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The Maximum Number of Zeros of Functions with Parameters and Application to Differential Equations*

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Abstract In this paper, we first study the problem of finding the maximum number of zeros of functions with parameters and then apply the results obtained to smooth or piecewise smooth planar autonomous systems and scalar periodic equations to study the number of limit cycles or periodic solutions, improving some fundamental results both on the maximum number of limit cycles bifurcating from an elementary focus of order k or a limit cycle of multiplicity k, or from a period annulus, and on the maximum number of periodic solutions for scalar periodic smooth or piecewise smooth equations as well.

Keywords Maximum number, Multiplicity, Limit cycle, Piecewise smooth equation.

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1. Introduction

As we know, in the study of differential equations a very important aspect is the number of limit cycles for planar systems or the number of periodic solutions for scalar periodic equations. There have been certain classical and fundamental theorems on this aspect. For example, in a family of C^∞ systems with parameters a limit cycle of multiplicity k generates at most k limit cycles, and an elementary focus of order k generates at most k limit cycles. In a family of C^∞ near-Hamiltonian systems with parameters, the total number of the first order Melnikov function can control the number of limit cycles bifurcating from a period annulus. These important results have many applications to polynomial systems. The proofs of the results are all very similar with two main steps. The first step is to establish a suitable bifurcation function. The second step is to analyze the number of zeros of the bifurcation function by using the method of contradiction. See Theorems 1.3.2,

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2.3.2 and 3.1.4 in Han [5], Theorem 2.4 in Part II of Christopher and Li [2] and Theorem 1.1 in Han [6]. However, from the proofs of these theorems, the multiplicity of the limit cycles bifurcated is not considered.

The aim of this paper is to further study the problem of finding the maximum number of zeros of functions with parameters and then improve some fundamental results on the maximum number of limit cycles bifurcating from an elementary focus of order k or a limit cycle of multiplicity k, or from a period annulus. We study the same problem for piecewise smooth systems on the plane and scalar periodic smooth or piecewise smooth equations as well. As a preliminary, we first study the maximum number of zeros of functions with parameters in Section 2. Then, based on the main results of Section 2, we study the maximum number of limit cycles in planar autonomous systems (smooth or piecewise smooth) or of periodic solutions in scalar periodic equations (smooth or piecewise smooth) in the rest sections.

2. The number of zeros of functions with parameters

Let $F : I \times D \to \mathbf{R}$ be a C^k function, where $I \subset \mathbf{R}$ is an open interval, $D = \{\lambda | \, |\lambda| < \varepsilon\} \subset \mathbf{R}^n, \varepsilon > 0, k \geq 1, n \geq 1$. As we know, for a fixed $\lambda \in D$ and an integer $1 \leq l \leq k$ we say that $x_0 \in I$ is a zero of F in x with multiplicity l if

$$\frac{\partial^l F}{\partial x^l}(x_0, \lambda) \neq 0, \ \frac{\partial^j F}{\partial x^j}(x_0, \lambda) = 0, \ j = 0, 1, \cdots, l-1.$$

In this section, we study the number of zeros of the function $F(x, \lambda)$ in x based on the number of zeros of the unperturbed function $F(x, 0) \equiv f(x)$. For the purpose, we need to present or establish some preparation lemmas.

First, from Section 1.3 of [7], we have the following lemma.

Lemma 2.1. *Let $x_0 \in I$, $1 \leq l \leq k$. Then, there exists a C^{k-l} function $\bar{R} : I \times D \to \mathbf{R}$ such that for $(x, \lambda) \in I \times D$*

$$F(x, \lambda) = \sum_{j=0}^{l-1} \frac{1}{j!} \frac{\partial^j F}{\partial x^j}(x_0, \lambda)(x - x_0)^j + (x - x_0)^l \bar{R}(x, \lambda)$$

with

$$\bar{R}(x_0, \lambda) = \frac{1}{l!} \frac{\partial^l F}{\partial x^l}(x_0, \lambda).$$

If $F \in C^\infty(I \times D)$, then $\bar{R} \in C^\infty(I \times D)$ for any given $l \geq 1$.

The conclusion of the above lemma is a simple improvement of the well-known Taylor formula. The key point is the smoothness of $\bar{R}$.

The following lemma looks obvious and its proof needs to use Lemma 2.1.

Lemma 2.2. *Let $x_0 \in I$, $1 \leq l \leq k$. Then, $x = x_0$ is a zero of the unperturbed function f with multiplicity l if and only if there exists a C^{k-l} function $g : I \to \mathbf{R}$ satisfying $g(x_0) \neq 0$ such that*

$$f(x) = (x - x_0)^l g(x), \ x \in I.$$

Proof. Suppose f has $x = x_0$ as a zero of multiplicity l. Then, we have

$$f^{(l)}(x_0) \neq 0,\ f^{(j)}(x_0) = 0,\ j = 0, \cdots, l-1.$$

By Lemma 2.1 it follows that

$$f(x) = (x - x_0)^l g(x),\ x \in I,$$

where $g \in C^{k-l}(I)$ with $g(x_0) = \frac{1}{l!} f^{(l)}(x_0) \neq 0$. Hence, the necessity part is proved.

