Journal of Nonlinear Modeling and Analysis
Volume 3, Number 1, March 2021, 71–92
http://jnma.ca; http://jnma.ijournal.cn
DOI:10.12150/jnma.2021.71

On a Special Generalized Mixture Class of Probabilistic Models

Christophe Chesneau[1,†] and Haitham M. Yousof[2]

Abstract In this paper, we develop a new mathematical strategy to create flexible lifetime distributions. This strategy is based on a special generalized mixture derived to the one involved in the so-called weighted exponential distribution. Thus, we introduce a new class of lifetime distributions called "special generalized mixture" class and discussed its qualities. In particular, a short list of new lifetime distributions is presented in details, with a focus on the one based on the Lomax distribution. Different mathematical properties are described, including distributional results, diverse moments measures, incomplete moments, characteristic function and bivariate extensions. Then, the applicability of the new class is investigated through the model parameters based on the Lomax distribution and the analysis of a practical data set.

Keywords Lifetime distribution, Weighted exponential distribution, Moments, Copula, Data analysis.

MSC(2010) 62N01, 62N02, 62E10.

1. Introduction

Despite certain qualities, the exponential distribution suffers from a lack of flexibility to model a large panel of lifetime phenomena. In order to improve it on this aspect, several extensions adding a shape parameter have been proposed, beginning with the notorious Weibull distribution. In particular, an alternative was proposed by the weighted exponential (WE) distribution introduced by [11]. It is based on the idea of [2], defining the related probability density function (PDF) by

$$f(x;\alpha,\lambda) = \frac{1}{P(\alpha X_1 > X_2)} f_*(x;\lambda) F_*(\alpha x;\lambda), \quad x > 0,$$

(and zero otherwise), where $\alpha > 0$, and X_1 and X_2 are two independent and identically distributed random variables defined on a certain probability space, say $(\Omega, \mathcal{A}, P)$, with the PDF specified by $f_*(x;\lambda) = \lambda e^{-\lambda x}$, $\lambda, x > 0$ and the cumulative distribution function (CDF) given by $F_*(x;\lambda) = 1 - e^{-\lambda x}$. In expanded form, $f(x;\alpha,\lambda)$ is expressed as

$$f(x;\alpha,\lambda) = \frac{1+\alpha}{\alpha} \lambda e^{-\lambda x}(1 - e^{-\alpha\lambda x}), \quad x > 0.$$

†the corresponding author.
Email address: christophe.chesneau@gmail.com(C. Chesneau), haitham.yousof@fcom.bu.edu.eg(C. H. M. Yousof)
[1]Université de Caen Normandie, LMNO, Campus II, Science 3, 14032, Caen, France
[2]Department of Statistics, Mathematics and Insurance, Benha University, Egypt

The corresponding CDF is obtained as

$$F(x;\alpha,\lambda)=\frac{1+\alpha}{\alpha}\left[1-e^{-\lambda x}-\frac{1}{1+\alpha}(1-e^{-\lambda x(1+\alpha)})\right],\quad x>0. \tag{1.1}$$

Then, it is shown in [11] that the WE distribution corresponds to a hidden truncation distribution, as well as the distribution of a sum of two independent random variables following exponential distributions with parameters λ and $\lambda(1+\alpha)$, respectively. Also, the importance of the new shape parameter α on the possible shapes of $f(x;\alpha,\lambda)$ is discussed, increasing the limited possibilities of the former exponential PDF. As important feature, the exponential distribution is a limiting distribution of the WE distribution; it is obtained by taking $\alpha\to+\infty$. For data analysis purposes, the WE model reveals to be an interesting alternative to various weighted exponential models, as the gamma, Weibull or generalized exponential models. Further details on the WE distribution can be found in [1], [19], [18], [8] and [6].

The following new remark is at the basis of this study. One can express the CDF given by (1.1) as the following generalized two-component mixture:

$$F(x;\alpha,\lambda)=w_1F_*(x;\lambda)+w_2F_*(x(1+\alpha);\lambda), \tag{1.2}$$

with $w_1=(1+\alpha)/\alpha>0$ and $w_2=-1/\alpha<0$ satisfying $w_1+w_2=1$, and $F_*(x;\lambda)$ and $F_*(x(1+\alpha);\lambda)$ are two valid CDFs, the second one being a scale version of the first one. For more details on the concept of generalized mixture distributions, we refer the reader to [21] and [12]. Motivated by the overall simplicity and great applicability of the WE distribution, our idea is to use the particular mixture structure of (1.2) to construct a general class of lifetime distributions, called the special generalized mixture-generated (SGM-G) class. As prime result, we consider a generic parent lifetime distribution with CDF denoted by $G(x;\xi)$ instead of $F_*(x;\lambda)$, and put the necessary conditions on it such that (1.2) remains a valid CDF. Note that only one extra parameter is introduced in comparison to the parent distribution, with the perspective of a significant gain in terms of statistical modelling. Then, we investigate the essential functions of the new class, and list new lifetime distributions of interest based on the half-Cauchy, half-logistic, half-normal and Lomax distributions, called SGMHC, SGMHL, SGMHN and SGMLx distributions, respectively. The general properties of the class are investigated in terms of those of the parent distribution, discussing distributional results, diverse moments measures including crude moments, variance, index of dispersion, skewness and kurtosis, also incomplete moments, characteristic function and bivariate extensions based on various copulas. As concrete application, these properties are applied to the SGMLx distribution. Then, the inferential issue of the SGMLx model is discussed through the maximum likelihood method. A complete analysis of practical data is performed for illustrative purposes, showing that the fit power of the SGMLx model is competitive in comparison to some existing lifetime models of the literature.

The sections making up the paper are as follows. Section 2 precises the SGM-G class, with description of its main functions and a short list of special distributions. Several properties of the SGM-G class are discussed in Section 3. Section 4 is devoted a data analysis, revealing the applicable potential of the SGMLx model. Some concluding notes are formulated in Section 5.

2. The SGM-G class

The essential elements for understanding the SGM-G class are described in this section.

2.1. Definition and elementary properties

The idea of the SGM-G class is based on the following new result.

Proposition 2.1. *Let $\alpha > 0$, $G(x;\xi)$ be a CDF of a continuous distribution with parameter(s) represented by ξ and $g(x;\xi)$ be the corresponding PDF. Let us suppose that $g(x;\xi)$ is decreasing with respect to x and with support $(0,+\infty)$. Then, the following function has the properties of a CDF:*

$$F(x;\alpha,\xi) = \frac{1+\alpha}{\alpha}\left[G(x;\xi) - \frac{1}{1+\alpha}G(x(1+\alpha);\xi)\right], \quad x > 0,$$

(and zero otherwise).

Proof. Firstly, $F(x;\alpha,\xi)$ is a continuous function since it is defined as a difference of two continuous functions. Then, since $g(x;\xi)$ is decreasing and $x(1+\alpha) \geq x$, we have $g(x;\xi) \geq g(x(1+\alpha);\xi)$. Therefore, for any $x > 0$, almost everywhere, we have

$$\frac{d}{dx}F(x;\alpha,\xi) = \frac{1+\alpha}{\alpha}\left[g(x;\xi) - g(x(1+\alpha);\xi)\right] \geq 0.$$

On the other side, we have $\lim_{x\to 0} F(x;\alpha,\xi) = 0$ and $\lim_{x\to+\infty} F(x;\alpha,\xi) = [(1+\alpha)/\alpha](1-1/(1+\alpha)\times 1) = 1$. The required properties to be a valid CDF are satisfied for $F(x;\alpha,\xi)$, ending the proof of Proposition 2.1. □

