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Existence of Periodic Solutions in Impulsive Differential Equations*

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Abstract In this paper, we are concerned with the problem of existence of periodic solutions for a class of second order impulsive differential equations. By Poincaré-Bohl theorem, we give several criteria to guarantee that the impulsive differential equation has periodic solutions under assumptions that the nonlinear term satisfies the linear growth conditions. Two specific examples are presented to illustrate the obtained results.

Keywords Impulse, Poincaré-Bohl theorem, Periodic solution, Existence.

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1. Introduction

In this paper, we consider the following second order impulsive differential equation

$$\begin{cases} x'' + g(x) = e(t, x, x'), & t \neq t_k,\ t \in \mathbb{R}, \\ x(t_k^+) = a_k x(t_k), & \\ x'(t_k^+) = b_k x'(t_k), & t = t_k,\ k \in \mathbb{Z}_+, \end{cases} \tag{1.1}$$

where $(x(t_k), x'(t_k)) = (x(t_k^-), x'(t_k^-))$, $g : \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz continuous function satisfying the linear growth condition

$$0 < l \leq \lim_{|x| \to +\infty} \frac{g(x)}{x} \leq L < +\infty,$$

$e : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous, bounded and 2π-periodic to the first variable, $0 < t_k < t_{k+1} \uparrow +\infty$, $a_k > 0, b_k > 0$ are constants and there exists a positive integer q such that $a_{k+q} = a_k, b_{k+q} = b_k$ and $t_{k+q} = t_k + 2\pi$ for $k \in \mathbb{Z}_+$, $\mathbb{Z}_+$ is the set of positive integers.

Impulsive effects widely exist in many evolution processes, in which their states are changed abruptly at some moments. Impulsive differential equation has been developed by many mathematicians. Please see the classical monographs [1, 13], and [8, 9, 15–19, 21–24] for the existence of periodic solutions. In addition, applications of the impulsive differential equation with/or without delays occur in biology, mechanics, engineering etc., see for example [11, 25–27] and the references therein.

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The continuous case of (1.1) without impulses is as follows

$$x'' + g(x) = e(t, x, x'). \tag{1.2}$$

This type of second order differential equation is one of the typical models both in ODE and forced vibrations. Particularly, the Duffing equation ($i.e.$ $x''+g(x) = e(t)$) is a class of mathematical and physical equations, it has many important applications in mechanical and electrical engineering. Recent decades, the existence of periodic solutions has been extensively studied for the Duffing equation under assumptions that the nonlinear term g satisfies superlinear, sublinear or semilinear conditions. The methods include fixed point theorem, variational method and topological degree theory [2–7, 10, 14].

When $e(t, x, x') = e(t)$, (1.2) is conservative. The Poincaré mapping is an area-preserving homeomorphism. By Poincaré-Birkhoff theorem, Jacobowitz [10] gave the first application to periodic solutions of second order differential equations, and proved the existence of infinitely many periodic solutions. See for example [4–6, 20] for some related researches and the references therein. Recently, the Poincaré-Birkhoff theorem has been applied to the impulsive differential equation [12, 17, 18]. In [12], Jiang et al. gave the first application of the Poincaré-Birkhoff theorem to the following impulsive Duffing equations at resonance

$$\begin{cases} x'' + g(x) = e(t), & t \neq t_k, \\ x(t_k^+) = a_k x(t_k), & \\ x'(t_k^+) = b_k x'(t_k), & k \in \mathbb{Z}_+, \end{cases} \tag{1.3}$$

under the superlinear condition. By using a method of comparing rotational inertia, the authors obtained the multiplicity of periodic solutions. In [18], Qian et al. used a geometric method to study the periodic solutions for a superlinear second order differential equation with general impulsive effects as follows

$$\begin{cases} x'' + g(x) = e(t, x, x'), & t \neq t_k, \\ \triangle x(t_k) = I_k(x(t_k), x'(t_k)), & \\ \triangle x'(t_k) = J_k(x(t_k), x'(t_k)), & k = \pm 1, \pm 2, \cdots, \end{cases} \tag{1.4}$$

and obtained the multiplicity when (1.4) is conservative. In [17], Niu and Li studied a conservative semilinear impulsive Duffing equation, in which their states occur one jump only in $[0, 2\pi]$. Similarly, by the Poincaré-Birkhoff theorem they proved the existence of infinitely many periodic solutions for autonomous and nonautonomous equations respectively. On the other hand, when (1.2) is nonconservative, Ding [7] developed a new twist fixed point theorem used for nonarea-preserving mappings. In [18], Qian et al. further studied the existence of periodic solutions for (1.4) being nonconservative.

In this paper, we study the existence of periodic solutions for a class of semilinear second order impulsive differential equations (1.1). Our aim is to estimate the time that any solution trajectory of the system rotates one circle on the phase plane. By using the Poincaré-Bohl theorem to obtain the existence results. As far as I know, there is no result on the existence of periodic solutions for the semilinear second order differential equations with the linear impulsive effects.

Now we recall an existence result from the Poincaré-Bohl theorem [4].

Theorem 1.1. *(two-dimension Poincaré-Bohl theorem) Suppose that $\mathcal{F} : \mathbb{D} \to \mathbb{R}^2$ is a continuous mapping, where $\mathbb{D}(\subset \mathbb{R}^2)$ denotes a closed bounded region including the origin O as an interior point, and the boundary $\partial\mathbb{D}$ is a piecewise smooth simple closed curve. For any $p \in \partial\mathbb{D}$, if the image $q = \mathcal{F}(p)$ satisfies $\overrightarrow{Oq} \neq \lambda\overrightarrow{Op}$, $\lambda > 0$ is a constant. Then, $\mathcal{F}$ has at least one fixed point in $\mathbb{D}$.*

It means that for any solution trajectory which starts from $p = (x(0), x'(0)) \in \partial\mathbb{D}$ and moves to $q = (x(2\pi), x'(2\pi)) \in \mathbb{R}^2$, the points p and q are not on the same ray starting from the origin.

This paper is organized as follows. In Section 2, we present some preliminaries. By estimating the time that any solution trajectory rotates one circle on the phase plane, we obtain the existence of periodic solutions via the two-dimension Poincaré-Bohl theorem (*i.e.* Theorem 1.1). Several existence criteria are presented by giving different assumptions to the growth speed of g. In Section 3, two specific examples with special impulses are given to illustrate the obtained results. Concluding remarks are outlined in section 4.

2. Preliminaries

Firstly, we recall some basic properties of the impulsive differential equation from [1]. Consider the following initial value problem

$$\begin{cases} u' = f(t,u), & t \neq t_k, \\ \Delta u(t_k) = I_k(u(t_k)), & k \in \mathbb{Z}_+, \end{cases} \tag{2.1}$$

$$u(0^+) = u_0, \tag{2.2}$$

where $\Delta u(t_k) = u(t_k^+) - u(t_k^-)$ with $u(t_k) = u(t_k^-)$, $k \in \mathbb{Z}_+$. Assume that

(i) $f : \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^n$ is continuous in $(t_k, t_{k+1}]\times\mathbb{R}^n$, locally Lipschitz with respect to the second variable and the limits $\lim_{t\to t_k^+,\nu\to u} f(t,\nu), k \in \mathbb{Z}_+$ exist;

(ii) $I_k : \mathbb{R}^n \to \mathbb{R}^n, k \in \mathbb{Z}_+$ are continuous;

(iii) f is 2π-periodic to the first variable, there exists an integer $q > 0$ satisfying $0 < t_1 < \cdots < t_q < 2\pi, t_{k+q} = t_k + 2\pi$ and $I_{k+q} = I_k$ for $k \in \mathbb{Z}_+$.

