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Dynamics Analysis of an SIS Epidemic Model with the Effects of Awareness*

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Abstract In this paper, an SIS model incorporating the effects of awareness spreading on epidemic is analyzed. Four kinds of equilibria of the model are given, and a new method is used to prove the stability of the equilibria. The threshold of awareness is R_1^a, which measures whether awareness spreads. When awareness does not spread, the basic reproduction number of disease is R_1^d, it is R_2^d when awareness spreads. The relationship among the three kinds of thresholds is discussed in details. Specially, the effects of various awareness parameters on epidemic are analyzed. Our theoretical results suggest that raising awareness can effectively reduce the basic reproduction number of disease and reduce the spread of disease. Furthermore, numerical simulations are performed to illustrate our results.

Keywords Awareness spreading, Epidemic model, Cooperative system, Stability.

MSC(2010) 34D05, 34D20, 34D23.

1. Introduction

In recent decades, many infectious diseases have broken out, including Ebola, swine flu, Zika virus and so on. At the end of 2019, an infectious disease named COVID-19 broke out in the world for the first time. It is highly infectious and no vaccine is available. In a short period of time, thousands of people were infected and many people died unfortunately. These epidemics seriously affect people's health and cause great economic losses. Due to the significant development of science and technology, the spread of awareness between people is becoming more and more convenient. Usually, the spread of disease information can raise people's awareness of the disease, which makes people take a series of protection measures [1]. When people try to prevent themselves from contracting an epidemic, the results may show lower susceptibility. Meanwhile, thanks to self-isolation or better sanitary conditions, the recovery time is shorter and the infectivity will also decrease [2]. Although the best way to control epidemic is to vaccinate, the cost of vaccination

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is very high. Worse still, there are some diseases that do not have vaccines (such as malaria, Hepatitis C, etc.). In this case, raising the awareness of the epidemic can control the epidemic more effectively. Especially, in developing and underdeveloped countries, mass media campaigns can play an important role in changing public health-related behaviors [3]. In view of the complex changes in mass behavior under awareness circumstances, it is very important to understand how awareness affects the spread of epidemics.

Many mathematical models are proposed to study the effects of awareness on epidemics. These models can be divided into two major classes: network-based models [4 – 6] and mean-field models [7 – 9]. So far, there are two ways to explain the effects of awareness: (i) by introducing a mass media compartment to represent the public interaction with mass media [10 – 14], (ii) by changing the rate of diseases transmission and taking preventive measures [15, 16]. For the choice of the affecting way of awareness, we choose the case (ii) in this paper. For the case (ii), there have also been a few studies in recent years. Kiss et al. [1] extended a simple SIRS model to account for the treatment class. They proved that although the awareness of the whole population did not affect the epidemic threshold, but it could reduce the prevalence of infection. An SIRS model that considering the effects of private and public awareness on epidemic was studied [8, 17].

Medical research had shown that many epidemics, including some bacterial and sexually transmitted diseases, do not have permanent immunity. These epidemics can be constructed as SIS models [18, 19]. In this paper, we establish an SIS model to study the effects of awareness spreading on epidemic. There have been some related previous studies. Samanta et al. [20] used an SIS model to study the effects of awareness program in epidemics outbreak - a slow fast dynamics. Liu et al. [21] investigated the stochastic diseases dynamics of an SIS epidemic model on two patches incorporating media coverage. Granell et al. [22] established an SIS-UAU model, the dynamical interplay between awareness and epidemic spreading in multiplex networks was investigated.

Our research is based on [23], there are still many differences. More practical factors are considered than previous work. Due to the effects of awareness, the two types of basic reproduction numbers of disease are given. Further, the relationship among three thresholds and the effects of various parameters on the epidemic threshold are analyzed. These have not been analyzed in previous works. In short, a comprehensive and detailed theoretical analysis is provided in this paper, the content of our research is more in line with the real life. Therefore, the research of this paper carries certain theoretic meaning and applied value.

The organization of this paper is as follows. The SIS model is described in Section 2. The four kinds of equilibria and three kinds of threshold expressions are given in Section 3. The stability of equilibria is analyzed in Section 4. Parametric analysis and numerical simulation are in Section 5. The main conclusions are summarized in Section 6.

2. The model description

We divide the overall population into four compartments: susceptible and unaware (S^n), susceptible and aware (S^a), infected and unaware (I^n), infected and aware (I^a). Specially, in this paper, the unaware individuals are the individuals without disease awareness and the individuals who have disease awareness but do not take

protective measures. The transition relations of all compartments are summarised in Figure 1. Dotted lines represent the infectious effect of the corresponding infected on the susceptible.

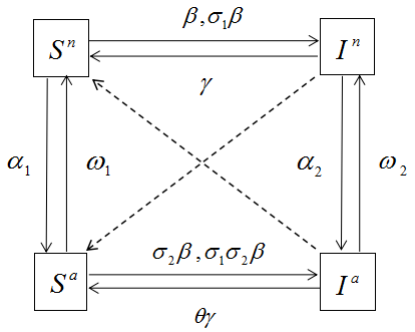


Figure 1. Classes and transitions in the model.

The remaining model parameters are as follows:

Table 1. Description of parameters

ω_1,ω_2	Awareness losing rate
α_1,α_2	Awareness spreading rate
γ	Recovery rate of unaware infected
$\theta\gamma$	Recovery rate of aware infected
β	Infection rate from unaware infected to unaware susceptible
$\sigma_1\beta$	Infection rate from aware infected to unaware susceptible
$\sigma_2\beta$	Infection rate from unaware infected to aware susceptible
$\sigma_1\sigma_2\beta$	Infection rate from aware infected to aware susceptible

where we assume $0 < \alpha_1 < \alpha_2 < 1$, $0 < \omega_2 < \omega_1 < 1$, $\theta > 1$ and $0 < \sigma_1, \sigma_2 < 1$ hold. Different from the previous work, in fact, susceptible individuals have a more significant reduction in the spread of the epidemic, so $\sigma_2 < \sigma_1$.

