Journal of Nonlinear Modeling and Analysis http://jnma.ca; http://jnma.ijournal.cn
Volume 3, Number 1, March 2021, 115–130 DOI:10.12150/jnma.2021.115

Homoclinic Orbits of a Quadratic Isochronous System by the Perturbation-incremental Method*

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Abstract In this paper, the perturbation-incremental method is presented for the analysis of a quadratic isochronous system. This method combines the remarkable characteristics of the perturbation method and the incremental method. The first step is the perturbation method. Assume that the parameter λ is small, i.e. $\lambda \approx 0$, the initial expression of the homoclinic orbit is obtained. The second step is the parameter incremental method. By extending the solution corresponding to small parameters to large parameters, we can get the analytical-expressions of homoclinic orbits.

Keywords Perturbation-incremental method, Homoclinic orbits, Quadratic isochronous system.

MSC(2010) 37G15, 34C37

1. Introduction

Considering the following plane system [7, 33]

$$\begin{cases} \dfrac{dx}{dt} = y + \lambda f(x, y), \\ \\ \dfrac{dy}{dt} = -g(x) + \lambda h(x, y), \end{cases} \tag{1.1}$$

where f, g and h are arbitrary nonlinear functions of their arguments, λ is a real parameter of arbitrary magnitude. If $f(0,0)$=$g(0)$=$h(0,0)$=0, the origin is a singular point. When $0 < \lambda < \overline{\lambda}$, equation (1.1) has a limit cycle around the origin. On the other hand, if λ is specified, then the analytical-expressions of limit cycles and

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*The authors were supported by National Natural Science Foundation of China (Nos.11701163, 11561022) and Foundation for Technological Base and Talents of Guangxi, China (Nos. GuiKe AD: 20159028).

homoclinic (heteroclinic) orbits will be calculated as given before in [5, 14, 31]. In practice, many quantitative methods are used to solve the analytical-expressions of limit cycles and homoclinic orbits such as Incremental Harmonic Balance method (IHB) [18, 19], Elliptic-Perturbation method (EP) [6, 10], Lindst-Poincaré method (LP) [11, 27], perturbation iteration method [4].

In recent years, many scholars have begun to pay attention to limit cycle bifurcations of isochronous systems [2, 22, 23, 26, 29]. Loud [24] has divided quadratic polynomial differential systems having an isochronous center into four classes S_1, S_2, S_3 and S_4. Yang [35] obtained the upper bounds of the number of limit cycles bifurcating from the period annuli of quadratic isochronous systems (S_1 and S_2) by using the Picard-Fuchs equation. Li [20] investigated the number of limit cycles which bifurcate from the period annulus of a class of quadratic isochronous system (S_3).

In this article, the perturbation-incremental method [5, 31] is given for the calculation of homoclinic orbits of quadratic isochronous differential systems [20, 35]. This method is especially suitable for some systems with parameters. When parameters of systems are small, the perturbation method is used to give the zero-order perturbation solution the analytical-expressions of homoclinic orbits. When parameters are gradually increasing, the parameter incremental method and the iterative method are used to extend the solution corresponding to small parameters to large parameters, and the analytical-expressions of homoclinic orbits satisfying the required accuracy are obtained.

Perturbation-incremental method is a new method combining the semi-analytical method with the numerical method, and has been developed for a long time. Xu et al. [32] applied the perturbation-incremental method to the calculation of limit cycles and homoclinic (heteroclinic) orbits of strong nonlinear oscillators in electrical engineering. Huang et al. [16, 17] used the perturbation-incremental method to discuss the limit cycles, homoclinic orbits and the quantitative analysis of parameters bifurcation of Bogdanov-Takens system. Chen et al. [8] and Lin [21] used the perturbation-incremental method to study the approximate solution of semi-stable limit cycles of Liénard equation, and as well calculation of multiple limit cycles with their bifurcation values. In the following years, the perturbation-incremental method has been widely used in the calculation of periodic solutions of nonlinear systems of delay differential equations [3, 12, 30], bifurcation of impulsive systems [28], calculation of limit cycles, homoclinic (heteroclinic) orbits and bifurcation of general dynamical systems [1, 9, 13, 15, 25].

Next, we will describe the main contents of this method and give an example.