Lemma 2.1. *Assume that (i)-(iii) hold. For any $u_0 \in \mathbb{R}^n$, there exists a unique solution $u(t) = u(t; 0, u_0)$ of (2.1)-(2.2). Further, $\mathcal{P}_t : u_0 \to u(t; 0, u_0)$ is continuous with respect to u_0 for $t \neq t_k, k \in \mathbb{Z}_+$.*

It is easy to show that solutions of (2.1) exist for $t \in \mathbb{R}$ provided that solutions of the corresponding differential equation without impulses exist for $t \in \mathbb{R}$. Moreover, if $\Phi_k : u_k \to u_k^+$ (where $u_k = u(t_k)$, $u_k^+ = u(t_k^+) = u_k + I_k(u_k)$), $k = 1, 2, \cdots, q$ are global homeomorphisms, $\mathcal{P}_t$ is a homeomorphism for $t \neq t_k, k \in \mathbb{Z}_+$. Denote by $\mathcal{P}_0 : u_0 \to u(t_1; 0, u_0)$, $\mathcal{P}_k : u_k^+ \to u(t_{k+1}; t_k, u_k^+), k = 1, \cdots, q-1$ and $\mathcal{P}_q : u_q^+ \to u(2\pi; t_q, u_q^+)$. Then, the Poincaré mapping $\mathcal{P}$ can be expressed by

$$\mathcal{P} : u_0 \to u(2\pi; 0, u_0), \qquad \mathcal{P} = \mathcal{P}_q \circ \Phi_q \circ \cdots \circ \mathcal{P}_1 \circ \Phi_1 \circ \mathcal{P}_0.$$

For $t \in [0, 2\pi]$, we consider the following equivalent system of (1.1)

$$\begin{cases} x' = y, \\ y' = -g(x) + e(t, x, y), & t \neq t_k,\ t \in [0, 2\pi], \\ x(t_k^+) = a_k x(t_k), \\ y(t_k^+) = b_k y(t_k), & t = t_k,\ k = 1, 2, \cdots, q, \end{cases} \tag{2.3}$$

and the initial value condition

$$x(0) = x_0, \qquad y(0) = y_0. \tag{2.4}$$

If $g(x)$ is assumed to be

$$(g_0): \lim_{x \to +\infty} g(x) = +\infty, \qquad \lim_{x \to -\infty} g(x) = -\infty,$$

solutions of (2.3) are defined on the whole t-axis. Let $x(t) = x(t; x_0, y_0)$, $y(t) = y(t; x_0, y_0)$ be the unique solution pair of (2.3)-(2.4). Then, the Poincaré mapping $\mathcal{P} : \mathbb{R}^2 \to \mathbb{R}^2$ is well defined by

$$\mathcal{P} : (x_0, y_0) \to (x(2\pi; x_0, y_0), y(2\pi; x_0, y_0)).$$

Defining the mappings $\mathcal{P}_i$ and Φ_i as follows

$$\begin{aligned} &\mathcal{P}_0 : (x_0, y_0) \to (x(t_1; x_0, y_0), y(t_1; x_0, y_0)) = (x_1, y_1), \\ &\Phi_0 : (x_1, y_1) \to (a_1 x_1, b_1 y_1) = (x_1^*, y_1^*), \\ &\mathcal{P}_i : (x_i^*, y_i^*) \to (x(t_{i+1}; x_0, y_0), y(t_{i+1}; x_0, y_0)) = (x_{i+1}, y_{i+1}), \\ &\Phi_i : (x_{i+1}, y_{i+1}) \to (a_{i+1} x_{i+1}, b_{i+1} y_{i+1}) = (x_{i+1}^*, y_{i+1}^*), \\ &\mathcal{P}_q : (x_q^*, y_q^*) \to (x(2\pi; x_0, y_0), y(2\pi; x_0, y_0)) = (x(2\pi), y(2\pi)), \qquad i = 1, 2, \cdots, q-1. \end{aligned}$$

Further, the Poincaré mapping $\mathcal{P}$ is of the form

$$\mathcal{P} = \mathcal{P}_q \circ \Phi_{q-1} \circ \cdots \circ \mathcal{P}_1 \circ \Phi_0 \circ \mathcal{P}_0.$$

Note that $\mathcal{P}$ is continuous to (x_0, y_0), and its fixed points correspond to the periodic solutions of (2.3).

Making the polar coordinates transformation

$$x(t) = r(t) \cos\theta(t), \qquad y(t) = r(t) \sin\theta(t),$$

then (2.3) becomes

$$\begin{cases} r'(t) = r \cos\theta \sin\theta + [e(t, r\cos\theta, r\sin\theta) - g(r\cos\theta)] \sin\theta, \\ \theta'(t) = -\sin^2\theta + \dfrac{1}{r}[e(t, r\cos\theta, r\sin\theta) - g(r\cos\theta)] \cos\theta, & t \neq t_k,\ t \in [0, 2\pi], \end{cases} \tag{2.5}$$

whenever $r \neq 0$, and for $t = t_k, k \in \mathbb{Z}_+$ one has that

$$\begin{cases} r(t_k^+) = \sqrt{a_k^2 x^2(t_k) + b_k^2 y^2(t_k)}, \\ \theta(t_k^+) = \arctan\Big(\dfrac{b_k}{a_k} \tan\theta(t_k)\Big). \end{cases} \tag{2.6}$$

Let $r(t) = r(t; r_0, \theta_0), \theta(t) = \theta(t; r_0, \theta_0)$ be the solution of (2.5) satisfying the initial value $(r(0), \theta(0)) = (r_0, \theta_0)$ and $x_0 = r_0 \cos\theta_0, y_0 = r_0 \sin\theta_0$. The Poincaré mapping is written in the polar coordinates form

$$r^* = r(2\pi; r_0, \theta_0), \qquad \theta^* = \theta(2\pi; r_0, \theta_0) + 2l\pi, \tag{2.7}$$

where l is an arbitrary integer. It can be easily seen that if

$$r(t; r_0, \theta_0) > 0, \qquad t \in [0, 2\pi],$$

then $\theta(2\pi; r_0, \theta_0)$ is well defined and it is continuous to (r_0, θ_0).

For any given $R_0 > 0$, denote by $\mathbb{S}_{R_0} = \{(r, \theta) : r = R_0\}$. Let $L(R_0)$ denote the solution trajectory of (2.5), which starts from $\mathbb{S}_{R_0}$ at $t = 0$. When $t \in [0, 2\pi]$, $L(R_0, [0, 2\pi])$ denotes the trajectory arc. Assume that $(r(t), \theta(t))$ is the solution of (2.5), which passes (r_0, θ_0) and satisfies $r_0 = R_0, \theta_0 \in [0, 2\pi]$. Under the assumption (g_0), it is easy to prove the following result.

Lemma 2.2. *Assume that (g_0) holds, then there exists $R_0 > 0$ sufficiently large such that when $r(t) > R_0$ we have that*

$$\theta'(t) < 0, \qquad t \in [0, 2\pi] \setminus \{t_k\}_{k=1}^q.$$

Proof. Denote by

$$E = \max_{\mathbb{R}^3}\{|e(t, x, y)|\}, \qquad \overline{E} = \max_{\mathbb{R}^3}\{e(t, x, y)\}, \qquad \underline{E} = \min_{\mathbb{R}^3}\{e(t, x, y)\}.$$

By (g_0), there exists $N > 0$ sufficiently large such that

$$g(x) > \overline{E}, \quad x \geq N, \qquad g(x) < \underline{E}, \quad x \leq -N.$$

Let $R_0 > N$ and consider $r(t) > R_0$. When $|r(t)\cos\theta(t)| \geq N, t \in [0, 2\pi]$, it follows that

$$\theta'(t) = -\sin^2\theta - \frac{[g(r\cos\theta) - e(t, r\cos\theta, r\sin\theta)]\cos^2\theta}{r\cos\theta} < 0$$

for $t \neq t_k, k = 1, 2, \cdots, q$. While for $|r(t)\cos\theta(t)| < N, t \in [0, 2\pi]$ then

$$\theta'(t) < -\sin^2\theta + \frac{|e(t, r\cos\theta, r\sin\theta) - g(r\cos\theta)|}{r} < -\sin^2\theta + \frac{\delta}{R_0} < -\frac{R_0^2 - N^2 - R_0\delta}{R_0^2}$$

for $t \neq t_k, k = 1, 2, \cdots, q$, where $\delta = E + \max_{|x| \leq N} |g(x)|$.

