Journal of Nonlinear Modeling and Analysis http://jnma.ca; http://jnma.ijournal.cn
Volume 3, Number 3, September 2021, 421–430 DOI:10.12150/jnma.2021.421

Travelling Wave Solutions and Conservation Laws of the (2+1)-dimensional Broer-Kaup-Kupershmidt Equation*

Lijun Zhang[1,†], Innocent Simbanefayi[2] and Chaudry Masood Khalique[2]

Abstract The travelling wave solutions and conservation laws of the (2+1)-dimensional Broer-Kaup-Kupershmidt (BKK) equation are considered in this paper. Under the travelling wave frame, the BKK equation is transformed to a system of ordinary differential equations (ODEs) with two dependent variables. Therefore, it happens that one dependent variable u can be decoupled into a second order ODE that corresponds to a Hamiltonian planar dynamical system involving three arbitrary constants. By using the bifurcation analysis, we obtain the bounded travelling wave solutions u, which include the kink, anti-kink and periodic wave solutions. Finally, the conservation laws of the BBK equation are derived by employing the multiplier approach.

Keywords The (2+1)-dimensional Broer-Kaup-Kupershmidt equation, Travelling wave solutions, Conservation laws, Multiplier method.

1. Introduction

A significant amount of mathematical research has been dedicated to developing tools for the treatment of nonlinear partial differential equations (NLPDEs). This is necessitated by the fact that NLPDEs have diverse applications in the physical world and their solutions help to shed light on the various phenomena with which we interact. The research has been in part vested in developing methods of obtaining their exact solutions. Here, we give a few of these methods: the Lie symmetry method [5, 18, 19], the inverse scattering transform method [1], the tanh-function method and extended tanh-function method [17], Symbolic methods [9], the Riccati equation method [16], the Jacobi elliptic function method [20], the exp-function method [8], the homogeneous balance method [23], Hirota's bilinear method [10], simple transformation method [13], F-expansion method [26], dynamical system method [27] and so on.

†the corresponding author.
Email address: li-jun0608@163.com (L. Zhang), 24536849@nwu.ac.za (I. Simbanefayi), Masood.Khalique@nwu.ac.za (C. M. Khalique)
[1]College of Mathematics and Systems Science, Shandong University of Science and Technology, Qingdao, Shandong 266590, China
[2]International Institute for Symmetry Analysis and Mathematical Modelling, Department of Mathematical Sciences, North-West University, Mafikeng Campus, Private Bag X 2046, Mmabatho 2735, South Africa
*This work was partially supported by National Nature Science Foundation of China (Nos. 12011530062, 11672270, 11872335).

The Broer-Kaup-Kupershmidt (BKK) system of equations

$$\begin{aligned} &u_{ty} - u_{xxy} + 2(uu_x)_y + 2v_{xx} = 0, \\ &v_t + v_{xx} + 2(uv)_x = 0. \end{aligned} \tag{1.1}$$

is one of the most popular system of NLPDEs to emerge in the past few decades. This is evidenced by the vast array of scholars who have researched various aspects of the system. It has applications in fluid dynamics where it models dispersive shallow water waves travelling in equal depth. By using a convenient scaling transformation, it has been shown in [6, 24] that the (2+1)-dimensional asymmetric Davey-Stewartson system [15]

$$\bar{q}_t + \frac{\bar{q}_{xx}}{2} + 2\bar{q}\partial_y^{-1}(\bar{q}\bar{r})_x = 0, \quad \bar{r}_t + \frac{\bar{r}_{xx}}{2} + 2\bar{r}\partial_y^{-1}(\bar{q}\bar{r})_x = 0,$$

transforms into (1) under the transformation

$$\bar{q} = \exp\left(-\int^x u\mathrm{dx}\right), \quad \bar{r} = -v\exp\left(\int^x u\mathrm{dx}\right).$$

Solitoff and dromion solutions are obtained in the same work [6]. Also, the Kadomtsev-Petviashvili equation transforms into the BKK equation under a symmetry constraint, see for example [15, 26]. In [26], the modified extended Fan sub-equation method is used to obtain soliton-like and Jacobi elliptic wave function-like solutions of (1). In [14], Bäcklund transformation and variable separation approach was used to obtain dromions, lumps and peakons through the introduction of an arbitrary function. Again, in [22, 25] an auxiliary equation method was utilised to obtain its exact travelling wave solutions. His semi-inverse method was applied in [28] to establish a variational principle of the BKK system. Several other researchers have utilised different ad hoc methods to establish different solutions of (1). Kassem and Rashed [11] came up with closed form solutions of (1) by using hidden symmetries of its Lie optimal systems. The most recent work on the BKK system was by Tang et al. [21] who presented the double Wronskian solutions by using Hirota's method and binary bell polynomials.