In the light of Proposition 2.1, we define the SGM-G class by the CDF given as

$$F(x;\alpha,\xi) = \frac{1+\alpha}{\alpha}\left[G(x;\xi) - \frac{1}{1+\alpha}G(x(1+\alpha);\xi)\right], \quad x > 0, \tag{2.1}$$

where $G(x;\xi)$ is a CDF of a parent continuous distribution with parameter(s) represented by ξ, having a decreasing PDF with respect to x and with support on $(0,+\infty)$, denoted by $g(x;\xi)$. As already mentioned previously, one can write $F(x;\alpha,\xi)$ as $F(x;\alpha,\xi) = w_1G(x;\xi) + w_2G(x(1+\alpha);\xi)$, with $w_1 = (1+\alpha)/\alpha$ and $w_2 = -1/\alpha$ such that $w_1 + w_2 = 1$, where $G(x;\xi)$ and $G(x(1+\alpha);\xi)$ are two CDFs, revealing the generalized mixture structure of the SGM-G class. With the choice of the exponential distribution as parent, i.e., with CDF and PDF specified by $G(x;\lambda) = 1 - e^{-\lambda x}$, $\lambda, x > 0$ and $g(x;\lambda) = \lambda e^{-\lambda x}$, respectively, $g(x;\lambda)$ being decreasing with respect to x, $F(x;\alpha,\lambda)$ becomes the CDF of the WE distribution proposed by [11], as described in (1.1). Also, the following limiting properties hold:

- $\lim_{\alpha\to+\infty} F(x;\alpha,\xi) = G(x;\xi)$, meaning that the parent distribution is a limiting distribution of the SGM-G class,
- $\lim_{\alpha\to 0} F(x;\alpha,\xi) = G(x;\xi) - xg(x;\xi)$, and this limiting function is a valid CDF provided that $\lim_{x\to 0} xg(x;\xi) = 0$ and $\lim_{x\to+\infty} xg(x;\xi) = 0$, with PDF given as $p(x;\xi) = -xdg(x;\xi)/dx$ (which is positive since $g(x;\xi)$ is decreasing by assumption).

Other interesting properties of the new class are discussed along the study.

2.2. Functions of interest

Based on (2.1), the survival function of the SGM-G class is given as

$$S(x;\alpha,\xi) = 1 - \frac{1+\alpha}{\alpha}\left[G(x;\xi) - \frac{1}{1+\alpha}G(x(1+\alpha);\xi)\right], \quad x > 0.$$

The corresponding PDF can be expressed as

$$f(x;\alpha,\xi) = \frac{1+\alpha}{\alpha}\left[g(x;\xi) - g(x(1+\alpha);\xi)\right], \quad x > 0.$$

Also, the associated hazard rate function (HRF) is given as

$$h(x;\alpha,\xi) = \frac{(1+\alpha)\left[g(x;\xi) - g(x(1+\alpha);\xi)\right]}{\alpha - [(1+\alpha)G(x;\xi) - G(x(1+\alpha);\xi)]}, \quad x > 0.$$

As functions related to the HRF, the reversed HRF is given as

$$r(x;\alpha,\xi) = \frac{g(x;\xi) - g(x(1+\alpha);\xi)}{G(x;\xi) - G(x(1+\alpha);\xi)/(1+\alpha)}, \quad x > 0$$

and the cumulative HRF is defined by

$$H(x;\alpha,\xi) = \log(\alpha) - \log\left\{\alpha - [(1+\alpha)G(x;\xi) - G(x(1+\alpha);\xi)]\right\}, \quad x > 0.$$

The quantile function can be obtained by inverting $F(x;\alpha,\xi)$. Hence, by denoting it as $Q(u;\xi)$, it satisfies $F(Q(u;\xi);\xi) = u$ for $u \in (0,1)$, i.e.,

$$(1+\alpha)G(Q(u;\xi);\xi) = \alpha u + G(Q(u;\xi)(1+\alpha);\xi).$$

In full generality, the analytical expression of $Q(u;\xi)$ is not possible, but a numerical evaluation is always possible if all the quantities involved are explicit.

2.3. Special distributions

In addition to the WE distribution, the SGM-G class contains a lot of lifetime distributions of potential interest for statistical purposes. Some of them are described below.

SGMHC distribution. The SGMHC distribution corresponds to the member of the SGM-G class defined with the half-Cauchy distribution as parent. One can define the half-Cauchy distribution by the CDF given as $G(x;b) = (2/\pi)\arctan(x/b), b, x > 0$ and with the PDF expressed as $g(x;b) = [2/(\pi b)][1+(x/b)^2]^{-2}$. Further details about the half-Cauchy distribution can be found in [20] and [4]. Then, since $g(x;b)$ is obviously decreasing with respect to x, Proposition 2.1 can be applied. Hence, the SGMHC distribution is defined by the CDF given as

$$F(x;\alpha,b) = \frac{2(1+\alpha)}{\alpha\pi}\left[\arctan\left(\frac{x}{b}\right) - \frac{1}{1+\alpha}\arctan\left(\frac{(1+\alpha)x}{b}\right)\right], \quad x > 0.$$

Also, the corresponding PDF is given by

$$f(x;\alpha,b) = \frac{2(1+\alpha)(\alpha+2)bx^2}{\pi(b^2+x^2)[(1+\alpha)^2x^2+b^2]}, \quad x > 0.$$

SGMHL distribution. The SGMHL distribution corresponds to the member of the SGM-G class defined with the half-Logistic distribution as parent. Rigorously, the half-Logistic distribution is defined by the CDF given by $G(x;b) = (1-e^{-bx})/(1+e^{-bx})$, $b, x > 0$ and with the PDF specified by $g(x;b) = 2be^{bx}/(1+e^{bx})^2$ (see [17]). Then, since $e^{bx} > 1$ for $x > 0$, we have

$$\frac{d}{dx}g(x;b) = -\frac{2b^2e^{bx}(e^{bx}-1)}{(e^{bx}+1)^3} < 0,$$

implying that $g(x;b)$ is decreasing with respect to x. Thus, Proposition 2.1 can be applied; the CDF of the SGMHL distribution is given by

$$F(x;\alpha,b) = \frac{1+\alpha}{\alpha}\left[\frac{1-e^{-bx}}{1+e^{-bx}} - \frac{1}{1+\alpha}\left(\frac{1-e^{-b(1+\alpha)x}}{1+e^{-b(1+\alpha)x}}\right)\right], \quad x > 0.$$

After some developments, we show that the corresponding PDF is obtained as

$$f(x;\alpha,b) = \frac{2(1+\alpha)(e^{(1+\alpha)bx}-e^{bx})(e^{b(\alpha+2)x}-1)}{\alpha(e^{bx}+1)^2(e^{(1+\alpha)bx}+1)^2}, \quad x > 0.$$

SGMHN distribution. The SGMHN distribution corresponds to the member of the SGM-G class based on the half-normal distribution. We may refer the reader to [13] for more details on this parent distribution, which is defined by the CDF given by $G(x;\sigma) = \operatorname{erf}(x/(\sigma\sqrt{2}))$, $\sigma, x > 0$, where $\operatorname{erf}(x) = (2/\sqrt{\pi})\int_0^x e^{-t^2}dt$ is the error function, and with the PDF specified by $g(x;\sigma) = [\sqrt{2}/(\sigma\sqrt{\pi})]e^{-x^2/(2\sigma^2)}$. It is clear that $g(x;b)$ is decreasing with respect to x. Hence, Proposition 2.1 can be applied; the SGMHN distribution has the following CDF:

$$F(x;\alpha,\sigma) = \frac{1+\alpha}{\alpha}\left[\operatorname{erf}\left(\frac{x}{\sigma\sqrt{2}}\right) - \frac{1}{1+\alpha}\operatorname{erf}\left(\frac{x(1+\alpha)}{\sigma\sqrt{2}}\right)\right], \quad x > 0.$$

Also, the corresponding PDF is obtained as

$$f(x;\alpha,\sigma) = \frac{\sqrt{2}(1+\alpha)}{\alpha\sigma\sqrt{\pi}}e^{-x^2/(2\sigma^2)}\left[1-e^{-x^2\alpha(2+\alpha)/(2\sigma^2)}\right], \quad x > 0.$$