Therefore, $\theta'(t) < 0$ for $t \in [0, 2\pi] \setminus \{t_k\}_{k=1}^q$ as long as $R_0 > \frac{1}{2}\delta + \sqrt{N^2 + \frac{1}{4}\delta^2}$. □

In this paper, we assume that the following condition holds.

$(H1)$ There exists $A_1 > 0$ such that

$$q_1 \leq \frac{g(x)}{x} \leq p_1, \quad x \leq -A_1, \qquad q_2 \leq \frac{g(x)}{x} \leq p_2, \quad x \geq A_1, \tag{2.8}$$

where p_1, q_1, p_2, q_2 are positive constants, and there is an integer $m > 0$ such that

$$\frac{1}{\sqrt{q_1}} + \frac{1}{\sqrt{q_2}} = \frac{2}{m}, \qquad \frac{1}{\sqrt{p_1}} + \frac{1}{\sqrt{p_2}} = \frac{2}{m+1}. \tag{2.9}$$

It is obvious that (2.8) implies (g_0) and $\overline{\lim}_{|x|\to+\infty}|\frac{g(x)}{x}|<+\infty$.

Lemma 2.3. *Assume that* $\overline{\lim}_{|x|\to+\infty}|\frac{g(x)}{x}|=b<+\infty$, *then there exist a constant* $R_1>0$ *and functions* $d_1,d_2:(0,+\infty)\to(0,+\infty)$ *such that when* $R_0\geq R_1$,

(1) $d_1(R_0)\leq r(t)\leq d_2(R_0), \qquad t\in[0,2\pi],\ r_0=R_0,$

(2) $\lim_{R_0\to+\infty}d_1(R_0)=\lim_{R_0\to+\infty}d_2(R_0)=+\infty,$

where $d_1(R_0)=\frac{\underline{M}R_0}{e^{(b+3)\pi}}, d_2(R_0)=\overline{M}R_0e^{(b+3)\pi}$ *with* $\overline{M}=\max_{1\leq k\leq q}\{1,a_k,b_k\},\underline{M}=\min_{1\leq k\leq q}\{1,a_k,b_k\}$.

Proof. For any given $\varepsilon=1$, there exists $A>0$ such that $|\frac{g(x)}{x}|<b+1$ for $|x|\geq A$. Let $r(t)>2\delta$ with $\delta=E+\max_{|x|\leq A}|g(x)|$. When $t\in[0,2\pi]\setminus\{t_k\},k=1,2,\cdots,q$,

$$\left|\frac{dr}{dt}\right|<\frac{1}{2}(b+3)r(t).$$

Choosing $R_1=2\delta e^{(b+3)\pi}$, when $R_0\geq R_1$ one has that

$$R_0e^{-(b+3)\pi}<r(t)<R_0e^{(b+3)\pi}.$$

For $t=t_k,k=1,2,\cdots,q$, it follows from (2.6) that

$$\min\{a_k,b_k\}r(t_k)\leq r(t_k^+)\leq\max\{a_k,b_k\}r(t_k).$$

Let $\overline{M}=\max_{k=1,\cdots,q}\{1,a_k,b_k\}$ and $\underline{M}=\min_{k=1,\cdots,q}\{1,a_k,b_k\}$. Then

$$d_1(R_0)\leq r(t)\leq d_2(R_0), \qquad t\in[0,2\pi],$$

where $d_1(R_0)=\underline{M}R_0e^{-(b+3)\pi}$ and $d_2(R_0)=\overline{M}R_0e^{(b+3)\pi}$. $\square$

The following result is mainly used to deal with the jumps at the impulsive time.

Lemma 2.4. *Let the function*

$$H(x)=\begin{cases}\dfrac{\arctan(a\tan x)}{x}, & x\neq 0,\\ a, & x=0,\end{cases}$$

where $a>0$ *is a constant. Then there exist* $\alpha(a)$ *and* $\beta(a)$ *such that*

$$0<\alpha(a)\leq H(x)\leq\beta(a)<\infty. \tag{2.10}$$

In fact, $\alpha(a)$ *and* $\beta(a)$ *can be of the form*

$$\alpha(a)=\begin{cases}a, & 0<a\leq 1,\\ \dfrac{1}{a}, & a\geq 1,\end{cases} \qquad \beta(a)=\begin{cases}\dfrac{1}{a}, & 0<a\leq 1,\\ a, & a\geq 1.\end{cases}$$

Proof. We only prove that $H(x)\leq\beta(a)$ for $x>0$, other cases are similar and so omitted. Suppose $f(x)=\arctan(a\tan x)-\beta(a)x$, it follows that

$$f'(x)=\frac{a\sec^2 x}{1+a^2\tan^2 x}-\beta(a)=\frac{a}{\cos^2 x+a^2\sin^2 x}-\beta(a).$$

So $f'(x)\leq 0$ for $a>0$. The conclusion holds due to $f(0)=0$. $\square$

To obtain the desirable results, we further assume the following conditions.

$(H2)$ There exist positive constants A_2 and B such that

$$(i) \quad g(x)-p_1x \geq B, \quad x \leq -A_2; \quad \text{or} \quad g(x)-p_2x \leq -B, \quad x \geq A_2,$$
$$(ii) \quad g(x)-q_1x \leq -B, \quad x \leq -A_2; \quad \text{or} \quad g(x)-q_2x \geq B, \quad x \geq A_2,$$

where $B > \frac{\max(1,p_1^2,p_2^2)d_2(R_0)}{\min(1,q_1^2,q_2^2)d_1(R_0)}E\pi$, and $d_1(R_0), d_2(R_0)$ are from Lemma 2.3.

$(H3)$ $\dfrac{E}{\min(1,q_1^2,q_2^2)d_1(R_0)} > \max\{M_\alpha, M_\beta\}$, where

$M_\alpha = \sum_{0<t_k<2\pi}\Big[1-\alpha\Big(\frac{b_k}{a_k}\Big)\Big]\prod_{0<t_j<t_k}\beta\Big(\frac{b_j}{a_j}\Big)$,

$M_\beta = \sum_{0<t_k<2\pi}\Big[\beta\Big(\frac{b_k}{a_k}\Big)-1\Big]\prod_{0<t_j<t_k}\beta\Big(\frac{b_j}{a_j}\Big)$.

Now we are ready to state and prove our main results.

Theorem 2.1. *Assume that* $(H1)$-$(H3)$ *hold, then* (1.1) *has at least one* 2π*-periodic solution.*

Proof. By $(H1)$, there exists $R_1' > 0$ such that $\theta'(t) < 0$ for $t \in [0,2\pi]\setminus\{t_k\}_{k=1}^q$. By Lemma 2.3, there exists $R_1 > 0$ such that for any $R_0 \geq R_1$ the trajectory arc $L(R_0,[0,2\pi])$ situates in $\mathcal{D} = \{d_1(R_0) \leq r \leq d_2(R_0)\}$. Let $R_0 > \max(\frac{\gamma R_1'}{\underline{M}}, R_1, \frac{\gamma A}{\underline{M}})$, where $\gamma = e^{(b+3)\pi}$ and $A \geq \max(A_1,A_2)$. In the following, we estimate the time of $L(R_0)$ rotating one circle in $\mathcal{D}$.