Conservation laws depict conserved quantities of physical interest. The most common physical quantities that are conserved are energy, charge, momentum and mass amongst others. Conservation laws also help in establishing the uniqueness, stability and existence of solutions of differential equations. There are several methods available for deriving conserved quantities, see for example [2–4, 7, 12].

Due to its undeniably vast applicability, continued study of the BKK system remains necessary. In this paper, we further explore the (2+1)-dimensional BKK system (1). Unlike most of the previous research on this NLPDE, we do not employ ad hoc methods to obtain its analytic solutions, but utilise a standard Lie based integration method [18]. Here, we provide a detailed outline of the derivation of the bounded travelling wave solutions, which include kink and anti-kink profiles. We also outline how periodic solutions of a snoidal nature are obtained. Moreover, the homotopy integral approach to finding conservation laws is explored in details. To the best of our knowledge, the literature is devoid of explicit applications of this approach, moreso for nonlinear partial differential equations with mixed derivatives, our work is novel in this regard.

2. Bounded travelling wave solutions of the BKK equations (1)

Let $\xi = x + by + ct$, then the BKK equation (1) becomes

$$\begin{aligned} &bcu'' - bu''' + 2b(uu')' + 2v'' = 0, \\ &cv' + v'' + 2(uv)' = 0. \end{aligned} \tag{2.1}$$

Integrating (2.1) once with respect to ξ and letting the constants of integration be zero, one gets

$$\begin{aligned} &bcu' - bu'' + 2buu' + 2v' = 0, \\ &(2u + c)v + v' = 0. \end{aligned} \tag{2.2}$$

Integrating the first equation of (2.2) once again, and then solving for v gives

$$v = \frac{1}{2}(bu' - bu^2 - bcu + g), \tag{2.3}$$

where g is a constant of integration. Substituting the expression of v from (2.3) into the second equation of system (2.2), we have for the case when $b \neq 0$, that

$$u'' = (2u + c)\left(u^2 + cu - \frac{g}{b}\right) + \frac{g_2}{b}. \tag{2.4}$$

Let $u' = y$, then equation (2.4) is equivalent to the following planar dynamical system:

$$u' = y, \quad y' = (2u + c)\left(u^2 + cu - \frac{g}{b}\right). \tag{2.5}$$

Suppose $u_0 > -\frac{c}{2}$ satisfies $g = bu_0(c + u_0)$, then system (2.5) has three equilibrium points $E_1(-c - u_0, 0)$, $E_2(-\frac{c}{2}, 0)$ and $E_3(u_0, 0)$, where $-c - u_0 < -\frac{c}{2} < u_0$. By determining the Jacobian determinant, we know that E_1 and E_3 are saddle points and E_2 is a center. Clearly, system (2.5) is a Hamiltonian system with Hamiltonian

$$H(u, y) = \frac{y^2}{2} - \frac{1}{2}u^4 - cu^3 - \frac{1}{2}\left(c^2 - 2\frac{g}{b}\right)u^2 + \frac{gc}{b}u. \tag{2.6}$$

Hence, $H(E_2) = h_2$ and $H(E_1) = H(E_3) = h_0$, where $h_1 = \frac{1}{2}u_0^2(u_0 + c)^2$ and $h_2 = -\frac{1}{32}c^2(c^2 + 8cu_0 + 8u_0^2)$. Therefore, we have the phase portrait of (2.5) (see Figure 1). It shows that there are two heteroclinic orbits connecting the two saddle points E_1 and E_3 which consist of the boundary of a family of periodic orbits. As is known to us all, the two heteroclinic orbits of (2.5) which are determined by $H(u, y) = h_1$ correspond to a kink and an anti-kink for u of (1) respectively and the periodic orbits given by $H(u, y) = h$ with $h_2 < h < h_1$ correspond to periodic travelling wave solutions. Hence, one sees easily from Figure 1 that there are a family of periodic wave solutions and a kink and an anti-kink for u of (1). Note that v is determined by (2.3) which is equivalent to

$$v = \frac{b}{2}(u' - u^2 - cu + u_0(c + u_0)), \tag{2.7}$$

so (u,v) is a periodic travelling wave solution of (1), if u is periodic.

