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Iterating a System of Variational-like Inclusion Problems in Banach Spaces

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Abstract In this manuscript, by using $(H, \varphi) - \eta$-monotone operators we study the existence of solution of a system of variational-like inclusion problems in Banach spaces. Further, we suggest an iterative algorithm for finding the approximate solution of this system and discuss the convergence criteria of the sequences generated by the iterative algorithm. The method used in this paper can be considered as an extension of methods for studying the existence of solution for various classes of variational inclusions considered and studied by many authors in Banach spaces.

Keywords System of variational-like inclusion problems, $(H, \varphi) - \eta$-monotone mappings, Resolvent operator, Cocoercive mappings, Banach spaces.

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1. Introduction

A widely studied problem known as variational inclusion problem have many applications in the fields of optimization and control, economics and transportation equilibrium, engineering sciences, etc. Several researchers used different approaches to develop iterative algorithms for solving various classes of variational inequality and variational inclusion problems. In details, we refer [2-5, 9-14, 20, 22-24] and the references therein. Recently Bhat and Shafi, Fang and Huang, Kazmi and Khan, and Lan *et al.* investigated several resolvent operators for generalized operators such as H-monotone [1, 3], H-accretive [4], (P, η)-proximal point [9], (P, η)-accretive [10], (H, η)-monotone [5], (A, η)-accretive [14] and mappings.

From the above results, in this manuscript, we intend to define the resolvent operator associated with $(H, \varphi) - \eta$-monotone mappings in Banach spaces. Using resolvent operator technique, we develop an iterative algorithm for solving the system of variational-like inclusion problems and prove that the sequences generated by the iterative algorithm converge strongly to a solution of the system. The results presented in this paper improve and extend many known results in the literature, see for example [6-8, 15-19, 23].

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2. Resolvent operator and formulation of problem

We need the following definitions and results from the literature.

Let X be a real Banach space equipped with norm $\|.\|$ and $X^\star$ be the topological dual space of X. Let $< .,. >$ be the dual pair between X and $X^\star$ and 2^X be the power set of X.

Definition 2.1 [21]. For $q > 1$, a mapping $J_q : X \to 2^{X^\star}$ is said to be generalized duality mapping, if it is defined by

$$J_q(x) = \{f \in X^\star : \langle x, f\rangle = \|x\|^q, \|x\|^{q-1} = \|f\|\}, \quad \forall x \in X.$$

In particular, J_2 is the usual normalized duality mapping on X, given as

$$J_q(x) = \|x\|^{q-2} J_2(x), \quad \forall x (\neq 0) \in X.$$

Note that if $X \equiv H$, a real Hilbert space, then J_2 becomes the identity mapping on X.

Definition 2.2 [21]. A Banach space X is said to be smooth if, for every $x \in X$ with $\|x\| = 1$, there exists a unique $f \in X^\star$ such that $\|f\| = f(x) = 1$.

The modulus of smoothness of X is the function $\rho_X : [0, \infty) \to [0, \infty)$, defined by

$$\rho_X(\sigma) = \sup\left\{\frac{\|x + y\| + \|x - y\|}{2} - 1 : x, y \in X, \|x\| = 1, \|y\| = \sigma\right\}.$$

Definition 2.3 [21]. A Banach space X is said to be

(i) uniformly smooth if $\lim\limits_{\sigma\to 0} \dfrac{\rho_X(\sigma)}{\sigma} = 0$,

(ii) q-uniformly smooth, for $q > 1$, if there exists a constant $c > 0$ such that $\rho_X(\sigma) \leq c\sigma^q,\ \sigma \in [0, \infty)$.

Note that if X is uniformly smooth, J_q becomes single-valued.

Lemma 2.1. *[21] Let $q > 1$ be a real number and let X be a smooth Banach space. Then, the following statements are equivalent:*

(i) X is q-uniformly smooth.

(ii) There is a constant $c_q > 0$ such that for every $x, y \in X$, the following inequality holds

$$\|x + y\|^q \leq \|x\|^q + q\langle y, J_q(x)\rangle + c_q\|y\|^q.$$

Definition 2.4. Let X be a real Banach space. Let $A : X \to X^\star$, $T : X \times X \to X^\star, \eta : X \times X \to X$ be single-valued mappings and $M : X \times X \to 2^{X^\star}$ be multi-valued mapping. Then

(i) A is said to be monotone, if

$$\Big\langle Ax - Ay, (x - y)\Big\rangle \geq 0, \quad \forall x, y \in X.$$

(ii) A is said to be η-monotone, if

$$\Big\langle Ax - Ay, \eta(x - y)\Big\rangle \geq 0, \quad \forall x, y \in X.$$

(iii) A is said to be strictly η-monotone, if A is η-monotone and equality holds if and only if $x = y$.

(iv) A is said to be δ-strongly η-monotone if there exists a constant $\delta > 0$ such that

$$\Big\langle Ax - Ay, \eta(x, y)\Big\rangle \geq \delta\|x - y\|^2, \quad \forall x, y \in X.$$

(v) A is said to be λ-Lipschitz continous if there exists a constant $\lambda > 0$ such that

$$\Big\|Ax - Ay\Big\| \leq \lambda\|x - y\|, \quad \forall x, y \in X.$$

(vi) $T(.,.)$ is said to be d_1-Lipschitz continuous in the first argument if there exists a constant $d_1 > 0$ such that

$$\Big\|T(x, z) - T(y, z)\Big\| \leq d_1\|x - y\|, \quad \forall x, y, z \in X,$$

In a similar way, we can define the Lipschitz continuity of the mapping $T(.,.)$ in the second argument.

(vii) η is said to be τ-Lipschitz continuous, if there exists a constant $\tau > 0$ such that

$$\Big\|\eta(x, y)\Big\| \leq \tau\|x - y\|, \quad \forall x, y \in X,$$

(viii) M is said to be η-monotone in first argument, if

$$\Big\langle u - v, \eta(x, y)\Big\rangle \geq 0, \ \forall x, y \in X, \ \forall u \in M(x, z), v \in M(y, z), \text{ for each fixed } z \in X,$$

(ix) M is said to be strictly η-monotone, if M is η- monotone and equality holds if and only if $x = y$.

Definition 2.5. Let $H : X \to X^\star, \varphi : X^\star \to X^\star, \eta : X \times X \to X$ be single-valued mappings and let $M : X \times X \to 2^{X^\star}$ be a multi-valued mapping. The mapping M is said to be $(H, \varphi) - \eta-$ monotone, if $\varphi \circ M(., t)$ is η-monotone in first argument and $(H + \varphi \circ M(., t))(X) = X^\star$, for each fixed $t \in X$.

Definition 2.6. Let $T : X \times X \to X^\star, p : X \to X^\star$ be single-valued mappings. Then the mapping T is called

(i) ϵ-p-cocoercive in the second argument if there exists a constant $\epsilon > 0$ such that $\forall x, y, u, v \in X$,

$$\langle T(x, u) - T(x, v), J_q{}^\star(p(u) - p(v))\rangle \geq \epsilon\|T(x, u) - T(x, v)\|^q.$$

where $J_q^\star : X^\star \to X^{\star\star}$ is the generalized mapping on $X^\star$.

Definition 2.7. Let $B, H : X \to X^\star$ and $g : X \to X$ be single-valued mappings. Then the mapping B is said to be λ-strongly accretive with respect to $H(g)$ if there exists a constant $\lambda > 0$, such that

$$\Big\langle Bx - By, J_q{}^\star(H(g(x)) - H(g(y)))\Big\rangle \geq \lambda\|x - y\|^q.$$

where $J_q^\star : X^\star \to X^{\star\star}$ is the generalized mapping on $X^\star$.