Under the above assumptions, the model for the spread of awareness and epidemic is as follows

$$\begin{cases} \dfrac{dS^n(t)}{dt} = -\dfrac{(I^n + \sigma_1 I^a)\,\beta S^n}{N} - \dfrac{\alpha_1\,(S^a + I^a)\,S^n}{N} + \omega_1 S^a + \gamma l^n, \\ \dfrac{dS^a(t)}{dt} = -\dfrac{(I^n + \sigma_1 I^a)\,\sigma_2\beta S^a}{N} + \dfrac{\alpha_1\,(S^a + I^a)\,S^n}{N} - \omega_1 S^a + \theta\gamma I^a, \\ \dfrac{dI^n(t)}{dt} = \dfrac{(I^n + \sigma_1 I^a)\,\beta S^n}{N} - \dfrac{\alpha_2\,(S^a + I^a)\,I^n}{N} + \omega_2 I^a - \gamma I^n, \\ \dfrac{dI^a(t)}{dt} = \dfrac{(I^n + \sigma_1 I^a)\,\sigma_2\beta S^a}{N} + \dfrac{\alpha_2\,(S^a + I^a)\,I^n}{N} - \omega_2 I^a - \theta\gamma I^a. \end{cases} \tag{2.1}$$

Since the system (2.1) does not include vital dynamics, so there are no births or deaths of the population, which means that the total population $S^n(t) + S^a(t) + I^n(t) + I^a(t) = N$ is constant. It is easy to show that the system (2.1) is well-posed, its solutions are non-negative for all $t \geq 0$. For more convenient calculation, we perform the following scale transformation on the system variables

$$S_n(t) = \frac{S^n(t)}{N}, S_a(t) = \frac{S^a(t)}{N}, I_n(t) = \frac{I^n(t)}{N}, I_a(t) = \frac{I^a(t)}{N}.$$

Simplify system (2.1) and substitute $S_n = 1 - S_a - I_n - I_a$ to obtain

$$\begin{cases} \dfrac{dS_a(t)}{dt} = -\left(I_n + \sigma_1 I_a\right)\sigma_2\beta S_a + \alpha_1\left(S_a + I_a\right)\left(1 - S_a - I_n - I_a\right) - \omega_1 S_a + \theta\gamma I_a, \\ \dfrac{dI_n(t)}{dt} = \left(I_n + \sigma_1 I_a\right)\beta\left(1 - S_a - I_n - I_a\right) - \alpha_2\left(S_a + I_a\right) I_n + \omega_2 I_a - \gamma I_n, \\ \dfrac{dI_a(t)}{dt} = \left(I_n + \sigma_1 I_a\right)\sigma_2\beta S_a + \alpha_2\left(S_a + I_a\right) I_n - \omega_2 I_a - \theta\gamma I_a, \end{cases} \tag{2.2}$$

where the set $\Gamma = \{(S_a, I_n, I_a) \in R_+^3 | I_n + S_a + I_a \leq 1\}$ is positively invariant for system (2.2).

3. Equilibria and basic reproduction number

3.1. Equilibria and their types

It is easy to obtain the system (2.2) has a disease-free awareness-free equilibrium

$$E_0^1 = \left(S_a^0, I_n^0, I_a^0\right) = (0, 0, 0),$$

a disease-free awareness-endemic equilibrium

$$E_0^2 = (S_a^0, I_n^0, I_a^0) = (\tfrac{\alpha_1 - \omega_1}{\alpha_1}, 0, 0),$$

and a disease-endemic awareness-free equilibrium $E_1 = (S_a^*, I_n^*, I_a^*) = (0, \frac{\beta - \gamma}{\beta}, 0)$.

The equilibrium E_0^2 and E_1 are only biologically feasible, providing the conditions

$$R_1^a = \frac{\alpha_1}{\omega_1} > 1 \text{ and } R_1^d = \frac{\beta}{\gamma} > 1$$

hold respectively.

Theorem 3.1. *There is only a disease-endemic awareness-endemic equilibrium for system (2.2), which is recorded as* $E_2 = (S_a^{**}, I_n^{**}, I_a^{**})$.

Proof. For more convenient calculation, we assume $\alpha_1 = \alpha_2 = \alpha$ and $\omega_1 = \omega_2 = \omega$. It is not a restriction, all our calculations can be made without this assumption. It is easy to know that the equilibrium E_2 satisfies

$$\begin{cases} -\left(I_n + \sigma_1 I_a\right)\sigma_2\beta S_a + \alpha\left(S_a + I_a\right)\left(1 - S_a - I_n - I_a\right) - \omega S_a + \theta\gamma I_a = 0, \\ \left(I_n + \sigma_1 I_a\right)\beta\left(1 - S_a - I_n - I_a\right) - \alpha\left(S_a + I_a\right) I_n + \omega I_a - \gamma I_n = 0, \\ \left(I_n + \sigma_1 I_a\right)\sigma_2\beta S_a + \alpha\left(S_a + I_a\right) I_n - \omega I_a - \theta\gamma I_a = 0. \end{cases} \tag{3.1}$$

From the first and the third equations of (3.1), we can obtain

$$I_n = \frac{\alpha S_a - \alpha I_a^2 - \left(2\beta S_a\sigma_2\sigma_1 + 2\alpha S_a - \gamma\theta - \alpha\right) I_a - \alpha S_a^2 - \omega S_a}{\beta S_a\sigma_2 + \alpha I_a + \alpha S_a}, \tag{3.2}$$

$$I_n = \frac{I_a\left(-\beta S_a\sigma_2\sigma_1 + \gamma\theta + \omega\right)}{\beta S_a\sigma_2 + \alpha I_a + \alpha S_a}, \tag{3.3}$$

by (3.2) and (3.3), we have $I_a = \frac{\alpha - \omega}{\alpha} - S_a$ and substitute it into equation (3.3), we obtain

$$I_n = \frac{(\alpha - \omega - \alpha S_a)(-\beta S_a \sigma_2 \sigma_1 + \gamma\theta + \omega)}{(\beta S_a \sigma_2 + \alpha - \omega)\alpha}.$$

Substituting the values of I_n and I_a into the second equation of (3.1), we obtain that S_a is the positive root of the quadratic equation

$$c_1 S_a^2 + c_2 S_a + c_3 = 0. \tag{3.4}$$

where
$c_1 = -\beta^2\alpha\sigma_2(\omega(\sigma_1+\sigma_2) + \sigma_1(\alpha-\omega)(\sigma_1-\sigma_2) + \gamma\sigma_1(\theta-\sigma_2))$;
$c_2 = \beta\alpha(\alpha\gamma(\theta\sigma_1-\sigma_2) + \omega(\alpha-\omega)(\sigma_1-\sigma_2) + \gamma^2(\theta^2-\sigma_2) + \omega\gamma\theta(2-\sigma_1)) + D$;
$c_3 = -\gamma(\alpha-\omega)\left(\alpha^2 + \alpha\gamma + (\sigma_1(\alpha-\omega) + \gamma\theta + \omega)\beta\theta\right)$;
$D = \beta\sigma_1\sigma_2(\alpha-\omega)^2(\beta\sigma_1+\alpha) + \beta^2\omega\sigma_2(\gamma\theta+\omega) + \omega^2\alpha\beta$
$+\beta(\alpha-\omega)[\beta\sigma_1\sigma_2(\gamma\theta+\omega) + \beta\omega\sigma_1\sigma_2 + \alpha\gamma\sigma_1\sigma_2]$.

