)^T \in \mathcal{C}_{12}$, $\gamma_2^u(0) = \gamma_3^s(0) = (0, -1, -1, 0)^T \in \mathcal{C}_{23}$, $\gamma_3^u(0) = \gamma_4^s(0) = (-1, 0, 0, 1)^T \in \mathcal{C}_{34}$, $\gamma_4^u(0) = \gamma_1^s(0) = (0, 1, 1, 0)^T \in \mathcal{C}_{41}$. In the following, to simplify notations, i_j is written as j for $j = 1, \cdots, 4$. Note that when $\varepsilon = 0$, all of the reset maps given in (5.2) are identities. Hence, assumptions (H1) and (H2) are satisfied for system (5.1-5.2). Since system (5.1-5.2) is of four dimensional, it is not possible to plot the heteroclinic cycle Γ in the phase space.

We smoothly extend the right hand side functions of system (5.1) and $\gamma_i^{u,s}(t)$ for $i = 1, \cdots, 4$, as is explained in Section 2. Let A_c be a 4×4 matrix given by

$$A_c = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Then, the corresponding linear variational systems $\dot{x} = Df_i(\gamma_i^{u,s}(t))x$ for $i = 1, \cdots, 4$ are given by

$$\dot{x} = Df_i(\gamma_i^{u,s}(t))x = A_c x, \quad i = 1, \cdots, 4, \tag{5.3}$$

where (5.3) is defined on $[-1,+\infty)$ for the superscript "s" and on $(-\infty,1]$ for the superscript "u". For $i=1,\cdots,4$, the fundamental matrix solutions $X_i^{u,s}(t)$ with $X_i^{u,s}(0)=I$ of (5.3) are given by $X_i^{u,s}(t)=\tilde{X}_i^{u,s}(t)(\tilde{X}_i^{u,s}(0))^{-1}$, where

$$\tilde{X}_i^{u,s}(t)=\begin{pmatrix} e^t & 0 & -e^{-t} & 0 \\ e^t & 0 & e^{-t} & 0 \\ 0 & e^t & 0 & -e^{-t} \\ 0 & e^t & 0 & e^{-t} \end{pmatrix}, \quad i=1,\cdots,4.$$

Here, $X_i^s(t)$ is defined on $[-1,+\infty)$ and $X_i^u(t)$ is defined on $(-\infty,1]$.

For each $i=1,\cdots,4$, we found that $P_i^{u,s}=\tilde{X}_i^{u,s}(0)\tilde{P}_i^{u,s}(\tilde{X}_i^{u,s}(0))^{-1}$, where $\tilde{P}_i^u=\tilde{P}_i^s=\text{diag}(0,0,1,1)$. Therefore,

$$\begin{aligned}\mathcal{N}P_i^u &= \text{span}\left\{(1,1,0,0)^T,(0,0,1,1)^T\right\},\\ \mathcal{R}P_i^s &= \text{span}\left\{(1,-1,0,0)^T,(0,0,1,-1)^T\right\}\end{aligned}$$

for $i=1,\cdots,4$. By direct computations, we have

$$\tilde{S}_1^u=\tilde{S}_3^u=\text{span}\left\{(1,1,0,0)^T\right\}, \quad \tilde{S}_2^u=\tilde{S}_4^u=\text{span}\left\{(0,0,1,1)^T\right\}.$$

It is easy to see that for each $i=1,\cdots,4$, $D_1\mathcal{R}_{i,i+1(\text{mod }4)}(\gamma_i^u(0),0): S_i^u\to\tilde{S}_i^u$ is an isomorphism, implying that $\dim(S_i^u)=\dim(\tilde{S}_i^u)$. Thus, we $\text{codim}(\mathcal{R}P_{i+1(\text{mod }4)}^s+\tilde{S}_i^u)=1$ for $i=1,\cdots,4$.

In summary, the unperturbed system of (5.1 - 5.2) has a heteroclinic orbit Γ that satisfies assumptions (H1 - H3).