To prove the sufficiency part let us suppose that $f(x) = (x-x_0)^l g(x)$ for a C^{k-l} function $g : I \to \mathbf{R}$ with $g(x_0) \neq 0$. Noting $k - l \geq 0$ the function g is continuous at x_0, and hence we have

$$f(x) = (x - x_0)^l g(x_0) + o(|x - x_0|^l) \tag{2.1}$$

for $|x - x_0|$ small. On the other hand, noting $f \in C^k(I)$, by Talor formula it holds for $|x - x_0|$ small

$$f(x) = \sum_{j=0}^{l} \frac{1}{j!} f^{(j)}(x_0)(x - x_0)^j + o(|x - x_0|^l). \tag{2.2}$$

Then, comparing (2.1) and (2.2) yields that

$$f^{(j)}(x_0) = 0,\ j = 0, \cdots, l-1,\ \text{and}\ f^{(l)}(x_0) = l! g(x_0) \neq 0.$$

This means that $x = x_0$ is a zero of f with multiplicity l. This finishes the proof. □

Lemma 2.3. *Let $1 \leq l \leq k$ and $h \in C^k(I)$. If the derivative h' has at most $l - 1$ zeros in I, multiplicity taken into account, then the function h has at most l zeros in I, multiplicity taken into account.*

Proof. We provide a proof by contradiction. If the conclusion is not true, then h has $l + 1$ zeros in I, multiplicity taken into account. Then, among them there are exactly r different ones, denoted by $x_1, \cdots, x_r$ for some integer r satisfying $1 \leq r \leq l + 1$. We can assume that $x_1 < \cdots < x_r$ and that they have multiplicity $n_1, \cdots, n_r$ respectively. Then

$$n_1 + \cdots + n_r \geq l + 1,\ n_1 \geq 1, \cdots,\ n_r \geq 1.$$

It is easy to see that $x_1, \cdots, x_r$ are zeros of h' with multiplicity $n_1 - 1, \cdots, n_r - 1$ respectively. Here, if $n_j - 1 = 0$ for some j then $h'(x_j) \neq 0$ and we say that x_j is a zero of h' with multiplicity zero. By Rolle theorem, the derivative function h' has $r - 1$ zeros, which are in $(x_1, x_2), \cdots, (x_{r-1}, x_r)$ respectively.

Hence, the total number of zeros of h', multiplicity taken into account, is at least

$$(n_1 - 1) + \cdots + (n_r - 1) + (r - 1) = n_1 + \cdots + n_r - 1 \geq l.$$

This contradicts to our assumption. Then, the proof is ended. □

Now, we are in a position to prove our first main result in this section as follows.

Theorem 2.1. *Consider the C^k function $F : I \times D \to \mathbf{R}$. Let the function $f(x) = F(x, 0)$ have a zero $x_0 \in I$ with multiplicity l, $1 \leq l \leq k$. Then, there exist $\varepsilon_0 \in (0, \varepsilon)$ and a neighborhood U of x_0 in I such that for all $|\lambda| < \varepsilon_0$ the function F has at most l zeros in U in x, multiplicity taken into account.*

Proof. Observe that if the theorem is true for $l = k$, then it is also true for any $l < k$. Thus, we need only to prove the conclusion for $l = k$. Then, we assume $l = k$ and proceed our proof by induction on l.

For the case of $l = 1$, we have $F \in C^1(I \times D)$ and

$$f(x_0) = 0,\ f'(x_0) \neq 0,$$

that is,

$$F(x_0, 0) = 0,\ F_x(x_0, 0) \neq 0.$$

Applying the implicit function theorem, the equation $F(x, \lambda) = 0$ in x has a unique zero $x = \varphi(\lambda) = x_0 + O(|\lambda|)$ for $|\lambda|$ sufficiently small. Thus, the conclusion is true for $l = k = 1$.

Suppose that the conclusion is true for $l = k = m$. That is, for any C^m function $G : I \times D \to \mathbf{R}$, if $g(x) \equiv G(x, 0)$ has a zero $\bar{x}_0 \in I$ with multiplicity m, then there exist $\bar{\varepsilon}_0 \in (0, \varepsilon)$ and a neighborhood V of $\bar{x}_0$ in I such that for all $|\lambda| < \bar{\varepsilon}_0$, the function G has at most m zeros in V in x, multiplicity taken into account.

Then, we want to use the above conclusion to prove that the conclusion of the theorem is true for $l = k = m + 1$. For the purpose, let us suppose that $F \in C^{m+1}(I \times D)$ and $F(x, 0) = f(x)$ has $x_0 \in I$ as a zero of multiplicity $m + 1$. It implies that x_0 is a zero of the derivate $f'(x)$ with multiplicity m. Note that $F_x \in C^m$ with $f'(x) = F_x(x, 0)$. Hence, by the inductive assumption, the function $F_x(x, \lambda)$ has at most m zeros in x in a neighborhood U of x_0 for $|\lambda|$ sufficiently small, multiplicity taken into account. Then, by Lemma 2.3 for fixed small $|\lambda|$, $F(x, \lambda)$ has at most $m + 1$ zeros in $x \in U$, multiplicity taken into account. Therefore, the conclusion is true for $l = k = m + 1$. This ends the proof. □

Remark 2.1. Let

$$b_j(\lambda) = \frac{\partial^j F}{\partial x^j}(x_0, \lambda),\ 0 \leq j \leq l.$$

If

$$b_l(0) \neq 0,\ b_j(0) = 0, 0 \leq j \leq l - 1,$$

$$\text{rank}\frac{\partial(b_0, \cdots, b_{l-1})}{\partial(\lambda_1, \cdots, \lambda_n)}|_{\lambda=0} = l,$$

then for any given $\mu > 0$ there exists a λ with $|\lambda| < \mu$ such that F has l simple zeros in $x \in (0, \mu)$.

To state our second main result of this section, we consider a family of C^k functions $F(x, a, \lambda)$ defined for $(x, a, \lambda) \in I \times V \times D$, where as before $I \subset \mathbf{R}$ is an open interval and $D = \{\lambda \in \mathbf{R}^m | |\lambda| < \varepsilon\}$, and V is a compact set in $\mathbf{R}^n$. Also, denote $F(x, a, 0)$ by $f(x, a)$, that is, $f(x, a) = F(x, a, 0)$. Then, the second main result can be stated as follows.