when $R_1^a = \frac{\alpha_1}{\omega_1} > 1$, $\alpha_1 = \alpha_2 = \alpha$, and $\omega_1 = \omega_2 = \omega$, we can obtain $\alpha > \omega$. Since $\theta > 1 > \sigma_2$, $\sigma_1 > \sigma_2$, so $c_1 < 0$, $c_2 > 0$ and $c_3 < 0$. After calculation, one root of equation (3.4) corresponds to the equilibrium $E_0^2 = \left(S_a^0, I_n^0, I_a^0\right) = (\frac{\alpha_1-\omega_1}{\alpha_1}, 0, 0)$. By the relationship between root and coefficient, it is easy to get that there are two positive roots of equation(3.4). If $R_2^d > 1$, the other positive root exists and makes $I_n > 0$, $S_a > 0$, $I_a > 0$, that is the equilibrium E_2. □

3.2. Threshold of awareness and disease

It can be seen from above that the threshold of awareness is $R_1^a = \frac{\alpha_1}{\omega_1}$, when awareness does not spread, the basic reproduction number of disease is $R_1^d = \frac{\beta}{\gamma}$. Next, we calculate the epidemic threshold when awareness spreads. According to the concepts of next generation matrix and the basic reproduction number presented in [25], we define

$$F = \begin{bmatrix} \beta(1 - 2I_n - S_a - \sigma_1 I_a - I_a) & \sigma_1\beta(1 - I_n - S_a - 2I_a) - \beta I_n \\ \sigma_2\beta S_a & \sigma_1\sigma_2\beta S_a \end{bmatrix},$$

$$V = \begin{bmatrix} \gamma + \alpha_2(S_a + I_a) & \alpha_2 I_n - \omega_2 \\ -\alpha_2(S_a + I_a) & -\alpha_2 I_n + \omega_2 + \theta\gamma \end{bmatrix},$$

$$FV^{-1}\left(E_0^2\right) = \begin{bmatrix} \omega_1 G_1 & \omega_1 G_2 \\ \sigma_2(\alpha_1 - \omega_1) G_1 & \sigma_2(\alpha_1 - \omega_1) G_2 \end{bmatrix},$$

where

$$G_1 = \frac{\beta\alpha_1(\gamma\theta + \omega_2) + \sigma_1\beta\alpha_2(\alpha_1 - \omega_1)}{\alpha_1\gamma(\alpha_1\alpha_2\theta + \alpha_1\gamma\theta - \alpha_2\theta\omega_1 + \alpha_1\omega_2)},$$
$$G_2 = \frac{\beta\alpha_1(\sigma_1\gamma + \omega_2) + \sigma_1\beta\alpha_2(\alpha_1 - \omega_1)}{\alpha_1\gamma(\alpha_1\alpha_2\theta + \alpha_1\gamma\theta - \alpha_2\theta\omega_1 + \alpha_1\omega_2)}.$$

Hence, a straightforward calculation of $\rho(FV^{-1})$ gives that the basic reproduction number of disease is $R_2^d = \psi_0$ when awareness spreads, where

$$\psi_0 = \frac{\beta\left[\sigma_1\sigma_2\alpha_2\left(\alpha_1-\omega_1\right)^2 + m\left(\alpha_1-\omega_1\right) + \alpha_1\omega_1\left(\gamma\theta+\omega_2\right)\right]}{\alpha_1\gamma\alpha_2\theta\left(\alpha_1-\omega_1\right) + \alpha_1^2\gamma\left(\gamma\theta+\omega_2\right)},$$

$$m = \alpha_1\sigma_2\gamma\theta + \alpha_1\sigma_2\omega_2 + \sigma_1\omega_1\alpha_2.$$

4. The stability of equilibria

4.1. Local stability of equilibria

Theorem 4.1. *If $R_1^a < 1$ and $R_1^d < 1$, then the disease-free awareness-free equilibrium E_0^1 is locally asymptotically stable.*

Proof. The Jacobian matrix of system (2.2) at E_0^1 is

$$J\left(E_0^1\right) = \begin{pmatrix} \alpha_1-\omega_1 & 0 & \alpha_1+\theta\gamma \\ 0 & \beta-\gamma & \sigma_1\beta+\omega_2 \\ 0 & 0 & -\omega_2-\theta\gamma \end{pmatrix}.$$

The characteristic equation of $J\left(E_0^1\right)$ is

$$\Phi_1(\lambda) = \left(\lambda-\alpha_1+\omega_1\right)\left(\lambda-\beta+\gamma\right)\left(\lambda+\omega_2+\theta\gamma\right) = 0.$$

where λ denotes the eigenvalue and

$$\lambda_1 = \alpha_1-\omega_1,\ \lambda_2 = \beta-\gamma,\ \lambda_3 = -\omega_2-\theta\gamma < 0.$$

Thus, if $R_1^a < 1$ and $R_1^d < 1$, then the $\lambda_1 = \alpha_1-\omega_1 < 0$, $\lambda_2 = \beta-\gamma < 0$. All roots of $\Phi_1(\lambda)$ have negative real parts. Hence, E_0^1 is locally asymptotically stable. □

Theorem 4.2. *If $R_1^a > 1$ and $R_2^d < 1$, then the disease-free awareness-endemic equilibrium E_0^2 is locally asymptotically stable.*

Proof. The Jacobian matrix of system (2.2) at E_0^2 is

$$J\left(E_0^2\right) = \begin{pmatrix} \omega_1-\alpha_1 & n_1 & n_2 \\ 0 & a & b \\ 0 & c & d \end{pmatrix},$$

where

$$n_1 = \frac{\left(\sigma_2\beta+\alpha_1\right)\left(\omega_1-\alpha_1\right)}{\alpha_1},\ n_2 = \frac{\left(\sigma_1\sigma_2\beta+\alpha_1\right)\left(\omega_1-\alpha_1\right)}{\alpha_1} + \omega_1 + \theta\gamma,$$

we use symbol M_1 and

$$M_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \dfrac{\beta\omega_1-\alpha_2\left(\alpha_1-\omega_1\right)}{\alpha_1}-\gamma & \dfrac{\beta\sigma_1\omega_1+\alpha_1\omega_2}{\alpha_1} \\ \dfrac{\left(\beta\sigma_2+\alpha_2\right)\left(\alpha_1-\omega_1\right)}{\alpha_1} & \dfrac{\sigma_1\sigma_2\beta\left(\alpha_1-\omega_1\right)}{\alpha_1}-\theta\gamma-\omega_2 \end{pmatrix}.$$

The eigenvalues of $J\left(E_0^2\right)$ are

$$\lambda_1 = \omega_1 - \alpha_1,$$
$$\lambda_{2,3} = \frac{1}{2}(tr(M_1) \pm \sqrt{(tr(M_1))^2 - 4det(M_1)}) = \frac{1}{2}[(a+d) \pm \sqrt{(a-d)^2 + 4bc}].$$