To derive the kink and anti-kink, we integrate along the heteroclinic orbits $H(u,y)=h_1$. That is,

$$y_{\pm}=\pm(u-u_0)(u+c+u_0).$$

Substituting the above equation into the first equation of (2.5), and then integrating the resulting differential equation yields

$$u_{\pm}(\xi)=-\frac{c}{2}\pm\frac{1}{2}(c+2u_0)\tanh\left(\frac{1}{2}(c+2u_0)(\xi-\xi_0)\right), \tag{2.8}$$

where ξ_0 is a constant of integration. It implies that for arbitrary constants ξ_0, c and $u_0>-\frac{c}{2}$, (2.8) determine a kink and an anti-kink for u of (1) respectively. Taking account of (2.7), we have the corresponding

$$v_{+}(\xi)=\frac{b}{4}(c+2x_0)^2\mathrm{sech}^2\left(\frac{1}{2}(c+2u_0)(\xi-\xi_0)\right) \tag{2.9}$$

and

$$v_{-}(\xi)=0.$$

That is to say, $(u_+(\xi),v_+(\xi))$ and $(u_-(\xi),v_-(\xi))$ are two travelling wave solutions to (1).

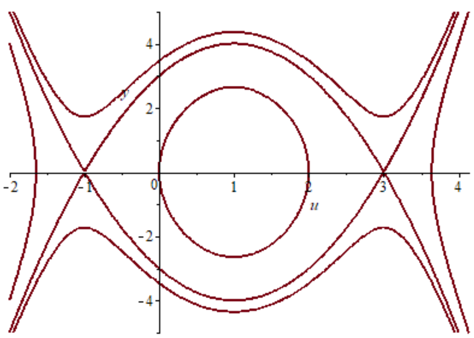


Figure 1. Phase orbit of (2.5) and the corresponding bounded travelling wave solutions.

For the periodic orbits determined by $H(u,y)=h$ with $h_2<h<h_1$, we know from (2.6) that the periodic orbit passing through $(u_2,0)$. For arbitrary $u_2\in(-c-u_0,-\frac{c}{2})$, is determined by

$$y=\pm\sqrt{(u-u_1)(u-u_2)(u_3-u)(u_4-u)}\quad(u_2\le u\le u_3), \tag{2.10}$$

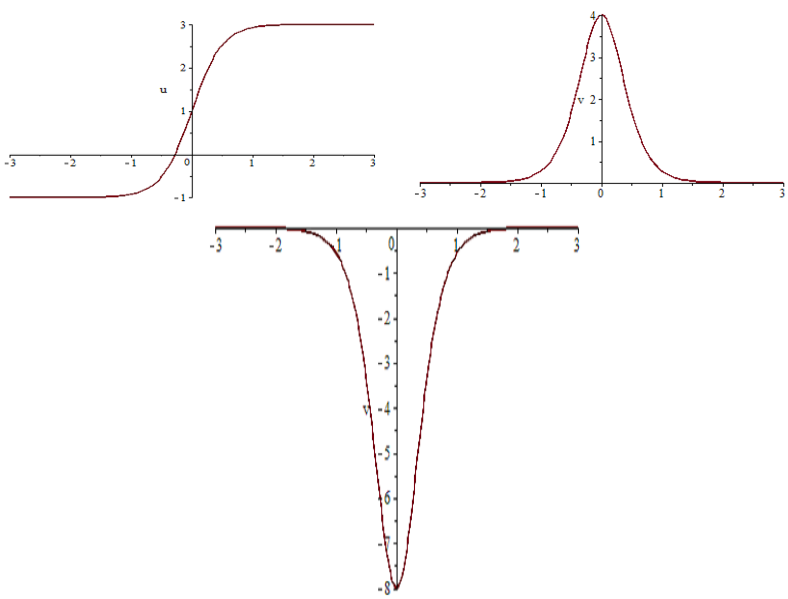
where u_1,u_3 and u_4 satisfying $u_1<-c-u_0$ and $-\frac{c}{2}<u_3<u_0<u_4$ are three real roots of the following cubic algebraic equation:

$$u^3+(2c+u_2)u^2+\left((c+u_2)^2-2u_0(c+u_0)\right)u+u_2(c+u_2)^2-2u_0(c+u_0)(c+u_2)=0.$$