Choosing $M_0(R_0,\theta_0) \in \mathcal{D}$, and consider $L(R_0)$ which starts from M_0 at $t=0$, where $\theta_0 = \theta(t=0) \in [0,2\pi]$. For convenience, we assume that $M_0 \in \{(x,y) : x > A\}$, $L(R_0)$ successively intersects with $\{x=A\}, \{x=-A\}, \{x=-A\}, \{x=A\}$ and $\{\theta=\theta_0\}$ at M_1, M_2, M_3, M_4, M_5 (see Figure 1 for example), and the corresponding moments and arguments are denoted by $\theta_1 = \theta(\bar{t}_1), \theta_2 = \theta(\bar{t}_2), \theta_3 = \theta(\bar{t}_3), \theta_4 = \theta(\bar{t}_4)$ and $\theta_5 = \theta(\bar{t}_5)$. Let

$$g(x) = \begin{cases} p_1x + f_1(x), & x < 0, \\ p_2x + f_2(x), & x \geq 0; \end{cases} \qquad g(x) = \begin{cases} q_1x + h_1(x), & x < 0, \\ q_2x + h_2(x), & x \geq 0. \end{cases} \tag{2.11}$$

Then, it follows from $(H2)$ that

$$(i) \quad f_1(x) \geq B, \quad x \leq -A; \quad \text{or} \quad f_2(x) \leq -B, \quad x \geq A.$$
$$(ii) \quad h_1(x) \leq -B, \quad x \leq -A; \quad \text{or} \quad h_2(x) \geq B, \quad x \geq A.$$

Next, we consider

$$f_2(x) \leq -B, \qquad h_2(x) \geq B, \qquad x \geq A. \tag{2.12}$$

By (2.5), (2.11)-(2.12) and Lemma 2.3, one has that

$$\begin{aligned}
-\theta' &= \sin^2\theta + \frac{1}{r}[g(r\cos\theta) - e(t,r\cos\theta,r\sin\theta)]\cos\theta \\
&= \sin^2\theta + p_2\cos^2\theta + \frac{1}{r}[f_2(r\cos\theta) - e(t,r\cos\theta,r\sin\theta)]\cos\theta \\
&\leq \sin^2\theta + p_2\cos^2\theta - \frac{B}{d_2(R_0)}\cos\theta + \frac{E}{d_1(R_0)}.
\end{aligned}$$

Moreover, when $\theta(t_k) \geq 0$ it follows from Lemma 2.4 that

$$0 < \alpha\Big(\frac{b_k}{a_k}\Big)\theta(t_k) \leq \theta(t_k^+) \leq \beta\Big(\frac{b_k}{a_k}\Big)\theta(t_k) < \infty,$$

and then by the impulsive integral inequality, one has that

$$\theta(t) \le \theta_0 \prod_{0<t_k<t} \beta\Big(\frac{b_k}{a_k}\Big) \le 2\pi \prod_{0<t_k<t} \beta\Big(\frac{b_k}{a_k}\Big).$$

Further, when $t \in (0, \bar{t}_1)$, one has that

$$\begin{aligned}
\bar{t}_1 &= \int_{\theta_0}^{\theta_1} \frac{d\theta}{\theta'} + \sum_{0<t_k<\bar{t}_1} \Big[\theta(t_k^+) - \theta(t_k)\Big] \\
&\ge \int_{\theta_1}^{\theta_0} \frac{d\theta}{\sin^2\theta + p_2\cos^2\theta - \frac{B}{d_2(R_0)}\cos\theta + \frac{E}{d_1(R_0)}} + \sum_{0<t_k<\bar{t}_1} \Big[\alpha\Big(\frac{b_k}{a_k}\Big) - 1\Big]\theta(t_k) \\
&\ge \int_{\theta_1}^{\theta_0} \frac{d\theta}{\sin^2\theta + p_2\cos^2\theta - \frac{B}{d_2(R_0)}\cos\theta + \frac{E}{d_1(R_0)}} \\
&\quad - \sum_{0<t_k<\bar{t}_1} \Big[1 - \alpha\Big(\frac{b_k}{a_k}\Big)\Big] 2\pi \prod_{0<t_j<t_k} \beta\Big(\frac{b_j}{a_j}\Big).
\end{aligned}$$

Similarly, when $t \in (\bar{t}_4, \bar{t}_5)$,

$$\begin{aligned}
\bar{t}_5 - \bar{t}_4 \ge& \int_{\theta_5}^{\theta_4} \frac{d\theta}{\sin^2\theta + p_2\cos^2\theta - \frac{B}{d_2(R_0)}\cos\theta + \frac{E}{d_1(R_0)}} \\
&- \sum_{\bar{t}_4<t_k<\bar{t}_5} \Big[1 - \alpha\Big(\frac{b_k}{a_k}\Big)\Big] 2\pi \prod_{\bar{t}_4<t_j<t_k} \beta\Big(\frac{b_j}{a_j}\Big).
\end{aligned}$$

Noting that $\theta_5 + 2\pi = \theta_0$, so we have that

$$\begin{aligned}
\bar{t}_1 + \bar{t}_5 - \bar{t}_4 &\ge \int_{\theta_1}^{\theta_4+2\pi} \frac{d\theta}{\sin^2\theta + p_2\cos^2\theta - \frac{B}{d_2(R_0)}\cos\theta + \frac{E}{d_1(R_0)}} - \bigwedge_{t_k\in(0,\bar{t}_1)\cup(\bar{t}_4,\bar{t}_5)} \Big(\frac{b_k}{a_k}\Big) \\
&= \int_{\theta_1}^{\theta_4+2\pi} \Big[\frac{1}{\sin^2\theta + p_2\cos^2\theta} + \frac{\frac{B\cos\theta}{d_2(R_0)} - \frac{E}{d_1(R_0)}}{(\sin^2\theta + p_2\cos^2\theta)^2}\Big] d\theta \\
&\quad + L(R_0) - \bigwedge_{t_k\in(0,\bar{t}_1)\cup(\bar{t}_4,\bar{t}_5)} \Big(\frac{b_k}{a_k}\Big) \\
&\ge \int_{\theta_1}^{\theta_4+2\pi} \frac{d\theta}{\sin^2\theta + p_2\cos^2\theta} + \frac{B}{\max(1,p_2^2)d_2(R_0)} \int_{-\frac{\pi}{2}+\psi}^{\frac{\pi}{2}-\psi} \cos\theta d\theta \\
&\quad - \frac{E\pi}{\min(1,p_2^2)d_1(R_0)} + L(R_0) - \bigwedge_{t_k\in(0,\bar{t}_1)\cup(\bar{t}_4,\bar{t}_5)} \Big(\frac{b_k}{a_k}\Big) \\
&= \int_{\theta_1}^{\theta_4+2\pi} \frac{d\theta}{\sin^2\theta + p_2\cos^2\theta} + \frac{2B}{\max(1,p_2^2)d_2(R_0)} - \frac{E\pi}{\min(1,p_2^2)d_1(R_0)} \\
&\quad + L(R_0) + o\Big(\frac{1}{R_0}\Big) - \bigwedge_{t_k\in(0,\bar{t}_1)\cup(\bar{t}_4,\bar{t}_5)} \Big(\frac{b_k}{a_k}\Big),
\end{aligned}$$

where

$$\bigwedge_{t_k\in(0,\bar{t}_1)\cup(\bar{t}_4,\bar{t}_5)} \Big(\frac{b_k}{a_k}\Big) = \sum_{t_k\in(0,\bar{t}_1)\cup(\bar{t}_4,\bar{t}_5)} \Big[1 - \alpha\Big(\frac{b_k}{a_k}\Big)\Big] 2\pi \prod_{t_j\in(0,t_k)\cup(\bar{t}_4,t_k)} \beta\Big(\frac{b_j}{a_j}\Big),$$

$$L(R_0) = \int_{\theta_1}^{\theta_4+2\pi} \frac{\left(\frac{B\cos\theta}{d_2(R_0)} - \frac{E}{d_1(R_0)}\right)^2 d\theta}{(\sin^2\theta + p_2\cos^2\theta)^2(\sin^2\theta + p_2\cos^2\theta - \frac{B\cos\theta}{d_2(R_0)} + \frac{E}{d_1(R_0)})},$$

and ψ is an angle as shown in Figure 1. It is easy to see that

$$\psi = \psi(R_0) = \arcsin\frac{A}{d_1(R_0)} = \frac{A}{d_1(R_0)} + o(\frac{1}{R_0^2}),$$

where $o()$ denotes the infinitesimal of higher order. Since

$$|L(R_0)| \le \frac{\pi}{R_0^2} \times \frac{\left(\frac{B}{M\gamma} + \frac{\gamma E}{M}\right)^2}{\min(1,p_2^2)(\min(1,p_2) - \frac{B}{d_2(R_0)})},$$

it follows that $L(R_0) = o\Big(\frac{1}{R_0}\Big)$.