Theorem 2.2. *Consider the C^k function $F(x, a, \lambda)$. If for each $a \in V$ the function $f(x, a)$ has at most l zeros in $x \in I$, multiplicity taken into account, where $1 \leq l \leq k$, then for any closed interval $I^* \subset I$ there exists an $\varepsilon^* > 0$ depending on I^* such that for all $a \in V$ and $|\lambda| < \varepsilon^*$ $F(x, a, \lambda)$ has at most l zeros in $x \in I^*$, multiplicity taken into account.*

Proof. We prove the conclusion by the method of contradiction. Let us suppose that the conclusion is not true. Then, there exists a closed interval $I^* \subset I$ such that for any $\widetilde{\varepsilon} > 0$ $F(x, a, \lambda)$ has $l+1$ zeros, multiplicity taken into account, in $x \in I^*$ for some (a, λ) with $a \in V$ and $|\lambda| < \widetilde{\varepsilon}$. Take $\widetilde{\varepsilon} = \frac{1}{n}$ with $n \geq 1$, and denote the corresponding (a, λ) by (a_n, λ_n) with $a_n \in V$ and $|\lambda_n| < \frac{1}{n}$. Then, $F(x, a_n, \lambda_n)$ has zeros $x_1^{(n)}, \cdots, x_{r_n}^{(n)} \in I^*$ with

$$\begin{gathered} x_1^{(n)} < \cdots < x_{r_n}^{(n)}, \quad 1 \leq r_n \leq l+1, \\ k_{1n} + \cdots + k_{r_n n} \geq l+1, \ k_{jn} \geq 1, \ j = 1, \cdots, r_n, \end{gathered} \tag{2.3}$$

where k_{jn} denotes the multiplicity of $x = x_j^{(n)}$.

We can suppose $r_n \to r$ as $n \to \infty$ with $1 \leq r \leq l+1$. Note that r_n and r are integers. Thus $r_n = r$ for large n. Since both V and I^* are compact, we can further suppose as $n \to \infty$

$$a_n \to \bar{a} \in V, \ x_j^{(n)} \to \bar{x}_j \in I^*, \ k_{jn} \to \bar{k}_j \geq 1, \ j = 1, \cdots, r.$$

By (2.3) it is easy to see

$$\bar{k}_1 + \cdots + \bar{k}_r \geq l+1. \tag{2.4}$$

Since k_{jn} and $\bar{k}_j$ are all integers, we have $k_{jn} = \bar{k}_j$ for large n. Thus, for $j = 1, \cdots, r$, we have

$$\frac{\partial^i F}{\partial x^i}(x_j^{(n)}, a_n, \lambda_n) = 0, \ i = 0, \cdots, \bar{k}_j - 1, \tag{2.5}$$

which give by letting $n \to \infty$ that

$$\frac{\partial^i F}{\partial x^i}(\bar{x}_j, \bar{a}, 0) = 0, \ i = 0, \cdots, \bar{k}_j - 1.$$

This means that each $\bar{x}_j \in I^*$ is a zero of the function $f(x, \bar{a})$ of multiplicity at least $\bar{k}_j$. If $\bar{x}_1, \cdots, \bar{x}_r$ are all different, then the total multiplicity of zeros of $f(x, \bar{a})$ in $x \in I^*$ is at least $\bar{k}_1 + \cdots + \bar{k}_r \geq l+1$ by (2.4). This contradicts to our assumption on f.

Next, we suppose that $\bar{x}_1, \cdots, \bar{x}_r$ are not different. In this case, we write the function F into the form

$$F(x, a, \lambda) = f(x, \bar{a}) + f_1(x, \mu) \equiv \bar{F}(x, \mu),$$

where $\mu = (a - \bar{a}, \lambda) \in \mathbf{R}^{n+m}$ and $f_1(x, 0) = 0$. Without loss of generality, we consider a simple case for the sake of convenience:

$$\bar{x}_1 = \bar{x}_2 < \cdots < \bar{x}_r. \tag{2.6}$$

The other cases can be discussed in the same way. Let $\mu_n = (a_n - \bar{a}, \lambda_n)$. Then, by (2.6) and (2.5), the function $\bar{F}(x, \mu_n)$ has $\bar{k}_1 + \bar{k}_2$ zeros near $\bar{x}_1$, multiplicity taken into account. Then, by Theorem 2.1 the function $f(x, \bar{a}) = \bar{F}(x, 0)$ has $\bar{x}_1$ as a zero of multiplicity at least $\bar{k}_1 + \bar{k}_2$. Hence, the total multiplicity of $f(x, \bar{a})$ is at least $(\bar{k}_1 + \bar{k}_2) + \cdots + \bar{k}_r \geq l+1$, a contradiction, too. Then, the proof is completed. □

Remark 2.2. Applying the implicit function theorem we see that if $f(x, a)$ has l simple zeros in $x \in I$ for some $a \in V$, then for all small $|\lambda|$ the function F has l simple zeros in $x \in I$.

Remark 2.3. One can easily see that if $F \in C^\infty(I \times D)$, then Theorems 2.1 and 2.2 hold for all $1 \le l < \infty$. In fact, in this case Theorem 2.1 has another proof: directly by using the well known Weierstrass-Malgrange preparation theorem, see Theorem 5.15 of the book [1]. However, our proof here is very elementary and easy to understand.