SGMLx distribution. The SGMLx distribution corresponds to the member of the SGM-G class defined with the Lomax distribution. Here, we define the Lomax distribution by the CDF given as $G(x;\beta,\lambda) = 1-(1+x/\lambda)^{-\beta}$, $\beta, \lambda, x > 0$ and the PDF given by $g(x;\beta,\lambda) = (\beta/\lambda)(1+x/\lambda)^{-\beta-1}$ (see [14]). Then, it is immediate that $g(x;\beta,\lambda)$ is decreasing with respect to x. By applying the Proposition 2.1, we legitimately define the SGMLx distribution by the following CDF:

$$F(x;\alpha,\beta,\lambda) = \frac{1+\alpha}{\alpha}\left\{1-\left(1+\frac{x}{\lambda}\right)^{-\beta} - \frac{1}{1+\alpha}\left[1-\left(1+\frac{x(1+\alpha)}{\lambda}\right)^{-\beta}\right]\right\},$$

$$x > 0. \tag{2.2}$$

Also, the corresponding PDF is given by

$$f(x;\alpha,\beta,\lambda) = \frac{(1+\alpha)\beta}{\alpha\lambda}\left[\left(1+\frac{x}{\lambda}\right)^{-\beta-1} - \left(1+\frac{x(1+\alpha)}{\lambda}\right)^{-\beta-1}\right], \quad x > 0. \tag{2.3}$$

2.4. On the SGMLx distribution

Thanks to preliminaries investigations on the shape properties, we now put a focus on the SGMLx distribution with CDF and PDF specified by (2.2) and (2.3), respectively. First of all, the corresponding HRF is given as

$$h(x;\alpha,\beta,\lambda)=\frac{(1+\alpha)\beta\left[(1+x/\lambda)^{-\beta-1}-(1+x(1+\alpha)/\lambda)^{-\beta-1}\right]}{\alpha\lambda-\lambda\left[\alpha-(1+\alpha)\left(1+x/\lambda\right)^{-\beta}+(1+x(1+\alpha)/\lambda)^{-\beta}\right]},\quad x>0. \tag{2.4}$$

Let us now investigate some asymptotic properties of $F(x;\alpha,\beta,\lambda)$, $f(x;\alpha,\beta,\lambda)$ and $h(x;\alpha,\beta,\lambda)$. When $x\to 0$, we have

$$F(x;\alpha,\beta,\lambda)\sim\frac{(1+\alpha)\beta(\beta+1)}{2\lambda^2}x^2,\quad f(x;\alpha,\beta,\lambda)\sim h(x;\alpha,\beta,\lambda)\sim\frac{(1+\alpha)\beta(\beta+1)}{\lambda^2}x.$$

Also, when $x\to+\infty$, we have

$$F(x;\alpha,\beta,\lambda)\sim 1-\frac{1+\alpha}{\alpha}\lambda^{\beta}\left[1-(1+\alpha)^{-\beta-1}\right]x^{-\beta},$$

$$f(x;\alpha,\beta,\lambda)\sim\frac{1+\alpha}{\alpha}\lambda^{\beta}\left[1-(1+\alpha)^{-\beta-1}\right]\beta x^{-\beta-1},\quad h(x;\alpha,\beta,\lambda)\sim\beta x^{-1}.$$

We observe that the parameter α mainly governs the main constant. The critical point(s) of $f(x;\alpha,\beta,\lambda)$ is(are) given as the solution(s) of the following differential equation: $df(x;\alpha,\beta,\lambda)/dx=0$, which is equivalent to

$$-\frac{(1+\alpha)\beta(\beta+1)}{\alpha\lambda^2}\left[\left(1+\frac{x}{\lambda}\right)^{-\beta-2}-(1+\alpha)\left(1+\frac{x(1+\alpha)}{\lambda}\right)^{-\beta-2}\right]=0.$$

After some algebraic manipulations, we find only one solution given as

$$x_0=\frac{\lambda\left[1-(1+\alpha)^{-1/(\beta+2)}\right]}{(1+\alpha)^{-1/(\beta+2)+1}-1}>0.$$

Since $df(x;\alpha,\beta,\lambda)/dx>0$ for $x<x_0$ and $df(x;\alpha,\beta,\lambda)/dx<0$ for $x>x_0$, the point x_0 is a maximum. We thus prove that the SGMLx distribution is unimodal, with mode given by x_0. We see that the parameters α and β have notable influences on the possible values of x_0.

The critical point(s) of $h(x;\alpha,\beta,\lambda)$ is(are) hard to obtain due to the complexity of the function. However, they can be determined numerically by using any mathematical software. We end this part by investigating the possible shapes of $f(x;\alpha,\beta,\lambda)$ and $h(x;\alpha,\beta,\lambda)$ through a graphical analysis.

Figures 1 and 2 plot several curves for $f(x;\alpha,\beta,\lambda)$ and $h(x;\alpha,\beta,\lambda)$, respectively, for selected values of α, β and λ.

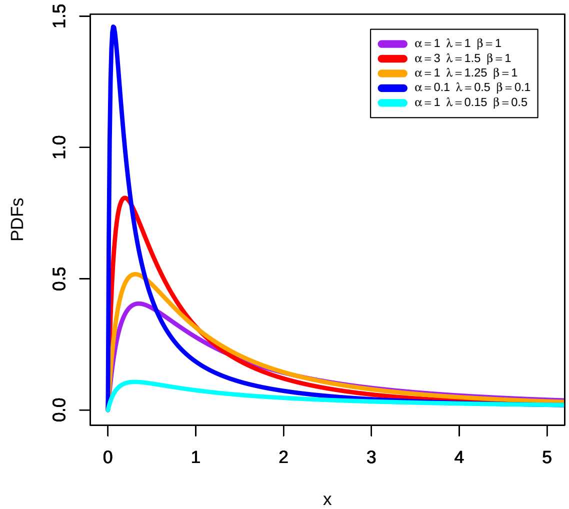

Figure 1. Plots of PDFs of the SGMLx distribution.

Based on Figure 1, we see that the new PDF is highly right-skewed, with possible spike shapes and a heavy-tail.

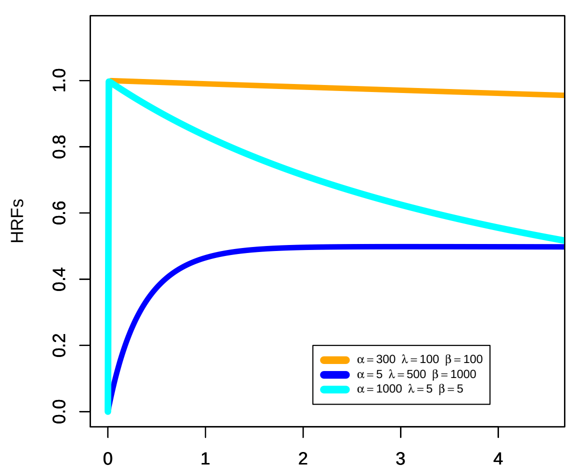

Figure 2. Plots of HRFs of the SGMLx distribution.

Figure 2 indicates that the new HRF can be "upside-down", "constant" and "increasing-constant". All these observations give us a precious indicator on the modelling ability of the SGMLx model; it is adequate for right skewed lifetime data with heavy tail, particularly adapted if one or more extreme values are present. Further properties of the SGMLx distribution will be discussed in the next.

3. Mathematical properties

In what follows, let Y be a random variable having $g(x;\xi)$ as PDF (PDF of the parent distribution) and X be a random variable having $f(x;\alpha,\xi)$ as PDF (PDF of the SGM-G class).

3.1. Distributional results

The following result presents some PDFs of sums of random variables involving the SGM-G class.

Proposition 3.1.