If $R_1^a > 1$, then $\alpha_1 > \omega_1$, so $\lambda_1 < 0$ and $bc > 0$. It is easy to show that $Re\lambda_{2,3} < 0$ if and only if $(a+d) + \sqrt{(a-d)^2 + 4bc} < 0$, that is $det(M_1) > 0$. Using that

$$det(M_1) = \frac{-\beta\left[\sigma_1\sigma_2\alpha_2\left(\alpha_1 - \omega_1\right)^2 + m\left(\alpha_1 - \omega_1\right) + \alpha_1\omega_1\left(\gamma\theta + \omega_2\right)\right]}{\alpha_1^2} + \frac{\gamma\alpha_2\theta\left(\alpha_1 - \omega_1\right) + \alpha_1\gamma\left(\gamma\theta + \omega_2\right)}{\alpha_1} > 0.$$

We can easily prove that if $R_1^a > 1$ and $R_2^d < 1$, then $detM_1 > 0$, so E_0^2 is locally asymptotically stable. □

Theorem 4.3. *If $R_1^a < 1$ and $R_1^d > 1$, then the disease-endemic awareness-free equilibrium E_1 is locally asymptotically stable.*

Proof. The Jacobian matrix of system (2.2) at E_1 is

$$J\left(E_1\right) = \frac{1}{\beta}\begin{pmatrix} (\gamma-\beta)\sigma_2\beta + \gamma\alpha_1 - \beta\omega_1 & 0 & \beta\theta\gamma + \gamma\alpha_1 \\ (\beta+\alpha_2)(\gamma-\beta) & \beta(\gamma-\beta) & \alpha_2(\gamma-\beta) + \sigma_1\gamma + \omega_2\beta \\ (\beta\sigma_2+\alpha_2)(\beta-\gamma) & 0 & \alpha_2(\beta-\gamma) - \theta\gamma\beta - \omega_2\beta \end{pmatrix}.$$

The characteristic equation of $J\left(E_1\right)$ is

$$\Phi_2(\lambda) = (\lambda - \gamma + \beta)\left|\lambda E - M_2\right| = 0,$$

where

$$M_2 = \begin{pmatrix} -\dfrac{(\beta-\gamma)\sigma_2\beta - \gamma\alpha_1 + \beta\omega_1}{\beta} & \dfrac{\beta\theta\gamma + \gamma\alpha_1}{\beta} \\ \dfrac{(\beta-\gamma)\sigma_2\beta - \gamma\alpha_2 + \beta\alpha_2}{\beta} & -\dfrac{\beta\theta\gamma + \gamma\alpha_2 + \beta\left(\omega_2 - \alpha_2\right)}{\beta} \end{pmatrix}.$$

The eigenvalues of $J\left(E_1\right)$ are

$$\lambda_1 = \gamma - \beta, \lambda_{2,3} = \frac{1}{2}(tr(M_2) \pm \sqrt{\left(tr(M_2)\right)^2 - 4det(M_2)}).$$

If $R_1^d > 1$, then $\lambda_1 < 0$, using the same argument in **Theorem 4.2**, the necessary and sufficient condition for $Re\lambda_{2,3} < 0$ is $det(M_2) > 0$. We define $M_2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, if $R_1^a < 1$, then $\alpha_1 < \omega_1, \alpha_2 < \omega_2$. Since $\alpha_1 < \alpha_2$, $\omega_2 < \omega_1$, so $|a| > |c|$, $|d| > |b|$, that is $det(M_2) > 0$, $Re\lambda_{2,3} < 0$. Hence, E_1 is locally asymptotically stable. □

4.2. Global stability of equilibria

In this subsection, for simplicity, we assume that $\alpha_1 = \alpha_2 = \alpha$ and $\omega_1 = \omega_2 = \omega$.

Theorem 4.4. *If $R_1^a < 1$ and $R_1^d < 1$, then the disease-free awareness-free equilibrium E_0^1 is globally asymptotically stable.*

Proof. Consider a Lyapunov function defined by

$$V_1 = S_a + \frac{\omega - \alpha}{\omega + \gamma\theta} I_n + I_a.$$

Differentiating V_1 along the solutions of system (2.2) leads to

$$\begin{aligned}
\dot{V}_1 =& -\alpha S_a^2 - \frac{\beta(\omega-\alpha)}{\omega+\gamma\theta} I_n^2 - \frac{(\beta\sigma_1+\alpha)(\omega-\alpha)}{\omega+\gamma\theta} I_a^2 - \frac{(\beta+\alpha)(\omega-\alpha)}{\omega+\gamma\theta} S_a I_n \\
& - \frac{(\omega-\alpha)(\gamma\theta-\beta\sigma_1)}{\omega+\gamma\theta} I_a - (\omega-\alpha) S_a - \frac{(\gamma-\beta)(\omega-\alpha)}{\omega+\gamma\theta} I_n - 2\alpha S_a I_a \\
& - \frac{\beta\sigma_1(\omega-\alpha)}{\omega+\gamma\theta} S_a I_a - \frac{(\omega-\alpha)(\beta\sigma_1+\beta+\alpha)}{\omega+\gamma\theta} I_a I_n \\
=& -\alpha S_a^2 - 2\alpha S_a I_a - \frac{\left(1-R_1^d\right)\left(1-R_1^a\right)\gamma\omega}{\omega+\gamma\theta} I_n - \left(1-R_1^a\right)\frac{\omega(\gamma\theta-\beta\sigma_1)}{\omega+\gamma\theta} I_a \\
& - \left(1-R_1^a\right)\left(\frac{(\beta+\alpha)\omega}{\omega+\gamma\theta} S_a I_n + \frac{\beta\sigma_1\omega}{\omega+\gamma\theta} S_a I_a + \frac{\omega(\beta\sigma_1+\beta+\alpha)}{\omega+\gamma\theta} I_a I_n\right) \\
& - \left(1-R_1^a\right)\left(\frac{\beta\omega}{\omega+\gamma\theta} I_n^2 + \frac{(\beta\sigma_1+\alpha)\omega}{\omega+\gamma\theta} I_a^2 + \omega S_a\right).
\end{aligned}$$

Therefore, if $R_1^a < 1$ and $R_1^d < 1$, then $\dot{V}_1 \le 0$. $\dot{V}_1 = 0$ if and only if $S_a = I_n = I_a = 0$. That is $\{(S_a, I_n, I_a) \,|\dot{V}_1 = 0\} = \left\{E_0^1\right\}$. Using the same argument in [26], by LaSalle's Invariance Principle[27], the E_0^1 is globally asymptotically stable. □

Theorem 4.5. *If $R_1^a < 1$ and $R_1^d > 1$, then the disease-endemic awareness-free equilibrium E_1 is globally asymptotically stable.*

Proof. Consider a Lyapunov function defined by

$$V_2 = S_a + \frac{3(\beta+\omega)}{\omega-\alpha} I_n + \frac{3(\beta+\omega)}{\omega-\alpha} I_a.$$