In the following we compute the Melnikov functions. The unitary vectors $\psi_i\in(\mathcal{R}P_{i+1(\text{mod }4)}^s+\tilde{S}_i^u)^\perp$ can be chosen as

$$\psi_1=\psi_3=\frac{1}{\sqrt{2}}(0,0,1,1)^T, \quad \psi_2=\psi_4=\frac{1}{\sqrt{2}}(1,1,0,0)^T.$$

The projections $R_{i,i+1(\text{mod }4)}^u w$, $R_{i,i+1(\text{mod }4)}^s w$ for $i=1,\cdots,4$ are given by

$$\begin{aligned}R_{12}^u w=R_{34}^u w&=(w_1,w_2,0,w_4-w_3)^T,\\ R_{12}^s w=R_{34}^s w&=(w_1,w_2,0,w_3+w_4)^T,\\ R_{23}^u w=R_{41}^u w&=(0,w_2-w_1,w_3,w_4)^T,\\ R_{23}^s w=R_{41}^s w&=(0,w_1+w_2,w_3,w_4)^T,\end{aligned}$$

where $w=(w_1,w_2,w_3,w_4)^T\in\mathbb{R}^4$. From (2.7), we obtain

$$\Psi_1^T(t)=\Psi_3^T(t)=\begin{cases}\frac{1}{\sqrt{2}}(0,0,e^{-t},e^{-t}),\ t\geq 0,\\ \frac{1}{\sqrt{2}}(0,0,-e^t,e^t),\ \ t\leq 0,\end{cases}$$

$$\Psi_2^T(t)=\Psi_4^T(t)=\begin{cases}\frac{1}{\sqrt{2}}(e^{-t},e^{-t},0,0),\ t\geq 0,\\ \frac{1}{\sqrt{2}}(-e^t,e^t,0,0),\ \ t\leq 0.\end{cases}$$

From (2.8 - 2.10), we have

$$\mathcal{M}_1(\alpha) = \frac{1}{\sqrt{2}}r + \sqrt{2}\varrho - \frac{1}{\sqrt{2}}\sum_{i=1}^{2} H_{1i}(\omega_i)\sin(\omega_i\alpha + \vartheta_{1i}), \tag{5.4}$$

$$\mathcal{M}_2(\alpha) = \frac{1}{\sqrt{2}}r + \frac{\sqrt{2}}{4}\varrho - \frac{1}{\sqrt{2}}\sum_{i=1}^{2} H_{2i}(\omega_i)\sin(\omega_i\alpha + \vartheta_{2i}), \tag{5.5}$$

$$\mathcal{M}_3(\alpha) = -\frac{1}{\sqrt{2}}r - \sqrt{2}\varrho - \frac{1}{\sqrt{2}}\sum_{i=1}^{2} H_{3i}(\omega_i)\sin(\omega_i\alpha + \vartheta_{3i}), \tag{5.6}$$

$$\mathcal{M}_4(\alpha) = -\frac{1}{\sqrt{2}}r + \sqrt{2}\varrho - \frac{1}{\sqrt{2}}\sum_{i=1}^{2} H_{4i}(\omega_i)\sin(\omega_i\alpha + \vartheta_{4i}), \tag{5.7}$$

where for $i = 1,\ 2$ and $j = 1, \cdots, 4$, $H_{ji}(\omega_i) = \sqrt{(\Xi^s_{ji}(\omega_i))^2 + (\Xi^c_{ji}(\omega_i))^2}$, $\vartheta_{ji} = \arctan(\Xi^c_{ji}(\omega_i)/\Xi^s_{ji}(\omega_i))$ and

$$\Xi^c_{1i}(\omega_i) = \frac{\omega_i(\sigma_{2i} - \sigma_{1i})}{1+\omega_i^2},\ \ \Xi^s_{1i}(\omega_i) = \frac{\sigma_{1i} + \sigma_{2i}}{1+\omega_i^2},$$
$$\Xi^c_{2i}(\omega_i) = \frac{\omega_i(\sigma_{3i} - \sigma_{2i})}{1+\omega_i^2},\ \ \Xi^s_{2i}(\omega_i) = \frac{\sigma_{2i} + \sigma_{3i}}{1+\omega_i^2},$$
$$\Xi^c_{3i}(\omega_i) = \frac{\omega_i(\sigma_{4i} - \sigma_{3i})}{1+\omega_i^2},\ \ \Xi^s_{3i}(\omega_i) = \frac{\sigma_{3i} + \sigma_{4i}}{1+\omega_i^2},$$
$$\Xi^c_{4i}(\omega_i) = \frac{\omega_i(\sigma_{1i} - \sigma_{4i})}{1+\omega_i^2},\ \ \Xi^s_{4i}(\omega_i) = \frac{\sigma_{1i} + \sigma_{4i}}{1+\omega_i^2}.$$

Now, we divide our discussion into the following two cases. In the following, we assume that $\varepsilon > 0$. Hence, $\varrho > 0$. We denote the set $(0, +\infty) \times (0, +\infty)$ by $\mathbb{R}^2_+$.

Case 1: System (5.1-5.2) is periodically excited.