- *Assume that X and Y are independent. Then, the PDF of $X+Y$ is given as*

$$f_{X+Y}(x;\alpha,\xi)=\frac{1+\alpha}{\alpha}\left[(g\star g)(x;\xi)-\frac{1}{1+\alpha}(g_\alpha\star g)(x;\xi)\right],\quad x>0,$$

  where $g_\alpha(x;\xi)$ is the scale PDF version of $g(x;\xi)$ given as $g_\alpha(x;\xi)=(1+\alpha)g(x(1+\alpha);\xi)$, and $(h\star k)(x;\xi)=\int_{-\infty}^{+\infty}h(t;\xi)k(x-t;\xi)dt$ is the standard convolution product of general functions $h(x;\xi)$ and $k(x;\xi)$ (provided that the convolution production exists).

- *Let X_1 and X_2 be two independent and identically distributed random variables from X. Then, the PDF of X_1+X_2 is given as*

$$f_{X_1+X_2}(x;\alpha,\xi)=$$
$$\frac{(1+\alpha)^2}{\alpha^2}\left[(g\star g)(x;\xi)-\frac{2}{1+\alpha}(g_\alpha\star g)(x;\xi)+\frac{1}{(1+\alpha)^2}(g\star g)_\alpha(x;\xi)\right],$$
$$x>0,$$

  where $(g\star g)_\alpha(x;\xi)$ is the scale PDF version of $(g\star g)(x;\xi)$ given as $(g\star g)_\alpha(x;\xi)=(1+\alpha)(g\star g)(x(1+\alpha);\xi)$.

Proof.

- Since X and Y are independent, with the use of their respective PDFs, the PDF of $X+Y$ is given as

$$\begin{aligned}f_{X+Y}(x;\alpha,\xi)&=\int_0^x f(t;\alpha,\xi)g(x-t;\xi)dt\\&=\frac{1+\alpha}{\alpha}\left[\int_0^x g(t;\xi)g(x-t;\xi)dt-\int_0^x g(t(1+\alpha);\xi)g(x-t;\xi)dt\right]\\&=\frac{1+\alpha}{\alpha}\left[(g\star g)(x;\xi)-\frac{1}{1+\alpha}(g_\alpha\star g)(x;\xi)\right],\end{aligned}$$

  proving the first point.

- Since X_1 and X_2 are independent and identically distributed, with the use of their respective PDFs, the PDF of X_1+X_2 is obtained as

$$f_{X_1+X_2}(x;\alpha,\xi)=\int_0^x f(t;\alpha,\xi)f(x-t;\alpha,\xi)dt$$

$$= \frac{(1+\alpha)^2}{\alpha^2}\left[\int_0^x g(t;\xi)g(x-t;\xi)dt - \int_0^x g(t;\xi)g((x-t)(1+\alpha);\xi)dt \right.$$
$$\left. - \int_0^x g(t(1+\alpha);\xi)g(x-t;\xi)dt + \int_0^x g(t(1+\alpha);\xi)g((x-t)(1+\alpha);\xi)dt\right].$$

With the use of the change of variable $y = x - t$ for the third integral and $y = t(1+\alpha)$ for last integral, we get

$$f_{X_1+X_2}(x;\alpha,\xi) =$$
$$\frac{(1+\alpha)^2}{\alpha^2}\left[(g\star g)(x;\xi) - \frac{2}{1+\alpha}(g_\alpha \star g)(x;\xi) + \frac{1}{(1+\alpha)^2}(g\star g)_\alpha(x;\xi)\right],$$

proving the second point.

This ends the proof of Proposition 3.1. □

One can note that $(g\star g)(x;\xi)$ corresponds to the PDF of the sum of two independent and identically distributed random variables from Y. Also, $(g_\alpha \star g)(x;\xi)$ corresponds to the PDF of the sum of two independent random variables, one is from Y and the other is from $Y/(1+\alpha)$. Also, the obtained PDFs in Proposition 3.1 are defined as generalized mixtures of PDFs, which are new in the literature as far as we know.

In the setting of the SGMLx distribution, the PDFs $(g\star g)(x;\xi)$ and $(g_\alpha\star g)(x;\xi)$ can be expressed as infinite series, as developed in [7].

3.2. Crude moments

Crude moments are of importance to study some precise characteristics of a distribution. Here, we express them for the proposed SGM-G class.

Proposition 3.2. *Let r be a positive integer. The r^{th} crude moment of X, say $\mu'_{r,X} = E(X^r)$, and the r^{th} crude moment of Y, say $\mu'_{r,Y} = E(Y^r)$, satisfy the following relation:*

$$\mu'_{r,X} = \frac{1+\alpha}{\alpha}\left[1 - \frac{1}{(1+\alpha)^{r+1}}\right]\mu'_{r,Y}.$$

(It is worth mentioning that $\mu'_{r,Y}$ don't always exist).

Proof. After development and the use of the change of variable $y = x(1+\alpha)$, we get

$$\mu'_{r,X} = \int_0^{+\infty} x^r f(x;\alpha,\xi)dx$$
$$= \frac{1+\alpha}{\alpha}\left[\int_0^{+\infty} x^r g(x;\xi)dx - \int_0^{+\infty} x^r g(x(1+\alpha);\xi)dx\right]$$
$$= \frac{1+\alpha}{\alpha}\left[1 - \frac{1}{(1+\alpha)^{r+1}}\right]\int_0^{+\infty} y^r g(y;\xi)dy = \frac{1+\alpha}{\alpha}\left[1 - \frac{1}{(1+\alpha)^{r+1}}\right]\mu'_{r,Y}.$$

This ends the proof of Proposition 3.2. □

Thanks to Proposition 3.2, the four first crude moments of X are given below:

$$\mu'_{1,X} = \frac{\alpha+2}{1+\alpha}\mu'_{1,Y}, \quad \mu'_{2,X} = \frac{\alpha^2+3\alpha+3}{(1+\alpha)^2}\mu'_{2,Y}, \quad \mu'_{3,X} = \frac{(\alpha+2)(\alpha^2+2\alpha+2)}{(1+\alpha)^3}\mu'_{3,Y}$$

and

$$\mu'_{4,X} = \frac{\alpha^4+5\alpha^3+10\alpha^2+10\alpha+5}{(1+\alpha)^4}\mu'_{4,Y}.$$

They are useful to derive various measures of X, as the mean, variance, standard deviation, index of dispersion, skewness and kurtosis coefficients, and so on. For instance, in the context of the SGMLx distribution, for $\beta > r$, we have $\mu_{r,Y} = \lambda^r r!/\prod_{u=1}^{r}(\beta - u)$, implying that

$$\mu'_{r,X} = \frac{1+\alpha}{\alpha}\left[1 - \frac{1}{(1+\alpha)^{r+1}}\right]\frac{\lambda^r r!}{\prod_{u=1}^{r}(\beta-u)}.$$

In particular, for $\beta > 2$, the mean, second crude moment and variance of X are given by, respectively,

$$\mu = \mathrm{E}(X) = \mu'_{1,X} = \frac{(\alpha+2)\lambda}{(1+\alpha)(\beta-1)}, \quad \mu'_{2,X} = \frac{2(\alpha^2+3\alpha+3)\lambda^2}{(1+\alpha)^2(\beta-2)(\beta-1)},$$

and

$$\mathrm{Var}(X) = \mu'_{2,X} - \mu^2 = \frac{\lambda^2(\alpha^2\beta+2\alpha\beta+2\alpha+2\beta+2)}{(1+\alpha)^2(\beta-2)(\beta-1)^2},$$

respectively. The index of dispersion of X is given as

$$\mathrm{IxDis}(X) = \frac{\mathrm{Var}(X)}{\mathrm{E}(X)} = \frac{\lambda(\alpha^2\beta+2\alpha\beta+2\alpha+2\beta+2)}{(1+\alpha)(\alpha+2)(\beta-2)(\beta-1)}.$$

When $\alpha \to +\infty$, we rediscover the corresponding measures of the parent Lomax distribution. Also, if $\beta > r$, the r^{th} central moment of X can be determined via the following relation:

$$\begin{aligned}\mu^*_{r,X} &= \mathrm{E}((X-\mu)^r)\\ &= \frac{(-1)^r r!\lambda^r}{\alpha(1+\alpha)^r}\sum_{k=0}^{r}\frac{1}{(r-k)!}(-1)^k\frac{(\alpha+2)^{r-k}}{(\beta-1)^{r-k}}\left[(1+\alpha)^{k+1}-1\right]\frac{1}{\prod_{u=1}^{k}(\beta-u)}.\end{aligned}$$

From this formula, if $\beta > 4$, we can express the skewness and kurtosis of X coefficients as

$$\mathrm{Skew}(X) = \frac{\mu^*_{3,X}}{\mathrm{Var}(X)^{3/2}}, \quad \mathrm{Kur}(X) = \frac{\mu^*_{4,X}}{\mathrm{Var}(X)^2},$$

respectively. These measures are collected in Table 1 for selected values of α, β and λ.