2. Perturbation-incremental method

The common nonlinear oscillators systems and quadratic systems can be reduced to the form of (1.1). We introduce a nonlinear time transformation of the form

$$\frac{d\varphi}{dt} = \Phi(\varphi), \quad \Phi(\varphi + 2\pi) = \Phi(\varphi), \tag{2.1}$$

where φ is the new time. In the φ domain, equation (1.1) has the form

$$\Phi \frac{dx}{d\varphi} = y + \lambda f(x, y), \quad \Phi \frac{dy}{d\varphi} = -g(x) + \lambda h(x, y). \tag{2.2}$$

From the solution of the first equation of (2.2), y is substituted into the second equation

$$\Phi\frac{d}{d\varphi}\Big(\Phi\frac{dx}{d\varphi}\Big)-\lambda\Phi\frac{df}{d\varphi}=-g(x)+\lambda h. \tag{2.3}$$

Assuming that $\lambda\approx 0$ and the origin of the (x,y) phase plane is an equilibrium point interior to the limit cycle, then the zero-order perturbation solution of limit cycles [34] can be supposed as

$$x=acos\varphi+b,\quad y=-a\Phi sin\varphi, \tag{2.4}$$

where a is the amplitude and b is the bias.

Multiplying both sides of equation (2.3) by $\frac{dx}{d\varphi}=-asin\varphi$ and then integrating, we have

$$\frac{1}{2}(\Phi sin\varphi)^2+\frac{v(acos\varphi+b)-v(a+b)}{a^2}+\lambda\frac{\Phi f sin\varphi}{a}-\lambda\int_0^{\varphi}\frac{\Big[f\frac{d(\Phi sin\theta)}{d\theta}-hsin\theta\Big]}{a}d\theta=0, \tag{2.5}$$

where

$$v(x)=\int_0^x g(u)du. \tag{2.6}$$

For simplicity, let

$$\widetilde{v}(a,b,\varphi)=\frac{v(acos\varphi+b)-v(a+b)}{a^2}, \tag{2.7}$$

$$\widetilde{m}(a,b,\Phi,\varphi)=\frac{\Phi f sin\varphi}{a} \tag{2.8}$$

and

$$\widetilde{f}(a,b,\Phi,\theta)=\frac{f\frac{d(\Phi sin\theta)}{d\theta}-hsin\theta}{a}. \tag{2.9}$$

By taking $\varphi=\pi$ and 2π respectively in equation (2.5), we obtain

$$\widetilde{v}(a,b,\pi)-\lambda\int_0^{\pi}\widetilde{f}(a,b,\Phi,\theta)d\theta=0 \tag{2.10}$$

and

$$\int_0^{2\pi}\widetilde{f}(a,b,\Phi,\theta)d\theta=0. \tag{2.11}$$

Suppose that the system of equation (1.1) has a saddle point $(h,0)$, if

$$\frac{ah}{|h|}+b-h=0, \tag{2.12}$$

then the homoclinic orbit appears.

From (2.5), (2.10), (2.11) and (2.12), a, b and Φ are obtained, and the analytical-expressions of the homoclinic orbit is obtained. The procedure of the perturbation-incremental method is divided into two steps.

1. The first step is the perturbation method. Supposing that the parameter λ is small, i.e. $\lambda \approx 0$, and the solution of equations (2.5), (2.10), (2.11) and (2.12) can be represented in the forms

$$a = a_0 + O(\lambda), \quad b = b_0 + O(\lambda), \quad \Phi = \Phi_0 + O(\lambda). \tag{2.13}$$

Let $\lambda \approx 0$, from (2.5) we have

$$\Phi_0(\varphi) = \frac{[2v(a_0 + b_0) - 2v(a_0 cos\varphi + b_0)]^{\frac{1}{2}}}{a_0|sin\varphi|}, \tag{2.14}$$

From (2.10), we have

$$v(-a_0 + b_0) - v(a_0 + b_0) = 0, \tag{2.15}$$

From (2.11) and (2.12), we obtain

$$\int_0^{2\pi} \tilde{f}(a_0, b_0, \Phi_0, \theta)d\theta = 0 \tag{2.16}$$

and

$$\frac{a_0 h}{|h|} + b_0 - h = 0. \tag{2.17}$$

The zero-order perturbation solution for the homoclinic orbit of equation (1.1) can be written as

$$x = a_0 cos\varphi + b_0, \quad \frac{dx}{dt} = -a_0\Phi_0(\varphi) sin\varphi. \tag{2.18}$$

2. The second step of the perturbation-incremental method is the parameter incremental method. Small increments are added to the current solution a_0, b_0 and Φ_0 (or the perturbation solution at the beginning of the procedure when $\lambda_0 = 0$) of equations (2.5), (2.10), (2.11) and (2.12), to obtain a neighbouring solution corresponding to

$$\lambda = \lambda_0 + \Delta\lambda, \quad a = a_0 + \Delta a, \quad b = b_0 + \Delta b, \quad \Phi = \Phi_0 + \Delta\Phi. \tag{2.19}$$

