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# Hopf Bifurcation Analysis for a Delayed Business Cycle Model-The Equivalence of Multiple Time Scales Versus Center Manifold Reduction Methods*

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**Abstract** In this paper, we study the Hopf bifurcation of a model with a second order term, which is the business cycle model with delay. Multiple time scales method, which is mainly used by the engineering researchers, and center manifold reduction method, which is mainly used by researchers from mathematical society, are used to derive the two types of normal forms near the Hopf critical point. A comparison between the two methods shows that the two normal forms are equivalent. Scholars can derive the normal form by choosing appropriate methods according to their actual demands. Moreover, bifurcation analysis and numerical simulations are given to verify the analytical predictions.

**Keywords** Business cycle, Hopf bifurcation, Normal form, Multiple time scales, Center manifold reduction.

**MSC(2010)** 34K18, 37L10.

## 1. Introduction

Over the past several decades, the bifurcation problems of delayed differential equations have been hot topics in studying nonlinear dynamical systems. As we all know, it is very important to compute normal forms of differential equations in study of bifurcation properties. There are two popular and effective approaches for determining the normal forms of bifurcations in nonlinear delayed dynamical systems, that is, multiple time scales (MTS) method [11, 12] and center manifold reduction (CMR) method [5, 6, 16].

The MTS method was originally used to study the Hopf bifurcation of one-dimensional second order ordinary differential vibration equation [11], and Nayfeh [12] extended this method to solve Hopf bifurcation of functional differential equations. The MTS method is mainly used by applied scientists and researchers from engineering society since it is simple without complicated computation [1,2,11–13,15], while this method has some limitations. For example, it cannot solve

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non-semisimple singularity bifurcation in functional differential equations, such as Bogdanov-Takens bifurcation.

The CMR method can be used to solve all kinds of functional differential equations [7, 9, 17], and it is widely used by researchers from mathematical society. However, this method needs the basic knowledge of functional analysis and algebra in mathematics major, and it needs large amount of calculation and complicated process. For CMR method in delayed differential systems, one needs to first change the delayed equations to an abstract ordinary differential equation in infinite dimensional image space, and then decompose the solution space of their linearized form into stable manifold and center manifold. Next, with adjoint operator equations, one computes the center manifold by projecting the whole space to the center manifold,and finally calculates the normal form restricted to the center manifold.

In fact, both of the two approaches combine the two steps involved in using center manifold theory and normal form theory into one unified step to obtain the normal form and nonlinear transformation simultaneously, thus, there may exist some relations between the two methods. Many authors considered some types of bifurcations by using the two methods at the same time. For example, Nayfeh [12] used both the MTS and CMR methods to derive equivalent normal forms of Hopf bifurcation for some simple delayed nonlinear dynamical systems, while the CMR method used in this paper had some differences with Faria's center manifold reduction method. Ding et al. [3, 4] applied the two methods to obtain the normal forms near Hopf-zero and double-Hopf critical points in delayed differential equations respectively, and showed their equivalence. Peng et al. [14] used two methods to study the Hopf bifurcation of van der Pol-Duffing equation with delay, while in this paper, Peng et al. only used Hassard's method [8] to derive the formulae for determining the stability of Hopf bifurcating periodic solutions and the direction of Hopf bifurcation, not showed the explicit normal form of Hopf bifurcation. Moreover, we find that the systems discussed in the above papers are all without quadratic terms. Actually, Yu et al. [18] proved that, if system does not contain second-order terms, the normal forms associated with the semisimple $n_1$-Hopf-$n_2$-zero singularity, derived by using the multiple time scales and center manifold reduction methods, are identical up to third order.

However, if system contains second-order terms, can we also obtain equivalent normal forms by using MTS and CMR methods? It is also the motivation of this study. In this paper, we consider the following delayed business cycle model, which contains quadratic term [10]. Then, we investigate the equivalence of two normal forms of Hopf bifurcation in this system, derived by using the multiple time scales and center manifold reduction methods, and the system shows as follows:

$$\ddot{x} + ax(t-\tau) - qx^3 = -v\dot{x}^3 - v\dot{x}^2 - u\dot{x}, \tag{1.1}$$

where $x$ represents gross national income(or output), $\dot{x}$ is the derivative of $x$ to time $t$; $0 < a < 1$ is marginal propensity to consume; $q > 0$ is fixed interest rate; $v > 0$ denotes the fixed rate of national income, we usually call it acceleration factor; $0 < u \leq 1$, $1/u$ is called Keynesian coefficient due to the time lag of investment process, $\tau$ represents the delay caused by the delay of investment decision. In the business cycle model, gross national income is an important target to reflect the overall economic activities, which is often used in the research of macroeconomic. Thus, the delay is introduced into gross national income, and the results show that the delay will cause the fluctuation of macroeconomics and effect the stability of the business cycle. Ma et al. [10] studied the Hopf bifurcation of system (1.1), the bifurcation direction coefficient, the direction and stability of the bifurcation periodic solution of the model (1.1) was calculated by using Hassard's method [8], while there was no specific normal form of Hopf bifurcation. In the present paper, the Hopf bifurcation normal form of the system (1.1) is derived by using MTS and CMR methods, and then the Hopf bifurcation direction and stability of periodic solution of the system are obtained, and it is proved that the two normal forms of deriving by using two methods are equivalent for this specific model.

The rest of the paper is organized as follows. In Section 2, the stability analysis of equilibria and Hopf bifurcation analysis of the system (1.1) are mainly carried out. In Section 3, we derive the normal form of Hopf bifurcation by using multiple time scales and center manifold reduction methods respectively, and the equivalent of the two normal forms is also considered in this section. Bifurcation analysis and numerical simulation are presented in Section 4. Finally, conclusions are drawn in the final part.

## 2. Stability of equilibrium and existence of Hopf bifurcation

In this part, we will discuss the stability of equilibria and the existence of Hopf bifurcation of system (1.1). In order to make the research more convenient, let $\dot{x} = y$, then model (1.1) can be shown as follows:

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -ax(t-\tau) + qx^3 - vy^3 - vy^2 - uy. \end{cases} \tag{2.1}$$

System (2.1) has three equilibria:

$$E_0 = (0,0), E_1 = (\sqrt{\frac{a}{q}}, 0), E_2 = (-\sqrt{\frac{a}{q}}, 0).$$

The characteristic equation of system (2.1), evaluated at $E_0$, is given as follows:

$$\lambda^2 + u\lambda + a\mathrm{e}^{-\lambda\tau} = 0. \tag{2.2}$$

When $\tau = 0$, the characteristic equation at $E_0$ is

$$\lambda^2 + u\lambda + a = 0.$$

Obviously, it has two characteristic values with negative real parts due to $a > 0$ and $u > 0$. Thus, equilibrium $E_0$ of system (2.1) is local asymptotically stable for $\tau = 0$.

The characteristic equation of system (2.1), evaluated at $E_{1,2}$ is given by

$$\lambda^2 + u\lambda - 3a + a\mathrm{e}^{-\lambda\tau} = 0. \tag{2.3}$$

When $\tau = 0$, it becomes

$$\lambda^2 + u\lambda - 2a = 0.$$

Obviously, it has one eigenvalue with positive real part due to $a > 0$ and $u > 0$. thus, equilibria $E_{1,2}$ of system (2.1) are unstable for $\tau = 0$.