Theorem 2.1. *Let X be a real Banach space. Let $\varphi : X^\star \to X^\star, \eta : X \times X \to X$ be single-valued mappings, $H : X \to X^\star$ be a strictly η-monotone mapping and $M : X \times X \to 2^{X^\star}$ be a $(H, \varphi)-\eta-$ monotone mapping. If $\langle u-v, \eta(x, y)\rangle \geq 0$ holds $\forall (y, v) \in Graph(\varphi \circ M(.,t))$, then $(x, u) \in Graph(\varphi \circ M(.,t))$, where $Graph(\varphi \circ M(.,t)) = \{(x, x^\star) \in X \times X^\star : x^\star \in \varphi \circ M(x, t)\}$, for each fixed $t \in X$.*

Proof. Suppose that there exists (x_0, u_0) such that

$$\langle u_0 - v, \eta(x_0, y)\rangle \geq 0, \quad \forall (y, v) \in Graph(\varphi \circ M(.,t)). \tag{2.1}$$

Since M is $(H, \varphi) - \eta-$ monotone, we have $(H + \varphi \circ M(.,t))(X) = X^\star$, so there exists $(x_1, u_1) \in Graph(\varphi \circ M(.,t))$ such that

$$H(x_1) + u_1 = H(x_0) + u_0. \tag{2.2}$$

From (2.1) and (2.2), it follows that

$$\langle u_0 - u_1, \eta(x_0, x_1)\rangle = -\langle H(x_0) - H(x_1), \eta(x_0, x_1)\rangle \geq 0.$$

Since H is strictly η-monotone mapping, it follows that $x_1 = x_0$. Also, from (2.2), we have $u_1 = u_0$. Hence, $(x_0, u_0) \in Graph(\varphi \circ M(.,t))$, that is, $u_0 \in \varphi \circ M(x_0, t)$. □

Theorem 2.2. *Let X be a real Banach space with its dual $X^\star$. Let $\varphi : X^\star \to X^\star, \eta : X \times X \to X$ be single-valued mappings, $H : X \to X^\star$ be a strictly η-monotone mapping and $M : X \times X \to 2^{X^\star}$ be a $(H, \varphi) - \eta-$ monotone mapping. Then, $(H + \varphi \circ M(.,t))^{-1}$, for each fixed $t \in X$ is a single-valued mapping.*

Proof. For any given $x^\star \in X^\star$, let $x, y \in (H + \varphi \circ M(.,t))^{-1}(x^\star)$. This implies

$$x^\star - H(x) \in (\varphi \circ M(x, t)) \;\text{ and }\; x^\star - H(y) \in (\varphi \circ M(y, t)).$$

Since $(\varphi \circ M(.,t))$ is $\eta-$monotone in the first argument, we have

$$\begin{aligned}&\langle x^\star - H(x) - (x^\star - H(y)), \eta(x, y)\rangle \\ &= -\langle H(x) - H(y), \eta(x, y)\rangle \geq 0.\end{aligned}$$

Therefore, it follows that $x = y$, this implies $(H+\varphi \circ M(.,t))^{-1}$ is a single-valued mapping. This completes the proof. □

Definition 2.8. Let X be a real Banach space with its dual $X^\star$. Le $\varphi : X^\star \to X^\star, \eta : X \times X \to X$ be single-valued mappings, $H : X \to X^\star$ be a strictly η-monotone mapping and $M : X \times X \to 2^{X^\star}$ be a $(H, \varphi) - \eta-$ monotone mapping. Then the resolvent operator $R^{H,\eta}_{M,\varphi} : X^\star \to X$ is defined by

$$R^{H,\eta}_{M(.,t),\varphi}(x^\star) = (H + \varphi \circ M(.,t))^{-1}(x^\star), \text{ for each fixed } t \in X \text{ and } \forall x^\star \in X^\star.$$

Theorem 2.3. *Let X be a real Banach space with its dual $X^\star$. Le $\varphi : X^\star \to X^\star$ be single-valued mapping, $\eta : X \times X \to X$ be τ-Lipschitz continuous mapping, $H : X \to X^\star$ be δ-strongly η-monotone mapping and $M : X \times X \to 2^{X^\star}$ be a $(H, \varphi)-\eta-$ monotone mapping. Then, the resolvent operator is Lipschitz continuous with constant $\frac{\tau}{\delta}$, that is,*

$$\left\| R^{H,\eta}_{M(.,t),\varphi}(x^\star) - R^{H,\eta}_{M(.,t),\varphi}(y^\star) \right\| \leq \frac{\tau}{\delta} \|x^\star - y^\star\|,$$

for each fixed $t \in X$ and $x^\star, y^\star \in X^\star$.

Proof. Let $x^\star, y^\star \in X^\star$. It follows that

$$R^{H,\eta}_{M(.,t),\varphi}(x^\star) = (H + \varphi \circ M(.,t))^{-1}(x^\star),$$

$$R^{H,\eta}_{M(.,t),\varphi}(y^\star) = (H + \varphi \circ M(.,t))^{-1}(y^\star),$$

and hence

$$x^\star - H(R^{H,\eta}_{M(.,t),\varphi}(x^\star)) \in \varphi \circ M(R^{H,\eta}_{M(.,t),\varphi}(x^\star), t)$$

$$y^\star - H(R^{H,\eta}_{M(.,t),\varphi}(y^\star)) \in \varphi \circ M(R^{H,\eta}_{M(.,t),\varphi}(y^\star), t).$$

Since $\varphi \circ M(.,t)$ is η-monotone in the first argument, we have

$$\Big\langle x^\star - H(R^{H,\eta}_{M(.,t),\varphi}(x^\star)) - \Big(y^\star - H(R^{H,\eta}_{M(.,t),\varphi}(y^\star))\Big), \eta(R^{H,\eta}_{M(.,t),\varphi}(x^\star), R^{H,\eta}_{M(.,t),\varphi}(y^\star))\Big\rangle \geq 0.$$

It follows that

$$\begin{aligned}
\tau\|x^\star - y^\star\| &\Big\|R^{H,\eta}_{M(.,t),\varphi}(x^\star) - R^{H,\eta}_{M(.,t),\varphi}(y^\star)\Big\| \\
&\geq \|x^\star - y^\star\| \Big\|\eta\Big(R^{H,\eta}_{M(.,t),\varphi}(x^\star), R^{H,\eta}_{M(.,t),\varphi}(y^\star)\Big)\Big\| \\
&\geq \Big\langle x^\star - y^\star, \eta\Big(R^{H,\eta}_{M(.,t),\varphi}(x^\star), R^{H,\eta}_{M(.,t),\varphi}(y^\star)\Big)\Big\rangle \\
&\geq \Big\langle H(R^{H,\eta}_{M(.,t),\varphi}(x^\star)) - H(R^{H,\eta}_{M(.,t),\varphi}(y^\star)), \\
&\quad \eta\Big(R^{H,\eta}_{M(.,t),\varphi}(x^\star), R^{H,\eta}_{M(.,t),\varphi}(y^\star)\Big)\Big\rangle \\
&\geq \delta\Big\|\Big(R^{H,\eta}_{M(.,t),\varphi}(x^\star) - R^{H,\eta}_{M(.,t),\varphi}(y^\star)\Big)\Big\|^2
\end{aligned}$$

Thus,

$$\Big\|R^{H,\eta}_{M(.,t),\varphi}(x^\star) - R^{H,\eta}_{M(.,t),\varphi}(y^\star)\Big\| \leq \frac{\tau}{\delta}\|x^\star - y^\star\|,$$

for each fixed $t \in X$ and $x^\star, y^\star \in X^\star$. □

Now, we formulate our main problem.