Here, we proceed instead by solving for Δa, Δb and $\Delta\Phi$ incrementally. For this purpose, we expand (2.5), (2.10), (2.11) and (2.12) in Taylor's series about the initial state and linearized incremental equations are derived by ignoring all the nonlinear terms of the small increments as follows

$$\Big[\Big(\frac{\partial\widetilde{v}}{\partial a}\Big)_0 + \lambda\Big(\frac{\partial\widetilde{m}}{\partial a}\Big)_0 - \lambda\int_0^{\varphi}\Big(\frac{\partial\widetilde{f}}{\partial a}\Big)_0 d\theta\Big]\Delta a + \Big[\Big(\frac{\partial\widetilde{v}}{\partial b}\Big)_0 + \lambda\Big(\frac{\partial\widetilde{m}}{\partial b}\Big)_0 - \lambda\int_0^{\varphi}\Big(\frac{\partial\widetilde{f}}{\partial b}\Big)_0 d\theta\Big]\Delta b$$

$$+\Big[\Big(\Phi_0 sin^2\varphi\Big)+\lambda\Big(\frac{\partial\widetilde{m}}{\partial\Phi}\Big)_0\Big]\Delta\Phi-\lambda\int_0^{\varphi}\Big(\frac{\partial\widetilde{f}}{\partial\Phi}\Big)_0\Delta\Phi d\theta$$

$$=-\frac{1}{2}(\Phi_0 sin\varphi)^2-\widetilde{v}(a_0,b_0,\varphi)-\lambda\widetilde{m}(a_0,b_0,\Phi_0,\varphi)+\lambda\int_0^{\varphi}\widetilde{f}(a_0,b_0,\Phi_0,\theta)d\theta, \tag{2.20}$$

$$\Big[\Big(\frac{\partial\widetilde{v}}{\partial a}\Big)_{0,\pi}-\lambda\int_0^{\pi}\Big(\frac{\partial\widetilde{f}}{\partial a}\Big)_0 d\theta\Big]\Delta a+\Big[\Big(\frac{\partial\widetilde{v}}{\partial b}\Big)_{0,\pi}-\lambda\int_0^{\pi}\Big(\frac{\partial\widetilde{f}}{\partial b}\Big)_0 d\theta\Big]\Delta b-\lambda\int_0^{\pi}\Big(\frac{\partial\widetilde{f}}{\partial\Phi}\Big)_0\Delta\Phi d\theta$$

$$=-\widetilde{v}(a_0,b_0,\pi)+\lambda\int_0^{\pi}\widetilde{f}(a_0,b_0,\Phi_0,\theta)d\theta, \tag{2.21}$$

$$\Delta a\int_0^{2\pi}\Big(\frac{\partial\widetilde{f}}{\partial a}\Big)_0 d\theta+\Delta b\int_0^{2\pi}\Big(\frac{\partial\widetilde{f}}{\partial b}\Big)_0 d\theta+\int_0^{2\pi}\Big(\frac{\partial\widetilde{f}}{\partial\Phi}\Big)_0\Delta\Phi d\theta=-\int_0^{2\pi}\widetilde{f}(a_0,b_0,\Phi_0,\theta)d\theta, \tag{2.22}$$

$$\frac{h}{|h|}\Delta a+\Delta b=h-\frac{h}{|h|}a_0-b_0, \tag{2.23}$$

where

$$\Big(\frac{\partial\widetilde{v}}{\partial a}\Big)_{0,\pi}=\frac{\partial}{\partial a}\widetilde{v}(a,b,\varphi)|_{a=a_0,b=b_0,\varphi=\pi}\text{ and similarly for }\Big(\frac{\partial\widetilde{v}}{\partial b}\Big)_{0,\pi}.$$

Any 2π periodic functions will have a Fourier expansion and it will be assumed that M harmonics will provide a sufficiently accurate representation. Therefore, we write

$$\Phi_0=\sum_{j=0}^{M}(P_j cosj\varphi+Q_j sinj\varphi),\quad Q_0=0. \tag{2.24}$$

Accordingly, the unknown $\Delta\Phi$ is expressed as

$$\Delta\Phi=\sum_{j=0}^{M}(\Delta P_j cosj\varphi+\Delta Q_j sinj\varphi),\quad \Delta Q_0=0. \tag{2.25}$$