Inserting equation (2.10) into the first equation of (2.5), and then integrating the resulting differential equation yields

$$u(\xi)=u_1+\frac{(u_2-u_1)(u_3-u_1)}{(u_3-u_1)-(u_3-u_2)sn^2(\Omega\xi,q)}, \tag{2.11}$$

where $\Omega = \frac{1}{2}\sqrt{(u_4-u_2)(u_3-u_1)}$ and $q = \sqrt{\frac{(u_3-u_2)(u_4-u_1)}{(u_3-u_1)(u_4-u_2)}}$. Therefore, we derive a family of periodic travelling wave solutions $u(\xi), v(\xi)$ to (1), where $u(\xi)$ is given by (2.11) and the associated $v(\xi)$ is derived by inserting (2.11) into (2.7) for arbitrary $u_2 \in (-c-u_0, -\frac{c}{2})$.



(1) $u = u_+(\xi)$ (2) $v = v_+(\xi)$ with $b = 1$ (3) $v = v_+(\xi)$ with $b = -2$

Figure 2. Bounded travelling wave solutions with $c = -2$ and $u_0 = 3$.

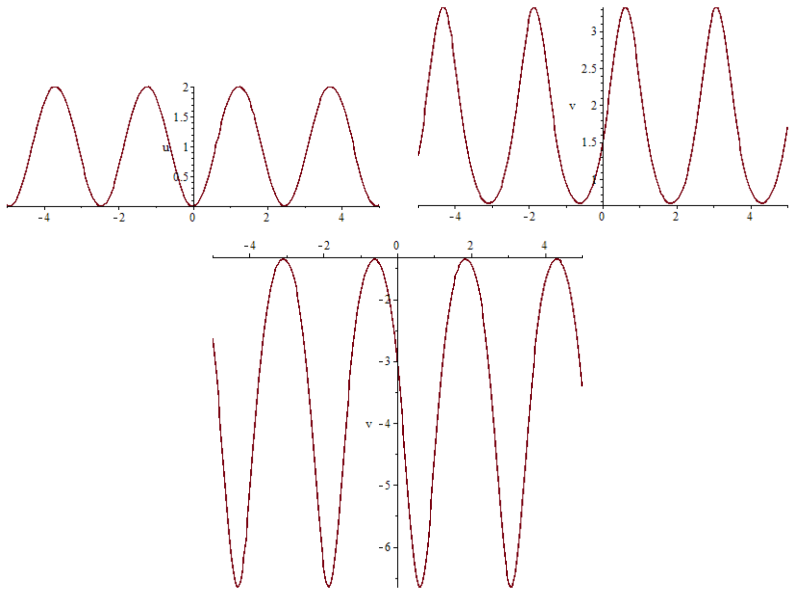
Theorem For arbitrary real numbers b, c and $u_0 > -\frac{c}{2}$, let $\xi = x + by + ct$. Then, the (2+1)-dimensional Broer-Kaup-Kupershmidt equation (1) has the following bounded travelling wave solutions:

(1) Two families of travelling wave solutions $(u_+(\xi), v_+(\xi))$ and $(u_-(\xi), v_-(\xi))$, where $u_+(\xi)$ is of kink shape but the associated $v_+(\xi)$ is of solitary shape whose amplitude is determined by b (see Figure 2). However, $u_-(\xi)$ is of anti-kink shape and $u_-(\xi) = 0$;

(2) A family of periodic travelling wave solutions $(u(\xi), v(\xi))$ given by (2.11), and the associated $v(\xi)$ is determined by (2.7) for arbitrary $u_2 \in (-c-u_0, -\frac{c}{2})$ (see Figure 3).