3. On the number of limit cycles

In this section, we use Theorems 2.1 and 2.2 to study the maximum number of limit cycles bifurcating from a multiple limit cycle, a weak focus or a period annulus.

First, consider a C^k planar system of the form

$$\dot{x} = f_0(x) + f_1(x, \mu), \ x \in \mathbf{R}^2, \tag{3.1}$$

where $\mu \in \mathbf{R}^n$ is a vector of parameters, $n \ge 1$, f_0 and f_1 are C^k functions with $k \ge 1$ and $f_1(x, 0) = 0$. Let $(3.1)|_{\mu=0}$ have a limit cycle L. Then, as in Chapter 1 of [5] one can define a C^k Poincare map of (3.1) near L, denoted by $P(a, \mu)$ such that for $|\mu|$ small (3.1) has a limit cycle near L if and only if $P(a, \mu) = a$ for $|a|$ small.

Set $d(a, \mu) = P(a, \mu) - a$, which is called a bifurcation function of (3.1) near L. Recall that for $1 \le l \le k$ the limit cycle L is called to have multiplicity l if $a = 0$ is a zero of $d(a, 0)$ of multiplicity l. Then, applying Theorem 2.1 to the function $d(a, \mu)$ we obtain the following bifurcation theorem immediately.

Theorem 3.1. *Consider the C^k system* (3.1). *If the limit cycle L of* $(3.1)|_{\mu=0}$ *has multiplicity l with $1 \le l \le k$, then there exist a neighborhood U of L and a constant $\varepsilon_0 > 0$ such that for all $|\mu| < \varepsilon_0$* (3.1) *has at most l limit cycles in U, multiplicity taken into account.*

Remark 3.1. The theorem above is a slight improvement of Theorem 1.3.2 in Han [5] on the maximum number of limit cycles. Also, if system (3.1) is of class C^∞, then the theorem is true for all $l \ge 1$.

Further, we study Hopf bifurcation for (3.1). Suppose $\mu = 0$ (3.1) has an elementary focus. Then, the focus keeps for $|\mu|$ small, denoted by O_μ. By removing O_μ to the origin and then normalizing the linear part, we can get the following C^k system

$$\dot{x} = \begin{pmatrix} \alpha(u) & -\beta(u) \\ \beta(u) & \alpha(u) \end{pmatrix} x + \begin{pmatrix} P(x, \mu) \\ Q(x, \mu) \end{pmatrix}, \ \beta(0) \neq 0, \tag{3.2}$$

where $P, Q \in C^k$ near the origin, and

$$P(x, \mu), Q(x, \mu) = o(|x|).$$

From [10], we know that (3.2) yields a C^k periodic equation below by letting $x = (r\cos\theta, r\sin\theta)$

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} = W(\theta, r, \mu). \tag{3.3}$$

Let $r(\theta, r_0, \mu)$ be the solution of (3.3) with the initial condition $r(0, r_0, \mu) = r_0$, which is C^r for $|r_0|$ small. Hence, (3.2) has a bifurcation function defined by

$$d(r_0, \mu) = r(2\pi, r_0, \mu) - r_0.$$

As we know, if

$$d(r_0, 0) = v_{2m+1} r_0^{2m+1} + o(r_0^{2m+1}),\ v_{2m+1} \neq 0$$

for some m satisfying $1 \leq 2m+1 \leq k$, then $(3.2)|_{\mu=0}$ has the origin as a focus of order m.

If $(3.2)|_{\mu=0}$ has a center at the origin, i.e., $d(r_0, 0) = 0$ for all $|r_0|$ small, then by Lemma 2.1 for $\mu \in \mathbf{R}$

$$d(r_0, \mu) = \mu d_1(r_0, \mu),\ d_1 \in C^{k-1}.$$

Hence, by Lemma 2.2 and Theorem 2.1, we obtain the following theorem.

Theorem 3.2. *Consider C^k system* (3.2). *If* $(3.2)|_{\mu=0}$ *has a weak focus of order m at the origin with $2m+1 \leq k$, then for all small $|\mu|$* (3.2) *has at most m limit cycles near the origin, multiplicity taken into account. If* $(3.2)|_{\mu=0}$ *has a center at the origin with $\mu \in \mathbf{R}$, and*

$$d_1(r_0, 0) = \bar{d}_m r_0^{2m+1} + o(r_0^{2m+1}),\ \bar{d}_m \neq 0$$

for some m with $1 \leq 2m+1 \leq k-1$, then for all small $|\mu|$ system (3.2) *has at most m limit cycles in a neighborhood of the origin, multiplicity taken into account.*

The above theorem is an improvement of Theorem 4 obtained in [10]. Also, it is true for all $m \geq 0$ if system (3.2) is of class C^∞.

Next, we consider a planar near-Hamiltonian system of the form

$$\begin{aligned} \dot{x} &= H_y(x, y) + \varepsilon f(x, y, \varepsilon, \delta), \\ \dot{y} &= -H_x(x, y) + \varepsilon g(x, y, \varepsilon, \delta), \end{aligned} \tag{3.4}$$

where $(x, y) \in \mathbf{R}^2, H \in C^{k+1}, f, g \in C^k$ with $k \geq 1$, ε is a small parameter, and $\delta \in V \subset \mathbf{R}^n$ with V a compact set. For (3.4), we have a fundamental assumption on the unperturbed system $(3.4)|_{\varepsilon=0}$ that it has a family of periodic orbits L_h defined by $H(x, y) = h, h \in J$ with J an open interval. The set $\Omega = \bigcup_{h \in J} L_h$ is called a period annulus. From [10], (3.4) has a bifurcation function of the form

$$F(h, \varepsilon, \delta) = M(h, \delta) + O(\varepsilon)$$

where

$$M(h, \delta) = \oint_{L_h} g\mathrm{d}x - f\mathrm{d}y|_{\varepsilon=0},\ h \in J$$

which is called the first order Melnikov function.