Expanding the periodic functions in equations (2.20)-(2.23) into Fourier series,

$$\Big(\frac{\partial\widetilde{v}}{\partial a}\Big)_0=\sum_{k\geq 0}\alpha_k cosk\varphi \tag{2.26}$$

and

$$\Big(\frac{\partial\widetilde{v}}{\partial b}\Big)_0=\sum_{k\geq 0}\beta_k cosk\varphi, \tag{2.27}$$

where the sine contributions will be identically zero by reference to equation (2.7). Also,

$$\widetilde{f}(a_0, b_0, \Phi_0, \varphi) = \sum_{k\geq 0}(\gamma_{1,k}cosk\varphi + \delta_{1,k}sink\varphi), \tag{2.28}$$

$$\widetilde{m}(a_0, b_0, \Phi_0, \varphi) = \sum_{k\geq 0}(m_{1,k}cosk\varphi + n_{1,k}sink\varphi), \tag{2.29}$$

$$\Big(\frac{\partial \widetilde{f}}{\partial a}\Big)_0 = \sum_{k\geq 0}(\gamma_{2,k}cosk\varphi + \delta_{2,k}sink\varphi), \tag{2.30}$$

$$\Big(\frac{\partial \widetilde{f}}{\partial b}\Big)_0 = \sum_{k\geq 0}(\gamma_{3,k}cosk\varphi + \delta_{3,k}sink\varphi), \tag{2.31}$$

$$\Big(\frac{\partial \widetilde{f}}{\partial \Phi}\Big)_0 = \sum_{k\geq 0}(\gamma_{4,k}cosk\varphi + \delta_{4,k}sink\varphi), \tag{2.32}$$

$$\Big(\frac{\partial \widetilde{m}}{\partial a}\Big)_0 = \sum_{k\geq 0}(m_{2,k}cosk\varphi + n_{2,k}sink\varphi), \tag{2.33}$$

$$\Big(\frac{\partial \widetilde{m}}{\partial b}\Big)_0 = \sum_{k\geq 0}(m_{3,k}cosk\varphi + n_{3,k}sink\varphi), \tag{2.34}$$

$$\Big(\frac{\partial \widetilde{m}}{\partial \Phi}\Big)_0 = \sum_{k\geq 0}(m_{4,k}cosk\varphi + n_{4,k}sink\varphi), \tag{2.35}$$

$$\Phi_0 sin^2\varphi = \sum_{k\geq 0}(\zeta_{1,k}cosk\varphi + \eta_{1,k}sink\varphi), \tag{2.36}$$

$$\frac{1}{2}(\Phi_0 sin\varphi)^2 + \widetilde{v}(a_0, b_0, \varphi) = \sum_{k\geq 0}(\zeta_{2,k}cosk\varphi + \eta_{2,k}sink\varphi), \tag{2.37}$$

where $\delta_{i,0}$, $\eta_{j,0}$ may be set to zero for all i, j. Substituting these expansions into equations (2.20)-(2.23) and employing the harmonic balance method, a system of linear equations with unknowns $\Delta a,\ \Delta b,\ \Delta P_j,\ \Delta Q_j$ in the form is as follows

$$A_n\Delta a + B_n\Delta b + A_{n,0}\Delta P_0 + \sum_{j=1}^{M}(A_{n,j}\Delta P_j + B_{n,j}\Delta Q_j) = R_n, \tag{2.38}$$

where $n = 0,\ 1,\ 2, ...,\ 2M+3$. The coefficients $A_n,\ B_n,\ A_{n,j},\ B_{n,j}$ and R_n are given in the Appendix. It should be noted that R_i, R_{M+i} are coefficients of the cosine and sine terms (for all i) in the Fourier expansion of the function on the right-hand side of (2.20), whereas R_{2M+1} and R_{2M+3} are the right-hand side values of equations (2.21-2.23). Thus, R_n in equation (2.38) is a residue term to prevent the drifting of the incremental process away from the actual solution. Equation (2.38) is solved by an equation solver such as the Gaussian elimination procedure. The values a_0, b_0 and Φ_0 can be updated by adding the original values to the corresponding incremental values. The iteration process continues until $R_n \to 0$ for all n (in fact, $|R_n|$ is less than the expected accuracy). The incremental process proceeds by adding the $\Delta\lambda$ incrementally to converged value of λ, using the previous solution as the zero-order perturbation solution until a new converged solution is obtained.

The a, b and $\Phi(\varphi)$ are found, the phase portrait of the homoclinic orbit is generated from

$$x = acos\varphi + b, \quad \frac{dx}{dt} = -a\Phi(\varphi)sin\varphi, \tag{2.39}$$

where φ varies from 0 to 2π.

In the next section, we will study a quadratic isochronous system using the perturbation-incremental method.