Due to $g(x) = p_1 x + f_1(x)$ and $f_1(x) \ge B$ for $x \le -A$, then

$$-\theta' \le \sin^2\theta + p_1\cos^2\theta + \frac{B}{d_2(R_0)}\cos\theta + \frac{E}{d_1(R_0)}.$$

Further when $t \in (\bar{t}_2, \bar{t}_3)$, one has that

$$\begin{aligned} \bar{t}_3 - \bar{t}_2 \ge & \int_{\theta_3}^{\theta_2} \frac{d\theta}{\sin^2\theta + p_1\cos^2\theta} + \frac{2B}{\max(1,p_1^2)d_2(R_0)} - \frac{E\pi}{\min(1,p_1^2)d_1(R_0)} \\ & - \bigwedge_{t_k\in(\bar{t}_2,\bar{t}_3)} \Big(\frac{b_k}{a_k}\Big) + o\Big(\frac{1}{R_0}\Big), \end{aligned}$$

where

$$\bigwedge_{t_k\in(\bar{t}_2,\bar{t}_3)} \Big(\frac{b_k}{a_k}\Big) = \sum_{t_k\in(\bar{t}_2,\bar{t}_3)} \Big[1 - \alpha\Big(\frac{b_k}{a_k}\Big)\Big] 2\pi \prod_{\bar{t}_2<t_j<t_k} \beta\Big(\frac{b_j}{a_j}\Big).$$

On the other hand, letting $f_1 = \max_{-A\le x\le 0}|f_1(x)|$ and $f_2 = \max_{0\le x\le A}|f_2(x)|$. Then,

$$\begin{aligned} -\theta' &= \sin^2\theta + p_2\cos^2\theta + \frac{1}{r}[f_2(r\cos\theta) - e(t, r\cos\theta, r\sin\theta)]\cos\theta \\ &\le \sin^2\theta + p_2\cos^2\theta + \frac{f_2+E}{d_1(R_0)}, \qquad 0 < x < A, \end{aligned}$$

and

$$-\theta' \le \sin^2\theta + p_1\cos^2\theta + \frac{f_1+E}{d_1(R_0)}, \qquad -A < x < 0.$$

Hence, when $t \in (\bar{t}_1, \bar{t}_2) \cup (\bar{t}_3, \bar{t}_4)$, one has that

$$\begin{aligned}
\bar{t}_2 - \bar{t}_1 + \bar{t}_4 - \bar{t}_3 =& \Big(\int_{-\frac{\pi}{2}}^{\theta_1} + \int_{\theta_4}^{-\frac{3\pi}{2}} + \int_{\theta_2}^{-\frac{\pi}{2}} + \int_{-\frac{3\pi}{2}}^{\theta_3} \Big) \frac{d\theta}{-\theta'} - \sum_{t_k \in (\bar{t}_1, \bar{t}_2) \cup (\bar{t}_3, \bar{t}_4)} \Big[\theta(t_k^+) - \theta(t_k) \Big] \\
\geq& \Big(\int_{-\frac{\pi}{2}}^{\theta_1} + \int_{\theta_4}^{-\frac{3\pi}{2}} \Big) \frac{d\theta}{\sin^2\theta + p_2 \cos^2\theta + \frac{f_2 + E}{d_1(R_0)}} \\
&+ \Big(\int_{\theta_2}^{-\frac{\pi}{2}} + \int_{-\frac{3\pi}{2}}^{\theta_3} \Big) \frac{d\theta}{\sin^2\theta + p_1 \cos^2\theta + \frac{f_1 + E}{d_1(R_0)}} - \bigwedge_{t_k \in (\bar{t}_1, \bar{t}_2) \cup (\bar{t}_3, \bar{t}_4)} \Big(\frac{b_k}{a_k} \Big) \\
=& \Big(\int_{-\frac{\pi}{2}}^{\theta_1} + \int_{\theta_4}^{-\frac{3\pi}{2}} \Big) \frac{d\theta}{\sin^2\theta + p_2 \cos^2\theta} + \Big(\int_{\theta_2}^{-\frac{\pi}{2}} + \int_{-\frac{3\pi}{2}}^{\theta_3} \Big) \frac{d\theta}{\sin^2\theta + p_1 \cos^2\theta} \\
&+ o\Big(\frac{1}{R_0} \Big) - \bigwedge_{t_k \in (\bar{t}_1, \bar{t}_2) \cup (\bar{t}_3, \bar{t}_4)} \Big(\frac{b_k}{a_k} \Big),
\end{aligned}$$

where

$$\bigwedge_{t_k \in (\bar{t}_1, \bar{t}_2) \cup (\bar{t}_3, \bar{t}_4)} \Big(\frac{b_k}{a_k} \Big) = \sum_{t_k \in (\bar{t}_1, \bar{t}_2) \cup (\bar{t}_3, \bar{t}_4)} \Big[1 - \alpha\Big(\frac{b_k}{a_k} \Big) \Big] 2\pi \prod_{t_j \in (\bar{t}_1, t_k) \cup (\bar{t}_3, t_k)} \beta\Big(\frac{b_j}{a_j} \Big).$$

By (2.9) and the above several inequalities, the time T that $L(R_0)$ rotates one circle in $\mathcal{D}$ satisfies

$$\begin{aligned}
T \geq& \int_{-\frac{3\pi}{2}}^{-\frac{\pi}{2}} \frac{d\theta}{\sin^2\theta + p_1 \cos^2\theta} + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d\theta}{\sin^2\theta + p_2 \cos^2\theta} - \bigwedge_{t_k \in (0, \bar{t}_5)} \Big(\frac{b_k}{a_k} \Big) + o\Big(\frac{1}{R_0} \Big) \\
&+ \frac{2B}{\max(1, p_2^2) d_2(R_0)} + \frac{2B}{\max(1, p_1^2) d_2(R_0)} - \frac{E\pi}{\min(1, p_1^2) d_1(R_0)} - \frac{E\pi}{\min(1, p_2^2) d_1(R_0)} \\
=& \frac{2\pi}{m+1} + \frac{2B}{\max(1, p_2^2) d_2(R_0)} + \frac{2B}{\max(1, p_1^2) d_2(R_0)} \\
&- \frac{E\pi}{\min(1, p_1^2) d_1(R_0)} - \frac{E\pi}{\min(1, p_2^2) d_1(R_0)} - \bigwedge_{t_k \in (0, \bar{t}_5)} \Big(\frac{b_k}{a_k} \Big) + o\Big(\frac{1}{R_0} \Big),
\end{aligned} \tag{2.13}$$

where

$$\bigwedge_{t_k \in (0, \bar{t}_5)} \Big(\frac{b_k}{a_k} \Big) = \sum_{t_k \in (0, \bar{t}_5)} \Big[1 - \alpha\Big(\frac{b_k}{a_k} \Big) \Big] 2\pi \prod_{t_j \in (0, t_k)} \beta\Big(\frac{b_j}{a_j} \Big).$$