By [10], we have that $M \in C^k$ in $h \in J$ for $\delta \in V$ and that for any closed interval $I \subset J$, $F \in C^k$ in $h \in I$ for ε small and $\delta \in V$. Thus, by Lemma 2.1 we have the following expansion in ε

$$F(h, \varepsilon, \delta) = \sum_{j=1}^{k} M_j(h, \delta)\varepsilon^{j-1} + o(\varepsilon^{k-1}) \tag{3.5}$$

for $h \in J$ and ε small, where

$$M_j(h, \delta) = \frac{1}{(j-1)!} \frac{\partial^{j-1} F}{\partial \varepsilon^{j-1}}(h, 0, \delta) \in C^{k-j+1}.$$

By applying Theorem 2.2 to (3.5), we have the following theorem immediately.

Theorem 3.3. *Consider C^k system* (3.4). *Suppose that there exist integers $r \geq 1$, $l \geq 1$ with $r + l \leq k + 1$ such that $M_1 = \cdots = M_{r-1} = 0, M_r \neq 0$. If M_r has at most l zeros in $h \in J$ for all $\delta \in V$, multiplicity taken into account, then for any compact set $D \subset \Omega(= \bigcup_{h \in J} L_h)$ there is $\varepsilon_0 = \varepsilon_0(D) > 0$ such that* (3.4) *has at most l limit cycles in D for $\delta \in V$ and $|\varepsilon| < \varepsilon_0$, multiplicity taken into account. In this case, we say that the period annulus Ω generates at most l limit cycles.*

In fact, by Lemma 2.1 under the condition of the above theorem, we have

$$F(h, \varepsilon, \delta) = \varepsilon^{r-1} F_1(h, \varepsilon, \delta),$$

where $F_1 \in C^{k-r+1}$ satisfying $F_1(h, 0, \delta) = M_r(h, \delta)$. Then, the conclusion is direct from Theorem 2.2.

We can introduce another small parameter in (3.4). More precisely, consider the following C^∞ system

$$\begin{aligned} \dot{x} &= H_y(x, y, \lambda) + \varepsilon f(x, y, \lambda, \delta), \\ \dot{y} &= -H_x(x, y, \lambda) + \varepsilon g(x, y, \lambda, \delta), \end{aligned} \tag{3.6}$$

where $0 < |\varepsilon| \ll |\lambda| \ll 1$, $\delta \in V \subset \mathbf{R}^n$. In this case, (3.6) has the following first order Melnikov function

$$M(h, \lambda, \delta) = \oint_{H(x,y,\lambda)=h} g\mathrm{d}x - f\mathrm{d}y, \;\; h \in J.$$

For $|\lambda|$ small, we have for any $k \geq 1$

$$M(h, \lambda, \delta) = \sum_{j=0}^{k} \bar{M}_j(h, \delta)\lambda^j + O(\lambda^{k+1}). \tag{3.7}$$

Han and Xiong [11] gave formulas of the functions $\bar{M}_1$ and $\bar{M}_2$.

In the same way by Theorem 2.2, we have the following theorem.

Theorem 3.4. *Let* (3.7) *satisfy the following*

$$\bar{M}_j = 0, \;\; j = 0, \cdots, k-1, \quad \bar{M}_k \neq 0$$

for some $k \geq 1$. If $\bar{M}_k$ has at most l zeros in $h \in J$ for all $\delta \in V$, multiplicity taken into account, then for any compact set $D \subset \Omega(= \bigcup_{h \in J} L_h)$ there is $\varepsilon_0 = \varepsilon_0(D, V) > 0$ such that (3.6) *has at most l limit cycles in D as $\delta \in V, 0 < |\varepsilon| \ll |\lambda| < \varepsilon_0$, multiplicity taken into account.*

We have studied the number of limit cycles for C^k or C^∞ systems on the plane. Next, we consider piecewise smooth systems on the plane.

4. Piecewise smooth systems

In this section, we study the problem of the number of limit cycles for piecewise smooth systems on the plane. For simplicity, we suppose the plane $\mathbf{R}^2$ is divided into two parts Ω^+ and Ω^- by a C^k smooth curve $\Sigma: x = \varphi(y), y \in \mathbf{R}$, where $k \geq 1$. Then,

$$\mathbf{R}^2 = \Omega^+ \cup \Omega^- \cup \Sigma,$$

where $\Omega^+ = \{(x, y)|x > \varphi(y), y \in \mathbf{R}\}$ and $\Omega^- = \{(x, y)|x < \varphi(y), y \in \mathbf{R}\}$.

Let $f^\pm(x, y)$ and $g^\pm(x, y)$ be four C^k functions defined on $\Omega^\pm \cup \Sigma$ respectively. Then, we have a piecewise C^k smooth system of the form

$$\dot{x} = f(x, y),\ \dot{y} = g(x, y) \tag{4.1}$$

where

$$f(x, y) = \begin{cases} f^+(x, y),\ (x, y) \in \Omega^+, \\ f^-(x, y),\ (x, y) \in \Omega^-, \end{cases} \quad g(x, y) = \begin{cases} g^+(x, y),\ (x, y) \in \Omega^+, \\ g^-(x, y),\ (x, y) \in \Omega^-. \end{cases}$$

Let (4.1) have a clockwise oriented closed orbit $L = L^+ \cup L^-$, which crosses the curve Σ twice. More precisely, the intersection $L \cap \Sigma$ is a set of two different points: $L \cap \Sigma = \{A, B\}$ with $L^+ = \widehat{AB} \subset (\Omega^+ \cup \Sigma)$ and $L^- = \widehat{BA} \subset (\Omega^- \cup \Sigma)$, and

$$\begin{vmatrix} \varphi' & 1 \\ f^\pm & g^\pm \end{vmatrix}_{A,B} \neq 0. \tag{4.2}$$

See Figure 1.