Next, we consider the existence of bifurcation periodic solutions near the equilibrium $E_0$ for $\tau > 0$. Let $\lambda = \mathrm{i}\omega_1(\mathrm{i}^2 = -1, \omega_1 > 0)$ be a root of characteristic equation (2.2), substituting the root into (2.2) and separating the real and imaginary parts, we obtain:

$$\omega_1^2 = a\cos(\omega_1\tau), \quad u\omega_1 = a\sin(\omega_1\tau),$$

that is,

$$\sin(\omega_1\tau) = \frac{u\omega_1}{a}, \quad \cos(\omega_1\tau) = \frac{\omega_1^2}{a}.$$

Let $z_1 = \omega_1^2$, thus

$$z_1^2 + u^2 z_1 - a^2 = 0,$$

then

$$\omega_1 = \sqrt{\frac{-u^2 + \sqrt{u^4 + 4a^2}}{2}}. \tag{2.4}$$

Due to $\frac{u\omega_1}{a} > 0$, then

$$\tau_1^{(j)} = \frac{1}{\omega_1}(\arccos(\frac{\omega_1^2}{a}) + 2j\pi), \ j = 0, 1, 2, \cdots. \tag{2.5}$$

When $\tau = \tau_1^{(j)}$, characteristic equation (2.2) have a pair of pure imaginary roots $\lambda = \pm\mathrm{i}\omega_1$. Calculating the transversality conditions, we have:

$$\mathrm{Re}(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau})^{-1}_{\tau=\tau_1^{(j)}} = \frac{2\omega_1^2 + u^2}{a^2} > 0, \quad j = 0, 1, 2, \cdots. \tag{2.6}$$

Next, we consider the possible bifurcation periodic solution near equilibria $E_{1,2}$. Suppose $\lambda = \mathrm{i}\omega_2(\mathrm{i}^2 = -1, \omega_2 > 0)$ is a root of characteristic equation (2.3), substituting it into (2.3) and separating the real and imaginary parts yields

$$\omega_2^2 + 3a = a\cos(\omega_2\tau), \quad u\omega_2 = a\sin(\omega_2\tau).$$

Then, we have

$$\sin(\omega_2\tau) = \frac{u\omega_2}{a}, \quad \cos(\omega_2\tau) = \frac{\omega_2^2 + 3a}{a}.$$

Let $z_2 = \omega_2^2$, we have

$$z_2^2 + (u^2 + 6a)z_2 + 8a^2 = 0.$$

It has no positive root due to $-u^2 - 6a < 0, 8a^2 > 0$, namely, equation (2.3) has no pure imaginary root. Thus, equation (2.1) does not occur Hopf bifurcation near equilibria $E_{1,2}$.

Combining above results, we obtain the following theorem.

**Theorem 2.1.** *System (2.1) undergoes a Hopf bifurcation at trivial equilibrium $E_0$ when $\tau = \tau_1^{(j)}(j = 0, 1, 2, \cdots)$, where $\tau_1^{(j)}$ is given by (2.5). The trivial equilibrium $E_0$ is local asymptotically stable for $\tau \in [0, \tau_1^{(0)})$, and unstable for $\tau > \tau_1^{(0)}$. System (2.1) does not undergo a Hopf bifurcation near equilibria $E_{1,2}$, and $E_{1,2}$ are always unstable for $\tau \geq 0$.*

# 3. Normal form in Hopf bifurcation

In order to normalize the delay, we first re-scale the time $t$ by using $t \mapsto t/\tau$, equation (2.1) is transformed into:

$$\begin{cases} \dfrac{dx}{dt} = \tau y(t), \\ \dfrac{dy}{dt} = -\tau[ax(t-1) + uy(t) + vy^2(t) - qx^3(t) + vy^3(t)]. \end{cases} \tag{3.1}$$

Equation (3.1) also can be written as:

$$\dot{Z}(t) = \tau N_1 Z(t) + \tau N_2 Z(t-1) + \tau F(Z(t), Z(t-1)), \tag{3.2}$$

where

$$Z(t) = (x(t), y(t))^{\mathrm{T}}, \quad Z(t-1) = (x(t-1), y(t-1))^{\mathrm{T}},$$
$$F(Z(t), Z(t-1)) = (0, qx^3(t) - vy^2(t) - vy^3(t))^{\mathrm{T}},$$

and

$$N_1 = \begin{pmatrix} 0 & 1 \\ 0 & -u \end{pmatrix}, N_2 = \begin{pmatrix} 0 & 0 \\ -a & 0 \end{pmatrix}.$$

In this section, we treat time delay $\tau$ as a bifurcation parameter, and denote the critical value $\tau = \tau_c$, at which system (3.2) undergoes a Hopf bifurcation at equilibrium $E_0$. First, we derive the normal form in the Hopf bifurcation of system (3.2) by using multiple time scales method. Then, we derive the normal form by using the center manifold reduction method. Finally, we compare the equivalence of the two methods associated with system (3.2).

## 3.1. Multiple time scales method

Let $h$ be eigenvector corresponding to eigenvalue $\lambda = \mathrm{i}\omega\tau$ of equation (3.2), and $h^*$ be the eigenvector corresponding to eigenvalue $\lambda = -\mathrm{i}\omega\tau$ of adjoint matrix of equation (3.2), satisfying

$$\langle h^*, h\rangle = \bar{h^*}^{\mathrm{T}} h = 1.$$

By calculating, it can be found that:

$$h = (1, \mathrm{i}\omega)^{\mathrm{T}}, h^* = d(u - \mathrm{i}w, 1)^{\mathrm{T}}, \tag{3.3}$$

where $d = \frac{1}{u - 2\mathrm{i}\omega}$.

We treat the delay $\tau$ as the bifurcation parameter, let $\tau = \tau_c + \varepsilon\mu$, where $\tau_c = \tau_1^{(j)} (j = 0, 1, 2, \cdots)$ is the Hopf bifurcation critical value, see formula (2.5) for $\tau_1^{(j)}$, $\mu$ is perturbation parameter, $\varepsilon$ is dimensionless scale parameter. Suppose system (3.2) undergoes a Hopf bifurcation from the trivial equilibrium at the critical point $\tau = \tau_c$, and then, by the MTS method, the solution of (3.2) is assumed as follows:

$$Z(t) = Z(T_0, T_1, T_2, \cdots) = \sum_{k=1}^{+\infty} \varepsilon^k Z_k(T_0, T_1, T_2, \cdots), \tag{3.4}$$

where

$$Z(T_0, T_1, T_2, \cdots) = (x(T_0, T_1, T_2, \cdots), y(T_0, T_1, T_2, \cdots))^{\mathrm{T}},$$
$$Z_k(T_0, T_1, T_2, \cdots) = (x_k(T_0, T_1, T_2, \cdots), y_k(T_0, T_1, T_2, \cdots))^{\mathrm{T}},$$

and the derivative with regard to $t$ is transformed into

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial T_0} + \varepsilon \frac{\partial}{\partial T_1} + \varepsilon^2 \frac{\partial}{\partial T_2} + \cdots = D_0 + \varepsilon D_1 + \varepsilon^2 D_2 + \cdots,$$

where $D_i$ is differential operator, and

$$D_i = \frac{\partial}{\partial T_i}, (i = 0, 1, 2, \cdots).$$

From (3.2), we have

$$\dot{Z}(t) = \varepsilon D_0 Z_1 + \varepsilon^2 D_1 Z_1 + \varepsilon^3 D_2 Z_1 + \varepsilon^2 D_0 Z_2 + \varepsilon^3 D_1 Z_2 + \varepsilon^3 D_0 Z_3 + \cdots. \quad (3.5)$$

We expand $x(T_0 - 1, \varepsilon(T_0 - 1), \varepsilon^2(T_0 - 1), \cdots)$ at $x(T_0 - 1, T_1, T_2, \cdots)$ by the Taylor expansion, we get

$$x(t-1) = \varepsilon x_{1,\tau_c} + \varepsilon^2 x_{2,\tau_c} + \varepsilon^3 x_{3,\tau_c} - \varepsilon^2 D_1 x_{1,\tau_c} - \varepsilon^3 D_2 x_{1,\tau_c} - \varepsilon^3 D_1 x_{2,\tau_c} + \cdots, \quad (3.6)$$

where $x_{j,\tau_c} = x_j(T_0 - 1, T_1, T_2, \cdots), j = 1, 2, \cdots$.