By using the right-side inequality of (2.10) and the similar arguments, we have that

$$\begin{aligned}
T \leq& \frac{2\pi}{m} - \Big[\frac{2B}{\max(1, q_2^2) d_2(R_0)} + \frac{2B}{\max(1, q_1^2) d_2(R_0)} - \frac{E\pi}{\min(1, q_1^2) d_1(R_0)} \\
&- \frac{E\pi}{\min(1, q_2^2) d_1(R_0)} \Big] + \sum_{t_k \in (0, \bar{t}_5)} \Big[\beta\Big(\frac{b_k}{a_k} \Big) - 1 \Big] 2\pi \prod_{0 < t_j < t_k} \beta\Big(\frac{b_j}{a_j} \Big) + o\Big(\frac{1}{R_0} \Big).
\end{aligned} \tag{2.14}$$

Since $p_1 \geq q_1, p_2 \geq q_2$ and $B > \frac{\max(1,p_1^2,p_2^2)d_2(R_0)}{\min(1,q_1^2,q_2^2)d_1(R_0)}E\pi$, it follows from $(H3)$ that

$$\begin{aligned}
&\frac{2B}{\max(1,p_2^2)d_2(R_0)} + \frac{2B}{\max(1,p_1^2)d_2(R_0)} \\
&- \frac{E\pi}{\min(1,p_1^2)d_1(R_0)} - \frac{E\pi}{\min(1,p_2^2)d_1(R_0)} - 2M_\alpha\pi > 0, \\
&\frac{2B}{\max(1,q_2^2)d_2(R_0)} + \frac{2B}{\max(1,q_1^2)d_2(R_0)} \\
&- \frac{E\pi}{\min(1,q_1^2)d_1(R_0)} - \frac{E\pi}{\min(1,q_2^2)d_1(R_0)} - 2M_\beta\pi > 0,
\end{aligned} \tag{2.15}$$

where $M_\alpha = \sum_{0<t_k<2\pi}\Big[1-\alpha\Big(\frac{b_k}{a_k}\Big)\Big]\prod_{0<t_j<t_k}\beta\Big(\frac{b_j}{a_j}\Big)$ and $M_\beta = \sum_{0<t_k<2\pi}\Big[\beta\Big(\frac{b_k}{a_k}\Big)-1\Big]\prod_{0<t_j<t_k}\beta\Big(\frac{b_j}{a_j}\Big)$.

Combining with (2.13)-(2.15), when $R_0 > 0$ is sufficiently large we have that

$$\frac{2\pi}{m+1} < T < \frac{2\pi}{m}. \tag{2.16}$$

This implies that the number of rotation of $L(R_0)$ in $[0, 2\pi]$ is greater than m but less than $m+1$. Hence, by Theorem 1.1, (1.1) has at least one 2π-periodic solution. □

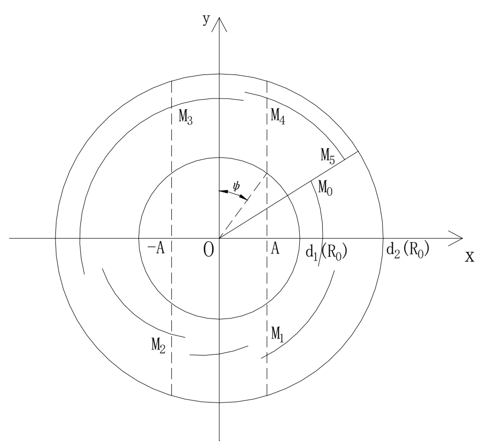

Figure 1. Schematic diagram

Remark 2.1. In (1.1), when $a_k = b_k, k \in \mathbb{Z}_+$, it follows from Lemma 2.4 that $M_\alpha = M_\beta = 0$. Further, by $(H1)$-$(H2)$, the time T that any solution trajectory of (1.1) rotates one circle satisfies

$$\begin{aligned}
&\frac{2\pi}{m+1} + \frac{2B}{\max(1,p_2^2)d_2(R_0)} - \frac{E\pi}{\min(1,p_1^2)d_1(R_0)} - \frac{E\pi}{\min(1,p_2^2)d_1(R_0)} + o\Big(\frac{1}{R_0}\Big) \\
&\leq T \leq \frac{2\pi}{m} - \Big[\frac{2B}{\max(1,q_2^2)d_2(R_0)} - \frac{E\pi}{\min(1,q_1^2)d_1(R_0)} - \frac{E\pi}{\min(1,q_2^2)d_1(R_0)}\Big] + o\Big(\frac{1}{R_0}\Big).
\end{aligned}$$

When $a_k = b_k = 1$ (*i.e.* $\overline{M} = \underline{M} = 1$), (1.1) degenerates into a continuous system without impulsive terms. Hence by $(H1)$-$(H2)$, it is easy to verify that

$$\begin{aligned}&\frac{2\pi}{m+1} + \Big[\frac{2B}{\max(1,p_2^2)\gamma} - \frac{\gamma E\pi}{\min(1,p_1^2)} - \frac{\gamma E\pi}{\min(1,p_2^2)}\Big]\frac{1}{R_0} + o\Big(\frac{1}{R_0}\Big)\\ &\leq T \leq \frac{2\pi}{m} - \Big[\frac{2B}{\max(1,q_2^2)\gamma} - \frac{\gamma E\pi}{\min(1,q_1^2)} - \frac{\gamma E\pi}{\min(1,q_2^2)}\Big]\frac{1}{R_0} + o\Big(\frac{1}{R_0}\Big),\end{aligned}$$

and (2.16) holds since $B > \frac{\max(1,p_1^2,p_2^2)}{\min(1,q_1^2,q_2^2)}\gamma^2 E\pi$ and $R_0 > 0$ is large sufficiently. About the periodic solution problem of the Duffing equations without impulses, there have been many interesting results, see [3–7,10,20] for example and the references therein.

Corollary 2.1. *Assume that* $(H1), (H3)$ *hold, and*

$$\begin{aligned}&\lim_{x\to-\infty}(g(x)-p_1x) = +\infty; \quad \text{or} \quad \lim_{x\to+\infty}(g(x)-p_2x) = -\infty,\\ &\lim_{x\to-\infty}(g(x)-q_1x) = -\infty; \quad \text{or} \quad \lim_{x\to+\infty}(g(x)-q_2x) = +\infty.\end{aligned} \tag{2.17}$$

Then, (1.1) *has at least one* 2π*-periodic solution.*

$(H1)'$ There exists $A_1 > 0$ such that

$$\frac{g(x)}{x} \leq p_1, \quad x \leq -A_1, \qquad \frac{g(x)}{x} \leq p_2, \quad x \geq A_1,$$

where $p_1 > 0, p_2 > 0$ are constants satisfying $\frac{1}{\sqrt{p_1}} + \frac{1}{\sqrt{p_2}} = 2$, and

$$\overline{\lim}_{x\to-\infty} g(x) < \underline{E} \leq \overline{E} < \underline{\lim}_{x\to+\infty} g(x).$$

$(H2)'$ There exist $A_2 > 0$ and $B > \frac{\max(1,p_1^2,p_2^2)d_2(R_0)}{\min(1,p_1^2,p_2^2)d_1(R_0)}E\pi$ such that

$$g(x) - p_1x \geq B, \quad x \leq -A_2; \quad \text{or} \quad g(x) - p_2x \leq -B, \quad x \geq A_2,$$

where $d_1(R_0), d_2(R_0)$ are from Lemma 2.3.