Differentiating V_2 along the solutions of system (2.2) leads to

$$\begin{aligned}
\dot{V}_2 =& -\frac{\left(\beta(1-I_n)-\gamma\right)^2}{\beta(\beta-\gamma)} - \frac{3\alpha(\beta+\omega)+\beta\sigma_1(\omega-\alpha)}{\omega-\alpha} I_a^2 - \frac{3\alpha(\beta+\omega)}{\omega-\alpha} S_a^2 \\
& - (\beta-\gamma)(\sigma_1+1+\frac{\alpha}{\beta}) I_n I_a - (\beta-\gamma)(1+\frac{\alpha}{\beta}) I_n S_a - \frac{6\alpha(\beta+\omega)}{\omega-\alpha} I_a S_a \\
& - \left(\beta\sigma_1(1-I_a-S_a)+\omega\right)\frac{I_a}{I_n} - (\omega-\alpha+\frac{\gamma(\alpha+\beta)+2\beta(\beta+\omega)}{\beta}) S_a \\
& - \frac{2\beta^2(1-\sigma_1)+\beta(\omega-\alpha+\gamma\sigma_1+\omega+\gamma)+\alpha\gamma}{\beta} I_a - \beta\sigma_1 I_a S_a \\
=& -\frac{\left(\beta(1-I_n)-\gamma\right)^2}{\beta\gamma\left(R_1^d-1\right)} - \frac{3\alpha(\beta+\omega)+\beta\sigma_1\omega\left(1-R_1^a\right)}{\omega\left(1-R_1^a\right)} I_a^2 - \frac{3\alpha(\beta+\omega)}{\omega\left(1-R_1^a\right)} S_a^2 \\
& - \gamma\left(R_1^d-1\right)(\sigma_1+1+\frac{\alpha}{\beta}) I_n I_a - \gamma\left(R_1^d-1\right)(1+\frac{\alpha}{\beta}) I_n S_a - \frac{6\alpha(\beta+\omega)}{\omega\left(1-R_1^a\right)} I_a S_a \\
& - \left(\beta\sigma_1(1-I_a-S_a)+\omega\right)\frac{I_a}{I_n} - (\omega\left(1-R_1^a\right)+\frac{\gamma(\alpha+\beta)+2\beta(\beta+\omega)}{\beta}) S_a
\end{aligned}$$

$$-\frac{2\beta^2(1-\sigma_1)+\beta(\omega(1-R_1^a)+\gamma\sigma_1+\omega+\gamma)+\alpha\gamma}{\beta}I_a-\beta\sigma_1 I_a S_a.$$

Therefore, if $R_1^a<1$ and $R_1^d>1$, then $\dot{V}_2\le 0$. $\dot{V}_2=0$ if and only if $S_a=I_a=0$, $I_n=\frac{\beta-\gamma}{\beta}$. That is $\{(S_a,I_n,I_a)\,|\dot{V}_2=0\}=\{E_1\}$. By LaSalle's Invariance Principle, the E_1 is globally asymptotically stable. □

Theorem 4.6. *If $R_1^a>1$ and $R_2^d<1$, then the disease-free awareness-endemic equilibrium E_0^2 is globally asymptotically stable. If $R_1^a>1$ and $R_2^d>1$, then the disease-endemic awareness-endemic equilibrium E_2 is globally asymptotically stable.*

Proof. If $R_1^a>1$, then the corresponding disease-free equilibrium is E_0^2. It shows that, as $t\to\infty$, we have $S_a(t)+I_a(t)\to\frac{\alpha-\omega}{\alpha}$, $S_n(t)+I_n(t)\to\frac{\omega}{\alpha}$, so the limiting system of system (2.2) is

$$\begin{cases}\dfrac{dI_n(t)}{dt}=(I_n+\sigma_1 I_a)\,\beta(\dfrac{\omega}{\alpha}-I_n)-(\alpha-\omega)I_n+\omega I_a-\gamma I_n,\\[2ex] \dfrac{dI_a(t)}{dt}=(I_n+\sigma_1 I_a)\,\sigma_2\beta(\dfrac{\alpha-\omega}{\alpha}-I_a)+(\alpha-\omega)I_n-\omega I_a-\theta\gamma I_a.\end{cases}\tag{4.1}$$

Since Γ is positively invariant with respect to system (2.2), the dynamics of system (4.1) can be explored on this restricted region

$$\Delta=\{(I_n,I_a)\in R_+^2|I_n\le\tfrac{\omega}{\alpha},I_a\le\tfrac{\alpha-\omega}{\alpha}\}.$$

Next, we study the globally asymptotic stability of the equilibrium E_0^2 and the equilibrium E_2 by studying the limit system (4.1). Motivated by the method in [28, 29], define

$$\begin{aligned}f(u)&=\begin{pmatrix}f_1(u_1,u_2)\\ f_2(u_1,u_2)\end{pmatrix}\\&=\begin{pmatrix}(u_1+\sigma_1u_2)\,\beta(\frac{\omega}{\alpha}-u_1)-(\alpha-\omega)u_1+\omega u_2-\gamma u_1\\ (u_1+\sigma_1u_2)\,\sigma_2\beta(\frac{\alpha-\omega}{\alpha}-u_2)+(\alpha-\omega)u_1-\omega u_2-\theta\gamma u_2\end{pmatrix}.\end{aligned}$$

Then $f:R_+^2\to R_+^2$ is a continuously differentiable map. Obviously, $f(0)=0$, $f_i(u)\ge 0$ for all $u\in\Delta$, $u_i=0$, $i=1,2$.

A straightforward calculation of $Df(u)$ gives

$$Df(u)=\begin{pmatrix}q_1 & \sigma_1\beta(\frac{\omega}{\alpha}-I_n)+\omega\\ \beta\sigma_2(\frac{\alpha-\omega}{\alpha}-I_a)+\alpha-\omega & q_2\end{pmatrix}.$$

Where

$$\begin{aligned}q_1&=\beta(\tfrac{\omega}{\alpha}-I_n)-\beta(I_a\sigma_1+I_n)-\alpha+\omega-\gamma,\\ q_2&=\beta\sigma_1\sigma_2(\tfrac{\alpha-\omega}{\alpha}-I_a)-\beta\sigma_2(I_a\sigma_1+I_n)-\gamma\theta-\omega.\end{aligned}$$

If $R_1^a>1$, then $\dfrac{\partial f_i}{\partial u_j}\ge 0(i\ne j)$, $u\in\Delta$, so f is cooperative on Δ. Obviously, $|Df(u)|\ne 0$, so $Df(u)$ is irreducible on $u\in\Delta$.