3. Conservation laws of the BKK equations (1.1)

A conservation law is a divergence expression $D_tT + D_xX + D_yY = 0$ subject to solutions of the BKK system (1). The conserved density T and the spatial flux (X, Y) are functions of t, x, y, u, v and the derivatives of u and v. The BKK system (1) does not possess a variational principle as it does not satisfy the Helmholtz conditions [3]. However, conserved quantities may be obtained by using a more direct approach which does not require the existence of a variational principle.



(1) $u = u(\xi)$ (2) $v = v(\xi)$ with $b = 1$ (3) $v = v(\xi)$ with $b = -2$

Figure 3. Periodic travelling wave solutions with $c = -2$, $u_0 = 3$ and $u_2 = 0$.

In this work, we employ the multiplier approach to compute conservation laws. This approach capitalises on the well-known correspondence between multipliers and conservation laws [4]. Furthermore, we will employ the first homotopy integral formula to compute conserved quantities. To update, the authors have not come across an explicit application of this formula to a system with mixed derivatives. Thus, This work will serve as an illustrative example, amongst other things. A determining condition to find multipliers of the BKK equations

$$
\begin{aligned}
&E_1 \equiv u_{ty} - u_{xxy} + 2(uu_x)_y + 2v_{xx} = 0, \\
&E_2 \equiv v_t + v_{xx} + 2(uv)_x = 0.
\end{aligned} \tag{3.1}
$$

is

$$
\begin{aligned}
&\frac{\delta}{\delta u}\left[\Lambda^1 E_1 + \Lambda^2 E_2\right] = 0, \\
&\frac{\delta}{\delta v}\left[\Lambda^1 E_1 + \Lambda^2 E_2\right] = 0,
\end{aligned} \tag{3.2}
$$

where $\delta/\delta u$ and $\delta/\delta v$ are Euler-Lagrange operators

$$
\begin{aligned}
\frac{\delta}{\delta u} &= \frac{\partial}{\partial u} - D_x\frac{\partial}{\partial u_x} - D_y\frac{\partial}{\partial u_y} + D_tD_y\frac{\partial}{\partial u_{ty}} + D_xD_y\frac{\partial}{\partial u_{xy}} - D_x^2D_y\frac{\partial}{\partial u_{xxy}}, \\
\frac{\delta}{\delta v} &= \frac{\partial}{\partial v} - D_t\frac{\partial}{\partial v_t} - D_x\frac{\partial}{\partial v_x} + D_x^2\frac{\partial}{\partial v_{xx}}.
\end{aligned} \tag{3.3}
$$

Furthermore, D_t, D_x and D_y are total derivatives given by

$$\begin{aligned}
D_t &= \frac{\partial}{\partial t} + u_t\frac{\partial}{\partial u} + v_t\frac{\partial}{\partial v} + u_{tt}\frac{\partial}{\partial u_t} + v_{tt}\frac{\partial}{\partial v_t} + u_{tx}\frac{\partial}{\partial u_x} + v_{tx}\frac{\partial}{\partial v_x} + \cdots ,\\
D_x &= \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + v_x\frac{\partial}{\partial v} + u_{xx}\frac{\partial}{\partial u_x} + v_{xx}\frac{\partial}{\partial v_x} + u_{xt}\frac{\partial}{\partial u_t} + v_{xt}\frac{\partial}{\partial v_t} + \cdots ,\\
D_y &= \frac{\partial}{\partial y} + u_y\frac{\partial}{\partial u} + v_y\frac{\partial}{\partial v} + u_{yy}\frac{\partial}{\partial u_y} + v_{yy}\frac{\partial}{\partial v_y} + u_{yt}\frac{\partial}{\partial u_t} + v_{yt}\frac{\partial}{\partial v_t} + \cdots .
\end{aligned}$$

Here, we seek to compute first order conservation law multipliers

$$\Lambda^\alpha = \Lambda^\alpha(t, x, u, v, u_x, v_x, u_y, v_y), \quad \alpha = 1, 2.$$

Expanding (3.2) and splitting on derivatives of u and v, we obtain the following system of fifteen multiplier determining equations:

