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A Note on the Bendixson-Dulac Theorem for Refracted Systems with Multiple Zones*

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Abstract This note extends the Bendixson-Dulac theorem to refracted systems with multiple zones. As an application, we prove that piecewise linear Duffing-type system has neither crossing limit cycles nor sliding limit cycles Therefore, it gives a positive answer to the Conjecture of [16].

Keywords Bendixson-Dulac theorem, Refracted system, Limit cycle.

MSC(2010) 34A36, 34C05, 34C07.

1. Introduction

In this paper, we focus on planar piecewise smooth differential system with multiple zones as follows.

$$Z^i(x,y) = \begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = X^i(x,y), \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = Y^i(x,y), \end{cases} \quad \text{if } (x,y) \in S^i, i = 1, 2, \cdots, n. \tag{1.1}$$

where the phase space S is partitioned into finitely many open sets S^i, and in each of which the system is smooth. We assume that 0 be a regular value of each functions $f_i : \mathbb{R}^2 \to \mathbb{R}$ for $i = 1, 2, \cdots, n-1$ and the discontinuous boundary $\Sigma_i = \{(x,y) | f_i(x,y) = 0\}$ between regions S^i and S^{i+1} to be a codimension-one switching manifold.

Definition 1.1. Let $Zf(p) = \langle Z(p), \nabla f(p) \rangle$. The discontinuous boundaries $\Sigma_i, i = 1, 2, \cdots, n-1$ can be divided into three open regions:

(i) *Crossing region* $\Sigma_i^c = \{p \in \Sigma_i | Z^i f_i(p) Z^{i+1} f_i(p) > 0\}$, see Figure 1.1;

(ii) *Attracting region* $\Sigma_i^a = \{p \in \Sigma_i | Z^i f_i(p) > 0, Z^{i+1} f_i(p) < 0\}$, see Figure 1.2;

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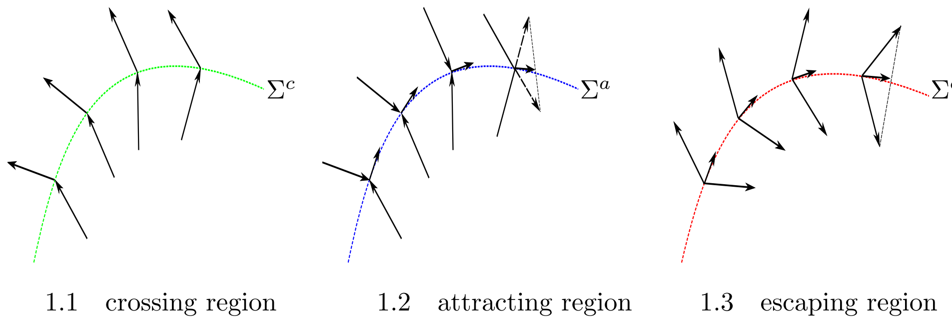

Figure 1. Definition of the vector field on Σ_i following Filippov's convention in the crossing, attracting and escaping regions

(iii) *Escaping region* $\Sigma_i^e = \{p \in \Sigma_i | Z^i f_i(p) < 0, Z^{i+1} f_i(p) > 0\}$, see Figure 1.3.

Note that if $q \in \Sigma_i^e$ for Z^i, then $q \in \Sigma_i^a$ for $-Z^i$. We say that $q \in \Sigma_i^a \bigcup \Sigma_i^e$ is a *sliding point*. If an isolated periodic orbit of systems (1.1) has sliding points, then it is called a *sliding limit cycle*. Otherwise, we call it a *crossing limit cycle*.

There are several papers [3, 9, 11, 12, 15] consider system (1.1) with two zones, that is $n = 2$. For the case $n = 3$ see [4, 13, 14]. As far as we know, there are few results about system (1.1) with $n \geqslant 4$ zones, see for instance [16–18].

For planar smooth differential systems there is a very developed qualitative theory nowadays [7]. This theory is based on several important results, including Existence and Uniqueness Theorem of solutions, Poincaré-Bnedixson Theorem and Bendixson-Dulac Theorem among others. Since piecewise smooth differential systems have become one of the most important frontiers in both Mathematics and Engineering [1], it is natural to know that if these results are true or false at the piecewise smooth differential systems scenario. It is not possible to guarantee the uniqueness of trajectories in sliding regions. Thus, most of the aforementioned classic results hold for piecewise smooth differential systems without sliding, see [2, 6, 8].