For every $k\in(0,1)$ and $u\in\Delta$, we have

$$\begin{aligned} f_1(ku_1, ku_2) &= (ku_1 + \sigma_1 ku_2)\,\beta(\frac{\omega}{\alpha} - ku_1) - (\alpha - \omega)ku_1 + \omega ku_2 - \gamma ku_1 \\ &> (ku_1 + \sigma_1 ku_2)\,\beta(\frac{\omega}{\alpha} - u_1) - (\alpha - \omega)ku_1 + \omega ku_2 - \gamma ku_1 \\ &= k f_1(u_1, u_2)\,. \end{aligned}$$

Using the same argument, we find that $f_2(ku_1, ku_2) > k f_2(u_1, u_2)$, so f is strictly sublinear on Δ.

A straightforward calculation of $Df(0)$ gives

$$Df(0) = \begin{pmatrix} \dfrac{\beta\omega}{\alpha} - \alpha + \omega - \gamma & \dfrac{\sigma_1\beta\omega}{\alpha} + \omega \\ \dfrac{\beta\sigma_2(\alpha-\omega)}{\alpha} + \alpha - \omega & \dfrac{\beta\sigma_1\sigma_2(\alpha-\omega)}{\alpha} - \gamma\theta - \omega \end{pmatrix}.$$

It is easy to show that when $\alpha_1 = \alpha_2 = \alpha$ and $\omega_1 = \omega_2 = \omega$, the $Df(0) = M_1$, so $s(Df(0)) := \max\{\mathrm{Re}\,\lambda : \Phi_3(\lambda)\}$, where $\Phi_3(\lambda)$ denotes the characteristic equation of $Df(0)$.

By **Corollary 3.2** in [28], and **Theorem 4.2** in this paper, we obtain the following results.

If $R_1^a > 1$, and $R_2^d < 1$, then

$$s(Df(0)) := \max\{\mathrm{Re}\,\lambda : \Phi_3(\lambda)\} < 0,$$

so the equilibrium $(0, 0)$ of system (4.1) is globally asymptotically stable. Since $S_a(t) \to \frac{\alpha-\omega}{\alpha}$, $I_n(t) \to 0$ and $I_a(t) \to 0$ as $t \to \infty$, while $E_0^2 = (\frac{\alpha-\omega}{\alpha}, 0, 0)$. Thus, based on the theory of asymptotic autonomous systems [30], E_0^2 is globally asymptotically stable.

If $R_1^a > 1$, and $R_2^d > 1$, then

$$s(Df(0)) := \max\{\mathrm{Re}\,\lambda : \Phi_3(\lambda)\} > 0,$$

so the equilibrium (I_n^*, I_a^*) of system (4.1) is globally asymptotically stable. Since $S_a(t) \to S_a^{**}$, $I_n(t) \to I_n^{**}$ and $I_a(t) \to I_a^{**}$ as $t \to \infty$, while $E_2 = (S_a^{**}, I_n^{**}, I_a^{**})$. By carrying out similar argument to the above, E_2 is globally asymptotically stable. □

5. Numerical simulation

5.1. Effect of awareness on system dynamics

If $R_1^a = \frac{\alpha_1}{\omega_1} < 1$, $R_1^d = \frac{\beta}{\gamma} < 1$, then the system approaches the equilibrium E_0^1;

If $R_1^a = \frac{\alpha_1}{\omega_1} < 1$, $R_1^d = \frac{\beta}{\gamma} > 1$, then the system approaches the equilibrium E_1;

In the above two cases, the awareness parameters have no effect on the epidemic threshold. The R_1^d increases with the increase of β and decreases with the increase of γ.

If

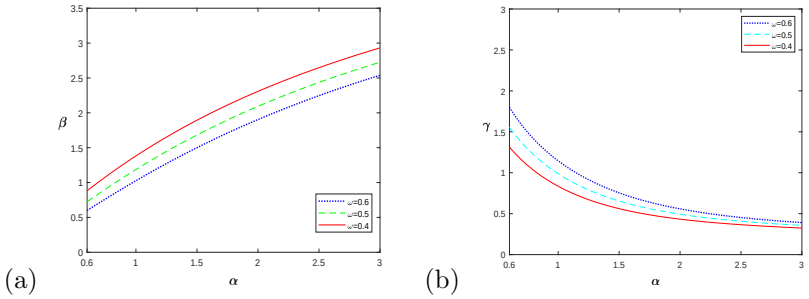
$$\begin{cases} R_1^a = \dfrac{\alpha_1}{\omega_1} > 1, \\ R_2^d = R_1^d \dfrac{\sigma_1\sigma_2\alpha_2\,(\alpha_1 - \omega_1)^2 + m\,(\alpha_1 - \omega_1) + \alpha_1\omega_1\,(\gamma\theta + \omega_2)}{\alpha_1\alpha_2\theta\,(\alpha_1 - \omega_1) + \alpha_1^2\,(\gamma\theta + \omega_2)} < 1, \end{cases}$$

where $m = (\alpha_1\sigma_2\gamma\theta + \alpha_1\sigma_2\omega_2 + \sigma_1\omega_1\alpha_2)$, then the system approaches the equilibrium E_0^2;

$$\begin{cases} \text{If} \\ R_1^a = \dfrac{\alpha_1}{\omega_1} > 1, \\ R_2^d = R_1^d \dfrac{\sigma_1\sigma_2\alpha_2\left(\alpha_1-\omega_1\right)^2 + m\left(\alpha_1-\omega_1\right) + \alpha_1\omega_1\left(\gamma\theta+\omega_2\right)}{\alpha_1\alpha_2\theta\left(\alpha_1-\omega_1\right) + \alpha_1^2\left(\gamma\theta+\omega_2\right)} > 1, \end{cases}$$

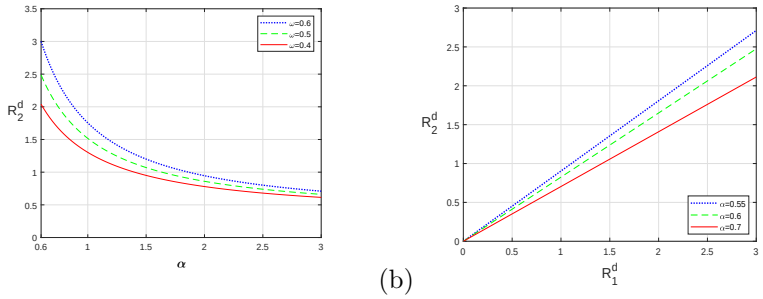
where $m = (\alpha_1\sigma_2\gamma\theta + \alpha_1\sigma_2\omega_2 + \sigma_1\omega_1\alpha_2)$, then the system approaches the equilibrium E_2;

Hence, when $R_1^a = \frac{\alpha_1}{\omega_1} > 1$, then the awareness spreading affects the epidemic threshold. Next, we learn about the effects of awareness spreading on epidemic.



(a) (b)

Figure 2. Relationship between awareness parameters and disease parameters.



(a) (b)

Figure 3. Relationship between awareness parameters and epidemic thresholds.