Substituting formulas (3.4)-(3.6) into equation (3.2), then comparing the coefficients of $\varepsilon$, $\varepsilon^2$ and $\varepsilon^3$ on both sides of the equation respectively. Then, we obtain the following expressions:

$$\begin{aligned} &D_0 x_1 - \tau_c y_1 = 0, \\ &D_0 y_1 + \tau_c u y_1 + \tau_c a x_{1,\tau_c} = 0. \end{aligned} \quad (3.7)$$

$$\begin{aligned} &D_0 x_2 - \tau_c y_2 = -D_1 x_1 + \mu y_1, \\ &D_0 y_2 + \tau_c u y_2 + \tau_c a x_{2,\tau_c} = \tau_c a D_1 x_{1,\tau_c} - D_1 y_1 - \mu a x_{1,\tau_c} - \mu u y_1 - \tau_c v y_1^2. \end{aligned} \quad (3.8)$$

$$\begin{aligned} D_0 x_3 - \tau_c y_3 &= -D_1 x_2 - D_2 x_1 + \mu y_2, \\ D_0 y_3 + \tau_c u y_3 + \tau_c a x_{3,\tau_c} &= a D_1(\mu x_{1,\tau_c} + \tau_c x_{2,\tau_c}) - D_1 y_2 + a\tau_c D_2 x_{1,\tau_c} \\ &\quad - D_2 y_1 + \tau_c q x_1^3 - \mu a x_{2,\tau_c} - \tau_c v y_1^3 \\ &\quad - 2\tau_c v y_1 y_2 - \mu(v y_1^2 + u y_2). \end{aligned} \quad (3.9)$$

Equation (3.7) has the solution with following form,

$$Z_1 = G h \mathrm{e}^{\mathrm{i}\omega\tau_c T_0} + \bar{G}\bar{h}\mathrm{e}^{-\mathrm{i}\omega\tau_c T_0}, \quad (3.10)$$

where $h$ is given by (3.3). Equation (3.8) is a linear non-homogeneous equation, and the non-homogeneous equation has a solution if and only if a solvability condition is satisfied. That is, the right-hand side of (3.8) be orthogonal to every solution of the adjoint homogeneous problem. Thus, substituting solution (3.10) into the right part of equation (3.8), and the coefficient vector of $\mathrm{e}^{\mathrm{i}\omega\tau_c T_0}$ is noted as $m_1$, by $\langle h^*, m_1\rangle = 0$, we can solve $\frac{\partial G}{\partial T_1}$, namely,

$$\frac{\partial G}{\partial T_1} = M\mu G, \quad (3.11)$$

where

$$M = b(-\omega^2 - a\mathrm{e}^{-\mathrm{i}\omega\tau_c}),$$

with

$$b = (u + 2\mathrm{i}\omega - \tau_c a\mathrm{e}^{-\mathrm{i}\omega\tau_c})^{-1}.$$

Solving equation (3.8), we obtain its solutions with following form:

$$\begin{aligned} x_2 &= f_1\mathrm{e}^{\mathrm{i}\omega\tau_c T_0} + g_1\mathrm{e}^{2\mathrm{i}\omega\tau_c T_0} + c.c. + l_1, \\ y_2 &= f_2\mathrm{e}^{\mathrm{i}\omega\tau_c T_0} + g_2\mathrm{e}^{2\mathrm{i}\omega\tau_c T_0} + c.c. + l_2, \end{aligned} \tag{3.12}$$

where $c.c.$ stands for the complex conjugate of the preceding terms, then substituting solutions (3.12) into (3.8), and we get

$$\begin{aligned} &f_1 = \frac{V + (\mathrm{i}\omega - M)J}{\mathrm{i}\omega\tau_c J}\mu G, \quad f_2 = \frac{V}{\tau_c J}\mu G, \quad g_1 = v\omega^2 SG^2, \quad g_2 = 2\mathrm{i}\omega^3 vG^2 S, \\ &l_1 = \frac{2v\omega^2}{a}G\bar{G}, \quad l_2 = 0, \end{aligned}$$

with

$$\begin{aligned} &J = -\omega^2 + \mathrm{i}\omega u + a\mathrm{e}^{-\mathrm{i}\omega\tau_c}, \quad M = b(-\omega^2 - a\mathrm{e}^{-\mathrm{i}\omega\tau_c}), \quad b = (u + 2\mathrm{i}\omega - \tau_c a\mathrm{e}^{-\mathrm{i}\omega\tau_c})^{-1}, \\ &V = \mathrm{e}^{-\mathrm{i}\omega\tau_c}(aM - 2a\mathrm{i}\omega + a\mathrm{i}\omega\tau_c M) + \omega^2(M + u), \quad S = \frac{1}{-4\omega^2 + 2\mathrm{i}\omega u + a\mathrm{e}^{-2\mathrm{i}\omega\tau_c}}. \end{aligned}$$

Next, substituting solutions (3.10) and (3.12) into (3.9), and the coefficient vector of $\mathrm{e}^{\mathrm{i}\omega\tau_c T_0}$ is denoted as $m_2$, by solvability condition, we have $\langle h^*, m_2\rangle = 0$. Note that $\mu$ is disturbance parameter, and $\mu^2$ has little influence for small unfolding parameter, thus, we ignore the $\mu^2 G$ term, then $\frac{\partial G}{\partial T_2}$ can be solved to yield

$$\frac{\partial G}{\partial T_2} = HG^2\bar{G}, \tag{3.13}$$

where

$$H = b\tau_c(3q - 3v\mathrm{i}\omega^3 - 4v^2\omega^4 S),$$

with

$$b = (u + 2\mathrm{i}\omega - \tau_c a\mathrm{e}^{-\mathrm{i}\omega\tau_c})^{-1}, \quad S = \frac{1}{-4\omega^2 + 2\mathrm{i}\omega u + a\mathrm{e}^{-2\mathrm{i}\omega\tau_c}}.$$

Let $G \to G/\varepsilon$, we obtain the normal form of Hopf bifurcation of system (3.2) truncated at the cubic order terms:

$$\dot{G} = M\mu G + HG^2\bar{G}, \tag{3.14}$$

where

$$M = b(-\omega^2 - a\mathrm{e}^{-\mathrm{i}\omega\tau_c}), \quad H = b\tau_c(3q - 3v\mathrm{i}\omega^3 - 4v^2\omega^4 S),$$

with

$$b = (u + 2\mathrm{i}\omega - \tau_c a\mathrm{e}^{-\mathrm{i}\omega\tau_c})^{-1}, \quad S = \frac{1}{-4\omega^2 + 2\mathrm{i}\omega u + a\mathrm{e}^{-2\mathrm{i}\omega\tau_c}}.$$