3. A quadratic isochronous system

In this section, we will discuss a quadratic isochronous system [26]

$$\begin{cases} \dfrac{dx}{dt} = -y + x^2 - y^2, \\ \\ \dfrac{dy}{dt} = x + 2xy. \end{cases} \tag{3.1}$$

First, transform it into a suitable form by $x \to y$, $y \to x$, and introduce the parameter $\lambda \in (0, 1]$, then we can change equation (3.1) to

$$\begin{cases} \dfrac{dx}{dt} = y + 2\lambda xy, \\ \\ \dfrac{dy}{dt} = -(x + x^2) + \lambda y^2. \end{cases} \tag{3.2}$$

Letting $f(x, y) = 2xy,\ h(x, y) = y^2,\ g(x) = x + x^2$, hence $v(x) = \dfrac{1}{2}x^2 + \dfrac{1}{3}x^3$. From equations (2.7), (2.8) and (2.9), we obtain

$$\widetilde{v}(a, b, \varphi) = -\Big(\frac{b}{a} + \frac{b^2}{a} + \frac{a}{3}\Big) + \Big(\frac{b}{a} + \frac{b^2}{a}\Big)cos\varphi - \Big(\frac{1}{2} + b\Big)sin^2\varphi + \frac{1}{3}acos^3\varphi, \tag{3.3}$$

$$\widetilde{m}(a, b, \Phi, \varphi) = -2\Phi^2 sin^2\varphi(acos\varphi + b) \tag{3.4}$$

and

$$\widetilde{f}(a,b,\Phi,\theta) = -2\Phi^2 sin\theta cos\theta(acos\theta + b) + \frac{2}{3}asin^3\theta(acos\theta + b) - a\Phi^2 sin^3\theta. \quad (3.5)$$

Equations (2.14) and (2.15) will give

$$b_0 = \frac{1}{2}\Big(-1 + \sqrt{1 - \frac{4}{3}a_0^2}\Big) \quad (3.6)$$

and

$$\Phi_0(\varphi) = \Big(1 + 2b_0 + \frac{2}{3}a_0 cos\varphi\Big)^{\frac{1}{2}}. \quad (3.7)$$

The fixed point $(-1, 0)$ in the phase plane is a saddle point, hence equation (2.17) becomes

$$-a_0 + b_0 + 1 = 0. \quad (3.8)$$

(1) The zero-order perturbation solution

When $\lambda \approx 0$, the two-dimensional streamline diagram of the system is given

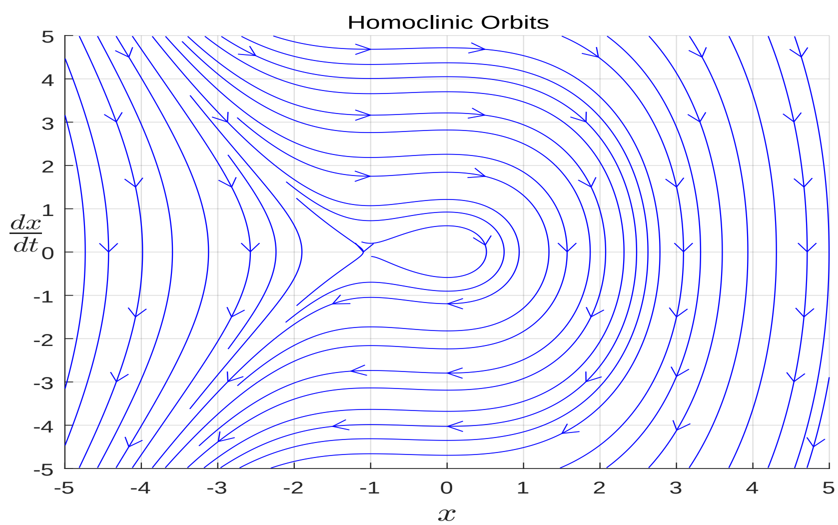

Figure 1. The two-dimensional streamline diagram for $\lambda = 0$.

From equations (2.16), (3.6), (3.7) and (3.8), we obtain

$$a_0 = \frac{3}{4}, \quad b_0 = -\frac{1}{4}, \quad \Phi_0 = \sqrt{\frac{1}{2} + \frac{1}{2}cos\varphi}. \quad (3.9)$$

Hence, for $\lambda \approx 0$, the zero-order perturbation solution of the equation (3.2)

$$x = \frac{3}{4}cos\varphi - \frac{1}{4}, \quad \frac{dx}{dt} = -\frac{3}{4}\sqrt{\frac{1}{2} + \frac{1}{2}cos\varphi}\ sin\varphi. \quad (3.10)$$