Figure 2(a) illustrates that when $R_1^a > 1$ and $R_2^d = 1$, β increases with rise in the value of α and decreases with increase in the value of ω. Figure 2(b) illustrates that when $R_1^a > 1$ and $R_2^d = 1$, γ decreases with increase in the value of α and increases with increase in the value of ω. Therefore, Figure 2 illustrates that the higher the awareness, the smaller recovery rate and the more infected individuals are allowed before the outbreak of the epidemic. In other words, raising awareness can effectively reduce the outbreak of epidemic. Here, the parameters are (a)$\gamma = 0.6$, (b)$\beta = 1.8$. Others are $\theta = 2$, $\sigma_1 = 0.5$, $\sigma_2 = 0.4$.

Figure 3(a) illustrates that R_2^d decreases with increase in the value of α when $R_1^a > 1$ holds. This shows that the stronger the α is, the fewer people are infected per unit time. Figure 3(b) illustrates the linear relationship between R_2^d and R_1^d is $R_2^d = kR_1^d$, where $k \in (0, 1)$. This shows that raising awareness can reduce the basic reproduction number of disease so as to reduce the spread of epidemic. Here, (b)$\omega_1 = 0.5$, $\omega_2 = 0.4$. Other parameters are $\beta = 1.8$, $\gamma = 0.6$, $\theta = 2$, $\sigma_1 = 0.5$, $\sigma_2 = 0.4$.

Next, we analyze the effects of awareness parameters σ_1, σ_2, and θ on epidemic threshold.

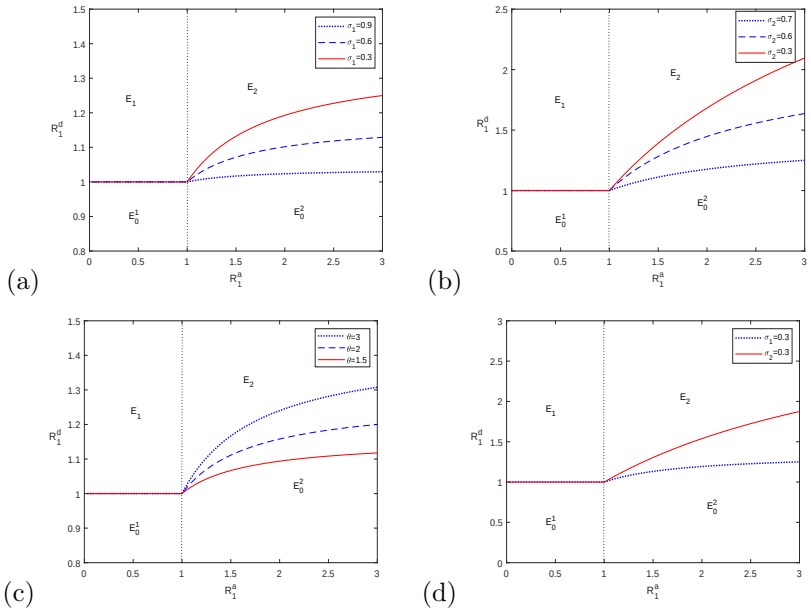


Figure 4. Effects of σ_1, σ_2 and θ on stable region of different equilibria.

When awareness has no effect on epidemic, that is $\sigma_1 = \sigma_2 = \theta = 1$, the epidemic threshold is given by

$$R_2^d = R_1^d.$$

Figure 4(a) illustrates when one considers *reduced infectivity*, where $0 < \sigma_1 < 1$, $\sigma_2 = \theta = 1$, the epidemic threshold is given by

$$R_2^d = R_1^d \frac{\omega\sigma_1 R_1^a + \omega + \gamma - \omega\sigma_1}{\omega R_1^a + \gamma}.$$

If $R_2^d > 1$, then $R_1^d > \dfrac{\omega R_1^a + \gamma}{\omega\sigma_1 R_1^a + \omega + \gamma - \omega\sigma_1} = 1 + \dfrac{\omega\left(R_1^a - 1\right)(1 - \sigma_1)}{\omega\sigma_1 R_1^a + \omega + \gamma - \omega\sigma_1}$. When the rate of awareness spreading is much higher than the rate of awareness losing, that is $R_1^a \to \infty$, then the inequality $R_1^d > \frac{1}{\sigma_1}$ holds. The parameters are $\gamma = 0.6$, $\omega_1 = 0.5$, $\omega_2 = 0.4$, $\alpha_2 = 0.6$.

Figure 4(b) illustrates when one considers *reduced susceptibility*, where $0 < \sigma_2 < 1$, $\sigma_1 = \theta = 1$, the epidemic threshold is given by

$$R_2^d = R_1^d \frac{\sigma_2\left(R_1^a - 1\right) + 1}{R_1^a}.$$

If $R_2^d > 1$, then $R_1^d > \dfrac{R_1^a}{\sigma_2\left(R_1^a - 1\right) + 1} = 1 + \dfrac{(1 - \sigma_2)\left(R_1^a - 1\right)}{\sigma_2\left(R_1^a - 1\right) + 1}$. Similar to the previous case, if $R_1^a \to \infty$, then the inequality $R_1^d > \frac{1}{\sigma_2}$ holds. The parameters are $\gamma = 0.6$, $\omega_2 = 0.4$, $\omega_1 = 0.5$, $\alpha_2 = 0.6$.

Figure 4(c) illustrates when one considers *faster recovery*, where $\theta > 1$, $\sigma_1 = \sigma_2 = 1$, the epidemic threshold is given by

$$R_2^d = R_1^d \frac{\omega R_1^a + \gamma\theta}{\omega R_1^a \theta - \omega\theta + \gamma\theta + \omega}.$$

If $R_2^d > 1$, then $R_1^d > \dfrac{\omega R_1^a \theta - \omega\theta + \gamma\theta + \omega}{\omega R_1^a + \gamma\theta} = 1 + \dfrac{\omega\left(R_1^a - 1\right)(\theta - 1)}{\omega R_1^a + \gamma\theta}$. If $R_1^a \to \infty$, then the inequality $R_1^d > \theta$ holds. The parameters are $\gamma = 0.6$, $\omega_1 = 0.5$, $\omega_2 = 0.4$, $\alpha_2 = 0.6$.

Figure 4(d) illustrates that when $R_1^a > 1$, the protective measures taken by the aware susceptible individuals can reduce the spread of the epidemic more than those taken by the aware infected individuals. The parameters are $\gamma = 0.6$, $\theta = 1$, $\alpha_2 = 0.6$, $\omega_1 = 0.5$, $\omega_2 = 0.4$.