## 3.2. Center manifold reduction method

In this subsection, we will use center manifold reduction method to derive the normal form of Hopf bifurcation of system (3.2) near Hopf bifurcation critical value $\tau_c$ when $\lambda = \mathrm{i}\omega\tau$. Let $\tau = \tau_c + \mu$, $\mu$ is a bifurcation parameter, system (3.2) can be written as

$$\dot{X}(t) = L(\mu)X_t + F(\mu, X_t), \tag{3.15}$$

where

$$L(\mu)X_t = \begin{pmatrix} (\tau_c+\mu)y_t(0) \\ (\tau_c+\mu)(-ax_t(-1) - uy_t(0)) \end{pmatrix},$$
$$F(\mu, X_t) = \begin{pmatrix} 0 \\ (\tau_c+\mu)(qx_t^3(0) - vy_t^3(0) - vy_t^2(0)) \end{pmatrix}.$$

Choose

$$\eta(\theta) = \begin{cases} \tau_c N_1, & \theta = 0, \\ 0, & \theta \in (-1, 0), \\ -\tau_c N_2, & \theta = -1, \end{cases}$$

where

$$N_1 = \begin{pmatrix} 0 & 1 \\ 0 & -u \end{pmatrix}, N_2 = \begin{pmatrix} 0 & 0 \\ -a & 0 \end{pmatrix}.$$

Then, the linearized equation of (3.15) is

$$\dot{X}(t) = L_0 X_t,$$

where $L_0\phi = \int_{-1}^{0} d\eta(\theta)\phi(\theta)$, $\phi \in \mathrm{C} = \mathrm{C}([-1,0], \ \mathrm{R}^2)$, and the bilinear form on $\mathrm{C}^* \times \mathrm{C}$ ($*$ stands for adjoint) is

$$\langle \psi(s), \phi(\theta) \rangle = \psi(0)\phi(0) - \int_{-1}^{0}\int_{0}^{\theta} \psi(\xi - \theta) d\eta(\theta)\phi(\xi) d\xi,$$

where $\phi \in C$, $\psi \in C^*$. Next, the phase space C is decomposed by $\Lambda = \{\pm \mathrm{i}\omega\tau_c\}$ as $\mathrm{C} = P \oplus Q$, where $Q = \{\phi \in \mathrm{C} : \ (\psi, \phi) = 0, \text{for all } \psi \in P^*\}$, and the bases for $P$ and its adjoint $P^*$ are given respectively by

$$\Phi(\theta) = \begin{pmatrix} \mathrm{e}^{\mathrm{i}\omega\tau_c\theta} & \mathrm{e}^{-\mathrm{i}\omega\tau_c\theta} \\ \mathrm{i}\omega\mathrm{e}^{\mathrm{i}\omega\tau_c\theta} & -\mathrm{i}\omega\mathrm{e}^{-\mathrm{i}\omega\tau_c\theta} \end{pmatrix}, \quad \Psi(s) = \begin{pmatrix} b(\mathrm{i}\omega+u)\mathrm{e}^{-\mathrm{i}\omega\tau_c s} & b\mathrm{e}^{-\mathrm{i}\omega\tau_c s} \\ \bar{b}(-\mathrm{i}\omega+u)\mathrm{e}^{\mathrm{i}\omega\tau_c s} & \bar{b}\mathrm{e}^{\mathrm{i}\omega\tau_c s} \end{pmatrix},$$

with $b = (u + 2\mathrm{i}\omega - \tau_c a\mathrm{e}^{-\mathrm{i}\omega\tau_c})^{-1}$, and they satisfy

$$(\Psi, \Phi) = I, \dot{\Phi} = \Phi B, -\dot{\Psi} = B\Psi,$$

where

$$B = \mathrm{diag}(\mathrm{i}\tau_c\omega, -\mathrm{i}\tau_c\omega).$$

Now, we consider the enlarged phase space BC of functions from $[-1,0]$ to $\mathrm{R}^2$, which are continuous on $[-1,0)$ with a possible jump discontinuity at zero. This space can be identified as $\mathrm{C}\times\mathrm{R}^2$. Thus, its elements can be written in the form $\psi=\phi+X_0c$, where $\phi\in\mathrm{C}$, $c\in\mathrm{R}^2$ and $X_0$ is a $2\times 2$ matrix-valued function, defined by $X_0(\theta)=0$ for $\theta\in[-1,0)$ and $X_0(0)=\mathrm{I}$. In the BC, (3.15) becomes an abstract ordinary differential equation,

$$\dot{w}=\mathcal{A}w+X_0\tilde{F}(w,\mu), \tag{3.16}$$

where $w\in\mathrm{C}$, and $\mathcal{A}$ is defined by

$$\mathcal{A}:\ \mathrm{C}^1\to\mathrm{BC},\quad \mathcal{A}w=\dot{w}+X_0[L_0w-\dot{w}(0)],$$

and

$$\tilde{F}(w,\mu)=[L(\mu)-L_0]w+F(w,\mu).$$

By the continuous projection $\pi:\ \mathrm{BC}\mapsto\mathrm{P}$, $\pi(\phi+X_0c)=\Phi[(\Psi,\phi)+\Psi(0)c]$, we can decompose the enlarged phase space by $\Lambda=\{\pm\mathrm{i}\omega\tau_c\}$ as $\mathrm{BC}=\mathrm{P}\oplus\mathrm{Ker}\pi$, where $\mathrm{Ker}\pi=\{\phi+X_0c:\ \pi(\phi+X_0c)=0\}$, denoting the Kernel under the projection $\pi$. Let $\eta=(\eta_1,\bar{\eta}_1)^{\mathrm{T}}$, $v_t\in Q^1:=Q\cap\mathrm{C}^1\subset\mathrm{Ker}\pi$, and $\mathcal{A}_{Q^1}$ is the restriction of $\mathcal{A}$ as an operator from $Q^1$ to the Banach space $\mathrm{Ker}\pi$. Further, denote $w_t=\Phi\eta+v_t$. Then, equation (3.16) is decomposed as

$$\begin{cases}\dot{\eta}=B\eta+\Psi(0)\tilde{F}(\Phi\eta+v_t,\mu),\\ \dfrac{\mathrm{d}v_t}{\mathrm{d}t}=\mathcal{A}_{Q^1}v_t+(\mathrm{I}-\pi)X_0\tilde{F}(\Phi\eta+v_t,\mu),\end{cases} \tag{3.17}$$

where

$$B=\mathrm{diag}(\mathrm{i}\tau_c\omega,-\mathrm{i}\tau_c\omega).$$

Next, let $M_2^1$ denote the operator defined in $V_2^3(\mathrm{R}^2\times\mathrm{Ker}\pi)$, with

$$M_2^1:V_2^3(\mathrm{R}^2)\mapsto V_2^3(\mathrm{R}^2),\quad (M_2^1p)(\eta,\mu)=D_\eta p(\eta,\mu)B\eta-Bp(\eta,\mu),$$

where $V_2^3(\mathrm{R}^2)$ represents the linear space of the second order homogeneous polynomials in three variables $(\eta_1,\bar{\eta}_1,\mu)$ with coefficients in $\mathrm{R}^2$. Then, it is easy to verify that one may choose the decomposition $V_2^3(\mathrm{R}^2)=\mathrm{Im}(M_2^1)\oplus\mathrm{Im}(M_2^1)^c$ with complementary space $\mathrm{Im}(M_2^1)^c$ spanned by the elements $\mu\eta_1e_1$ and $\mu\bar{\eta}_1e_2$, where $e_i$ $(i=1,2)$ are unit vectors.