Definition 1.2. If $Z^i f_i(p) = Z^{i+1} f_i(p)$ for any $p \in \Sigma^i$ and $i = 1, 2, \cdots, n-1$, then system (1.1) is called *refracted system*.

It is worth noting that the refracted system has neither attracting region nor escaping region. There are several papers investigating the dynamics of refracted system, see for instance [3, 15].

The Bendixson-Dulac theorem is an important tool to investigate the number of limit cycles for smooth differential systems, see for instance the textbook [7]. This theorem has been generalized to multiple connected regions, see [5, 10]. Recently, the authors [6] extended the Bendixson-Dulac theorem to refracted systems with two zones. Thus, it provides a criterion to find upper bounds for the number of limit cycles in refracted systems.

2. Statement of the main results

In this paper, we generalized the Bendixson-Dulac Theorem of [6] with two zones to multiple zones. In order to state our main results, we need to introduce some notations and definitions.

Definition 2.1. Let $B^i(x,y)$ a $\mathcal{C}^1$ piecewise functions defined in S^i and

$$Z_B^i(x,y)=\begin{cases}\dfrac{\mathrm{d}x}{\mathrm{d}t}=B^i(x,y)X^i(x,y),\\ \dfrac{\mathrm{d}y}{\mathrm{d}t}=B^i(x,y)Y^i(x,y),\end{cases}\qquad \text{if } (x,y)\in S^i, i=1,2,\cdots,n, \tag{2.1}$$

a refracted system. Assume that $\mathrm{div}(\mathrm{Z_B^i})\geqslant(\leqslant)0, \mathrm{i}=1,2,\cdots,\mathrm{n}$ and $\sum\limits_{i=1}^{n}\mathrm{div}(\mathrm{Z_B^i})\neq 0$, we call $B^i(x,y)$ is a $\mathcal{C}^1-$Dulac piecwise function for system (1.1) in the region S.

Theorem 2.1. *Let $S\subset\mathbb{R}^2$ be an l connected region with a boundary defined by a finite number of smooth pieces. If there exists a $\mathcal{C}^1-$ Dulac piecewise function, B^i for system* (1.1) *in region S, then system* (1.1) *has at most l limit cycles in S.*

As an application of Theorem 2.1, we investigate the number of limit cycles for the following piecewise linear Duffing-type system

$$\begin{cases}\dfrac{\mathrm{d}x}{\mathrm{d}t}=y,\\ \dfrac{\mathrm{d}y}{\mathrm{d}t}=-x+\varepsilon g(x,y,\varepsilon),\end{cases} \tag{2.2}$$

where $g(x,y,\varepsilon)=\alpha_i h^{'}(x)+\varepsilon y$ with $x\in(a_i,a_{i+1}], i=0,1,\cdots,n$, and $h^{'}(x)$ is the derivative of a differentiable function $h(x)$.

The authors in [16] conjectured that the system (2.2) has no limit cycles using the second variation of the displacement function expressed for continuous functions. We give a positive answer to this conjecture as follows.

Theorem 2.2. *Planar discontinuous system* (2.2) *has neither crossing limit cycles nor sliding limit cycles.*

3. Proof of the main results

3.1. Proof of Theorem 2.1

We prove Theorem 2.1 by contradiction. Suppose that system (1.1) has a periodic orbit γ which ranged m zones, we assume that $\gamma=\gamma_1\cup\prod\limits_{i=2}^{m-1}\gamma_i^{\pm}\cup\gamma_m$ without loss of generality, see Figure 2 for $m=4$. We emphasize that the proof of Theorem 2.1 is inspired by [6].

First, we consider the case $l=0$, that is S is a simple connected region.