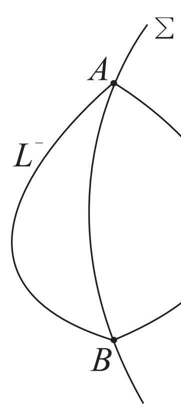

Figure 1. The crossing closed orbit L

Obviously, L^+ and L^- are the orbits of the right subsystem

$$\dot{x} = f^+(x, y),\ \dot{y} = g^+(x, y) \tag{4.3}$$

and the left subsystem

$$\dot{x} = f^-(x, y),\ \dot{y} = g^-(x, y) \tag{4.4}$$

respectively.

Let $a_0, b_0 \in \mathbf{R}$ be such that

$$A = (\varphi(a_0), a_0),\ B = (\varphi(b_0), b_0).$$

Then, for a near a_0 (4.3) has an orbit $L_1^+(a)$ starting from $A_1(\varphi(a), a)$ and arriving at $B_1(\varphi(b), b)$. Similarly, (4.4) has an orbit $L_1^-(b)$ starting from B_1 and arriving at $A_2(\varphi(c), c)$. Note that both $L_1^+ = \widehat{A_1B_1}$ and $L_1^- = \widehat{B_1A_2}$ are C^k smooth and that φ is a C^k smooth function. By (4.2) and the implicit function theorem, we can see that $b = P_1(a) \in C^k$ and $c = P_2(b) \in C^k$. Then, we define the Poincaré map of (4.1) near L as

$$c = P_2(P_1(a)) \equiv P_L(a) \tag{4.5}$$

for a near a_0, where $P_L = P_2 \circ P_1 \in C^k$. Since L is closed, we have

$$P_1(a_0) = b_0,\ P_2(b_0) = a_0,\ P_L(a_0) = a_0.$$

As in the smooth case, we can use P_L to define the stability and the multiplicity of L. For example, for the multiplicity of L we give the following definition.

Definition 4.1. Let $d(a) = P_L(a) - a$. If there exists an integer l with $1 \le l \le k$ such that

$$d^{(l)}(a_0) \ne 0,\ d^{(j)}(a_0) = 0 \ \text{ for } \ 0 \le j \le l-1,$$

then we say that L is a crossing limit cycle of (4.1) with multiplicity l or a limit cycle of multiplicity l for short. When $l = 1$, we say that L is simple or hyperbolic.

The following lemma is fundamental for system (4.1).

Lemma 4.1. *For the Poincaré map P_L of (4.1) near L, we have*

$$P_L'(a_0) = \frac{K_1}{K_2}\exp(I(L)), \tag{4.6}$$

where

$$\begin{aligned}
K_1 &= (f^+(A) - \varphi'(a_0)g^+(A))(f^-(B) - \varphi'(b_0)g^-(B)),\\
K_2 &= (f^+(B) - \varphi'(b_0)g^+(B))(f^-(A) - \varphi'(a_0)g^-(A)),\\
I(L) &= \int_{\widehat{AB}} \operatorname{tr}\frac{\partial(f^+, g^+)}{\partial(x,y)}\mathrm{d}t + \int_{\widehat{BA}} \operatorname{tr}\frac{\partial(f^-, g^-)}{\partial(x,y)}\mathrm{d}t.
\end{aligned}$$

Thus, L is a limit cycle of multiplicity 1 if and only if $\frac{K_1}{K_2}\exp(I(L)) \ne 1$, and it is stable (resp., unstable) if $\frac{K_1}{K_2}\exp(I(L)) < 1$ (resp., $\frac{K_1}{K_2}\exp(I(L)) > 1$).

Proof. Let

$$u = x - \varphi(y)$$

so that system (4.1) becomes

$$\begin{aligned}
\dot{u} &= f(u + \varphi(y), y) - \varphi'(y)g(u + \varphi(y), y) \equiv \hat{f}(u, y),\\
\dot{y} &= g(u + \varphi(y), y) \equiv \hat{g}(u, y).
\end{aligned}$$

We need only to find the derivative of the Poincaré map at $a = a_0$ for the new system. For simplicity, we still use P_L to denote the Poincaré map, and use P_1, P_2 to denote the Poincaré map of the right subsystem

$$\dot{u} = \hat{f}^+(u, y), \quad \dot{y} = \hat{g}^+(u, y)$$

for $(u, y) \in \hat{\Omega}^+$, and the left subsystem

$$\dot{u} = \hat{f}^-(u, y), \quad \dot{y} = \hat{g}^-(u, y)$$

for $(u, y) \in \hat{\Omega}^-$, respectively, where

$$\hat{\Omega}^+ = \{(u, y) | u > 0, y \in \mathbf{R}\}, \ \hat{\Omega}^- = \{(u, y) | u < 0, y \in \mathbf{R}\}.$$

By a well known result (see [18] or Lemma 2.3 in [14]), we have

$$P_1'(a_0) = \frac{\Delta_1}{\Delta_2} \exp\left(\int_{\widehat{AB}} \operatorname{tr} \frac{\partial(\hat{f}^+, \hat{g}^+)}{\partial(u, y)} \mathrm{d}t \right), \tag{4.7}$$

where

$$\Delta_1 = \begin{vmatrix} \hat{f}^+(\hat{A}) & 0 \\ \hat{g}^+(\hat{A}) & 1 \end{vmatrix}, \quad \Delta_2 = \begin{vmatrix} \hat{f}^+(\hat{B}) & 0 \\ \hat{g}^+(\hat{B}) & -1 \end{vmatrix}, \quad \hat{A} = (0, a_0), \ \hat{B} = (0, b_0).$$