Consequently, the normal form of (3.17) on the center manifold associated with the origin equilibrium near $\mu=0$ has the form

$$\frac{\mathrm{d}\eta}{\mathrm{d}t}=B\eta+\frac{1}{2}g_2^1(\eta,0,\mu)+\text{h.o.t.},$$

where $g_2^1$ is the function giving the quadratic terms in $(\eta_1,\bar{\eta}_1,\mu)$ for $v_t=0$, and is determined by $g_2^1(\eta,0,\mu)=\mathrm{Proj}_{(\mathrm{Im}(M_2^1))^c}\times f_2^1(\eta,0,\mu)$, where $f_2^1(\eta,0,\mu)$ is the function giving the quadratic terms in $(\eta,\mu)$ for $v_t=0$ defined by the first equation of (3.17). Thus, the normal form, truncated at the quadratic order terms, is given by

$$\dot{\eta}_1=\mathrm{i}\omega\tau_c\eta_1+M^*\mu\eta_1+h.o.t., \tag{3.18}$$

where

$$M^* = b(-\omega^2 - a\mathrm{e}^{-\mathrm{i}\omega\tau_c}),$$

with

$$b = (u + 2\mathrm{i}\omega - \tau_c a\mathrm{e}^{-\mathrm{i}\omega\tau_c})^{-1}.$$

To find the normal form up to third order, similarly, let $M_3^1$ denote the operator defined in $V_3^4(\mathrm{R}^4 \times \mathrm{Ker}\pi)$, with

$$M_3^1 : V_3^2(\mathrm{R}^2) \mapsto V_3^2(\mathrm{R}^2), \quad (M_3^1 p)(\eta, \mu) = D_\eta p(\eta, \mu)B\eta - Bp(\eta, \mu),$$

where $V_3^2(\mathrm{R}^2)$ denotes the linear space of the third-order homogeneous polynomials in two variables $(\eta_1, \bar{\eta}_1)$ with coefficients in $\mathrm{R}^2$. Then, one may choose the decomposition $V_3^2(\mathrm{R}^2) = \mathrm{Im}(M_3^1) \oplus \mathrm{Im}(M_3^1)^c$ with complementary space $\mathrm{Im}(M_3^1)^c$ spanned by the elements $\eta_1^2\bar{\eta}_1 e_1$ and $\eta_1\bar{\eta}_1^2 e_2$, where $e_i$ $(i = 1, 2)$ are unit vectors.

Then, we obtain

$$\frac{\mathrm{d}\eta}{\mathrm{d}t} = B\eta + \frac{1}{2!}g_2^1(\eta, 0, \mu) + \frac{1}{3!}\tilde{f}_3^1(\eta, 0, \mu) + \text{h.o.t.}, \tag{3.19}$$

where

$$\begin{aligned}\tilde{f}_3^1(\eta, 0, \mu) =& f_3^1(\eta, 0, \mu) + \frac{3}{2}((D_{(\eta, v_t)} f_2^1(\eta, v_t, \mu))_{v_t=0} U_2(\eta) \\ &- D_\eta U_2^1(\eta) g_2^1(\eta, 0, \mu)).\end{aligned}$$

Notice that

$$D_\eta h(\eta)B\eta - \dot{h}(\eta) + X_0(\dot{h}(\eta)(0) - L_0(h(\eta))) = (X_0 - \Phi\Psi(0))f_2^1,$$

where

$$X_0(\theta) = \begin{cases} 0, & \theta \in (-1, 0), \\ I, & \theta = 0, \end{cases} \quad (I - \pi)X_0 = \begin{cases} -\Phi\Psi(0), & -1 \le \theta < 0, \\ I - \Phi(0)\Psi(0), & \theta = 0, \end{cases}$$

we have

$$\begin{aligned} h_{20}^{(1)}(\theta) &= \frac{1}{2\mathrm{i}\omega\tau_c} C_1 \mathrm{e}^{2\mathrm{i}\omega\tau_c\theta} + \mathrm{i}\omega v(b\mathrm{e}^{\mathrm{i}\omega\tau_c\theta} + \bar{b}\mathrm{e}^{-\mathrm{i}\omega\tau_c\theta}), \\ h_{20}^{(2)}(\theta) &= \frac{1}{2\mathrm{i}\omega\tau_c} C_2 \mathrm{e}^{2\mathrm{i}\omega\tau_c\theta} - v\omega^2(b\mathrm{e}^{\mathrm{i}\omega\tau_c\theta} - \bar{b}\mathrm{e}^{-\mathrm{i}\omega\tau_c\theta}), \\ h_{11}^{(1)}(\theta) &= -4\mathrm{i}\omega v(-b\mathrm{e}^{\mathrm{i}\omega\tau_c\theta} + \bar{b}\mathrm{e}^{-\mathrm{i}\omega\tau_c\theta}) + C_3, \\ h_{11}^{(2)}(\theta) &= -4\omega^2 v(b\mathrm{e}^{\mathrm{i}\omega\tau_c\theta} + \bar{b}\mathrm{e}^{-\mathrm{i}\omega\tau_c\theta}), \end{aligned} \tag{3.20}$$

where

$$\begin{aligned} C_1 &= \frac{4\mathrm{i}\omega^3\tau_c v(1 - \mathrm{i}\omega b + \mathrm{i}\omega\bar{b}) + 2a\omega^2\tau_c v(b\mathrm{e}^{-\mathrm{i}\omega\tau_c} + \bar{b}\mathrm{e}^{\mathrm{i}\omega\tau_c})}{-4\omega^2 + 2\mathrm{i}\omega u + a\mathrm{e}^{-2\mathrm{i}\omega\tau_c}}, \\ C_2 &= 2\mathrm{i}\omega C_1 + 2\omega^2\tau_c v(\mathrm{i}\omega b - \mathrm{i}\omega\bar{b}), \\ C_3 &= \frac{4\omega^2 v}{a}[u(b + \bar{b}) + \mathrm{i}\omega(b - \bar{b}) - 1] + 4\mathrm{i}\omega v(-b\mathrm{e}^{-\mathrm{i}\omega\tau_c} + \bar{b}\mathrm{e}^{\mathrm{i}\omega\tau_c}). \end{aligned}$$

Due to

$$\frac{1}{3!}g_3^1(\eta, 0, 0) = \frac{1}{3!}(I - P_{I,3}^1)\tilde{f}_3^1(\eta, 0, 0),$$

finally, the normal form of Hopf bifurcation on the center manifold arising from (3.19) becomes

$$\dot{\eta}_1 = \mathrm{i}\omega\tau_c\eta_1 + M^*\mu\eta_1 + H^*\eta_1^2\bar{\eta}_1, \tag{3.21}$$

where

$$M^* = b(-\omega^2 - a\mathrm{e}^{-\mathrm{i}\omega\tau_c}),\ \ H^* = b\tau_c[3q - 3\mathrm{i}\omega^3 v - 2\mathrm{i}\omega v(h_{11}^{(2)}(0) - h_{20}^{(2)}(0)) + 9\mathrm{i}v^2\omega^3(b - \bar{b})],$$

with

$$\begin{aligned}
h_{20}^{(2)}(0) &= \frac{4\mathrm{i}\omega^3 v(1 - \mathrm{i}\omega b + \mathrm{i}\omega\bar{b})}{-4\omega^2 + 2\mathrm{i}\omega u + a\mathrm{e}^{-2\mathrm{i}\omega\tau_c}} + \frac{2a\omega^2 v(b\mathrm{e}^{-\mathrm{i}\omega\tau_c} + \bar{b}\mathrm{e}^{\mathrm{i}\omega\tau_c})}{-4\omega^2 + 2\mathrm{i}\omega u + a\mathrm{e}^{-2\mathrm{i}\omega\tau_c}},\\
h_{11}^{(2)}(0) &= -4v\omega^2(b + \bar{b}),\\
b &= (u + 2\mathrm{i}\omega - \tau_c a\mathrm{e}^{-\mathrm{i}\omega\tau_c})^{-1}.
\end{aligned}$$

## 3.3. Comparison of the MTS and CMR methods

Equation (3.14) is the normal form derived by using the MTS method, and equation (3.21) is the normal form derived by using the CMR method.