Table 1. E(X), Var(X), IxDis(X), Skew(X) and Kur(X) of the SGMLx distribution.

α	β	λ	E(X)	Var(X)	IxDis(X)	Skew(X)	Kur(X)
0.0001	100	1000	20.20101	210.2865	10.4097	1.47922	6.378828
0.001			20.19193	210.0976	10.40503	1.479221	6.378831
0.1			19.28375	192.0491	9.959119	1.483942	6.406767
1			15.15152	131.1819	8.658009	1.67259	7.495892
5			11.78451	107.3517	9.10956	1.98955	9.170701
10			11.01928	105.1623	9.543478	2.038319	9.40061
50			10.29907	104.1927	10.11671	2.060445	9.499061
100			10.20094	104.1451	10.20936	2.061205	9.502409
200			10.15150	104.1208	10.25669	2.061612	9.504135
500			10.12121	104.1164	10.28695	2.061556	9.503903
1000			10.11111	104.1146	10.29705	2.061547	9.503866
2000			10.10606	104.1137	10.30210	2.061545	9.503856
10000			10.10202	104.1129	10.30614	2.061544	9.503853
50000			10.10121	104.1127	10.30695	2.061544	9.503853
500	5	1000	250.4996	104248.4	416.1618	4.64057	69.34955
	10		71.57142	5888.52	82.27474	2.482542	13.63037
	25		41.75000	1887.372	45.20652	2.267132	11.36265
	50		20.44898	433.8800	21.21769	2.126377	10.06007
	100		10.12121	104.1164	10.28695	2.061556	9.503903
	150		6.724832	45.65274	6.788681	2.040692	9.33049
1.5	7.5	50	10.76923	102.2055	9.490509	2.922870	21.51898
		100	21.53846	408.8219	18.98102	2.922870	21.51898
		200	43.07692	1635.288	37.96204	2.922870	21.51898
		500	107.6923	10220.55	94.90509	2.922870	21.51896
		1000	215.3846	40882.19	189.8102	2.922869	21.51878
		2000	430.7692	163528.8	379.6204	2.922863	21.51689
		5000	1076.923	1022052	949.0484	2.922556	21.47721
		10000	2153.843	4087870	1897.942	2.918038	21.18396
150	5.5	150	33.55385	1750.328	52.16474	4.147928	49.97932
300	10	500	55.74067	3860.570	69.25948	2.811081	17.82874
500	50	1000	20.44898	433.8800	21.21769	2.126377	10.06007

Based on Table 1, for the considered values of the parameters, we see that Skew(X) is positive. Further, the spread for Kur(X) is alway greater than 3. Also, IxDis(X) is superior to 1, so the SGMLx model may be used as an "over-dispersed" model.

3.3. Incomplete moments

The incomplete moments allow to define various mean deviations and functions of importance. Here, we express the incomplete moments in the setting of the SGM-G class.

Proposition 3.3. *Let r be a positive integer. The r^{th} incomplete moment of X at $t \geq 0$, say $\mu'_{r,X}(t) = E(X^r 1_{X\leq t})$, and the r^{th} incomplete moment of Y at $t \geq 0$, say $\mu'_{r,Y}(t) = E(Y^r 1_{Y\leq t})$, satisfy the following relation:*

$$\mu'_{r,X}(t) = \frac{1+\alpha}{\alpha}\left[\mu'_{r,Y}(t) - \frac{1}{(1+\alpha)^{r+1}}\mu'_{r,Y}\left(t(1+\alpha)\right)\right].$$

Proof. Following the lines of the proof of Proposition 3.2, we arrive at

$$\mu'_{r,X}(t) = \int_0^t x^r f(x;\alpha,\xi)dx = \frac{1+\alpha}{\alpha}\left[\int_0^t x^r g(x;\xi)dx - \int_0^t x^r g(x(1+\alpha);\xi)dx\right]$$
$$= \frac{1+\alpha}{\alpha}\left[\int_0^t x^r g(x;\xi)dx - \frac{1}{(1+\alpha)^{r+1}}\int_0^{t(1+\alpha)} y^r g(y;\xi)dy\right]$$
$$= \frac{1+\alpha}{\alpha}\left[\mu'_{r,Y}(t) - \frac{1}{(1+\alpha)^{r+1}}\mu'_{r,Y}\left(t(1+\alpha)\right)\right].$$

This completes the proof of Proposition 3.3. □

For most of the parent distributions, $\mu'_{r,Y}(t)$ exists and is well known, allowing to express $\mu'_{r,X}(t)$ in a simple way. That is, we can derive the mean deviation, mean residual life function, Lorenz curve, and so on. For instance, in the context of the SGMLx distribution, we have $\mu'_{r,Y}(t) = \lambda^r \beta B_{\lambda t/(1+\lambda t)}(r+1, \beta - r)$, where $B_x(a,b) = \int_0^x t^{a-1}(1-t)^{b-1}dt$, $x \geq 0$ and $a, b > 0$ is the so-called incomplete beta function, implying that

$$\mu'_{r,X}(t) = \frac{1+\alpha}{\alpha}\lambda^r\beta\times$$
$$\left[B_{\lambda t/(1+\lambda t)}(r+1,\beta-r) - \frac{1}{(1+\alpha)^{r+1}}B_{\lambda t(1+\alpha)/[1+\lambda t(1+\alpha)]}(r+1,\beta-r)\right].$$

As examples of application, one can express the mean deviation of X about μ via $\mu'_{1,X}(t)$ as

$$\delta = \mathrm{E}(|X-\mu|) = 2\mu F(\mu;\alpha,\beta,\lambda) - 2\mu'_{1,X}(\mu)$$
$$= \frac{2(\alpha+2)\lambda}{\alpha(\beta-1)}\left\{1 - \left(1 + \frac{(\alpha+2)}{(1+\alpha)(\beta-1)}\right)^{-\beta} - \frac{1}{1+\alpha}\left[1 - \left(1 + \frac{\alpha+2}{\beta-1}\right)^{-\beta}\right]\right\}$$
$$- 2\frac{1+\alpha}{\alpha}\lambda\beta\times$$
$$\left[B_{(\alpha+2)\lambda^2/[(1+\alpha)(\beta-1)+(\alpha+2)\lambda^2]}(2,\beta-1) - \frac{1}{(1+\alpha)^2}B_{(\alpha+2)\lambda^2/[\beta-1+(\alpha+2)\lambda^2]}(2,\beta-1)\right]$$

and the mean waiting time as

$$M(t) = \mathrm{E}(t - X \mid X \leq t)$$

$$= t - \frac{\lambda\beta\left[B_{\lambda t/(1+\lambda t)}(2,\beta-1) - B_{\lambda t(1+\alpha)/[1+\lambda t(1+\alpha)]}(2,\beta-1)/(1+\alpha)^2\right]}{1-(1+t/\lambda)^{-\beta} - \left[1-(1+t(1+\alpha)/\lambda)^{-\beta}\right]/(1+\alpha)}.$$

Further details and applications on these probabilistic objects, also defined with $\mu'_{r,X}(t)$ with $r > 1$, can be found in [5].