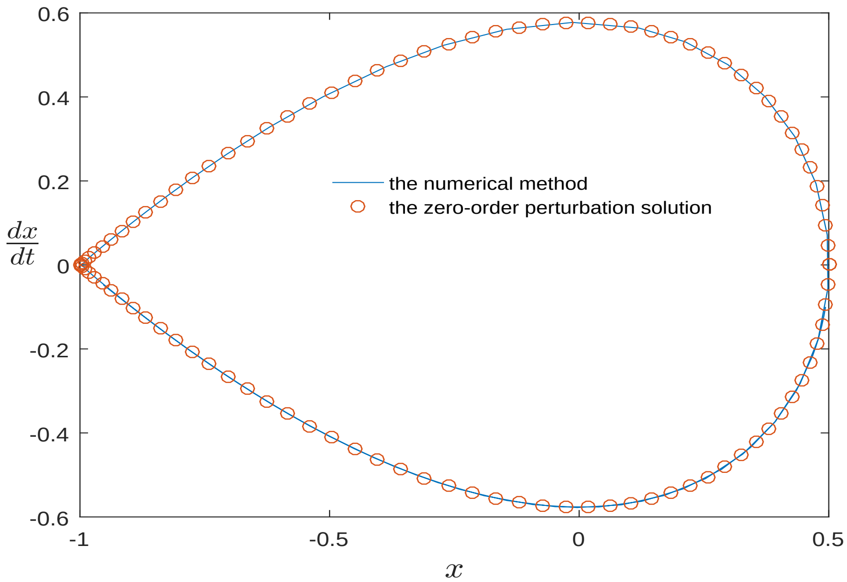

Figure 2. Homoclinic orbits phase for $\lambda = 0$.

(2) The parameter incremental method

Letting $\lambda_0 = 0$, $\Delta\lambda = 0.002$, and $M = 4$. Table 1 shows the results of 25 successive increments of $\Delta\lambda$ staring from the zero-order perturbation solution, and the phase diagrams of homoclinic orbits are given as shown in Figure 3 and Figure 4 when $\lambda = 0.040$ and $\lambda = 0.050$.

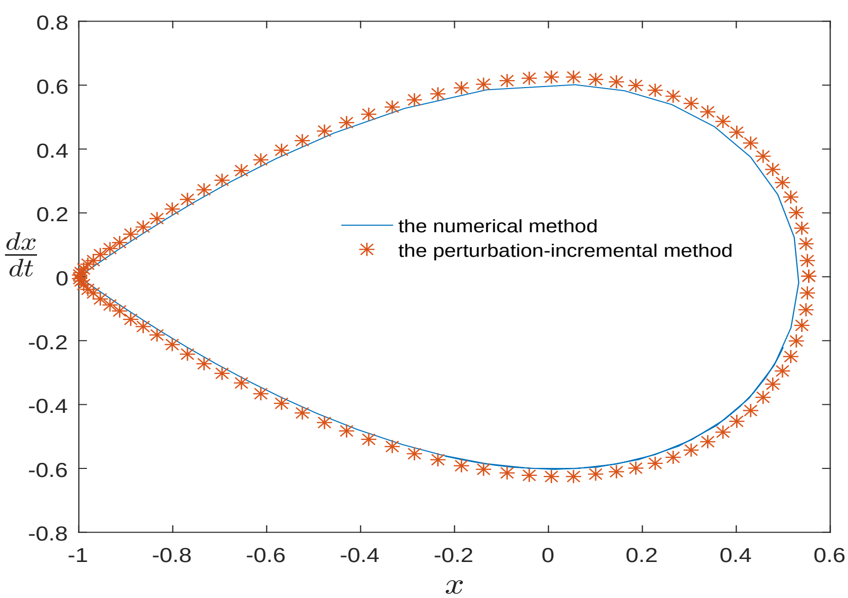

Figure 3. Homoclinic orbits phase for $\lambda = 0.04$.

Table 1. Values of a, b at the homoclinic orbits for various λ of equation (3.2)

λ	a	b	λ	a	b
0.002	0.7489482203	-0.25105177970	0.028	0.8560995146	-0.14390048520
0.004	0.7534021728	-0.24659782720	0.030	0.8243394614	-0.17566053840
0.006	0.8895746752	-0.11042532480	0.032	0.8043431373	-0.19565686250
0.008	1.1313867140	0.131386714200	0.034	0.7897776816	-0.21022231820
0.010	1.3063483960	0.306348396200	0.036	0.7829275613	-0.21707243850
0.012	1.3616782120	0.361678211700	0.038	0.7763406032	-0.22365939660
0.014	1.3429087140	0.342908713400	0.040	0.7768658016	-0.22313419820
0.016	1.2873910520	0.287391051600	0.042	0.7739786557	-0.22602134410
0.018	1.1184562601	0.208491196900	0.044	0.7756616431	-0.22433835670
0.020	1.1184562600	0.118456260100	0.046	0.7772558541	-0.22274414570
0.022	1.0318717410	0.031871741160	0.048	0.7757369549	-0.22426304490
0.024	0.9580452789	-0.04195472097	0.050	0.7764955465	-0.22350445330
0.026	0.8985727515	-0.10142724830			