Let denotes $(BX)^i$ and $(BY)^i$ for $B^i(x,y)X^i(x,y)$ and $B^i(x,y)Y^i(x,y)$ respectively. On the discontinuous boundaries Σ_i, from $f_i(x,y)=0$ we have

$$\frac{\partial f_i(x,y)}{\partial x}\mathrm{d}x+\frac{\partial f_i(x,y)}{\partial y}\mathrm{d}y=0. \tag{3.1}$$

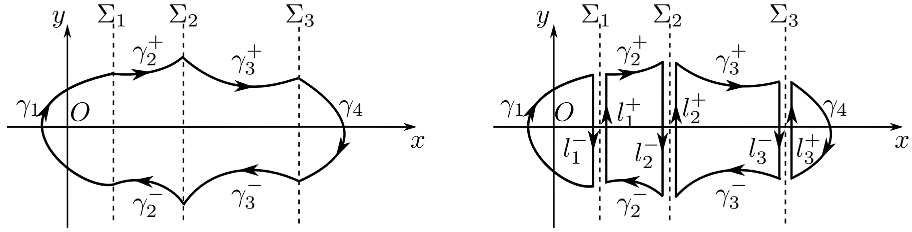

Figure 2. A crossing periodic orbit of system (1.1) with simple connected region

We denote $l_i^- \in S^i$ and $l_i^+ \in S^{i+1}$ at a distance $\varepsilon > 0$ of Σ_i, and consequently

$$\begin{aligned}&\int_{l_i^-} -(BY)^i \mathrm{d}x + (BX)^i \mathrm{d}y \\ &= \int_{l_i^-} \frac{1}{\dfrac{\partial f_i(x,y)}{\partial x}} \left(\frac{\partial f_i(x,y)}{\partial x} (BX)^i + \frac{\partial f_i(x,y)}{\partial y} (BY)^i \right) \mathrm{d}y.\end{aligned} \tag{3.2}$$

Similarly, we get

$$\begin{aligned}&\int_{l_i^+} -(BY)^{i+1} \mathrm{d}x + (BX)^{i+1} \mathrm{d}y \\ &= \int_{l_i^+} \frac{1}{\dfrac{\partial f_i(x,y)}{\partial x}} \left(\frac{\partial f_i(x,y)}{\partial x} (BX)^{i+1} + \frac{\partial f_i(x,y)}{\partial y} (BY)^{i+1} \right) \mathrm{d}y.\end{aligned} \tag{3.3}$$

From (3.2) and (3.3), and recalling $l_i^- = -l_i^+$ we deduce that

$$\begin{aligned}&\int_{l_i^-} -(BY)^i \mathrm{d}x + (BX)^i \mathrm{d}y \\ &+ \int_{l_i^+} -(BY)^{i+1} \mathrm{d}x + (BX)^{i+1} \mathrm{d}y \\ &= \int_{l_i^+} \frac{1}{\dfrac{\partial f_i(x,y)}{\partial x}} \left((BX)^{i+1} f_i - (BX)^i f_i \right) \mathrm{d}y = 0,\end{aligned} \tag{3.4}$$

the last equality holds because system (1.1) is a refracted system.

On one hand, using the Green formula, we have

$$\begin{aligned}
&\int_{\gamma_1} -(BY)^1\mathrm{d}x+(BX)^1\mathrm{d}y\\
&+\sum_{i=2}^{m-1}\int_{\gamma_i^+\cup\gamma_i^-} -(BY)^i\mathrm{d}x+(BX)^i\mathrm{d}y\\
&+\int_{\gamma_m} -(BY)^m\mathrm{d}x+(BX)^m\mathrm{d}y\\
&=\oint_{\gamma_1\cup l_1^-} -(BY)^1\mathrm{d}x+(BX)^1\mathrm{d}y\\
&+\sum_{i=2}^{m-1}\oint_{\gamma_i^+\cup\gamma_i^-\cup l_{i-1}^+\cup l_i^-} -(BY)^i\mathrm{d}x+(BX)^i\mathrm{d}y\\
&+\oint_{\gamma_m\cup l_{m-1}^+} -(BY)^m\mathrm{d}x+(BX)^m\mathrm{d}y\\
&=\iint_{S^i}\mathrm{div}(\mathrm{Z_B^i})\mathrm{dxdy}>0.
\end{aligned} \tag{3.5}$$

On the other hand, since γ is a periodic orbit of system (1.1), it is obvious that

$$\begin{aligned}
&\int_{\gamma_1} -(BY)^1\mathrm{d}x+(BX)^1\mathrm{d}y\\
&+\sum_{i=2}^{m-1}\int_{\gamma_i^+\cup\gamma_i^-} -(BY)^i\mathrm{d}x+(BX)^i\mathrm{d}y\\
&+\int_{\gamma_m} -(BY)^m\mathrm{d}x+(BX)^m\mathrm{d}y=0.
\end{aligned} \tag{3.6}$$

Thus, we have a contradiction.