For each $i = 1, 2, j \in \{1, 2\} \setminus i$, let X_i be a real Banach space with norm $\|.\|_i$ and let $X_i^\star$ be its dual space with norm $\|.\|_{\star i}$. Let $g_i : X_i \to X_i, H_i : X_i \to X_i^\star, \varphi_i : X_i^\star \to X_i^\star, \eta_i : X_i \times X_i \to X_i$, $P_i : X_i \to X_j, Q_i : X_j \to X_i, S_i : X_j \times X_i \to X_i^\star, p_i : X_j \to X_i^\star$ be single-valued mappings and let $M_i : X_i \times X_j \to 2^{X_i^\star}$ be $(H_i, \varphi_i) - \eta_i-$ monotone mappings respectively. Then, the system of variational-like inclusion problems (SVLIP) is to find $(x, y) \in X_1 \times X_2$ such that

$$\left.\begin{aligned}
&0 \in S_1(P_1(x), Q_1(y)) + p_1(y) + M_1(g_1(x), y), \\
&0 \in S_2(P_2(y), Q_2(x)) + p_2(x) + M_2(g_2(y), x).
\end{aligned}\right\} \tag{2.3}$$

Special Cases:

I. *If in problem (2.3), $X_1 = X_2 \equiv X$, a real Banach space, $S_1 = S_2 = S : X \times X \to X^\star, P_1 = P_2 = P, Q_1 = Q_2 = Q$ and $p_1 = p_2 = p$ be such that $P, Q : X \to X$ and $p : X \to X^\star$ and $M_1 = M_2 = M : X \times X \to 2^{X^\star}$, then problem (2.3) reduces to the following problem: Find $x, y \in X$ such that*

$$0 \in S(P(x), Q(y)) + p(x) + M(g(x), y), \tag{2.4}$$

which is an important generalization of the problem considered and studied by Luo and Huang [15].

II. *If in problem (2.3), $X_1 = X_2 \equiv H$, a real Hilbert space, $S_1 = S_2 \equiv 0$, (a zero mapping) $p_1 = p_2 = p : H \to H$, $M_1 = M_2 = M : H \to 2^H$, then problem (2.3) reduces to the following problem: Find $x \in H$ such that*

$$0 \in p(x) + M(g(x)). \tag{2.5}$$

This type of problem (2.5) has been considered and studied by Qing-Bang Zhang [23].

We remark that for appropriate and suitable choices of the above defined mappings, SVLIP (2.3) includes a number of variational and variational-like inclusions as special cases, see for example [3,6-8,17,18] and the related references cited therein.

3. Iterative algorithm

First, we give the following technical lemma:

Lemma 3.1. *Let X_i be a real Banach space, let $\varphi_i : X_i^\star \to X_i^\star$ be a single-valued mapping satisfying $\varphi_i(t + t') = \varphi_i(t) + \varphi_i(t'), \quad \forall t, t' \in X_i^\star$ and $ker(\varphi_i) = \{0\}, (i.e., ker(\varphi_i) = \{t \in X_i^\star, \varphi_i(t) = 0\})$. Let $g_i : X_i \to X_i, \eta_i : X_i \times X_i \to X_i, P_i : X_i \to X_j, Q_i : X_j \to X_i, S_i : X_j \times X_i \to X_i^\star, p_i : X_j \to X_i^\star$ be single-valued mappings and $H_i : X_i \to X_i^\star$ be a strictly η_i-monotone mapping and let $M_i : X_i \times X_j \to 2^{X_i^\star}$ be $(H_i, \varphi_i) - \eta_i-$ monotone mappings, respectively. Then $(x, y) \in X_1 \times X_2$ is a solution of (2.3) if and only if*

$$g_1(x) = R^{H_1,\eta_1}_{M_1(.,y),\varphi_1}\Big(H_1(g_1(x)) - \varphi_1 \circ S_1(P_1(x), Q_1(y)) - \varphi_1 \circ p_1(y)\Big) \tag{3.1}$$

$$g_2(y) = R^{H_2,\eta_2}_{M_2(.,x),\varphi_2}\Big(H_2(g_2(y)) - \varphi_2 \circ S_2(P_2(y), Q_2(x)) - \varphi_2 \circ p_2(x)\Big) \tag{3.2}$$

where $R^{H_1,\eta_1}_{M_1(.,y),\varphi_1} = (H_1 + \varphi_1 \circ M_1(., y))^{-1}$, $R^{H_2,\eta_2}_{M_2(.,x),\varphi_2} = (H_2 + \varphi_2 \circ M_2(., x))^{-1}$ are the resolvent operators.

Proof. Let $(x, y) \in X_1 \times X_2$ is a solution of (2.3), then we have

$$g_1(x) = R^{H_1,\eta_1}_{M_1(.,y),\varphi_1}\Big(H_1(g_1(x)) - \varphi_1 \circ S_1(P_1(x), Q_1(y)) - \varphi_1 \circ p_1(y)\Big)$$

$$\Longleftrightarrow g_1(x) = (H_1 + \varphi_1 \circ M_1(.,y))^{-1}\Big(H_1(g_1(x)) \\ -\varphi_1 \circ S_1(P_1(x), Q_1(y)) - \varphi_1 \circ p_1(y)\Big)$$

$$\Longleftrightarrow H_1(g_1(x)) + \varphi_1 \circ M_1(g_1(x), y) = H_1(g_1(x)) \\ -\varphi_1 \circ \Big(S_1(P_1(x), Q_1(y)) + p_1(y)\Big)$$

$$\Longleftrightarrow 0 \in \varphi_1 \circ \Big(S_1(P_1(x), Q_1(y)) + p_1(y)\Big) + \varphi_1 \circ M_1(g_1(x), y)$$

$$\Longleftrightarrow 0 \in \varphi_1 \circ \Big(S_1(P_1(x), Q_1(y)) + p_1(y) + M_1(g_1(x), y)\Big)$$

$$\Longleftrightarrow 0 \in S_1(P_1(x), Q_1(y)) + p_1(y) + M_1(g_1(x), y).$$

Since $\varphi_i(t + t') = \varphi_i(t) + \varphi_i(t')$ and $ker(\varphi_i) = \{0\}$.
Proceeding likewise by using (3.2), we have

$$g_2(y) = R^{H_2,\eta_2}_{M_2(.,x),\varphi_2}\Big(H_2(g_2(y)) - \varphi_2 \circ S_2(P_2(y), Q_2(x)) - \varphi_2 \circ p_2(x)\Big)$$

$$\Longleftrightarrow 0 \in S_2(P_2(y), Q_2(x)) + p_2(x) + M_2(g_2(y), x).$$

□

Lemma 3.1 is very important from the numerical point of view as it allows us to suggest the following iterative algorithm for finding the approximate solution of SVLIP (2.3).