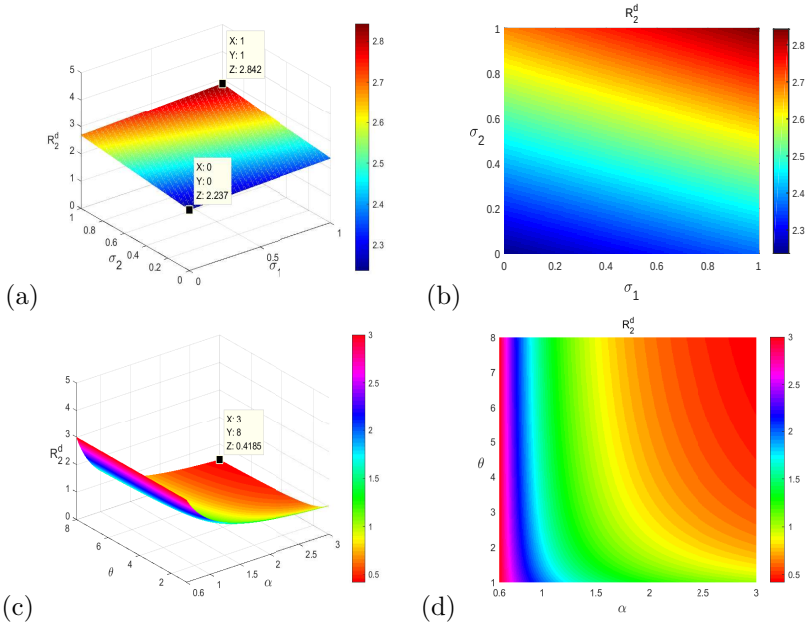


Figure 5. Effects of various parameters on the epidemic threshold R_2^d when $R_1^a > 1$.

Figure 5(a,b) illustrates that when $R_1^a > 1(\alpha = 0.6, \omega = 0.5, \beta = 1.8, \gamma = 0.6, \theta = 2)$, the effects of σ_1 and σ_2 on epidemic threshold R_2^d. It can be noted that R_2^d decreases with the decrease of σ_1 and σ_2, while the effect of the change of σ_2 on R_2^d is more obvious. Therefore, we should increase awareness education for individuals, especially those unaware susceptible individuals so as to reduce the spread of disease to a greater extent. Figure 5(c,d) illustrates that when $R_1^a > 1(\omega = 0.6, \beta = 1.8, \gamma = 0.6, \sigma_1 = 0.6, \sigma_2 = 0.5)$, the effects of α and θ on epidemic threshold R_2^d. It can be seen that R_2^d decreases with the increase of α and θ. When α is relatively small, the change of θ has fewer effects on the epidemic threshold.

5.2. Dynamic behavior of equilibria

In this subsection, we analyze the time series dynamics of different classes, we take the initial values of each variable as: $S_n(0) = 0.1, S_a(0) = 0.1, I_n(0) = 0.4, I_a(0) = 0.4$. For the rest of the parameter values, we refer to reference [8, 23]. Figure 6(a) shows that when $R_1^a < 1$ and $R_1^d < 1(\beta = 0.5, \gamma = 0.6, \alpha_1 = 0.3, \omega_1 = 0.5, \alpha_2 = 0.4)$, the system (2.2) has only one disease-free awareness-free equilibrium E_0^1, and E_0^1 is globally asymptotically stable; Figure 6(b) shows that when $R_1^a > 1$ and $R_2^d < 1(\beta = 0.5, \gamma = 0.6, \alpha_1 = 0.6, \omega_1 = 0.5, \alpha_2 = 0.7)$, the system (2.2)

has only one disease-free awareness-endemic equilibrium E_0^2, and E_0^2 is globally asymptotically stable; Figure 6(c) shows that when $R_1^a < 1$ and $R_1^d > 1(\beta = 1.8$, $\gamma = 0.6$, $\alpha_1 = 0.3$, $\omega_1 = 0.5$, $\alpha_2 = 0.4)$, the system (2.2) has only one disease-endemic awareness-free equilibrium E_1, and E_1 is globally asymptotically stable; Figure 6(d) shows that when $R_1^a > 1$ and $R_2^d > 1(\beta = 1.8$,$\gamma = 0.6$, $\alpha_1 = 0.6$, $\omega_1 = 0.5$, $\alpha_2 = 0.7)$, the system (2.2) has only one disease-endemic awareness-endemic equilibrium E_2, and E_2 is globally asymptotically stable. Other parameter values are $\omega_2 = 0.4$, $\sigma_1 = 0.6$, $\sigma_2 = 0.5$, $\theta = 2$.

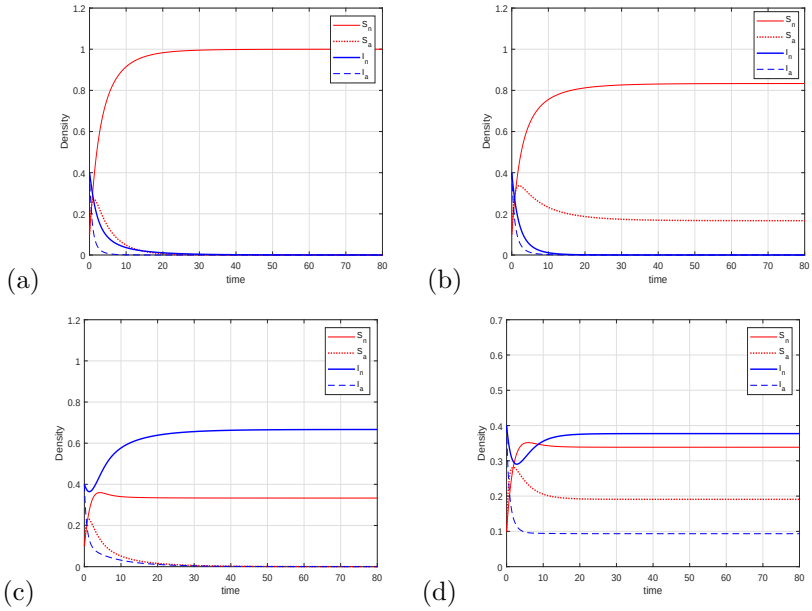


Figure 6. The stability of different equilibria.

6. Discussion

In this paper, we study an SIS model for the effects of awareness spreading on epidemic based on previous studies. The main feature of our model is that it is more comprehensive and more in line with the practical life. The dynamic process of the model is fully analyzed in details, and the effects of various aspects of awareness on epidemic are considered. Based on the analysis, we find that raising awareness can effectively reduce the basic regeneration number of disease. In addition, awareness can not only reduce the infection rate of epidemic, but also raise the recovery rate of individuals and shorten the period of epidemic.

The way of awareness spreading studied in this paper only consider the communication between individuals, while the way of awareness losing only take the loss of individual autonomy into account. However, in fact, people can also obtain awareness through Internet searching and lose awareness through contacting others. Moreover, although awareness may improve epidemic prevention in some cases, it may cause public panic in other cases, leading to further spread of epidemic. These factors will be taken into account in the subsequent researches. Though we only studied the one way of awareness spreading and losing, our research results also play

a crucial role in preventing and controlling the spread of some infectious diseases.

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