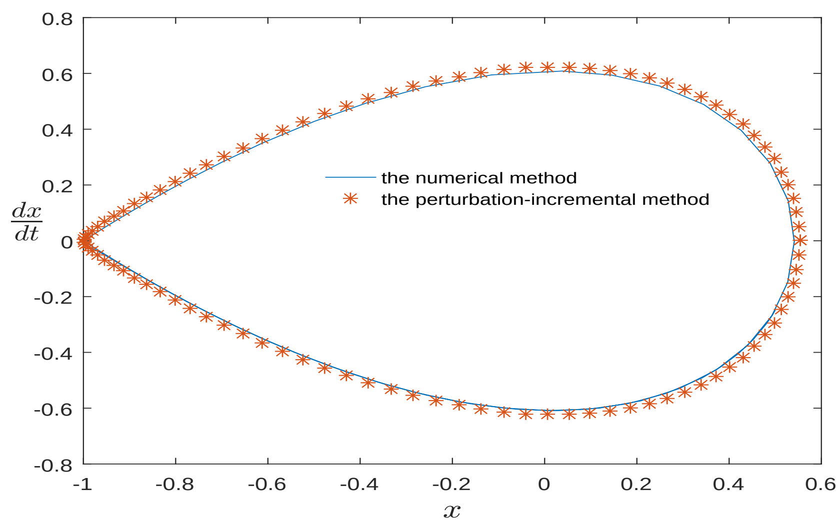

Figure 4. Homoclinic orbits phase for $\lambda = 0.05$.

where $\lambda = 0.040$,

$$\Phi = \sqrt{0.5537316036 + 0.5179105344\ cos\varphi},\ \ P_0 = \frac{2}{\pi} + 0.1588290610,$$

$$P_1 = \frac{4}{3\pi} - 0.1646340531,\ \ P_2 = -\frac{4}{15\pi} + 0.1848655365,$$

$$P_3 = \frac{4}{35\pi} - 0.09899845246,\ P_4 = -\frac{4}{63\pi} + 0.07909531348,$$
$$Q_1 = Q_2 = Q_3 = Q_4 = 0$$

and $\lambda = 0.050$,

$$\Phi = \sqrt{0.5529910934 + 0.5176636977\ cos\varphi},\ P_0 = \frac{2}{\pi} + 0.1101298576,$$
$$P_1 = \frac{4}{3\pi} - 0.09019901266,\ P_2 = -\frac{4}{15\pi} + 0.1293934705,$$
$$P_3 = \frac{4}{35\pi} - 0.0692587667,\ P_4 = -\frac{4}{63\pi} + 0.05467317834,$$
$$Q_1 = Q_2 = Q_3 = Q_4 = 0.$$

4. Conclusion

The perturbation-incremental method is an effective method to find the analytical-expressions of homoclinic orbits of quadratic isochronous systems. Homoclinic orbits obtained by using the perturbation-incremental method are compared with those from the numerical method and they are in good agreement. The advantages of the method are that

(1) The zero-order perturbation solution of the homoclinic orbit is determined by perturbation method as shown in Figure 2.

(2) Satisfactory results for arbitrary parameters can be obtained by the parameter incremental method, as shown in Figure 3 and Figure 4.

(3) Within the range of accuracy required, the result of $\lambda = 0.050$ is more accurate than $\lambda = 0.040$.

It will be my further research to apply the perturbation-incremental method to the cubic isochronous systems, and to consider the two parameters with parameters bifurcation problems.

Acknowledgements

The author gratefully acknowledges the financial supports by National Natural Science Foundation of China (Nos.11701163, 11561022) and Foundation for Technological Base and Talents of Guangxi, China (Nos. GuiKe AD: 20159028). The author thanks the reviewer for reviewing this paper and giving valuable comments, which have greatly helped improve the presentation of this paper. The author would also like to take this opportunity to express his sincere gratitude for the generous support of professors and classmates H. L. Wang, Z. X. Li, Z. S. Chu and Z. Chen.

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Appendix

Coefficients of terms in equation (2.38)

$$A_0 = \alpha_0 + \lambda m_{2,0} - \lambda \sum_{k=1}^{M} \frac{1}{k}\delta_{2,k},$$

$$A_i = \alpha_i + \lambda m_{2,i} + \lambda \frac{1}{i}\delta_{2,i},$$

$$A_{m+i} = \lambda n_{2,i} - \lambda \frac{1}{i} r_{2,i},$$

$$A_{2m+1} = \sum_{k=1}^{M} (-1)^k \alpha_k - \lambda \sum_{k=1}^{M} \frac{1}{k}[1 - (-1)^k]\delta_{2,k},$$

$$A_{2m+2} = r_{2,0},$$

$$A_{2m+3} = sign(h),$$

$$B_0 = \beta_0 + \lambda m_{3,0} - \lambda \sum_{k=1}^{M} \frac{1}{k}\delta_{3,k},$$