Iterative Algorithm 3.2. *For arbitrary point $(x_0, y_0) \in X_1 \times X_2$, compute the sequences $\{x_n\} \in X_1, \{y_n\} \in X_2$ by the iterative scheme:*

$$x_{n+1} = x_n - g_1(x_n) + R^{H_1,\eta_1}_{M_1(.,y_n),\varphi_1}\Big(H_1(g_1(x_n)) - \varphi_1 \circ S_1(P_1(x_n), Q_1(y_n)) - \varphi_1 \circ p_1(y_n)\Big)$$

and

$$g_2(y_n) = R^{H_2,\eta_2}_{M_2(.,x_n),\varphi_2}\Big(H_2(g_2(y_n)) - \varphi_2 \circ S_2(P_2(y_n), Q_2(x_n)) - \varphi_2 \circ p_2(x_n)\Big), \ \forall n = 1, 2, \cdots .$$

4. Existence of solution and convergence analysis

Now, we prove the existence of solution and show that the sequences generated by Iterative Algorithm 3.2 converge strongly to a solution of SVLIP (2.3).

Theorem 4.1. *For $i \in \{1, 2\}, j \in \{1, 2\} \setminus i$, let $X_i^\star$ be q_i-uniformly smooth Banach space, $P_i : X_i \to X_j$, $Q_i : X_j \to X_i$ and $p_i : X_j \to X_i^\star$ be ξ_i, h_i and θ_i-Lipschitz continuous, respectively. Let $S_i : X_j \times X_i \to X_i^\star$ be α_i-Lipschitz continuous in the first argument and β_i-Lipschitz continuous in the second argument, $g_i : X_i \to X_i$ be σ_i-Lipschitz continuous and r_i-strongly accretive, $\eta_i : X_i \times X_i \to X_i$ be τ_i-Lipschitz* *continuous and $H_i : X_i \to X_i^\star$ be γ_i-Lipschitz continuous. Suppose $\varphi_i : X_i^\star \to X_i^\star$ be a single-valued mapping satisfying $\varphi_i(t+t') = \varphi_i(t) + \varphi_i(t')$, $\forall t, t' \in X_i^\star$ and $ker(\varphi_i) = \{0\}$ such that φ_i be μ_i-Lipschitz continuous and let $\varphi_i \circ S_i$ be $\epsilon_i - \varphi_i \circ p_i$-cocoercive in the second argument and λ_i-strongly accretive with respect to $H_i(g_i)$ in the first argument. In addition, if*

$$0 < (\Delta_1 + \Delta_2 + \Delta_3\Delta_4) < 1, \tag{4.1}$$

where

$$\Delta_1 = (1 - q_1 r_1 + c_{q_1}\sigma_1^{q_1})^{\frac{1}{q_1}},$$

$$\Delta_2 = \frac{\tau_1}{\delta_1}(\gamma_1^{q_1}\sigma_1^{q_1} - q_1\lambda_1 + c_{q_1}\mu_1^{q_1}\alpha_1^{q_1}\xi_1^{q_1})^{\frac{1}{q_1}},$$

$$\Delta_3 = \Big(a_2 + \frac{\tau_1}{\delta_1}\Big(\mu_1^{q_1}\theta_1^{q_1} + (c_{q_1} - q_1\epsilon_1)\mu_1^{q_1}\beta_1^{q_1}h_1^{q_1}\Big)^{\frac{1}{q_1}}\Big),$$

$$\Delta_4 = \frac{\Big(a_1 + \frac{\tau_2}{\delta_2}\Big(\mu_2^{q_2}\theta_2^{q_2} + (c_{q_2} - q_2\epsilon_2)\mu_2^{q_2}\beta_2^{q_2}h_2^{q_2}\Big)^{\frac{1}{q_2}}\Big)}{\Big[r_2 - \frac{\tau_2}{\delta_2}(\gamma_2^{q_2}\sigma_2^{q_2} - q_2\lambda_2 + c_{q_2}\mu_2^{q_2}\alpha_2^{q_2}\xi_2^{q_2})^{\frac{1}{q_2}}\Big]},$$

c_{q_i} is constant of smoothness of Banach space X_i and

$$\left.\begin{aligned} &\Big\|R^{H_1,\eta_1}_{M_1(.,y_1),\varphi_1}(b_1) - R^{H_1,\eta_1}_{M_1(.,y_2),\varphi_1}(b_1)\Big\|_1 \\ &\qquad \le a_2\|y_1 - y_2\|_2, \quad \forall\ b_1 \in X_1^\star, y_1, y_2 \in X_2, \\ &\Big\|R^{H_2,\eta_2}_{M_2(.,x_1),\varphi_2}(b_2) - R^{H_2,\eta_2}_{M_2(.,x_2),\varphi_2}(b_2)\Big\|_2 \\ &\qquad \le a_1\|x_1 - x_2\|_1, \quad \forall\ \ b_2 \in X_2^\star, x_1, x_2 \in X_1. \end{aligned}\right\} \tag{4.2}$$

Then, the iterative sequence $\{x_n\}, \{y_n\}$ generated by Iterative Algorithm 3.2 converge strongly to a solution $(x, y) \in X_1 \times X_2$ of SVLIP (2.3).

Proof. Let $(x, y) \in X_1 \times X_2$ be a solution of SVLIP (2.3). By Iterative Algorithm 3.2 and above conditions, we have

$$\begin{aligned} &\|x_{n+1} - x_n\|_1 \\ &= \Big\|x_n - g_1(x_n) + R^{H_1,\eta_1}_{M_1(.,y_n),\varphi_1}\Big(H_1(g_1(x_n)) \\ &\quad -\varphi_1 \circ S_1(P_1(x_n), Q_1(y_n)) - \varphi_1 \circ p_1(y_n)\Big) \\ &\quad -\Big\{x_{n-1} - g_1(x_{n-1}) + R^{H_1,\eta_1}_{M_1(.,y_{n-1}),\varphi_1}\Big(H_1(g_1(x_{n-1})) \\ &\quad -\varphi_1 \circ S_1(P_1(x_{n-1}), Q_1(y_{n-1})) - \varphi_1 \circ p_1(y_{n-1})\Big)\Big\}\Big\|_1 \end{aligned}$$

$$\le \|x_n - x_{n-1} - (g_1(x_n) - g_1(x_{n-1}))\|_1$$

$$+\Big\|R^{H_1,\eta_1}_{M_1(.,y_n),\varphi_1}\Big(H_1(g_1(x_n)) - \varphi_1 \circ S_1(P_1(x_n), Q_1(y_n)) - \varphi_1 \circ p_1(y_n)\Big)$$
$$-R^{H_1,\eta_1}_{M_1(.,y_n),\varphi_1}\Big(H_1(g_1(x_{n-1})) - \varphi_1 \circ S_1(P_1(x_{n-1}), Q_1(y_{n-1}))$$
$$-\varphi_1 \circ p_1(y_{n-1})\Big)\Big\|_1$$

$$+\Big\|R^{H_1,\eta_1}_{M_1(.,y_n),\varphi_1}\Big(H_1(g_1(x_{n-1})) - \varphi_1 \circ S_1(P_1(x_{n-1}), Q_1(y_{n-1})) - \varphi_1 \circ p_1(y_{n-1})\Big)$$
$$-R^{H_1,\eta_1}_{M_1(.,y_{n-1}),\varphi_1}\Big(H_1(g_1(x_{n-1})) - \varphi_1 \circ S_1(P_1(x_{n-1}), Q_1(y_{n-1}))$$
$$-\varphi_1 \circ p_1(y_{n-1})\Big)\Big\|_1. \tag{4.3}$$