3.4. Characteristic function

The characteristic function is useful to determine several distributional properties involving the SGM-G class.

Proposition 3.4. *The characteristic function of X at $t \in \mathbb{R}$, say $\varphi_X(t) = E(e^{itX})$ with $i^2 = -1$, and the characteristic function of Y at $t \in \mathbb{R}$, say $\varphi_Y(t) = E(e^{itY})$, satisfy the following relation:*

$$\varphi_X(t) = \frac{1+\alpha}{\alpha}\left[\varphi_Y(t) - \frac{1}{1+\alpha}\varphi_Y\left(\frac{t}{1+\alpha}\right)\right].$$

Proof. By following the lines of the proof of Proposition 3.2, we get

$$\varphi_X(t) = \int_0^{+\infty} e^{itx} f(x;\alpha,\xi)dx = \frac{1+\alpha}{\alpha}\left[\int_0^{+\infty} e^{itx} g(x;\xi)dx - \int_0^{+\infty} e^{itx} g(x(1+\alpha);\xi)dx\right]$$
$$= \frac{1+\alpha}{\alpha}\left[\int_0^{+\infty} e^{itx} g(x;\xi)dx - \frac{1}{1+\alpha}\int_0^{+\infty} e^{ity/(1+\alpha)} g(y;\xi)dy\right]$$
$$= \frac{1+\alpha}{\alpha}\left[\varphi_Y(t) - \frac{1}{1+\alpha}\varphi_Y\left(\frac{t}{1+\alpha}\right)\right].$$

This completes the proof of Proposition 3.4. □

As an application of Proposition 3.4, the crude moment X can be obtained via the following formula: $\mu'_{r,X} = i^{-r}\varphi_X(t)^{(r)}\mid_{t=0}$. Also, the PDF of the sum of n independent and identically random variables $X_1, \ldots, X_n$ from X can be derived by the inversion formula:

$$f_{X_1+\ldots+X_n}(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-itx}[\varphi_X(t)]^n dt$$
$$= \frac{(1+\alpha)^n}{2\pi\alpha^n}\sum_{k=0}^{n}\binom{n}{k}(-1)^k\frac{1}{(1+\alpha)^k}\int_{-\infty}^{+\infty} e^{-itx}\left[\varphi_Y\left(\frac{t}{1+\alpha}\right)\right]^k [\varphi_Y(t)]^{n-k}dt.$$

This function can be expressed for some simple distributions on Y, the integral term corresponding to the Fourier transform of the function $[\varphi_Y(t/(1+\alpha))]^k[\varphi_Y(t)]^{n-k}$.

3.5. Bivariate extensions

Some bivariate extensions of the new class is now discussed, with the objective to applied it for the analysis of bivariate data in future studies. In this regard, diverse directions can be explored. For instance, one can consider the bivariate extension of the SGM-G class defined by the bivariate CDF specified by

$$F(x, y; \alpha_1, \alpha_2, \xi) =$$

$$\frac{(1+\alpha_1)(1+\alpha_2)}{\alpha_1+\alpha_2+\alpha_1\alpha_2}\left[G(x,y;\xi)-\frac{1}{(1+\alpha_1)(1+\alpha_2)}G(x(1+\alpha_1),y(1+\alpha_2);\xi)\right],$$
$$(x,y)\in(0,+\infty)^2,$$

(and zero otherwise), where $\alpha_1, \alpha_2 > 0$, $G(x, y; \xi)$ is a bivariate CDF of a parent distribution, with parameter(s) represented by ξ, and with decreasing PDF with respect to both x and y. A more clear structure of dependence between the two subjacent marginal random variables can be imposed by the use of the copulas (see [16]). Two examples are described below.

- By applying the Clayton copula, a bivariate extension of the SGM-G class is proposed by the following bivariate CDF:

  $$F(x,y;\omega,\alpha_1,\alpha_2,\xi_1,\xi_2)=\left[\max\left([F(x;\alpha_1,\xi_1)]^{-\omega}+[F(y;\alpha_2,\xi_2)]^{-\omega}-1,0\right)\right]^{-1/\omega},$$
  $$(x,y)\in(0,+\infty)^2,$$

  where and $\omega \neq 0$ and $\omega \geq -1$, $F(x; \alpha_1, \xi_1)$ and $F(y; \alpha_2, \xi_2)$ are the CDFs defined as (2.1).

- By applying the so-called Farlie-Gumbel-Morgenstern copula (see [15] and [9]), a bivariate extension of the SGM-G class is proposed by the following bivariate CDF:

  $$F(x,y;\omega,\alpha_1,\alpha_2,\xi_1,\xi_2)=F(x;\alpha_1,\xi_1)F(y;\alpha_2,\xi_2)\times$$
  $$\{1+\omega\,[1-F(x;\alpha_1,\xi_1)]\,[1-F(y;\alpha_2,\xi_2)]\},\quad (x,y)\in(0,+\infty)^2,$$

  where $\omega \in [-1, 1]$. Then, the parameter ω governs the independence of the subjacent marginal random variables; they are independent if $\omega = 0$.

Other interesting bivariate extensions can be derived from other various copulas.

4. Data analysis

In this section, we illustrate the importance and flexibility of the SGMLx model in a concrete data analysis.

4.1. Relief times data

We consider the data of [10] given as {1.1, 1.4, 1.3, 1.7, 1.9, 1.8, 1.6, 2.2, 1.7, 2.7, 4.1, 1.8, 1.5, 1.2, 1.4, 3, 1.7, 2.3, 1.6, 2}. This data set represents the lifetime's data relating to relief times (in minutes) of 20 patients receiving an analgesic. Figures 3 and 4 show the Box plot and the total time test (TTT) of the data, respectively.

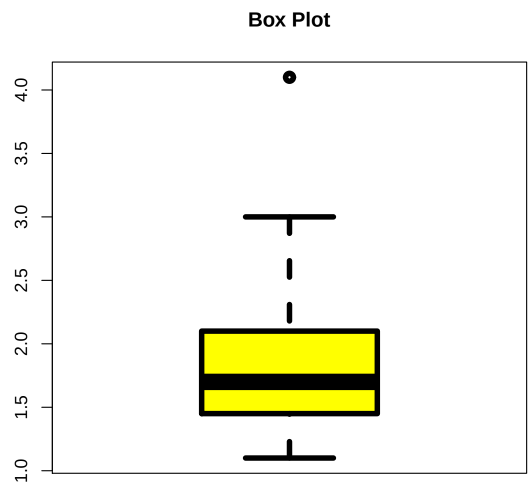

Figure 3. Box plot of the relief times data.

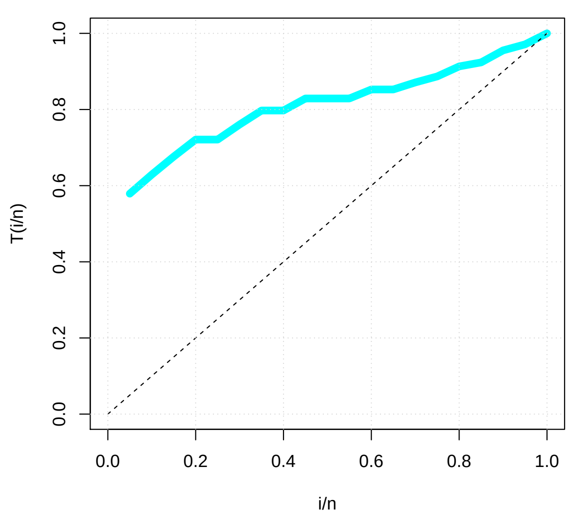

Figure 4. TTT plot of the relief times data.