It is obvious that

$$\hat{f}^+(\hat{A}) = f^+(A) - \varphi'(a_0) g^+(A), \ \hat{f}^+(\hat{B}) = f^+(B) - \varphi'(b_0) g^+(B),$$

and

$$\hat{f}_u^+ + \hat{g}_y^+ = (f_x^+ - \varphi'(y) g_x^+) + (g_x^+ \varphi'(y) + g_y^+) = f_x^+ + g_y^+.$$

Thus, it follows from (4.7)

$$P_1'(a_0) = -\frac{f^+(A) - \varphi'(a_0) g^+(A)}{f^+(B) - \varphi'(b_0) g^+(B)} \exp\left(\int_{\widehat{AB}} \operatorname{tr} \frac{\partial(f^+, g^+)}{\partial(x, y)} \mathrm{d}t \right). \tag{4.8}$$

Similarly, we have

$$P_2'(b_0) = -\frac{f^-(B) - \varphi'(b_0) g^-(B)}{f^-(A) - \varphi'(a_0) g^-(A)} \exp\left(\int_{\widehat{BA}} \operatorname{tr} \frac{\partial(f^-, g^-)}{\partial(x, y)} \mathrm{d}t \right). \tag{4.9}$$

Note that by (4.5)

$$P_L'(a_0) = P_2'(b_0) P_1'(a_0).$$

Then, (4.6) follows from (4.8) and (4.9). This finishes the proof. □

Now, we consider a perturbed system of the form

$$\dot{x} = F(x, y, \mu), \quad \dot{y} = G(x, y, \mu), \tag{4.10}$$

where

$$(F(x, y, \mu), G(x, y, \mu)) = \begin{cases} (F^+(x, y, \mu), G^+(x, y, \mu)), \ x > \varphi(y), \\ (F^-(x, y, \mu), G^-(x, y, \mu)), \ x < \varphi(y), \end{cases}$$

with $\mu \in \mathbf{R}^n, n \geq 1$, $F^\pm, G^\pm \in C^k$ in (x, y, μ), satisfying

$$F^\pm(x, y, 0) = f^\pm(x, y), \quad G^\pm(x, y, 0) = g^\pm(x, y).$$

As before, we can define the Poincaré map $\tilde{P}(a,\mu)$ of the above system to be the composition of $P_1(a,\mu)$ and $P_2(a,\mu)$ of the right subsystem

$$\dot{x} = F^+(x,y,\mu),\ \dot{y} = G^+(x,y,\mu)$$

and the left subsystem

$$\dot{x} = F^-(x,y,\mu),\ \dot{y} = G^-(x,y,\mu),$$

respectively. That is, $\tilde{P}(a,\mu) = P_2(P_1(a,\mu),\mu)$. Obviously, $\tilde{P}(a,0) = P_L(a)$. Then, using Theorem 2.1 to the function $\tilde{P}(a,\mu) - a$, we obtain the following theorem immediately.

Theorem 4.1. *Suppose that the unperturbed system* (4.1) *has a crossing limit cycle* L *of multiplicity* l *with* $1 \le l \le k$. *Then, there exist a neighborhood* U *of* L *and a constant* $\varepsilon_0 > 0$ *such that for all* $|\mu| < \varepsilon_0$ (4.10) *has at most* l *limit cycles in* U, *multiplicity taken into account.*

Now, we study Hopf bifurcation for (4.10). A basic assumption is that the unperturbed system (4.1) of (4.10) has a "singular point" like a focus. For convenience we suppose that it is at the origin with $\varphi(0) = 0$. More precisely, we give the following definition.

Definition 4.2. Consider the piecewise C^k smooth system (4.1) with $\varphi(0) = 0$. Suppose that there exist constants $\varepsilon_2 > \varepsilon_1 > 0$ such that

(i) for any $a \in (0,\varepsilon_1)$, the positive orbit of (4.3) starting from $(\varphi(a),a)$ intersects the curve Σ at a point $(\varphi(b),b)$ with $b = P_1(a) \in (-\varepsilon_2,0)$ and $P_1(0+) = 0$;

(ii) for any $b \in (-\varepsilon_1,0)$, the positive orbit of (4.4) starting from $(\varphi(b),b)$ intersects the curve Σ at a point $(\varphi(c),c)$ with $c = P_2(b) \in (0,\varepsilon_2)$ and $P_2(0-) = 0$.

Let $P_0(a) = P_2(P_1(a))$. If $P_0(a) \ne a$ for all $a \in (0,\varepsilon_1)$, we call the origin as a focus of (4.1). If $P_0(a) = a$ for all $a \in (0,\varepsilon_1)$, we call the origin as a center of (4.1).

Further, we say the focus at the origin to be stable (resp., unstable) if $P_0(a) < a$ (resp., $P_0(a) > a$) for all $a \in (0,\varepsilon_1)$.

For the sake of convenience, we suppose $\varphi = 0$ and $F^\pm, G^\pm \in C^\infty$ in (x,y,μ). In this case, for (4.1) Liu and Han [15] gave a definition of the succession function or the displacement function $d(a)$ for $0 < |a| \ll 1$ as follows:

$$d(a) = \begin{cases} P_2(P_1(a)) - a, & 0 < a \ll 1, \\ 0, & a = 0, \\ P_1(P_2(a)) - a, & 0 < -a \ll 1. \end{cases}$$

and proved that the function $d(a)$ has a positive zero near $a = 0$ if and only if it has a negative zero near $a = 0$.