Since g_i is r_i-strongly-accretive and σ_i-Lipschitz continuous, then using Lemma 2.1, we have

$$\|x_n - x_{n-1} - (g_1(x_n) - g_1(x_{n-1}))\|_1^{q_1}$$
$$\le \|x_n - x_{n-1}\|_1^{q_1} - q_1\Big\langle g_1(x_n) - g_1(x_{n-1}), J_{q_1}(x_n - x_{n-1})\Big\rangle_1$$
$$+c_{q_1}\|g_1(x_n) - g_1(x_{n-1})\|_1^{q_1}$$
$$\le (1 - q_1 r_1 + c_{q_1}\sigma_1^{q_1})\ \|x_n - x_{n-1}\|_1^{q_1}.$$

This implies

$$\|x_n - x_{n-1} - (g_1(x_n) - g_1(x_{n-1}))\|_1$$
$$\le (1 - q_1 r_1 + c_{q_1}\sigma_1^{q_1})^{\frac{1}{q_1}}\ \|x_n - x_{n-1}\|_1. \tag{4.4}$$

Using Theorem 2.3 ,we have the following estimate,

$$\Big\|R^{H_1,\eta_1}_{M_1(.,y_n),\varphi_1}\Big(H_1(g_1(x_n)) - \varphi_1 \circ S_1(P_1(x_n), Q_1(y_n)) - \varphi_1 \circ p_1(y_n)\Big)$$

$$-R^{H_1,\eta_1}_{M_1(.,y_n),\varphi_1}\Big(H_1(g_1(x_{n-1})) - \varphi_1 \circ S_1(P_1(x_{n-1}), Q_1(y_{n-1})) - \varphi_1 \circ p_1(y_{n-1})\Big)\Big\|_1$$

$$\le \frac{\tau_1}{\delta_1}\Big\|\Big(H_1(g_1(x_n)) - \varphi_1 \circ S_1(P_1(x_n), Q_1(y_n)) - \varphi_1 \circ p_1(y_n)\Big)$$

$$-\Big(H_1(g_1(x_{n-1})) - \varphi_1 \circ S_1(P_1(x_{n-1}), Q_1(y_{n-1})) - \varphi_1 \circ p_1(y_{n-1})\Big)\Big\|_1$$

$$\le \frac{\tau_1}{\delta_1}\Big\|H_1(g_1(x_n)) - H_1(g_1(x_{n-1}))$$

$$-\Big(\varphi_1 \circ S_1(P_1(x_n), Q_1(y_n)) - \varphi_1 \circ S_1(P_1(x_{n-1}), Q_1(y_n))\Big)\Big\|_1$$

$$+\frac{\tau_1}{\delta_1}\Big\|\varphi_1 \circ S_1(P_1(x_{n-1}), Q_1(y_n)) - \varphi_1 \circ S_1(P_1(x_{n-1}), Q_1(y_{n-1}))$$

$$-\Big(\varphi_1 \circ p_1(y_n) - \varphi_1 \circ p_1(y_{n-1})\Big)\Big\|_1. \qquad (4.5)$$

Since H_1, g_1, φ_1 and P_1 is γ_1,σ_1,μ_1 and ξ_1-Lipschitz continuous, respectively, $S_1 : X_2 \times X_1 \to X_1^\star$ is α_1-Lipschitz continuous in the first argument and $\varphi_1 \circ S_1$ is λ_1-strongly accretive with respect to $H_1(g_1)$ in the first argument, then using Lemma 2.1, we have

$$\Big\|H_1(g_1(x_n)) - H_1(g_1(x_{n-1})) - \Big(\varphi_1 \circ S_1(P_1(x_n), Q_1(y_n))$$

$$-\varphi_1 \circ S_1(P_1(x_{n-1}), Q_1(y_n))\Big)\Big\|_1^{q_1}$$

$$\leq \Big\|H_1(g_1(x_n)) - H_1(g_1(x_{n-1}))\|_1^{q_1}$$

$$-q_1\Big\langle \varphi_1 \circ S_1(P_1(x_n), Q_1(y_n)) - \varphi_1 \circ S_1(P_1(x_{n-1}), Q_1(y_n)),$$

$$J_{q_1}^\star\Big(H_1(g_1(x_n)) - H_1(g_1(x_{n-1}))\Big)\Big\rangle_1$$

$$+c_{q_1}\Big\|\varphi_1 \circ S_1(P_1(x_n), Q_1(y_n)) - \varphi_1 \circ S_1(P_1(x_{n-1}), Q_1(y_n))\Big\|_1^{q_1}$$

$$\leq (\gamma_1^{q_1}\sigma_1^{q_1} - q_1\lambda_1 + c_{q_1}\mu_1^{q_1}\alpha_1^{q_1}\xi_1^{q_1})\ \|x_n - x_{n-1}\|_1^{q_1}.$$

This implies

$$\Big\|H_1(g_1(x_n)) - H_1(g_1(x_{n-1}))$$

$$-\Big(\varphi_1 \circ S_1(P_1(x_n), Q_1(y_n)) - \varphi_1 \circ S_1(P_1(x_{n-1}), Q_1(y_n))\Big)\Big\|_1$$

$$\leq (\gamma_1^{q_1}\sigma_1^{q_1} - q_1\lambda_1 + c_{q_1}\mu_1^{q_1}\alpha_1^{q_1}\xi_1^{q_1})^{\frac{1}{q_1}}\|x_n - x_{n-1}\|_1. \qquad (4.6)$$

Also, since φ_1, p_1 and Q_1 is μ_1,θ_1 and h_1-Lipschitz continuous respectively, $S_1 : X_2 \times X_1 \to X_1^\star$ is β_1-Lipschitz continuous in the second argument and $\varphi_1 \circ S_1$ is $\epsilon_1 - \varphi_1 \circ p_1$-cocoercive in the second argument, then using Lemma 2.1, we have

$$\Big\|\varphi_1 \circ S_1(P_1(x_{n-1}), Q_1(y_n)) - \varphi_1 \circ S_1(P_1(x_{n-1}), Q_1(y_{n-1}))$$

$$-\Big(\varphi_1 \circ p_1(y_n) - \varphi_1 \circ p_1(y_{n-1})\Big)\Big\|_1^{q_1}$$

$$\leq \|\varphi_1 \circ p_1(y_n) - \varphi_1 \circ p_1(y_{n-1})\|_1^{q_1}$$

$$-q_1\Big\langle \varphi_1 \circ S_1(P_1(x_{n-1}), Q_1(y_n)) - \varphi_1 \circ S_1(P_1(x_{n-1}), Q_1(y_{n-1})),$$

$$J^\star_{q_1}\Big(\varphi_1 \circ p_1(y_n) - \varphi_1 \circ p_1(y_{n-1})\Big)\Big\rangle_1$$

$$+c_{q_1}\Big\|\varphi_1 \circ S_1(P_1(x_{n-1}), Q_1(y_n)) - \varphi_1 \circ S_1(P_1(x_{n-1}), Q_1(y_{n-1}))\Big\|_1^{q_1}$$

$$\leq \mu_1^{q_1}\theta_1^{q_1}\|y_n - y_{n-1}\|_2^{q_1}$$

$$-q_1\epsilon_1\Big\|\varphi_1 \circ S_1(P_1(x_{n-1}), Q_1(y_n)) - \varphi_1 \circ S_1(P_1(x_{n-1}), Q_1(y_{n-1}))\Big\|_1^{q_1}$$

$$+c_{q_1}\Big\|\varphi_1 \circ S_1(P_1(x_{n-1}), Q_1(y_n)) - \varphi_1 \circ S_1(P_1(x_{n-1}), Q_1(y_{n-1}))\Big\|_1^{q_1}$$

$$\leq \mu_1^{q_1}\theta_1^{q_1}\|y_n - y_{n-1}\|_2^{q_1}$$

$$+(c_{q_1} - q_1\epsilon_1)\Big\|\varphi_1 \circ S_1(P_1(x_{n-1}), Q_1(y_n)) - \varphi_1 \circ S_1(P_1(x_{n-1}), Q_1(y_{n-1}))\Big\|_1^{q_1}$$

$$\leq \Big(\mu_1^{q_1}\theta_1^{q_1} + (c_{q_1} - q_1\epsilon_1)\mu_1^{q_1}\beta_1^{q_1}h_1^{q_1}\Big)\|y_n - y_{n-1}\|_2^{q_1}.$$