With the polar coordinate: $G = r_1\mathrm{e}^{\mathrm{i}\theta_1}$, substituting that expression into (3.14), we obtain the amplitude equation of (3.14) on the center manifold as

$$\dot{r}_1 = \mathrm{Re}(M)\mu r_1 + \mathrm{Re}(H)r_1^3, \tag{3.22}$$

where $M$ and $H$ are given by (3.14).

With the polar coordinate: $\eta_1 = r_2\mathrm{e}^{\mathrm{i}\theta_2}$, substituting that expression into (3.21), we obtain the amplitude equation of (3.21) on the center manifold as

$$\dot{r}_2 = \mathrm{Re}(M^*)\mu r_2 + \mathrm{Re}(H^*)r_2^3, \tag{3.23}$$

where $M^*$ and $H^*$ are given by (3.21).

Note that $G$ in (3.14), used to represent the normal form in the MTS method, corresponds to $\eta_1$ in (3.21), used to denote the normal form in the CMR method. Next, neglecting the difference in the notations, we consider the equivalence of the two normal forms.

Firstly, the two normal forms are identical up to the second order, that is, $M = M^*$. The coefficients of cubic terms in the above two normal forms are $H$ and $H^*$ respectively. As long as $\mathrm{Re}(H)\mathrm{Re}(H^*) > 0$, equation (3.22) can be transformed into (3.23) by linear transformation $r_1 = \sqrt{\frac{\mathrm{Re}(H^*)}{\mathrm{Re}(H)}}r_2$. Thus, we obtain the following theorem:

**Theorem 3.1.** *Consider the Hopf bifurcation of system (2.1), the polar coordinate normal forms (3.22) and (3.23) derived by MTS and CMR methods respectively, are equivalent as long as* $\mathrm{Re}(H)\mathrm{Re}(H^*) > 0$.

**Remark 3.1.** MTS and CMR methods are two methods for deriving normal form. Thus, the expressions associated with coefficients of cubic terms $H$ and $H^*$ are slightly different. Actually, both of the two normal forms are local equivalent topologically to system (3.2) near equilibrium $E_0$. If there exist some linear transformation such that the two normal forms are equivalent, both of the two normal forms derived by MTS and CMR methods are equivalent. Then, we can determine the stability of Hopf bifurcating periodic solutions and the direction of Hopf bifurcation by using any one of the two normal forms (3.22) and (3.23).

# 4. Bifurcation analysis and numerical simulations

In this section, we first give a bifurcation analysis based on the normal form (3.22) (or (3.23)) of Hopf bifurcation associated with system (2.1), and then present some numerical simulation results.

The nontrivial equilibrium of (3.22) (or (3.23)) corresponds to the periodic solution of system (2.1). In order to analyze the stability of periodic solution of equation (2.1), we can directly discuss the stability of nontrivial equilibrium of system (3.22) (or (3.23)).

For the stability of periodic solutions of Hopf bifurcation of system (2.1), we have the following theorem:

**Theorem 4.1.** *When* $\frac{\mathrm{Re}(M)\mu}{\mathrm{Re}(H)} < 0$ *(or* $\frac{\mathrm{Re}(M^*)\mu}{\mathrm{Re}(H^*)} < 0$*), system (2.1) exists periodic solutions: If* $\mathrm{Re}(M)\mu < 0$ *(or* $\mathrm{Re}(M^*)\mu < 0$*), the periodic solution is unstable; if* $\mathrm{Re}(M)\mu > 0$ *(or* $\mathrm{Re}(M^*)\mu > 0$*), the periodic solution is stable.*

In order to give a more clear bifurcation picture, we consider the actual meaning of parameters and choose

$$a = 0.5,\ q = 0.1,\ v = 0.25,\ u = 0.6,$$

by a simple calculation from (2.4) and (2.5), we obtain

$$\omega = 0.5928, \quad \tau_1^{(0)} = 1.3351.$$

Therefore, from Theorem 2.1, the trivial equilibrium $E_0$ is local asymptotically stable for $\tau \in [0, \tau_1^{(0)}) = [0, 1.3351)$, and unstable for $\tau > \tau_1^{(0)} = 1.3351$.

From expressions (3.14) and (3.21), we have

$$M = M^* = 0.1797 + 0.4374\mathrm{i}, \quad H = -0.1021 - 0.2665\mathrm{i}, \quad H^* = -0.0758 - 0.3682\mathrm{i}.$$

Obviously, $\mathrm{Re}(H)\mathrm{Re}(H^*) > 0$, from Theorem 3.1, the two normal forms are equivalent. By Theorem 4.1, system (2.1) exists stable periodic solutions near the bifurcation critical value for $\mu > 0$. We choose three groups of parameter values: $\tau = 0$, $\tau = 1 \in (0, \tau_1^{(0)})$ and $\tau = 1.5 > \tau_1^{(0)}$, with initial values being all $(0.1, 0.1)$, corresponding to a stable fixed point shown in Figure 1, a stable fixed point depicted in Figure 2 and a stable periodic solution as shown in Figure 3 respectively. It is clear that the numerical simulations agree with the analytical predictions.