Based on Figure 3, it is noted that the relief times data has an extreme value. Also, based on Figure 4, we see that the relief times data has an increasing HRF. As already mentioned before, these features can be captured by the SGMLx model.

4.2. Statistical methodology

The maximum likelihood methodology is adopted. In the setting of the SGMLx distribution, by denoting $x_1, \ldots, x_n$ some generic data, it consists in determining the maximum likelihood estimates given as

$$(\hat{\alpha}, \hat{\beta}, \hat{\lambda}) = \text{argmax}_{(\alpha,\beta,\lambda)\in(0,+\infty)^3}\, \ell_{(\alpha,\beta,\lambda)},$$

where $\ell_{(\alpha,\beta,\lambda)}$ denotes the log-likelihood function given as

$$\ell_{(\alpha,\beta,\lambda)} = \sum_{i=1}^{n} \log[f(x_i;\alpha,\beta,\lambda)] = n\log(1+\alpha) + n\log\beta - n\log\alpha - n\log\lambda \\ + \sum_{i=1}^{n} \log\left[\left(1+\frac{x_i}{\lambda}\right)^{-\beta-1} - \left(1+\frac{x_i(1+\alpha)}{\lambda}\right)^{-\beta-1}\right].$$

Then, we can plug-in the estimates $\hat{\alpha}$, $\hat{\beta}$ and $\hat{\lambda}$ into the functions of the SGMLx distribution to obtained estimates of them. For instance, the estimated CDF and PDF of the SGMLx distribution are given as $\hat{F}(x) = F(x;\hat{\alpha},\hat{\beta},\hat{\lambda})$ and $\hat{f}(x) = f(x;\hat{\alpha},\hat{\beta},\hat{\lambda})$, respectively. This methodology can be transposed to any other models, this is what we will do in the next study

Here, we compare the fits of the SGMLx model with those of the standard two parameters Lx (2PLx) model (see [14]), one-parameter Lx (1PLx) model (see [14]), exponential (Exp) model, one-parameter Weibull (W) model (see [22]), one-parameter Log-logistic (1PLL) model (see [3]), three-parameter Burr-Hatke Lx (3PBHLx) model (new based on [23]), two-parameter Burr-Hatke Lx (2PBHLx) model (new based on [23]), two-parameter Burr-Hatke Exp (2PBHExp) model (see [23]) and one-parameter Burr-Hatke Exp (1PBHExp) model (see [23]).

In order to compare these models, the following goodness of-fit statistics are considered:

1. Akaike Information Criteria (AIC)

$$\text{AIC} = -2\hat{\ell} + 2n_{[p]},$$

2. Consistent AIC (CAIC):

$$\text{CAIC} = -2\hat{\ell} + \frac{2nn_{[p]}}{n - n_{[p]} - 1},$$

3. Bayesian IC (BIC):

$$\text{BIC} = -2\hat{\ell} + n_{[p]}\log n,$$

4. Hannan-Quinn IC (HQIC):

$$\text{HQIC} = -2\hat{\ell} + \frac{2n_{[p]}}{\log(\log n)},$$

where $n_{[p]}$ is the number of the model parameters, n is the number of data (here, $n = 20$), and $\hat{\ell}$ is the log-likelihood function determined at the estimated maximum likelihood parameter(s). The better model is the one having the smaller value of $-\hat{\ell}$, AIC, CAIC, BIC and HQIC.

4.3. Numerical and graphical studies

With the above methodology, we now present complete numerical and graphical studies based on the relief times data. Table 2 provides the maximum likelihood estimates (MLEs), with standard errors (SEs) in parentheses.

Table 2. MLEs and SEs for the relief times data.

Model	Estimates		
SGMLx(α, β, λ)	0.5017448	1.22929×10^7	1.50076×10^7
3PBHLx(α, β, λ)	1.3126×10	1.42083×10^5	5.35617×10^5
2PBHLx(α, β)	20332130	72914938	-
2PBHExp(α, β)	0.007196	72.11277	-
2PLx(α, β)	25119398	4772482	-
1PBHExp(α)	0.279000	-	-
Exp(α)	0.526172	-	-
W(α)	9.995036	-	-
Lx(α)	0.960352	-	-
1PLL(α)	2.491406	-	-

Among others, from Table 2, for the SGMLx model, we see that the MLEs of α, β and λ, are given as $\hat{\alpha} = 0.5017448$, $\hat{\beta} = 1.22929 \times 10^7$ and $\hat{\lambda} = 1.50076 \times 10^7$, respectively. Table 3 completes Table 2 by giving the corresponding goodness-of-fits statistics.

Table 3. $-\hat{\ell}$, AIC, CAIC, BIC and HQIC for the relief times data.

Model	$-\hat{\ell}$	AIC	CAIC	BIC	HQIC
SGMLx	**26.42983**	**58.85966**	**60.35966**	**61.84686**	**59.44279**
3PBHLx	34.09499	74.18999	75.68999	77.17719	74.77312
2PBHLx	34.35456	72.70911	73.41500	74.70058	73.09787
2PBHExp	32.83867	69.67735	70.38323	71.66881	70.06610
2PLx	32.83708	69.67416	70.38004	71.66562	70.06291
1PBHExp	34.35456	70.70911	70.93133	71.70484	70.90349
Exp	32.83708	67.67416	67.89638	68.66989	67.86853
W	31.78509	65.57018	65.7924	66.56591	65.76456
1PLx	41.63307	85.26614	85.48837	86.26188	85.46052
1PLL	32.80248	67.60496	67.82719	68.60070	67.79934

Based on Table 3, the SGMLx model provides an adequate fit, more than the standard two-parameter Lomax model, one-parameter Lomax model, exponential model, one-parameter Weibull model, one-parameter Log-logistic model, three parameter Burr-Hatke Lomax model, two-parameter Burr-Hatke Lomax model, two-parameter Burr-Hatke exponential model and one-parameter Burr-Hatke exponential model.

Based on the results for the SGMLx model, Figures 5, 6, 7 and 8 give the estimated PDF (EPDF), estimated CDF (ECDF), probability-probability (P-P) plot and estimated HRF (EHRF), respectively.

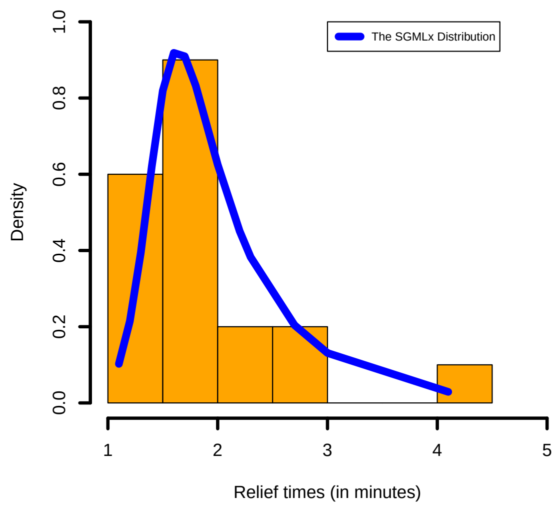

Figure 5. Plot of the EPDF of the SGMLx model over the histogram for the relief times data.

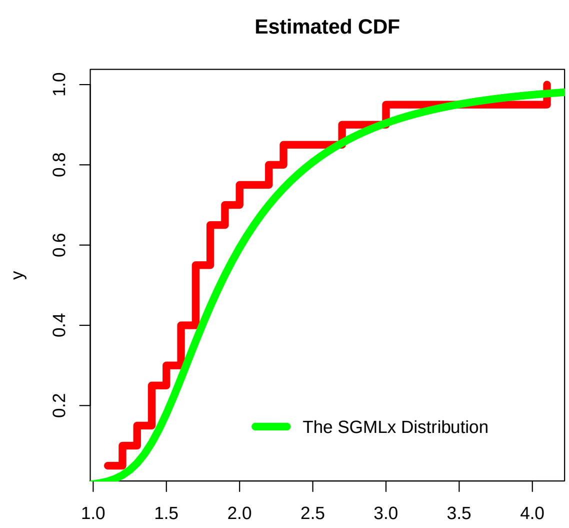

Figure 6. Plot of the ECDF of the SGMLx model over the empirical CDF for the relief times data.