$(H3)'$ $\dfrac{E}{\min(1,p_1^2,p_2^2)d_1(R_0)} > M_\alpha$, where $M_\alpha = \sum_{0<t_k<2\pi}\Big[1-\alpha\Big(\frac{b_k}{a_k}\Big)\Big]\prod_{0<t_j<t_k}\beta\Big(\frac{b_j}{a_j}\Big)$.

Theorem 2.2. *Assume that* $(H1)'$-$(H3)'$ *hold, then* (1.1) *has at least one* 2π*-periodic solution.*

Proof. By the similar analysis to Theorem 2.1, it follows from $(H1)'$-$(H3)'$ that

$$\begin{aligned}T \geq &\int_{-\frac{3\pi}{2}}^{-\frac{\pi}{2}} \frac{d\theta}{\sin^2\theta + p_1\cos^2\theta} + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d\theta}{\sin^2\theta + p_2\cos^2\theta} - 2\pi M_\alpha + o\Big(\frac{1}{R_0}\Big)\\ &+ \frac{2B}{\max(1,p_2^2)d_2(R_0)} + \frac{2B}{\max(1,p_1^2)d_2(R_0)} - \frac{E\pi}{\min(1,p_1^2)d_1(R_0)} - \frac{E\pi}{\min(1,p_2^2)d_1(R_0)} > 2\pi.\end{aligned}$$

Then, the number of rotation of any solution trajectory rotating in $\mathcal{D}$ for $t \in [0, 2\pi]$ is less than one. Hence, by Theorem 1.1, the conclusion holds. □

Corollary 2.2. *Assume that* $(H1)', (H3)'$ *hold, and*

$$\lim_{x\to-\infty}(g(x)-p_1x) = +\infty; \qquad \text{or} \qquad \lim_{x\to+\infty}(g(x)-p_2x) = -\infty.$$

Then, (1.1) *has at least one* 2π*-periodic solution.*

Theorem 2.3. *Assume that* $(H3)$ *holds, and*

$(H4)$ *there exists* $A_1 > 0$ *such that*

$$q_1 \leq \frac{g(x)}{x} \leq p_1, \quad x \leq -A_1, \qquad q_2 \leq \frac{g(x)}{x} \leq p_2, \quad x \geq A_1,$$

where p_1, q_1, p_2, q_2 *are positive constants, and there is an integer* $m > 0$ *such that*

$$\frac{2}{m+1} < \frac{1}{\sqrt{p_1}} + \frac{1}{\sqrt{p_2}} < \frac{1}{\sqrt{q_1}} + \frac{1}{\sqrt{q_2}} < \frac{2}{m}. \tag{2.18}$$

Then, (1.1) *has at least one* 2π*-periodic solution.*

Proof. By $(H4)$, there exist positive constants $p_1^*(> p_1), p_2^*(> p_2), q_1^*(< q_1)$ and $q_2^*(< q_2)$ such that

$$\frac{1}{\sqrt{q_1^*}} + \frac{1}{\sqrt{q_2^*}} = \frac{2}{m}, \qquad \frac{1}{\sqrt{p_1^*}} + \frac{1}{\sqrt{p_2^*}} = \frac{2}{m+1},$$

and then $(H1)$ holds for such $p_1^*, p_2^*, q_1^*, q_2^*$ and A_1, m. Moreover, it follows that

$$g(x) - p_2^* x \leq (p_2 - p_2^*)x, \quad g(x) - q_2^* x \geq (q_2 - q_2^*)x, \quad x \geq A_1.$$

Hence, (2.17) is satisfied, and by Corollary 2.1 the conclusion holds. □

Corollary 2.3. *Assume that* $(H3)'$ *holds, and*

$(H4)'$ *there exists* $A_1 > 0$ *such that*

$$\frac{g(x)}{x} \leq p_1, \quad x \leq -A_1, \qquad \frac{g(x)}{x} \leq p_2, \quad x \geq A_1,$$

where p_1, p_2 *are positive constants satisfying* $\frac{1}{\sqrt{p_1}} + \frac{1}{\sqrt{p_2}} > 2$*, and*

$$\overline{\lim}_{x\to-\infty} g(x) < \underline{E} \leq \overline{E} < \underline{\lim}_{x\to+\infty} g(x).$$

Then, (1.1) *has at least one* 2π*-periodic solution.*

Proof. By $(H4)'$, there exist $p_1^*(> p_1)$ and $p_2^*(> p_2)$ such that $\frac{1}{\sqrt{p_1^*}} + \frac{1}{\sqrt{p_2^*}} = 2$. Moreover, $g(x) - p_2^* x \leq (p_2 - p_2^*)x$ for $x \geq A_1$. So all conditions of Corollary 2.2 hold. □

Remark 2.2. In $(H4)$, if (2.18) is substituted into the following condition.
For any integer $n > 1$, there exists an integer $m > 0$ satisfying $(n, m) = 1$ such that

$$\frac{2n}{m+1} < \frac{1}{\sqrt{p_1}} + \frac{1}{\sqrt{p_2}} < \frac{1}{\sqrt{q_1}} + \frac{1}{\sqrt{q_2}} < \frac{2n}{m}. \tag{2.19}$$

Then, all conclusions on 2π-periodic solutions can be changed as the existence of $2n\pi$-periodic solutions which are not $2l\pi$-periodic for $1 \leq l < n$.

In fact, by the similar arguments to Theorem 2.1, for any integer $n > 1$ then the time T that any solution trajectory rotates one circle in $\mathcal{D}$ for $t \in [0, 2n\pi]$ satisfies

$$\frac{2n\pi}{m+1} < T < \frac{2n\pi}{m}.$$

Hence, by Theorem 1.1, there exists at least one $2n\pi$-periodic solution. Moreover, it follows from $(n, m) = 1$ that the periodic solution is not $2l\pi$-periodic for $1 \leq l < n$.

3. Examples

In this section, we give two special impulses to (1.1).

Example 3.1. Consider the following impulses form

$$\begin{cases} \Delta x(t_k) = cx(t_k), \\ \Delta y(t_k) = cy(t_k), \qquad k \in \mathbb{Z}_+, \end{cases} \tag{3.1}$$

where $\Delta x(t_k) = x(t_k^+) - x(t_k)$, $\Delta y(t_k) = y(t_k^+) - y(t_k)$ and $(x(t_k), y(t_k)) = (x(t_k^-), y(t_k^-))$, $c > 0$ is a constant, $0 < t_1 < \cdots < t_q < 2\pi$ satisfying $t_{k+q} = t_k + 2\pi$, $k \in \mathbb{Z}_+$.

The equivalent system of (1.1) with (3.1) is as follows

$$\begin{cases} x' = y, \\ y' = -g(x) + e(t, x, y), \qquad t \neq t_k,\ t \in \mathbb{R}, \\ \Delta x(t_k) = cx(t_k), \\ \Delta y(t_k) = cy(t_k), \qquad k \in \mathbb{Z}_+. \end{cases} \tag{3.2}$$

By the transformation $x(t) = r(t)\cos\theta(t),\ y(t) = r(t)\sin\theta(t)$, when $t \in [0, 2\pi]$ (3.2) becomes

$$\begin{cases} \theta'(t) = -\sin^2\theta + \dfrac{1}{r}[e(t, r\cos\theta, r\sin\theta) - g(r\cos\theta)]\cos\theta, \\ r'(t) = r\cos\theta\sin\theta + [e(t, r\cos\theta, r\sin\theta) - g(r\cos\theta)]\sin\theta, \qquad t \neq t_k,\ t \in [0, 2\pi], \\ \Delta\theta(t_k) = \theta(t_k^+) - \theta(t_k) = 0, \\ \Delta r(t_k) = r(t_k^+) - r(t_k) = cr(t_k), \qquad k = 1, \cdots, q. \end{cases} \tag{3.3}$$

Noting that $\theta(t)$ is continuous with respect to t, so Lemma 2.4 is invalid. We give the following result.