Without lose of generality, we assume that $\sigma_{j1} \neq 0$ and $\sigma_{j2} = 0$ for $j = 1, \cdots, 4$. It is easy to see that for all $j = 1, \cdots, 4$, $\mathcal{M}_j(\alpha)$ are all periodic of the same period $2\pi/\omega_1$. From (5.4 - 5.7), we obtain that $\mathcal{M}_1(\alpha) = 0$ has simple zeros, if and only if

$$r + 2\varrho < H_{11}(\omega_1) = \frac{1}{1+\omega_1^2}\sqrt{(1+\omega_1^2)(\sigma_{11}^2 + \sigma_{21}^2) + 2(1-\omega_1^2)\sigma_{11}\sigma_{21}}. \tag{5.8}$$

$\mathcal{M}_2(\alpha) = 0$ has simple zeros, if and only if

$$r + \frac{1}{2}\varrho < H_{21}(\omega_1) = \frac{1}{1+\omega_1^2}\sqrt{(1+\omega_1^2)(\sigma_{21}^2 + \sigma_{31}^2) + 2(1-\omega_1^2)\sigma_{21}\sigma_{31}}. \tag{5.9}$$

$\mathcal{M}_3(\alpha) = 0$ has simple zeros, if and only if

$$r + 2\varrho < H_{31}(\omega_1) = \frac{1}{1+\omega_1^2}\sqrt{(1+\omega_1^2)(\sigma_{31}^2 + \sigma_{41}^2) + 2(1-\omega_1^2)\sigma_{31}\sigma_{41}}. \tag{5.10}$$

$\mathcal{M}_4(\alpha) = 0$ has simple zeros, if and only if

$$|r - 2\varrho| < H_{41}(\omega_1) = \frac{1}{1+\omega_1^2}\sqrt{(1+\omega_1^2)(\sigma_{11}^2 + \sigma_{41}^2) + 2(1-\omega_1^2)\sigma_{11}\sigma_{41}}. \tag{5.11}$$

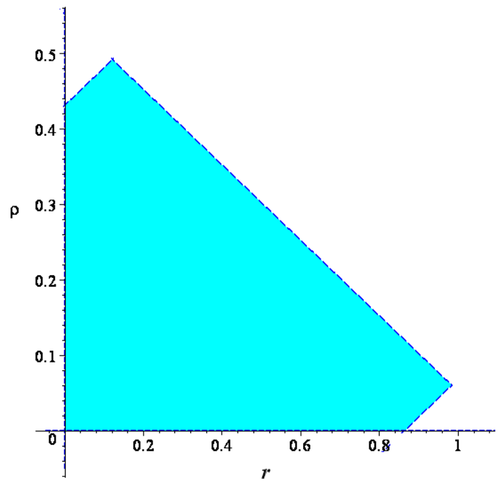

Figure 3. The shaded area represents the set $\mathcal{H}_p$ in the (r, ϱ) parameter space.

When (5.8-5.11) are all satisfied, then each of $\mathcal{M}_j(\alpha)(j = 1, \cdots, 4)$ has exactly two simple zeros in $[0, 2\pi/\omega_1]$ and system (5.1-5.2) is chaotic for $\varepsilon > 0$ sufficiently small.

For example, take $\omega_1 = 1$, $\sigma_{11} = 1$, $\sigma_{21} = 1.2$, $\sigma_{31} = 1.4$ and $\sigma_{41} = 0.7$, then (5.8 - 5.11) are all satisfied, if and only if $(r, \varrho) \in \mathcal{H}_p$, where

$$\mathcal{H}_p = \left\{(r, \varrho) \in \mathbb{R}^2_+ : r + 2\varrho < \sqrt{1.22}, |r - 2\varrho| < \sqrt{0.745}\right\}.$$

Please see the shaded area in Figure 3. Take $(r, \varrho) = (0.4, 0.2) \in \mathcal{H}_p$. Then,

$$\mathcal{M}_1(\alpha) = \frac{2\sqrt{2}}{5} - \frac{\sqrt{61}}{10} \sin\left(\alpha + \arctan\left(\frac{1}{11}\right)\right),$$
$$\mathcal{M}_2(\alpha) = \frac{\sqrt{2}}{4} - \frac{\sqrt{170}}{10} \sin\left(\alpha + \arctan\left(\frac{1}{13}\right)\right),$$
$$\mathcal{M}_3(\alpha) = -\frac{2\sqrt{2}}{5} - \frac{7\sqrt{5}}{5} \sin\left(\alpha - \arctan\left(\frac{2}{21}\right)\right),$$
$$\mathcal{M}_4(\alpha) = -\frac{\sqrt{298}}{10} \sin\left(\alpha + \arctan\left(\frac{3}{17}\right)\right).$$