$\Lambda^1_{ty} = 0,\ \Lambda^2_{ty} = 0,\ 2\Lambda^1_{xx} - \Lambda^2_t = 0,\ \Lambda^1_{xy} = 0,\ \Lambda^2_x = 0,\ \Lambda^1_u = 0,\ \Lambda^2_u = 0,\ \Lambda^2_{u_x} = 0,$
$\Lambda^1_v = 0,\ \Lambda^2_v = 0,\ \Lambda^2_v = 0,\ \Lambda^1_{u_x} = 0,\ \Lambda^2_{u_x} = 0,$
$\Lambda^1_{v_x} = 0,\ \Lambda^2_{v_x} = 0,\ \Lambda^1_{u_y} = 0,\ \Lambda^2_{u_y} = 0.$

Solving the above system for Λ^1 and Λ^2, we obtain

$$\begin{aligned}
\Lambda^1 &= \frac{1}{4}F_2'(t)\,x^2 + F_3(t)\,x + F_4(y) + F_5(t)\,,\\
\Lambda^2 &= F_1(y) + F_2(t)
\end{aligned} \tag{3.4}$$

where F_i, $i = 1, \cdots, 5$ are arbitrary functions of their respective arguments. Thus, we have the following five multipliers

$$\begin{aligned}
&\Lambda^1_1 = 0, && \Lambda^2_1 = F_1(y);\\
&\Lambda^1_2 = \frac{1}{4}F_2'(t)x^2, && \Lambda^2_2 = F_2(t);\\
&\Lambda^1_3 = F_3(t)x, && \Lambda^2_3 = 0;\\
&\Lambda^1_4 = F_4(y), && \Lambda^2_4 = 0;\\
&\Lambda^1_5 = F_5(t), && \Lambda^2_5 = 0.
\end{aligned} \tag{3.5}$$

The homotopy integral formula [3] is a revolutionary approach for computing conserved vectors, and is given by

$$\Phi = \int_0^1 \sum_{j=1}^{k} \partial_\lambda \partial^{j-1} u^m_{(\lambda)} \left(\sum_{l=j}^{k} (-D)^{l-j} \cdot \left(\frac{\partial E_m \Lambda^m}{\partial u}\right)\Big|_{u^m = u^m_{(\lambda)}} \right) \mathrm{d}\lambda,$$

where $m = 1, \cdots, n$, and n is the number of dependent variables. Also, $\Phi = (T, X)$ is a conserved quantity composed of conserved density T and spatial flux X. In accordance with system (3.1) and multipliers (3.4) we have, in explicit form

$$
\begin{aligned}
T =& \int_0^1 \left(u D_y \left(\frac{\partial E_1 \Lambda_1}{\partial u_{ty}} \right) |_{u=u_{(\lambda)}} + v \left(\frac{\partial E_2 \Lambda_2}{\partial v_t} \right) |_{v=v_{(\lambda)}} \right) \mathrm{d}\lambda, \\
X =& \int_0^1 \left\{ u \left\{ \left(\frac{\partial E_1 \Lambda_1}{\partial u_x} \right) |_{u=u_{(\lambda)}} + \left(\frac{\partial E_2 \Lambda_2}{\partial u_x} \right) |_{u=u_{(\lambda)}} - D_y \left(\frac{\partial E_1 \Lambda_1}{\partial u_{xy}} \right) |_{u=u_{(\lambda)}} \right.\right. \\
& \left. + D_x D_y \left(\frac{\partial E_1 \Lambda_1}{\partial u_{xxy}} \right) |_{u=u_{(\lambda)}} \right\} - u_x D_y \left(\frac{\partial E_1 \Lambda_1}{\partial u_{xxy}} \right) |_{u=u_{(\lambda)}} \\
& + v \left\{ \left(\frac{\partial E_2 \Lambda_2}{\partial v_x} \right) |_{v=v_{(\lambda)}} - D_x \left(\frac{\partial E_1 \Lambda_1}{\partial v_{xx}} \right) |_{v=v_{(\lambda)}} - D_x \left(\frac{\partial E_2 \Lambda_2}{\partial v_{xx}} \right) |_{v=v_{(\lambda)}} \right\} \\
& \left. + v_x \left(\frac{\partial E_2 \Lambda_2}{\partial v_{xx}} \right) |_{v=v_{(\lambda)}} \right\} \mathrm{d}\lambda \\
Y =& \int_0^1 \left\{ u \left(\frac{\partial E_1 \Lambda_1}{\partial u_y} \right) |_{u=u_{(\lambda)}} + u_t \left(\frac{\partial E_1 \Lambda_1}{\partial u_{ty}} \right) |_{u=u_{(\lambda)}} + u_x \left(\frac{\partial E_1 \Lambda_1}{\partial u_{xy}} \right) |_{u=u_{(\lambda)}} \right. \\
& \left. + u_{xx} \left(\frac{\partial E_1 \Lambda_1}{\partial u_{xxy}} \right) |_{u=u_{(\lambda)}} \right\} \mathrm{d}\lambda
\end{aligned}
\tag{3.6}
$$