This implies

$$\Big\|\varphi_1 \circ S_1(P_1(x_{n-1}), Q_1(y_n)) - \varphi_1 \circ S_1(P_1(x_{n-1}), Q_1(y_{n-1}))$$
$$-\Big(\varphi_1 \circ p_1(y_n) - \varphi_1 \circ p_1(y_{n-1})\Big)\Big\|_1$$
$$\leq \Big(\mu_1^{q_1}\theta_1^{q_1} + (c_{q_1} - q_1\epsilon_1)\mu_1^{q_1}\beta_1^{q_1}h_1^{q_1}\Big)^{\frac{1}{q_1}}\|y_n - y_{n-1}\|_2, \tag{4.7}$$

where $J^\star_{q_1} : X_1^\star \to X_1^{\star\star}$ is the generalized mapping on $X_1^\star$.
Using (4.6) and (4.7) in (4.5), we have

$$\Big\|R^{H_1,\eta_1}_{M_1(.,y_n),\varphi_1}\Big(H_1(g_1(x_n)) - \varphi_1 \circ S_1(P_1(x_n), Q_1(y_n)) - \varphi_1 \circ p_1(y_n)\Big)$$
$$-R^{H_1,\eta_1}_{M_1(.,y_n),\varphi_1}\Big(H_1(g_1(x_{n-1})) - \varphi_1 \circ S_1(P_1(x_{n-1}), Q_1(y_{n-1}))$$
$$-\varphi_1 \circ p_1(y_{n-1})\Big)\Big\|_1$$

$$\leq \frac{\tau_1}{\delta_1}(\gamma_1^{q_1}\sigma_1^{q_1} - q_1\lambda_1 + c_{q_1}\mu_1^{q_1}\alpha_1^{q_1}\xi_1^{q_1})^{\frac{1}{q_1}}\|x_n - x_{n-1}\|_1$$

$$+\frac{\tau_1}{\delta_1}\Big(\mu_1^{q_1}\theta_1^{q_1}+(c_{q_1}-q_1\epsilon_1)\mu_1^{q_1}\beta_1^{q_1}h_1^{q_1}\Big)^{\frac{1}{q_1}}\|y_n-y_{n-1}\|_2. \tag{4.8}$$

Also,

$$\Big\|R^{H_1,\eta_1}_{M_1(.,y_n),\varphi_1}\Big(H_1(g_1(x_{n-1}))-\varphi_1\circ S_1(P_1(x_{n-1}),Q_1(y_{n-1}))$$

$$-\varphi_1\circ p_1(y_{n-1})\Big)$$

$$-R^{H_1,\eta_1}_{M_1(.,y_{n-1}),\varphi_1}\Big(H_1(g_1(x_{n-1}))-\varphi_1\circ S_1(P_1(x_{n-1}),Q_1(y_{n-1}))$$

$$-\varphi_1\circ p_1(y_{n-1})\Big)\Big\|_1$$

$$\leq a_2\|y_n-y_{n-1}\|_2. \tag{4.9}$$

Using (4.4)-(4.9) in (4.3), we have

$$\|x_{n+1}-x_n\|_1$$

$$\leq\Big((1-q_1r_1+c_{q_1}\sigma_1^{q_1})^{\frac{1}{q_1}}\;+\frac{\tau_1}{\delta_1}(\gamma_1^{q_1}\sigma_1^{q_1}-q_1\lambda_1+c_{q_1}\mu_1^{q_1}\alpha_1^{q_1}\xi_1^{q_1})^{\frac{1}{q_1}}\Big)$$
$$\times\|x_n-x_{n-1}\|_1$$

$$+\Big(a_2+\frac{\tau_1}{\delta_1}\Big(\mu_1^{q_1}\theta_1^{q_1}+(c_{q_1}-q_1\epsilon_1)\mu_1^{q_1}\beta_1^{q_1}h_1^{q_1}\Big)^{\frac{1}{q_1}}\Big)\|y_n-y_{n-1}\|_2. \tag{4.10}$$

Again, since g_2 is r_2-strongly accretive, we have

$$\|g_2(y_n)-g_2(y_{n-1})\|_2\;\|y_n-y_{n-1}\|_2^{q_2-1}$$

$$\geq\Big\langle g_2(y_n)-g_2(y_{n-1}),J_{q_2}(y_n-y_{n-1})\Big\rangle_2$$

$$\geq r_2\|y_n-y_{n-1}\|_2^{q_2}.$$

This implies

$$\|y_n-y_{n-1}\|_2\leq\frac{1}{r_2}\|g_2(y_n)-g_2(y_{n-1})\|_2. \tag{4.11}$$

Now,

$$\|g_2(y_n)-g_2(y_{n-1})\|_2$$

$$
\begin{aligned}
&\leq \Big\|R^{H_2,\eta_2}_{M_2(.,x_n),\varphi_2}\Big(H_2(g_2(y_n)) - \varphi_2 \circ S_2(P_2(y_n), Q_2(x_n)) \\
&\quad -\varphi_2 \circ p_2(x_n)\Big) \\
&\quad -R^{H_2,\eta_2}_{M_2(.,x_{n-1}),\varphi_2}\Big(H_2(g_2(y_{n-1})) - \varphi_2 \circ S_2(P_2(y_{n-1}), Q_2(x_{n-1})) \\
&\quad -\varphi_2 \circ p_2(x_{n-1})\Big)\Big\|_2 \\
&\leq \Big\|R^{H_2,\eta_2}_{M_2(.,x_n),\varphi_2}\Big(H_2(g_2(y_n)) - \varphi_2 \circ S_2(P_2(y_n), Q_2(x_n)) \\
&\quad -\varphi_2 \circ p_2(x_n)\Big) \\
&\quad -R^{H_2,\eta_2}_{M_2(.,x_n),\varphi_2}\Big(H_2(g_2(y_{n-1})) - \varphi_2 \circ S_2(P_2(y_{n-1}), Q_2(x_{n-1})) \\
&\quad -\varphi_2 \circ p_2(x_{n-1})\Big)\Big\|_2 \\
&\quad +\Big\|R^{H_2,\eta_2}_{M_2(.,x_n),\varphi_2}\Big(H_2(g_2(y_{n-1})) - \varphi_2 \circ S_2(P_2(y_{n-1}), Q_2(x_{n-1})) \\
&\quad -\varphi_2 \circ p_2(x_{n-1})\Big) \\
&\quad -R^{H_2,\eta_2}_{M_2(.,x_{n-1}),\varphi_2}\Big(H_2(g_2(y_{n-1})) - \varphi_2 \circ S_2(P_2(y_{n-1}), Q_2(x_{n-1})) \\
&\quad -\varphi_2 \circ p_2(x_{n-1})\Big)\Big\|_2. \qquad (4.12)
\end{aligned}
$$