Clearly, $\mathcal{M}_1(\alpha)$, $\mathcal{M}_2(\alpha)$, $\mathcal{M}_3(\alpha)$, $\mathcal{M}_4(\alpha)$ are all periodic functions of period 2π. In $[0, 2\pi]$, $\mathcal{M}_1(\alpha)$ has two simple zeros $\alpha_{11} \approx 0.7193383634$, $\alpha_{12} \approx 2.240934516$; $\mathcal{M}_2(\alpha)$ has two simple zeros $\alpha_{21} \approx 0.3167922441$, $\alpha_{22} \approx 2.671256627$; $\mathcal{M}_3(\alpha)$ has two simple zeros $\alpha_{31} \approx 4.044399052$, $\alpha_{32} \approx 5.570282321$; $\mathcal{M}_4(\alpha)$ has two simple zeros $\alpha_{41} \approx 2.966920455$, $\alpha_{42} \approx 6.108513108$. Thus, system (5.1-5.2) is chaotic for $\varepsilon > 0$ sufficiently small.

Case 2: System (5.1-5.2) is quasiperiodically excited.

In this case, ω_1/ω_2 is irrational and $\sigma_{j1}\sigma_{j2} \neq 0$ for $j = 1, \cdots, 4$. Then, $\mathcal{M}_1(\alpha)$, $\cdots$, $\mathcal{M}_4(\alpha)$ are all quasiperiodic with fundamental frequencies ω_1, ω_2.

From (5.4-5.7), we obtain that when

$$r + 2\varrho < H_{11}(\omega_1) + H_{12}(\omega_2). \tag{5.12}$$

$\mathcal{M}_1(\alpha) = 0$ has simple zeros, when

$$r + \frac{1}{2}\varrho < H_{21}(\omega_1) + H_{22}(\omega_2), \tag{5.13}$$

$\mathcal{M}_2(\alpha) = 0$ has simple zeros, when

$$r + 2\varrho < H_{31}(\omega_1) + H_{32}(\omega_2). \tag{5.14}$$

$\mathcal{M}_3(\alpha) = 0$ has simple zeros, when

$$|r - 2\varrho| < H_{41}(\omega_1) + H_{42}(\omega_2). \tag{5.15}$$

$\mathcal{M}_4(\alpha) = 0$ has simple zeros.

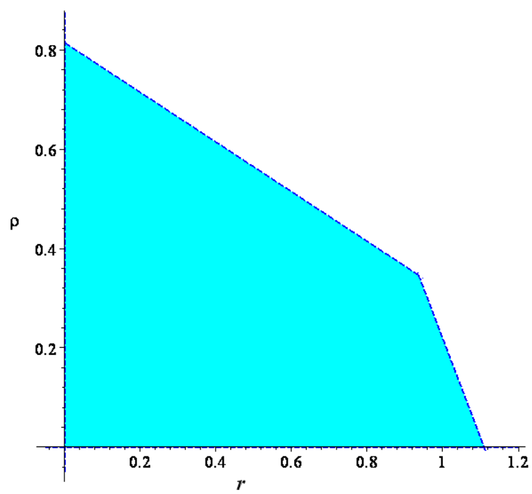

Figure 4. The shaded area represents the set $\mathcal{H}_{qp}$ in the (r, ϱ) parameter space.

As a concrete example, take $\omega_1 = 1$, $\omega_2 = \sqrt{3}$, $\sigma_{11} = 1.4$, $\sigma_{21} = 0.8$, $\sigma_{31} = 0.4$, $\sigma_{41} = 1$ and $\sigma_{12} = 4$, $\sigma_{22} = 0.5$, $\sigma_{32} = 1.1$, $\sigma_{42} = 2$. For this set of parameters, it is easy to see that conditions (5.12 - 5.14) hold implies that (5.15) holds. Thus, (5.12 - 5.15) are all satisfied, if and only if $(r, \varrho) \in \mathcal{H}_{qp}$, where

$$\mathcal{H}_{qp} = \left\{ (r, \varrho) \in \mathbb{R}_+^2 : r + 2\varrho < \frac{2\sqrt{58} + \sqrt{301}}{20}, r + \frac{1}{2}\varrho < \frac{4\sqrt{10} + \sqrt{91}}{20} \right\}.$$