Choosing the homotopy $u_{(\lambda)} = \lambda u$, and $v_{(\lambda)} = \lambda v$ as is usually the case, we have for Λ_1^1 and Λ_1^2

$$
\begin{aligned}
T_1 =& \int_0^1 v F_1 \mathrm{d}\lambda \\
=& v F_1, \\
X_1 =& \int_0^1 (4\lambda u v F_1 + v_x F_1) \, \mathrm{d}\lambda \\
=& 2uvF_1 + v_x F_1, \\
Y_1 =& 0.
\end{aligned}
$$

$T_1 = vF_1$ is mass density, and $X_1 = 2uv + v_x F_1$ is mass flux or momentum.

Similarly, the complete set of flux densities and spatial fluxes are derived to obtain

$$
\begin{aligned}
T_2 =& \frac{1}{4} x^2 u_y F_2' + v F_2, \\
X_2 =& \frac{1}{2} u u_y F_2' x^2 + 2\, uvF_2 + \frac{1}{2} x u_y F_2' - \frac{1}{4} x^2 u_{xy} F_2' - xvF_2' + \frac{1}{2} x^2 v_x F_2' + v_x F_2, \\
Y_2 =& -\frac{1}{4} u \; (x^2 F_2'' + 2xuF_2' + 2\, F_2' \;); \\
T_3 =& = x u_y \, F_3, \\
X_3 =& 2xuu_y F_3 + u_y \, F_3 - u_{xy} F_3 x - 2\, vF_3 + 2xv_x \, F_3, \\
Y_3 =& - xuF_3' + uF_3 \\
T_4 =& u_y \, F_4, \\
X_4 =& = 2\, uu_y F_4 - u_{xy} F_4 + 2\, v_x \, F_4, \\
Y_4 =& 0; \\
T_5 =& u_y \, F_5,
\end{aligned}
$$

$$X_5 = 2\,uF_5u_y - u_{xy}\,F_5 + 2\,v_x\,F_5,$$
$$Y_5 = -\,uF_5';$$

The conservation laws obtained here are local and infinitely many for the presence of arbitrary functions. Due the order of the dependencies of the adjoint symmetries being of a lower order than the leading derivatives of the underlying system, the conserved quantities are of low-order.

References

[1] M. J. Ablowitz and P. A. Clarkson, *Solitons, Nonlinear Evolution Equations and Inverse Scattering*, Cambridge University Press, Cambridge, 1991.

[2] S. C. Anco, *Symmetry properties of conservation laws*, International Journal of Modern Physics B, 2016, 30, Article ID 1640003, 14 pages.

[3] S. C. Anco, *Generalization of Noether's Theorem in Modern Form to Non-variational Partial Differential Equations. In: Melnik R., Makarov R., Belair J. (eds) Recent Progress and Modern Challenges in Applied Mathematics, Modeling and Computational Science. Fields Institute Communications*, Volume 79, Springer, New York, 2017.

[4] S. C. Anco, *Symmetry-invariant conservation laws of partial differential equations*, European Journal of Applied Mathematics, 2018, 29, 78–117.

[5] G. W. Bluman and S. Kumei, *Symmetries and Differential Equations*, Springer-Verlag, New York, 1989.

[6] C. Chen and Y. Li, *The exact solutions of Some (2+1)-dimensional integrable systems*, Communications in Theoretical Physics, 2002, 38, 129–132.

[7] M. L. Gandarias, S. C. Anco and E. Recio, *Conservation laws, symmetries, and line soliton solutions of generalized KP and Boussinesq Equations with p-power nonlinearities in two dimensions*, Theoretical and Mathematical Physics, 2018, 197, 1393–1411.