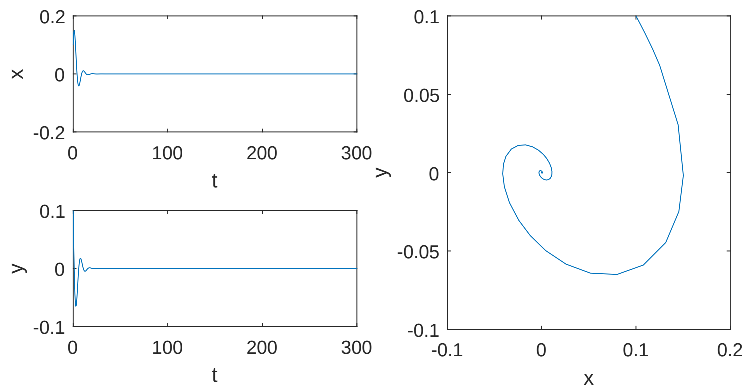


**Figure 1.** Equilibrium point $E_0 = (0, 0)$ of system (2.1) with $\tau = 0$ is locally asymptotically stable.

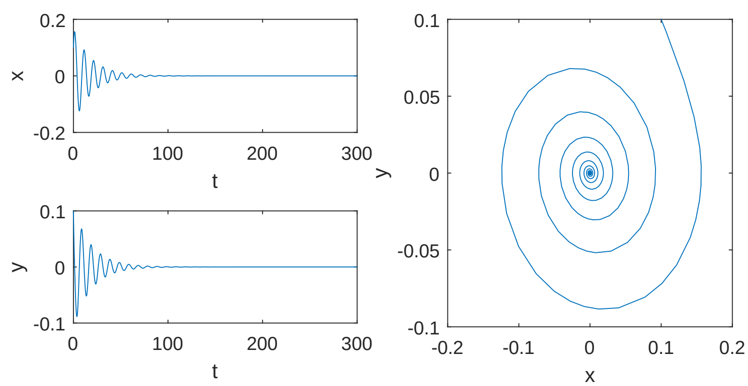


**Figure 2.** Equilibrium point $E_0 = (0, 0)$ of system (2.1) with $\tau = 1$ is locally asymptotically stable.

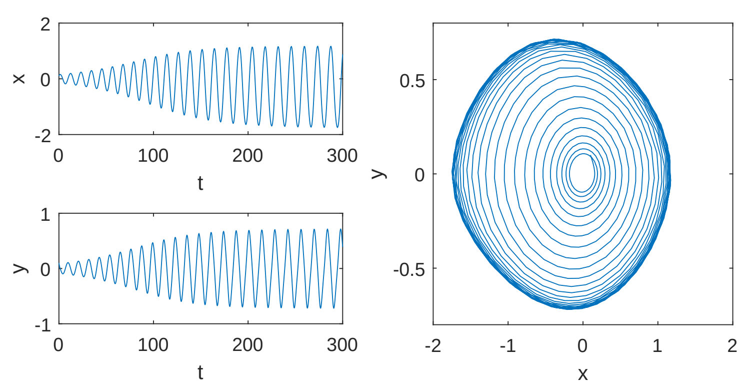


**Figure 3.** System (2.1) with $\tau = 1.5$ has a stable periodic solution.

Next, we choose reasonable area of parameters for further discussion and simulation, and consider the universal applicability associated with the results of Theorem

3.1. First, we fix three variables among of parameters $a$, $q$, $v$ and $u$, and show the value of $\mathrm{Re}(H)\mathrm{Re}(H^*)$ varies with the remaining one parameter (see Figure 4). Figure 4(a) shows $q = 0.1$, $v = 0.25$ and $u = 0.6$, in the domain $a \in (0.35, 0.9)$, and Figure 4(b) shows $a = 0.5$, $v = 0.25$ and $u = 0.6$ in the domain $q \in (0, 0.5)$, and Figure 4(c) shows $a = 0.5$, $q = 0.1$ and $u = 0.6$ in the domain $v \in (0, 0.98)$, and Figure 4(d) shows $a = 0.5$, $q = 0.1$ and $v = 0.25$ in the domain $u \in (0, 0.85)$, obviously, $\mathrm{Re}(H)\mathrm{Re}(H^*)$ are always positive.

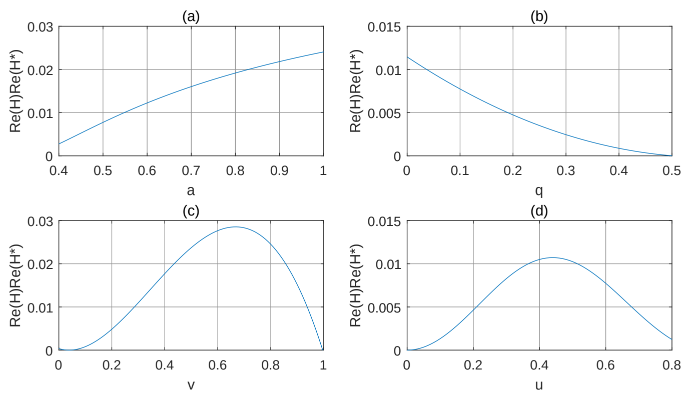


**Figure 4.** The value of $\mathrm{Re}(H)\mathrm{Re}(H^*)$ varies with parameters $a$, $q$, $v$ and $u$ in system (2.1) respectively.

Further, we fix two variables among of parameters $a$, $q$, $v$ and $u$, and show the image of $\mathrm{Re}(H)\mathrm{Re}(H^*)$ varies with the remaining two parameter (see Figures 5-7). Figure 5(a) shows $a = 0.5$, $u = 0.6$, in the domain $q \in (0.1, 0.4), v \in (0.2, 0.5)$, Figure 5(b) shows $a = 0.5$, $q = 0.1$, in the domain $u \in (0, 0.6), v \in (0, 1)$, Figure 6(a) shows $q = 0.1$, $u = 0.6$, in the domain $a \in (0.4, 1), v \in (0.5, 1)$, Figure 6(b) shows $q = 0.1$, $v = 0.25$, in the domain $a \in (0.5, 1), u \in (0.4, 0.7)$, Figure 7(a) shows $a = 0.5$, $v = 0.25$, in the domain $q \in (0, 0.6), u \in (0, 0.5)$, Figure 7(b) shows $u = 0.6$, $v = 0.25$, in the domain $a \in (0.4, 0.43), q \in (0.4, 0.5)$, obviously, $\mathrm{Re}(H)\mathrm{Re}(H^*)$ are always positive.

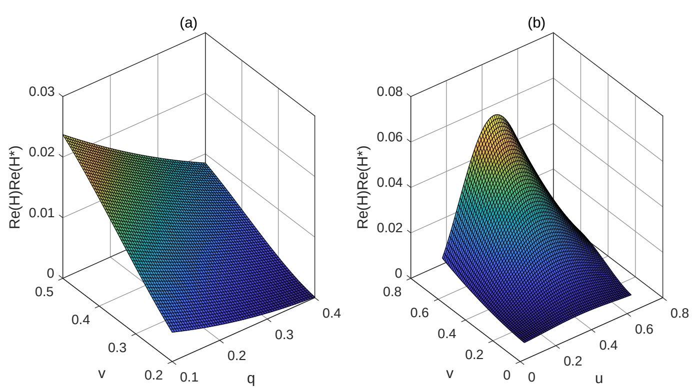


**Figure 5.** (a) The images of $\mathrm{Re}(H)\mathrm{Re}(H^*)$ varies with parameters $q$ and $v$ in system (2.1), (b) The images of $\mathrm{Re}(H)\mathrm{Re}(H^*)$ varies with parameters $u$ and $v$ in system (2.1).

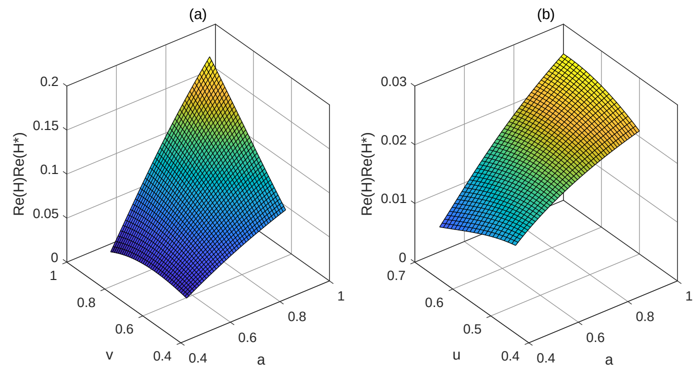


**Figure 6.** (a) The images of $\mathrm{Re}(H)\mathrm{Re}(H^*)$ varies with parameters $a$ and $v$ in system (2.1), (b) The images of $\mathrm{Re}(H)\mathrm{Re}(H^*)$ varies with parameters $a$ and $u$ in system (2.1).