Secondly, we consider that system (1.1) defined on the l-connected region. We just prove the case $l=1$. Since the general case $l\geqslant 2$ can be deduced similarly, we omit it here. We assume that system (1.1) has two crossing periodic orbits $\gamma=\gamma_1\cup\prod_{i=2}^{m-1}\gamma_i^{\pm}\cup\gamma_m$ and $\tilde{\gamma}=\tilde{\gamma}_1\cup\prod_{i=2}^{m-1}\tilde{\gamma}_i^{\pm}\cup\tilde{\gamma}_m$, see Figure 3.

Similar to the previous case, we complement the linear segments $l_{i,j}^{\pm}, i=1,2,\cdots,m-1, j=1,2$, such that

$$\begin{aligned}
D_1 &=\mathrm{int}\left(\gamma_1\bigcup \mathrm{l}_{11}^-\bigcup\tilde{\gamma}_1\bigcup \mathrm{l}_{12}^-\right);\\
D_i^+ &=\mathrm{int}\left(\gamma_\mathrm{i}^+\bigcup \mathrm{l}_{\mathrm{i-1,1}}^+\bigcup\tilde{\gamma}_\mathrm{i}^+\bigcup \mathrm{l}_{\mathrm{i,1}}^-\right),\mathrm{i}=2,3,\cdots,\mathrm{m}-1;\\
D_i^- &=\mathrm{int}\left(\gamma_\mathrm{i}^-\bigcup \mathrm{l}_{\mathrm{i-1,2}}^+\bigcup\tilde{\gamma}_\mathrm{i}^-\bigcup \mathrm{l}_{\mathrm{i,2}}^-\right),\mathrm{i}=2,3,\cdots,\mathrm{m}-1;\\
D_m^+ &=\mathrm{int}\left(\gamma_\mathrm{m}\bigcup \mathrm{l}_{\mathrm{m-1,1}}^+\bigcup\tilde{\gamma}_\mathrm{m}\bigcup \mathrm{l}_{\mathrm{m-1,2}}^+\right).
\end{aligned} \tag{3.7}$$

Note that we reverse the orientation of $\tilde{\gamma}$ in order to keep the above boundary of regions clock-wise oriented, see Figure 3 again. We can obtain a similar contradiction as the previous case.

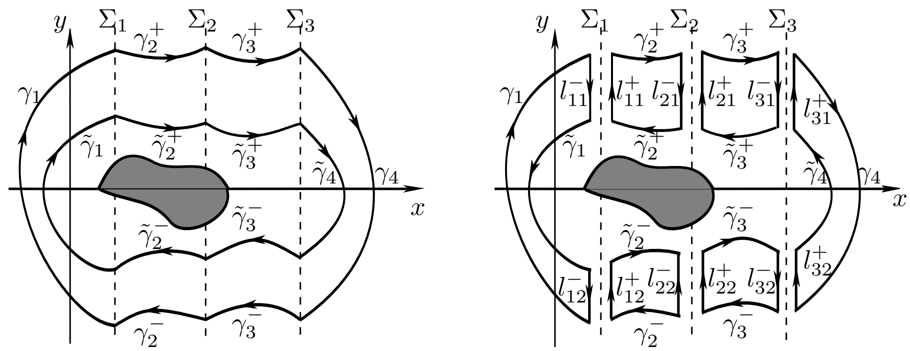

Figure 3. Two crossing periodic orbits of system (1.1) with 1-connected region

3.2. Proof of Theorem 2.2

Recall that $\Sigma_i : \{x = a_i\}, i = 1, 2, \cdots, n-1$. It is easy to check that system (2.2) has no sliding region on $\Sigma = \bigcup_{i=1}^{n-1} \Sigma_i$, so that system (2.2) has no sliding limit cycles.

In the following, we prove that system (2.2) has no crossing limit cycles. Since $\mathrm{div}(\mathrm{Z^i}) = \varepsilon^2 > 0$, and that system (2.2) is defined in $\mathbb{R}^2$, which is a simple connected region. According to Theorem 2.1, we conclude that system (2.2) has no crossing limit cycle.

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