[8] J. He and X. Wu, *Exp-function method for nonlinear wave equations*, Chaos, Solitons & Fractals, 2006, 30, 700–708.

[9] W. Hereman and A. Nuseir, *Symbolic methods to construct exact solutions of nonlinear partial differential equations*, Mathematics and Computers in Simulation, 1997, 43, 13–27.

[10] R. Hirota, *The direct method in soliton theory*, Cambridge University Press, Cambridge, 2004.

[11] M. M. Kassem and A. S. Rashed, *N-solitons and cuspon waves solutions of (2+1)-dimensional Broer-Kaup-Kaupershmidt equations via hidden symmetries of Lie optimal system*, Chinese Journal of Physics, 2019, 57, 90–104.

[12] C. M. Khalique and I. Simbanefayi, *A symbolic computational approach to finding solutions and conservation laws for (3 + 1)-dimensional modified BBM models*, Alexandria Engineering Journal, 2020, 59, 1799–1809.

[13] N. A. Kudryashov, *Simplest equation method to look for exact solutions of nonlinear differential equations*, Chaos, Solitons & Fractals, 2005, 24, 1217–1231.

[14] H. Li, *Rich localized coherent structures of the (2+1)-dimensional BroerKaup Kupershmidt equation*, Communications in Theoretical Physics, 2003, 39, 513–518.

[15] S. Lou and X. Hu, *Infinitely many Lax pairs and symmetry constraints of the KP equation*, Journal of Mathematical Physics, 1997, 38, 6401–6427.

[16] W. Ma and B. Fuchssteiner, *Explicit and exact solutions to a Kolmogorov-Petrovskii-Piskunov equation*, International Journal of Nonlinear Mechanics, 1996, 31, 329–338.

[17] W. Malfliet and W. Hereman, *The tanh method I: exact solutions of nonlinear evolution and wave equations*, Physica Scripta, 1996, 54, 563–568.

[18] P. J. Olver, *Applications of Lie Groups to Differential Equations*, Second Edition, Springer-Verlag, Berlin, 1993.

[19] L. V. Ovsiannikov, *Group Analysis of Differential Equations*, Academic Press, New York, 1982.

[20] E. J. Parkes, B. R. Duffy and P. C. Abbott, *The Jacobi elliptic-function method for finding periodic-wave solutions to nonlinear evolution equations*, Physics Letters A, 2002, 295, 280–286.

[21] Y. Tang, W. Yuen and L. Zhang, *Double Wronskian solutions to the (2+1)-dimensional Broer-Kaup-Kupershmidt equation*, Applied Mathematics Letters, 2020, 105, Article ID 106285.

[22] G. Xu and Z. Li, *Exact travelling wave solutions of the Whitham-Broer-Kaup and Broer-Kaup- Kupershmidt equations*, Chaos, Solitons & Fractals, 2005, 24, 549–556.

[23] M. Wang, Y. Zhou and Z. Li, *Application of a homogeneous balance method to exact solutions of nonlinear equations in mathematical physics*, Physics Letters A, 1996, 21, 67–75.

[24] T. Wu, J. Zhang, L. P. Look, V. Rogthurd and M. Tulin,*On modeling nonlinear long wave, in Mathematics is for Solving Problems*, SIAM, 1996, 233–236.

[25] X. Yang and J. Tang, *New travelling wave solutions fro combined KdV-mKdV equation and (2+1)-dimesional Broer-kaup-Kupershmidt system*, Chinese Physics, 2007, 16, 310–318.

[26] E. Yomba, *The modified extended Fan sub-equation method and its application to the (2+1)-dimensional Broer-Kaup-Kupershmidt equation*, Chaos, Solitons & Fractals, 2006, 27, 187–196.

[27] L. Zhang, Y. Wang, C. M. Khalique and Y. Bai, *Peakon and cuspon solutions of a generalized Camassa-Holm-Novikov equation*, Journal of Applied Analysis and Computation, 2018, 8, 1938–1958.

[28] X. Zhou, Variational approach to the Broer-Kaup-Kupershmidt equation, Physics Letters A, 2007, 363, 108–109.