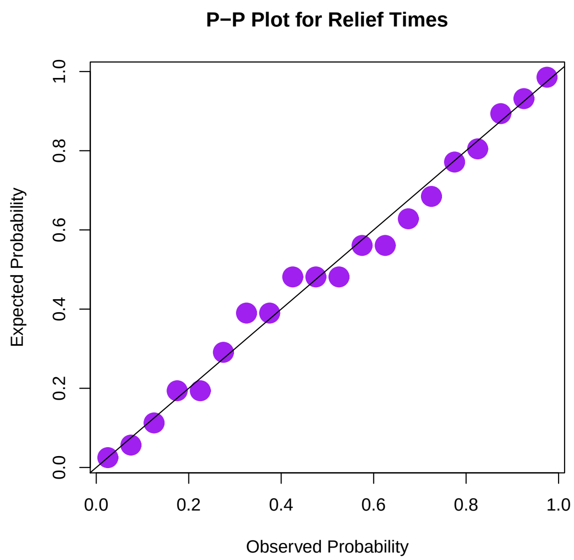

Figure 7. P-P plot of the SGMLx model for the relief times data.

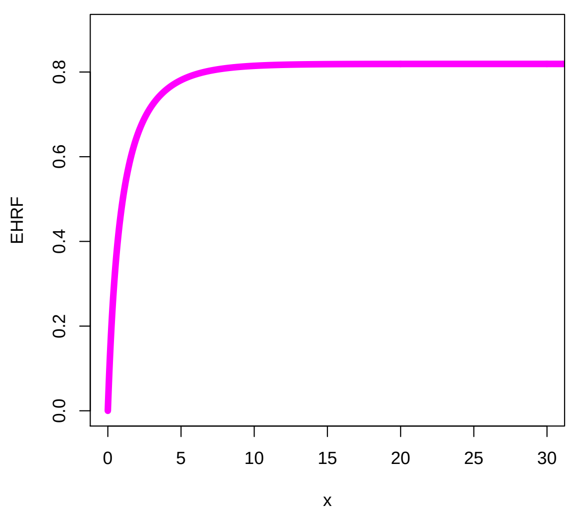

Figure 8. Plot of the EHRF of the SGMLx model for the relief times data.

The fits of the SGMLx model displayed in Figures 5, 6 and 7 are quite satisfying, showing that the main part of the data has been captured, as well as the extreme value. The EHRF depicts in Figure 8 has the expected form, in view of the related TTT plot.

5. Conclusion

The foundation of this study was the generalized two-component mixture involved in the construction of the famous weighted exponential distribution introduced by [11]. We have developed a new strategy to extend this special generalized mixtures, relaxing the choice of the exponential distribution as parent, and thus proposing a new class of distributions. We have shown several of its mathematical and practical properties, with a focus on the special distribution using the Lomax distribution as parent. Then, the analysis of a medical data sets shows that the proposed methodology can be applied quite efficiency for data fitting objectives.

As perspectives of work, one can think to apply the proposed bivariate extensions to appropriate real data sets, which can find nice issues in the frameworks of the simple regression or clustering. Also, in view of the recent works on the former weighted exponential distribution, possible interesting extensions of the new class can be its power version defined by a CDF of the form: $F(x;\alpha,\gamma,\xi) = F(x^{\gamma};\alpha,\xi)$, exponentiated version defined by a CDF of the form: $F(x;\alpha,\upsilon,\xi) = F(x;\alpha,\xi)^{\upsilon}$ and Pareto-like distributions with CDF of the form: $F(x;\alpha,x_0,\xi) = F(x-x_0;\alpha,\gamma,\xi)$ for $x > x_0$. These extensions deserve further treatments that we leave for future works.

Acknowledgements

We would like to thank the reviewers for the constructive comments which have helped to improve the paper.

References

[1] D. K. Al-Mutairi, M. E. Ghitany and D. Kundu, *A new bivariate distribution with weighted exponential marginals and its multivariate generalization*, Statistical Papers, 2011, 52, 921–936.

[2] A. Azzalini, *A class of distributions which includes the normal ones*, Scandinavian Journal of Statistics, 1985, 12(2), 171–178.

[3] I. W. Burr, *Cumulative frequency functions*, Annals of Mathematical Statistics, 1942, 13(2), 215–232.

[4] G. M. Cordeiro and A. J. Lemonte, *The beta-half Cauchy distribution*, Journal of probability and Statistics, 2011, Article ID 904705, 18 pages.

[5] G. M. Cordeiro, R. B. Silva and A. D. C. Nascimento, *Recent Advances in Lifetime and Reliability Models*, Bentham books, 2020.
DOI: 10.2174/97816810834521200101

[6] S. Das and D. Kundu, *On weighted exponential distribution and its length biased version*, Journal of the Indian Society of Probability and Statistics, 2016, 17, 57–77.

[7] D. Dyer, *The convolution of generalized F distributions*, Journal of the American Statistical Association, 1982, 77(377), 184–189.

[8] Z. S. M. Farahani and E. Khorram, *Bayesian statistical inference for weighted exponential distribution*, Communications in Statistics - Simulation and Computation, 2014, 43(6), 1362–1382.

[9] D. J. G. Farlie, *The performance of some correlation coefficients for a general bivariate distribution*, Biometrika, 1960, 47(3-4), 307–323.

[10] A. J. Gross and V. A. Clark, *Survival Distributions: Reliability Applications in the Biometrical Sciences*, John Wiley, New York, 1975.

[11] R. D. Gupta and D. Kundu, *A new class of weighted exponential distributions*, Statistics, 2009, 43(6), 621–634.

[12] R. Jiang, M. J. Zuo and H. X. Li, *Weibull and inverse Weibull mixture models allowing negative weights*, Reliability Engineering and System Safety, 1999, 66(3), 227–234.

[13] F. C. Leone, L. S. Nelson and R. B. Nottingham, *The folded normal distribution*, Technometrics, 1961, 3(4), 543–550.

[14] K. S. Lomax, *Business failures: Another example of the analysis of failure data*, Journal of the American Statistical Association, 1954, 49(268), 847–852.

[15] D. Morgenstern, *Einfache beispiele zweidimensionaler verteilungen*, Mitteilingsblatt fur Mathematische Statistik, 1956, 8, 234–235.

[16] R. B. Nelsen, *An Introduction to Copulas*, 2nd Edition, Springer-Verlag, Berlin, 2006.
DOI: 10.1007/0-387-28678-0

[17] A. K. Olapade, *On characterizations of the half-logistic distribution*, Interstat, 2003, 2(2).

[18] S. Roy and M. A. S. Adnan, *Wrapped weighted exponential distributions*, Statistics and Probability Letters, 2012, 82(1), 77–83.

[19] M. K. Shakhatreh, *A two-parameter of weighted exponential distribution*, Statistics and Probability Letters, 2012, 82(2), 252–261.

[20] M. W. Shaw, *Simulation of population expansion and spatial pattern when individual dispersal distributions do not decline exponentially with distance*, Proceedings of the Royal Society B, 1995, 259(1356), 243–248.

[21] D. M. Titterington, A. F. M. Smith and U. E. Makov, *Statistical Analysis of Finite Mixture Distribution*, Wiley, New York, 1985.

[22] W. Weibull, *A statistical distribution function of wide applicability*, Journal of Applied Mechanics, 1951, 18(3), 293–297.

[23] H. M. Yousof, E. Altun, T. G. Ramires, M. Alizadeh and M. Rasekhi, *A new family of distributions with properties, regression models and applications*, Journal of Statistics and Management Systems, 2018, 21(1), 163–188.