$$B_i = \beta_i + \lambda m_{3,i} + \lambda \frac{1}{i}\delta_{3,i},$$

$$B_{m+i} = \lambda n_{3,i} - \lambda \frac{1}{i} r_{3,i},$$

$$B_{2m+1} = \sum_{k=1}^{M} (-1)^k \beta_k - \lambda \sum_{k=1}^{M} \frac{1}{k}[1 - (-1)^k]\delta_{3,k},$$

$$B_{2m+2} = r_{3,0},$$

$$B_{2m+3} = 1,$$

$$A_{0,j} = \frac{1}{2}(\zeta_{1,-j} + \zeta_{1,j}) + \frac{1}{2}\lambda(m_{4,-j} + m_{4,j}) - \frac{1}{2}\lambda \sum_{k=1}^{M} \frac{1}{k}(\delta_{4,k-j} - \delta_{4,j-k} + \delta_{4,j+k}),$$

$$A_{i,j} = \frac{1}{2}(\zeta_{1,i-j} + \zeta_{1,j-i} + \zeta_{1,j+i}) + \frac{1}{2}\lambda(m_{4,i-j} + m_{4,j-i} + m_{4,j+i})$$
$$+\frac{1}{2i}\lambda(\delta_{4,i-j} - \delta_{4,j-i} + \delta_{4,j+i}),$$

$$A_{m+i,j} = \frac{1}{2}(\eta_{1,i-j} - \eta_{1,j-i} + \eta_{1,j+i}) + \frac{1}{2}\lambda(n_{4,i-j} - n_{4,j-i} + n_{4,j+i})$$
$$-\frac{1}{2i}\lambda(r_{4,i-j} + r_{4,j-i} + r_{4,j+i}),$$

$$A_{2m+1,j} = -\frac{1}{2}\lambda\sum_{k=1}^{M}\frac{[1-(-1)^k]}{k}(\delta_{4,k-j} - \delta_{4,j-k} + \delta_{4,j+k}),$$

$$A_{2m+2,j} = \frac{1}{2}(r_{4,-j} + r_{4,j}),$$

$$A_{2m+3,j} = 0,$$

$$B_{0,j} = \frac{1}{2}(\eta_{1,-j} + \eta_{1,j}) + \frac{1}{2}\lambda(n_{4,-j} + n_{4,j})$$
$$-\frac{1}{2}\lambda\sum_{k=1}^{M}\frac{1}{k}(r_{4,k-j} + r_{4,j-k} - r_{4,j+k}),$$

$$B_{i,j} = \frac{1}{2}(\eta_{1,j-i} + \eta_{1,j+i} - \eta_{1,i-j}) + \frac{1}{2}\lambda(n_{4,j-i} + n_{4,j+i} + n_{4,i-j})$$
$$+\frac{1}{2i}\lambda(r_{4,i-j} + r_{4,j-i} - r_{4,j+i}),$$

$$B_{m+i,j} = \frac{1}{2}(\zeta_{1,i-j} + \zeta_{1,j-i} - \zeta_{1,j+i}) + \frac{1}{2}\lambda(m_{4,i-j} + m_{4,j-i} - m_{4,j+i})$$
$$-\frac{1}{2i}\lambda(\delta_{4,j-i} + \delta_{4,j+i} - \delta_{4,i-j}),$$

$$B_{2m+1,j} = -\frac{1}{2}\lambda\sum_{k=1}^{M}\frac{[1-(-1)^k]}{k}(r_{4,k-j} + r_{4,j-k} - r_{4,j+k}),$$

$$B_{2m+2,j} = \frac{1}{2}(\delta_{4,j} - \delta_{4,-j}),$$

$$B_{2m+3,j} = 0,$$

$$R_0 = -\zeta_{2,0} - \lambda m_{1,0} + \lambda\sum_{k=1}^{M}\frac{1}{k}\delta_{1,k},$$

$$R_i = -\zeta_{2,i} - \lambda m_{1,i} - \lambda\frac{1}{i}\delta_{1,i},$$

$$R_{m+i} = -\eta_{2,i} - \lambda n_{1,i} + \lambda \frac{1}{i} r_{1,i},$$

$$R_{2m+1} = -\sum_{k=1}^{M} (-1)^k \zeta_{2,k} + \lambda \sum_{k=1}^{M} \frac{[1-(-1)^k]}{k} \delta_{1,k},$$

$$R_{2m+2} = -r_{1,0},$$

$$R_{2m+3} = h - sign(h) a_0 - b_0,$$

where

$$i = 1, 2, ..., M, \quad j = 0, 1, 2, ..., M,$$

$$\zeta_{1,k} = \eta_{1,k} = r_{4,k} = \delta_{4,k} = 0, \ for \ k < 0.$$