Theorem 3.1. *Assume that* $(H1)$ *holds, and there exist* $A_2 > 0$, $B > \frac{\max(1, p_1^2, p_2^2)(1+c)\gamma^2 E\pi}{\min(1, q_1^2, q_2^2)}$ *such that*

(i) $g(x) - p_1 x \geq B, \quad x \leq -A_2;$ or $g(x) - p_2 x \leq -B, \quad x \geq A_2,$

(ii) $g(x) - q_1 x \leq -B, \quad x \leq -A_2;$ or $g(x) - q_2 x \geq B, \quad x \geq A_2.$

Then, (3.2) *has at least one* 2π*-periodic solution.*

Proof. With the similar analysis, then the time T that any solution trajectory of (3.2) rotates one circle during $[0, 2\pi]$ satisfies

$$\begin{aligned} &\frac{2\pi}{m+1} + \Big[\frac{2B}{\max(1, p_2^2)(1+c)\gamma} - \frac{E\gamma\pi}{\min(1, p_1^2)} - \frac{E\gamma\pi}{\min(1, p_2^2)}\Big]\frac{1}{R_0} + o\Big(\frac{1}{R_0}\Big) \\ &\leq T \leq \frac{2\pi}{m} - \Big[\frac{2B}{\max(1, q_2^2)(1+c)\gamma} - \frac{E\gamma\pi}{\min(1, q_1^2)} - \frac{E\gamma\pi}{\min(1, q_2^2)}\Big]\frac{1}{R_0} + o\Big(\frac{1}{R_0}\Big). \end{aligned}$$

Hence, (2.16) holds when $R_0 > 0$ is sufficiently large. □

Example 3.2. Assume that $g(x)$ is given by

$$g(x) = \begin{cases} \mu_1 x + \nu, \qquad x < 0, \\ \mu_2 x + \nu, \qquad x \geq 0, \end{cases} \tag{3.4}$$

and consider the following impulses

$$\begin{cases} \Delta x(t_k) = 0, \\ \Delta y(t_k) = dy(t_k), \qquad k \in \mathbb{Z}_+, \end{cases} \tag{3.5}$$

where $\mu_1 > 0, \mu_2 > 0, d > 0$ and ν are constants. Then, the equivalent system of (1.1) with (3.4)-(3.5) is as follows

$$\begin{cases} x' = y, \\ y' = -g(x) + e(t, x, y), \qquad t \neq t_k,\ t \in \mathbb{R}, \\ \Delta x(t_k) = 0, \\ \Delta y(t_k) = dy(t_k), \qquad k \in \mathbb{Z}_+. \end{cases} \tag{3.6}$$

Let $(x(t), y(t)) = (x(t; x_0, y_0), y(t; x_0, y_0))$ be the solution of (3.6) satisfying the initial value $(x(0), y(0)) = (x_0, y_0)$, and assume that $x(t_k) = 0, k \in \mathbb{Z}_+$. Denote by

$$\Phi_k : \ (0, y_k) \to (0, y_k^+),$$

where $y_k = y(t_k),\ y_k^+ = y(t_k^+)$. We choose $M_0(x_0, y_0) \in \{(x, y) : x > 0, y > 0\}$, and consider the solution trajectory L_{M_0} of (3.6) which starts from M_0 at $t = 0$ (see Figure 2 for example). When $|(x_0, y_0)|$ is sufficiently large, L_{M_0} moves clockwise in $\{x > 0\}$ and $\{x < 0\}$ respectively. Assume that L_{M_0} intersects with $\{x = 0, y < 0\}$ at $M_1(0, y_1)$ and Φ maps it to $M_1^+(0, y_1^+)$. Later L_{M_0} starting from M_1^+ enters into $\{x < 0\}$. It intersects with $\{x = 0, y > 0\}$ at $M_2(0, y_2)$, Φ maps M_2 to $M_2^+(0, y_2^+)$, and next intersects with $\{\theta = \theta_0\}$ (where $\theta_0 = \arctan \frac{y_0}{x_0}$) at $M_3(x(t^*), y(t^*))$. By the similar analysis, we estimate the time $T = t^*$ that L_{M_0} rotates one circle on (x, y)-plane. Noting that

$$\mathcal{P}_{t^*} : \ (x_0, y_0) \to (x(t^*; x_0, y_0), y(t^*; x_0, y_0)), \qquad \mathcal{P}_{t^*} = \mathcal{P}_2 \circ \Phi_2 \circ \mathcal{P}_1 \circ \Phi_1 \circ \mathcal{P}_0,$$

where $(x_0, y_0) \xrightarrow{\mathcal{P}_0} (0, y_1), (0, y_1^+) \xrightarrow{\mathcal{P}_1} (0, y_2), (0, y_2^+) \xrightarrow{\mathcal{P}_2} (x(t^*), y(t^*))$ and $(0, y_i) \xrightarrow{\Phi_i} (0, y_i^+), i = 1, 2$.

Theorem 3.2. *Assume that* $\frac{1}{\sqrt{\mu_1}} + \frac{1}{\sqrt{\mu_2}} \neq \frac{2}{m}$ *for any positive integer* m*, and letting* $d > \frac{1}{2}[-1 + \sqrt{1 + \frac{4Ee^{(b+3)\pi}}{\min(1, \mu_1^2, \mu_2^2) R_0}}]$*, where* $b = \max(\mu_1, \mu_2)$*. Then* (3.6) *has at least one* 2π*-periodic solution.*

Proof. Choosing $p_1 = q_1 = \mu_1$ and $p_2 = q_2 = \mu_2$, there must be an integer $m > 0$ such that

$$\frac{2}{m+1} < \frac{1}{\sqrt{p_1}} + \frac{1}{\sqrt{p_2}} = \frac{1}{\sqrt{q_1}} + \frac{1}{\sqrt{q_2}} < \frac{2}{m},$$

or

$$\frac{1}{\sqrt{p_1}} + \frac{1}{\sqrt{p_2}} > 2.$$

Due to $a_k = 1, b_k = d + 1$, it follows from Lemma 2.4 that

$$\sum_{0<t_k<t^*} \Big[1 - \alpha\Big(\frac{b_k}{a_k}\Big)\Big] \prod_{0<t_j<t_k} \beta\Big(\frac{b_j}{a_j}\Big) = d,$$

$$\sum_{0<t_k<t^*} \Big[\beta\Big(\frac{b_k}{a_k}\Big) - 1\Big] \prod_{0<t_j<t_k} \beta\Big(\frac{b_j}{a_j}\Big) = d(d+1).$$

Let $g^*(x) = g(x) - \nu$ and $e^*(t, x, y) = e(t, x, y) - \nu$. When $A_1 > 0$ is large sufficiently, all conditions of Theorem 2.3 or Corollary 2.3 hold. $\square$

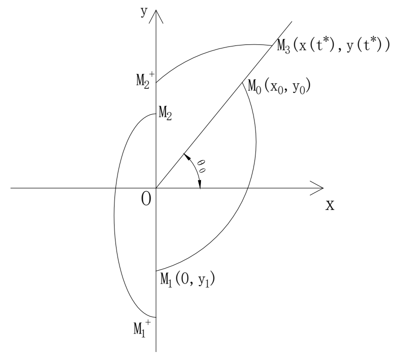

Figure 2. Schematic diagram

4. Concluding remarks

In this paper, we have studied the existence of periodic solutions for a semilinear second order differential equation with linear impulsive effects. We first analyzed the properties of solutions, and estimated the time that any solution trajectory rotates one circle on the phase plane. Then, by the two-dimension Poincaré-Bohl theorem (*i.e.* Theorem 1.1), we obtained some existence criteria of periodic solutions. Finally, two examples with special impulses are presented to illustrate the effectiveness of the obtained results.

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