Please see the shaded area in Figure 4. Take $(r, \varrho) = (0.5, 0.1) \in \mathcal{H}_{qp}$. Then,

$$\begin{aligned} \mathcal{M}_1(\alpha) &= \frac{7\sqrt{2}}{20} - \frac{\sqrt{65}}{10} \sin\left(\alpha - \arctan\left(\frac{3}{11}\right)\right) \\ &\quad - \frac{\sqrt{114}}{4} \sin\left(\sqrt{3}\alpha - \arctan\left(\frac{7\sqrt{3}}{9}\right)\right), \\ \mathcal{M}_2(\alpha) &= \frac{11\sqrt{2}}{40} - \frac{\sqrt{5}}{5} \sin\left(\alpha - \arctan\left(\frac{1}{3}\right)\right) \end{aligned}$$

$$
\begin{aligned}
&-\frac{\sqrt{181}}{40}\sin\left(\sqrt{3}\alpha+\arctan\left(\frac{3\sqrt{3}}{8}\right)\right),\\
\mathcal{M}_3(\alpha) &= -\frac{7\sqrt{2}}{20}-\frac{1}{5}\sin\left(\alpha+\arctan\left(\frac{3}{7}\right)\right)\\
&-\frac{\sqrt{602}}{40}\sin\left(\sqrt{3}\alpha-\arctan\left(\frac{9\sqrt{3}}{31}\right)\right),\\
\mathcal{M}_4(\alpha) &= -\frac{3}{10}-\frac{\sqrt{6}}{5}\sin\left(\alpha+\arctan\left(\frac{1}{6}\right)\right)\\
&-3\sin\left(\sqrt{3}\alpha-\arctan\left(\frac{7\sqrt{3}}{3}\right)\right).
\end{aligned}
$$

Clearly, $\mathcal{M}_1(\alpha)$, $\mathcal{M}_2(\alpha)$, $\mathcal{M}_3(\alpha)$, $\mathcal{M}_4(\alpha)$ are all quasi-periodic functions of fundamental frequencies $\omega_1 = 1$ and $\omega_2 = \sqrt{3}$. Hence, all of them have infinitely many simple zeros. For example, in $[0, 2\pi]$, $\mathcal{M}_1(\alpha)$ has four simple zeros $\alpha_{11} \approx 0.6289728376$, $\alpha_{12} \approx 2.427783136$, $\alpha_{13} \approx 4.942184708$, $\alpha_{14} \approx 5.251063458$; $\mathcal{M}_2(\alpha)$ has two simple zeros $\alpha_{21} \approx 0.4539411143$, $\alpha_{22} \approx 1.531829215$; $\mathcal{M}_3(\alpha)$ has four simple zeros $\alpha_{31} \approx 2.524171802$, $\alpha_{32} \approx 3.006146459$, $\alpha_{33} \approx 5.423432222$, $\alpha_{34} \approx 6.099300212$; $\mathcal{M}_4(\alpha)$ has three simple zeros $\alpha_{41} \approx 2.524171802$, $\alpha_{42} \approx 3.006146459$, $\alpha_{43} \approx 5.423432222$. Thus, system (5.1-5.2) is chaotic for $\varepsilon > 0$ sufficiently small.

6. Concluding remarks

In this paper, we extended the results of [37] on transversal heteroclinic bifurcation of PWS systems. Therefore, it is applicable to more general systems such as systems with more general types of switching manifolds and systems with impacts like the ones considered in [23, 35]. More precisely, we studied heteroclinic bifurcation and the appearance of chaos in n-dimensional time-perturbed PWS hybrid systems. We assume that the unperturbed system has an orbit connecting hyperbolic saddles of the unperturbed system that crosses every switching manifold transversally, possibly multiple times. Unlike the systems considered in [37], we do not require the switching manifolds intersect each other at a connected $(n-2)$-dimensional submanifold and impacts are also allowed. By applying the functional analytical method developed by Battelli and Fečkan in [4–8, 19], we obtained a set of Melnikov type functions and show that their zeros correspond to the occurrence of chaos of the system. Finally, we applied our results to a four-dimensional quasiperiodically excited system with impacts formed by two linked rocking blocks.

Although there have been lots of works on homoclinic and heteroclinic bifurcations and chaos of PWS systems so far, many problems still need to be solved. For example, to the best of our knowledge, there are still no results on the study of sliding or grazing heteroclinic orbits for PWS systems. It is worth mentioning that in this paper, we assume that the switching manifolds of system (2.1-2.2) are all of codimension-1. Recently, Hosham has investigated bifurcation of limit cycles in PWS systems with the phase space being split into four regions that are separated by codimension-2 manifolds in [29]. Then, it is natural to study homoclinic and heteroclinic bifurcations for such systems. In our future work, we plan to investigate those problems, which are interesting and more difficult.

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