Proceeding likewise by using (4.5)-(4.8), we have

$$
\begin{aligned}
&\Big\|R^{H_2,\eta_2}_{M_2(.,x_n),\varphi_2}\Big(H_2(g_2(y_n)) - \varphi_2 \circ S_2(P_2(y_n), Q_2(x_n)) - \varphi_2 \circ p_2(x_n)\Big) \\
&\quad -R^{H_2,\eta_2}_{M_2(.,x_n),\varphi_2}\Big(H_2(g_2(y_{n-1})) - \varphi_2 \circ S_2(P_2(y_{n-1}), Q_2(x_{n-1})) \\
&\quad -\varphi_2 \circ p_2(x_{n-1})\Big)\Big\|_2 \\
&\leq \frac{\tau_2}{\delta_2}\Big\|\Big(H_2(g_2(y_n)) - \varphi_2 \circ S_2(P_2(y_n), Q_2(x_n)) - \varphi_2 \circ p_2(x_n)\Big) \\
&\quad -\Big(H_2(g_2(y_{n-1})) - \varphi_2 \circ S_2(P_2(y_{n-1}), Q_2(x_{n-1})) - \varphi_2 \circ p_2(x_{n-1})\Big)\Big\|_2 \\
&\leq \frac{\tau_2}{\delta_2}\Big\|H_2(g_2(y_n)) - H_2(g_2(y_{n-1})) \\
&\quad -\Big(\varphi_2 \circ S_2(P_2(y_n), Q_2(x_n)) - \varphi_2 \circ S_2(P_2(y_{n-1}), Q_2(x_n))\Big)\Big\|_2 \\
&\quad +\frac{\tau_2}{\delta_2}\Big\|\varphi_2 \circ S_2(P_2(y_{n-1}), Q_2(x_n)) - \varphi_2 \circ S_2(P_2(y_{n-1}), Q_2(x_{n-1}))
\end{aligned}
$$

$$-(\varphi_2 \circ p_2(x_n) - \varphi_2 \circ p_2(x_{n-1}))\Big\|_2. \tag{4.13}$$

Since H_2, g_2, φ_2 and P_2 is γ_2,σ_2, μ_2 and ξ_2-Lipschitz continuous, respectively, $S_2 : X_1 \times X_2 \to X_2^\star$ is α_2-Lipschitz continuous in the first argument and $\varphi_2 \circ S_2$ is λ_2-strongly accretive with respect to $H_2(g_2)$ in the first argument, then using Lemma 2.1, we have

$$\begin{aligned}
&\Big\|H_2(g_2(y_n)) - H_2(g_2(y_{n-1}))\\
&-\Big(\varphi_2 \circ S_2(P_2(y_n), Q_2(x_n)) - \varphi_2 \circ S_2(P_2(y_{n-1}), Q_2(x_n))\Big)\Big\|_2^{q_2}\\
\le\ &\Big\|H_2(g_2(y_n)) - H_2(g_2(y_{n-1}))\Big\|_2^{q_2}\\
&-q_2\Big\langle \varphi_2 \circ S_2(P_2(y_n), Q_2(x_n))\\
&-\varphi_2 \circ S_2(P_2(y_{n-1}), Q_2(x_n)), J_{q_2}^\star\Big(H_2(g_2(y_n)) - H_2(g_2(y_{n-1}))\Big)\Big\rangle_2\\
&+c_{q_2}\Big\|\varphi_2 \circ S_2(P_2(y_n), Q_2(x_n)) - \varphi_2 \circ S_2(P_2(y_{n-1}), Q_2(x_n))\Big\|_2^{q_2}\\
\le\ &(\gamma_2^{q_2}\sigma_2^{q_2} - q_2\lambda_2 + c_{q_2}\mu_2^{q_2}\alpha_2^{q_2}\xi_2^{q_2})\|y_n - y_{n-1}\|_2^{q_2}.
\end{aligned}$$

This implies

$$\begin{aligned}
&\Big\|H_2(g_2(y_n)) - H_2(g_2(y_{n-1}))\\
&-\Big(\varphi_2 \circ S_2(P_2(y_n), Q_2(x_n)) - \varphi_2 \circ S_2(P_2(y_{n-1}), Q_2(x_n))\Big)\Big\|_2\\
&\le (\gamma_2^{q_2}\sigma_2^{q_2} - q_2\lambda_2 + c_{q_2}\mu_2^{q_2}\alpha_2^{q_2}\xi_2^{q_2})^{\frac{1}{q_2}}\|y_n - y_{n-1}\|_2.
\end{aligned} \tag{4.14}$$

Also, since φ_2, p_2 and Q_2 is μ_2,θ_2 and h_2-Lipschitz continuous, respectively, $S_2 : X_1 \times X_2 \to X_2^\star$ is β_2-Lipschitz continuous in the second argument and $\varphi_2 \circ S_2$ is $\epsilon_2 - \varphi_2 \circ p_2$-cocoercive in the second argument, then using Lemma 2.1, we have

$$\Big\|\varphi_2 \circ S_2(P_2(y_{n-1}), Q_2(x_n)) - \varphi_2 \circ S_2(P_2(y_{n-1}), Q_2(x_{n-1}))$$

$$
\begin{aligned}
&-(\varphi_2 \circ p_2(x_n) - \varphi_2 \circ p_2(x_{n-1}))\Big\|_2^{q_2} \\
\leq\ & \|\varphi_2 \circ p_2(x_n) - \varphi_2 \circ p_2(x_{n-1})\|_2^{q_2} \\
&-q_2\Big\langle \varphi_2 \circ S_2(P_2(y_{n-1}), Q_2(x_n)) - \varphi_2 \circ S_2(P_2(y_{n-1}), Q_2(x_{n-1})), \\
&J_{q_2}^{\star}\Big(\varphi_2 \circ p_2(x_n) - \varphi_2 \circ p_2(x_{n-1})\Big)\Big\rangle_2 \\
&+c_{q_2}\Big\|\varphi_2 \circ S_2(P_2(y_{n-1}), Q_2(x_n)) - \varphi_2 \circ S_2(P_2(y_{n-1}), Q_2(x_{n-1}))\Big\|_2^{q_2} \\
\leq\ & \mu_2^{q_2}\theta_2^{q_2}\|x_n - x_{n-1}\|_1^{q_2} \\
&-q_2\epsilon_2\Big\|\varphi_2 \circ S_2(P_2(y_{n-1}), Q_2(x_n)) - \varphi_2 \circ S_2(P_2(y_{n-1}), Q_2(x_{n-1}))\Big\|_2^{q_2} \\
&+c_{q_2}\Big\|\varphi_2 \circ S_2(P_2(y_{n-1}), Q_2(x_n)) - \varphi_2 \circ S_2(P_2(y_{n-1}), Q_2(x_{n-1}))\Big\|_2^{q_2} \\
\leq\ & \mu_2^{q_2}\theta_2^{q_2}\|x_n - x_{n-1}\|_1^{q_2} \\
&+(c_{q_2} - q_2\epsilon_2)\Big\|\varphi_2 \circ S_2(P_2(y_{n-1}), Q_2(x_n)) - \varphi_2 \circ S_2(P_2(y_{n-1}), Q_2(x_{n-1}))\Big\|_2^{q_2} \\
\leq\ & \Big(\mu_2^{q_2}\theta_2^{q_2} + (c_{q_2} - q_2\epsilon_2)\mu_2^{q_2}\beta_2^{q_2}h_2^{q_2}\Big)\|x_n - x_{n-1}\|_1^{q_2}, \\
\Longrightarrow\ & \Big\|\varphi_2 \circ S_2(P_2(y_{n-1}), Q_2(x_n)) - \varphi_2 \circ S_2(P_2(y_{n-1}), Q_2(x_{n-1})) \\
&-(\varphi_2 \circ p_2(x_n) - \varphi_2 \circ p_2(x_{n-1}))\Big\|_2 \\
\leq\ & \Big(\mu_2^{q_2}\theta_2^{q_2} + (c_{q_2} - q_2\epsilon_2)\mu_2^{q_2}\beta_2^{q_2}h_2^{q_2}\Big)^{\frac{1}{q_2}}\|x_n - x_{n-1}\|_1,
\end{aligned} \tag{4.15}
$$