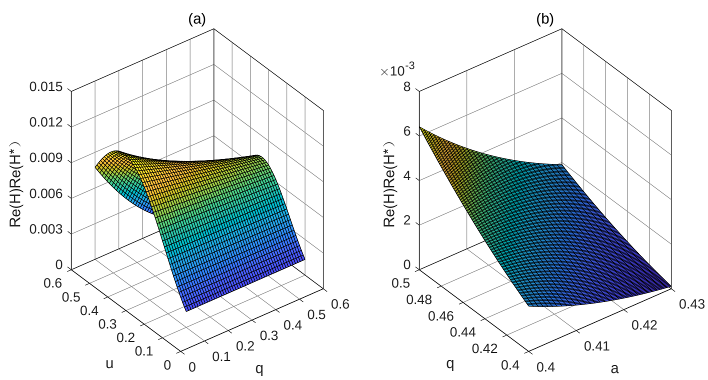


**Figure 7.** (a) The images of $\mathrm{Re}(H)\mathrm{Re}(H^*)$ varies with parameters $q$ and $u$ in system (2.1), (b) The images of $\mathrm{Re}(H)\mathrm{Re}(H^*)$ varies with parameters $a$ and $q$ in system (2.1).

**Remark 4.1.** By a comparison between the MTS and CMR methods, we find that the coefficients of second order terms $\mathrm{Re}(M)$ and $\mathrm{Re}(M^*)$ in the two normal forms (3.22) and (3.23) are identical, while the coefficients of third order terms $\mathrm{Re}(H)$ and $\mathrm{Re}(H^*)$ are slightly different. Figures 4-7 show the images of $\mathrm{Re}(H)\mathrm{Re}(H^*)$ vary with one or two parameters in system (2.1), and $\mathrm{Re}(H)\mathrm{Re}(H^*) > 0$ in these reasonable areas of parameters. Thus, we can introduce a linear transformation such that the two normal forms (3.22) and (3.23) are equivalent from Theorem 3.1. Namely, the two normal forms of Hopf bifurcation in system (2.1), derived by using the multiple time scales and center manifold reduction methods, are equivalent.

**Remark 4.2.** As far as we know, there is no paper to study the equivalence of MTS and CMR methods for systems with second-order terms. The feature of this paper is to discuss the equivalence of the two methods when the system have second-order terms. Some scholars have proved the equivalence of MTS and CMR, but the models without second-order terms. For example, Nayfeh [12] used two methods to calculate the Hopf bifurcation normal forms of some systems; Ding et al. [3, 4] also used two methods to calculate the normal forms of Hopf-zero bifurcation and double Hopf bifurcation; Yu et al. [18] gave a general proof, that is, for systems without second-order terms, the normal forms of Hopf bifurcation obtained by the two methods must be equivalent. For general delayed differential systems, both MTS and CMR methods can calculate the normal form of Hopf bifurcation. The two methods have their own advantages. Obviously, MTS is more convenient, the calculation process is relatively simple, and it is easier for the scholars in the engineering society to understand, but at present, the MTS has not given a general method for calculating the Bogdanov-Takens bifurcation. On the other hand, the CMR can be used to calculate the Bogdanov-Takens bifurcation, but this method requires a lot of calculation and complex process, and also requires a lot of basic knowledge in mathematics major. Thus, researchers can take appropriate methods according to their own needs.

## 5. Conclusions

In this paper, we have studied Hopf bifurcation of a model with a second order term, which is the business cycle system with time delay. We have obtained two normal forms by using multiple time scales and center manifold reduction methods respectively. A comparison between the two methods shows that the two normal forms are equivalent when there exists a second order term in the system. Thus, we can derive the same results associated with Hopf bifurcation direction and the stability of periodic solution of the system (2.1) by using the two normal forms. Moreover, bifurcation analysis near the Hopf bifurcation critical point is given, showing that the system (2.1) may exhibit a stable fixed point and periodic solutions. Numerical simulations are given to verify the analytical predictions. We have also given the images of $\mathrm{Re}(H)\mathrm{Re}(H^*)$ vary with one or two variables in system (2.1), and further verify the equivalence of the normal forms obtained by the two methods from Theorem 3.1 due to $\mathrm{Re}(H)\mathrm{Re}(H^*) > 0$ in reasonable areas of parameters.

We can also see that the MTS method, unlike the CMR method which involves solving differential equations, only involves algebraic manipulations with explicit algebraic formulas. Actually, we can introduce a linear transformation such that the two normal forms are equivalent. Subsequent work is that we will plan to prove the equivalence of two methods for general systems.

## References

[1] M. Chen, R. Wu, B. Liu and L. Chen, *Spatiotemporal dynamics in a ratio-dependent predator-prey model with time delay near the Turing–Hopf bifurcation point*, Communications in Nonlinear Science and Numerical Simulation, 2019, 77, 141–167.

[2] S. L. Das and A. Chatterjee, *Multiple scales without center manifold reductions for delay differential equations near Hopf bifurcations*, Nonlinear Dynamics, 2002, 30, 323–335.

[3] Y. Ding, W. Jiang and P. Yu, *Hopf-zero bifurcation in a generalized Gopalsamy neural network model*, Nonlinear Dynamics, 2012, 70, 1037–1050.

[4] Y. Ding, W. Jiang and P. Yu, *Double Hopf bifurcation in delayed van der Pol-Duffing equation*, International Journal of Bifurcation and Chaos, 2013, 23, Article ID 1350014.

[5] T. Faria and L. T. Magalhães, *Normal form for retarded functional differential equations with parameters and applications to Hopf bifurcation*, Journal of Differential Equations, 1995, 122, 181–200.

[6] T. Faria and L. T. Magalhães, *Normal form for retarded functional differential equations and applications to Bogdanov–Takens singularity*, Journal of Differential Equations, 1998, 122, 201–224.

[7] P. Hao, X. Wang and J. Wei, *Hopf bifurcation analysis of a diffusive single species model with stage structure and strong Allee effect*, Mathematics and Computers in Simulation, 2018, 153, 1–14.

[8] B. D. Hassard, N. D. Kazarinoff and Y. H. Wan, *Theory and Applications of Hopf Bifurcation*, Cambridge University Press, Cambridge, 1981.

[9] W. Jiang, Q. An and J. Shi, *Formulation of the normal form of Turing-Hopf bifurcation in partial functional differential equations*, Journal of Differential Equations, 2020, 268, 6067–6102.

[10] J. Ma and Q. Gao, *Stability and Hopf bifurcations in a business cycle model with delay*, Applied Mathematics and Computation, 2009, 215, 829–834.

[11] A. H. Nayfeh, *Introduction to Perturbation Techniques*, Wiley-Interscience, New York, 1981.

[12] A. H. Nayfeh, *Order reduction of retarded nonlinear systems-the method of multiple scales versus center manifold reduction*, Nonlinear Dynamics, 2008, 51, 483–500.

[13] L. Pei and S. Wang, *Double Hopf bifurcation of differential equation with linearly state-dependent delays via MMS*, Applied Mathematics and Computation, 2019, 341, 256–276.

[14] M. Peng, Z. Zhang, Z. Qu and Q. Bi, *Qualitative analysis in a delayed van der Pol oscillator*, Physica A, 2020, 544, Article ID 123482.

[15] A. Shooshtari and A. A. Pasha Zanoosi, *A multiple times scale solution for nonlinear vibration of mass grounded system*, Applied Mathematical Modelling, 2010, 34, 1918–1929.

[16] S. Wiggins, *Introduction to Applied Nonlinear Dynamical Systems and Chaos*, Springer, New York, 1990.

[17] X. Xu and J. Wei, *Bifurcation analysis of a spruce budworm model with diffusion and physiological structures*, Journal of Differential Equations, 2017, 262, 5206–5230.

[18] P. Yu, Y. Ding and W. Jiang, *Equivalence of the MTS method and CMR method for differential equations associated with semisimple singularity*, International Journal of Bifurcation and Chaos, 2014, 24, Article ID 1450003.