where $J_{q_2}^{\star} : X_2^{\star} \to X_2^{\star\star}$ is the generalized mapping on $X_2^{\star}$.
Using (4.14) and (4.15) in (4.13), we have

$$
\begin{aligned}
&\Big\|R^{H_2,\eta_2}_{M_2(.,x_n),\varphi_2}\Big(H_2(g_2(y_n)) - \varphi_2 \circ S_2(P_2(y_n), Q_2(x_n)) - \varphi_2 \circ p_2(x_n)\Big) \\
&-R^{H_2,\eta_2}_{M_2(.,x_n),\varphi_2}\Big(H_2(g_2(y_{n-1})) - \varphi_2 \circ S_2(P_2(y_{n-1}), Q_2(x_{n-1})) \\
&-\varphi_2 \circ p_2(x_{n-1})\Big)\Big\|_2 \\
&\leq \frac{\tau_2}{\delta_2}(\gamma_2^{q_2}\sigma_2^{q_2} - q_2\lambda_2 + c_{q_2}\mu_2^{q_2}\alpha_2^{q_2}\xi_2^{q_2})^{\frac{1}{q_2}}\|y_n - y_{n-1}\|_2 \\
&+\frac{\tau_2}{\delta_2}\Big(\mu_2^{q_2}\theta_2^{q_2} + (c_{q_2} - q_2\epsilon_2)\mu_2^{q_2}\beta_2^{q_2}h_2^{q_2}\Big)^{\frac{1}{q_2}}\|x_n - x_{n-1}\|_1.
\end{aligned} \tag{4.16}
$$

Again,

$$
\begin{aligned}
&\Big\|R^{H_2,\eta_2}_{M_2(.,x_n),\varphi_2}\Big(H_2(g_2(y_{n-1})) - \varphi_2 \circ S_2(P_2(y_{n-1}), Q_2(x_{n-1})) - \varphi_2 \circ p_2(x_{n-1})\Big) \\
&-R^{H_2,\eta_2}_{M_2(.,x_{n-1}),\varphi_2}\Big(H_2(g_2(y_{n-1})) - \varphi_2 \circ S_2(P_2(y_{n-1}), Q_2(x_{n-1})) - \varphi_2 \circ p_2(x_{n-1})\Big)\Big\|_2
\end{aligned}
$$

$$\leq a_1\|x_n - x_{n-1}\|_1. \tag{4.17}$$

Using (4.13)-(4.17) in (4.12), we have

$$\|g_2(y_n) - g_2(y_{n-1})\|_2$$

$$\leq \frac{\tau_2}{\delta_2}(\gamma_2^{q_2}\sigma_2^{q_2} - q_2\lambda_2 + c_{q_2}\mu_2^{q_2}\alpha_2^{q_2}\xi_2^{q_2})^{\frac{1}{q_2}}\|y_n - y_{n-1}\|_2$$

$$+\Big(a_1 + \frac{\tau_2}{\delta_2}\Big(\mu_2^{q_2}\theta_2^{q_2} + (c_{q_2} - q_2\epsilon_2)\mu_2^{q_2}\beta_2^{q_2}h_2^{q_2}\Big)^{\frac{1}{q_2}}\Big)\|x_n - x_{n-1}\|_1. \tag{4.18}$$

Combining (4.11) and (4.18), we have

$$\|y_n - y_{n-1}\|_2$$

$$\leq \frac{1}{r_2}\|g_2(y_n) - g_2(y_{n-1})\|_2$$

$$\leq \frac{1}{r_2}\Big(\frac{\tau_2}{\delta_2}(\gamma_2^{q_2}\sigma_2^{q_2} - q_2\lambda_2 + c_{q_2}\mu_2^{q_2}\alpha_2^{q_2}\xi_2^{q_2})^{\frac{1}{q_2}}\Big)\|y_n - y_{n-1}\|_2$$

$$+\frac{1}{r_2}\Big(a_1 + \frac{\tau_2}{\delta_2}\Big(\mu_2^{q_2}\theta_2^{q_2} + (c_{q_2} - q_2\epsilon_2)\mu_2^{q_2}\beta_2^{q_2}h_2^{q_2}\Big)^{\frac{1}{q_2}}\Big)\|x_n - x_{n-1}\|_1,$$

or

$$\Big\{r_2 - \frac{\tau_2}{\delta_2}(\gamma_2^{q_2}\sigma_2^{q_2} - q_2\lambda_2 + c_{q_2}\mu_2^{q_2}\alpha_2^{q_2}\xi_2^{q_2})^{\frac{1}{q_2}}\Big\}\|y_n - y_{n-1}\|_2$$

$$\leq \Big(a_1 + \frac{\tau_2}{\delta_2}\Big(\mu_2^{q_2}\theta_2^{q_2} + (c_{q_2} - q_2\epsilon_2)\mu_2^{q_2}\beta_2^{q_2}h_2^{q_2}\Big)^{\frac{1}{q_2}}\Big)\|x_n - x_{n-1}\|_1.$$

This implies

$$\|y_n - y_{n-1}\|_2$$

$$\leq \frac{\Big(a_1 + \frac{\tau_2}{\delta_2}\Big(\mu_2^{q_2}\theta_2^{q_2} + (c_{q_2} - q_2\epsilon_2)\mu_2^{q_2}\beta_2^{q_2}h_2^{q_2}\Big)^{\frac{1}{q_2}}\Big)}{\Big\{r_2 - \frac{\tau_2}{\delta_2}(\gamma_2^{q_2}\sigma_2^{q_2} - q_2\lambda_2 + c_{q_2}\mu_2^{q_2}\alpha_2^{q_2}\xi_2^{q_2})^{\frac{1}{q_2}}\Big\}}\ \|x_n - x_{n-1}\|_1. \tag{4.19}$$

Using (4.19) in (4.10), we have

$$\|x_{n+1} - x_n\|_1 \leq (\Delta_1 + \Delta_2 + \Delta_3\Delta_4)\|x_n - x_{n-1}\|_1$$

$$\leq \omega\|x_n - x_{n-1}\|_1, \tag{4.20}$$

where $\Delta_1, \Delta_2, \Delta_3, \Delta_4$ is as defined in (4.1) and $\omega = (\Delta_1 + \Delta_2 + \Delta_3\Delta_4)$, so that $0 \leq \omega < 1$ from (4.1). Therefore, it follows from (4.20) that $\{x_n\}$ is a Cauchy sequence in X_1. Hence, there exists $x \in X_1$ such that $x_n \to x$ as $n \to \infty$. Now, since $\{x_n\}$ is a Cauchy sequence in X_1, it follows from (4.19) that $\{y_n\}$ is a Cauchy sequence in X_2. Therefore, there exists $y \in X_2$ such that $y_n \to y$ as $n \to \infty$. Thus, the approximate solution $\{x_n\}, \{y_n\}$ generated by Iterative Algorithm 3.2 converge strongly to (x, y) a solution of SVLIP (2.3). □

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