As we know, there are four types of foci: PP, FF, PF and FP, where "P" means parabolic and "F" means "focus" (See [4], [3]). Then, Han and Zhang [12] gave a definition for elementary focus as follows.

Definition 4.3. Suppose that the origin is a focus of system (4.1) with the orbits near the origin being oriented clockwise.

(i) The origin is called an elementary focus of PP type of system (4.1), if

$$H_P^\pm:\ f^\pm(0,0) = 0,\ f_y^\pm(0,0) > 0,\ \pm g^\pm(0,0) < 0. \tag{4.11}$$

(ii) The origin is called an elementary focus of FF type of system (4.1), if

$$H_F^{\pm}:\ \begin{cases} f^{\pm}(0,0)=0,\ g^{\pm}(0,0)=0,\ f_y^{\pm}(0,0)>0, \\ (f_x^{\pm}(0,0)-g_y^{\pm}(0,0))^2+4f_y^{\pm}(0,0)g_x^{\pm}(0,0)<0. \end{cases} \tag{4.12}$$

(iii) The origin is called an elementary focus of FP or PF type of system (4.1), if H_P^- in (4.11) and H_F^+ in (4.12) hold or H_P^+ in (4.11) and H_F^- in (4.12) hold.

Han and Zhang [12] proved that the function d is C^∞ for $0 \le a \ll 1$, if the origin is an elementary focus. Hence, in this case we can write

$$d(a)=\sum_{i\ge 1} V_i a^i,\quad 0\le a\ll 1. \tag{4.13}$$

Similarly, if the origin is an elementary focus of (4.10) for all small $|\mu|$, then we can define a displacement function $\tilde{d}(a,\mu)=O(a)$ for $0<|a|\ll 1$ for the system which is C^∞ for $0\le a\ll 1$. Hence, we can write $\tilde{d}(a,\mu)=ad_1(a,\mu)$ with $d_1(a,\mu)\in C^\infty$ for $0\le |a|\ll 1$. Note that PF type can be changed to FP type by exchanging x and y. We need only to consider the focus of FF, PP and FP types.

First, for the focus of FF type, Liu and Han [15] proved that if (i) $d(a)=V_k a^k+O(a^{k+1})$ with $V_k\ne 0$ and $k\ge 2$ and (ii) the origin is an elementary focus of (4.10) for all small $|\mu|$, then there are at most $k-1$ limit cycles of (4.10) bifurcated from the origin for all small $|\mu|$. Then, Liang and Han [13] obtained a very similar result for the case of FP type.

In fact, by using Theorem 2.1 to the function $d_1(a,\mu)$, we can obtain immediately the following theorem which improve slightly the related results obtained in Liu and Han [15] and Liang and Han [13].

Theorem 4.2. *Let the following two conditions be satisfied: (i) $d(a)=V_ka^k+O(a^{k+1})$ with $V_k\ne 0$ and $k\ge 2$; (ii) the origin is an elementary focus of FF or FP type of* (4.10) *for all small $|\mu|$, then there are at most $k-1$ limit cycles of* (4.10) *bifurcated from the origin for all small $|\mu|$, multiplicity taken into account.*

Suppose that the origin is a focus of PP type of system (4.1). Liang and Han [13] introduced the following function

$$F_0(a)=\begin{cases} P_1(a)-P_2^{-1}(a),\ a>0, \\ 0, \qquad\qquad\qquad a=0, \\ P_1^{-1}(a)-P_2(a),\ a<0, \end{cases}$$

and proved that $F_0(a)=O(a^2)\in C^\infty$ for all $|a|$ small and that the coefficients in the expansion

$$F_0(a)=\sum_{i\ge 2} V_i^* a^i,\ |a|\ll 1 \tag{4.14}$$

satisfy

$$V_{2k+1}^*=O(|V_2^*,V_4^*,\cdots,V_{2k}^*|),\ k\ge 1.$$

Further, the coefficients V_i in (4.13) and those in (4.14) have the relationship

$$V_i=(-1)^{i+1}V_i^*+O(|V_2^*,V_3^*,\cdots,V_{i-1}^*|),\ i\ge 3,$$

which give

$$V_{2k+1} = O(|V_2, V_4, \cdots, V_{2k}|), \ k \geq 1.$$

If the origin is an elementary focus of PP type of (4.10) for all small $|\mu|$, then we can define a C^∞ function $\tilde{F}(a,\mu)$ with $\tilde{F}(a,0) = F_0(a)$ and

$$\tilde{F}(a,\mu) = \sum_{i\geq 2} \tilde{V}_i(\mu)a^i, \ |a| \ll 1,$$

where

$$\tilde{V}_{2j+1}(\mu) = O(|\tilde{V}_2(\mu), \tilde{V}_4(\mu), \cdots, \tilde{V}_{2j}(\mu)|), \ j \geq 1.$$

Then, note that the function $\tilde{F}(a,\mu)$ has a positive (or negative) zero near $a = 0$ if and only if system (4.10) has a limit cycle near the origin. By using Theorem 2.1 to the function $\tilde{F}$, we have the following theorem.

Theorem 4.3. *Suppose that the origin is an elementary focus of PP type of* (4.10) *for $|\mu|$ small. If for some $k \geq 1$*

$$V_2 = V_4 = \cdots = V_{2k} = 0, \ V_{2k+2} \neq 0,$$

then for small $|\mu|$ system (4.10) *has at most k limit cycles near the origin, multiplicity taken into account.*

The above theorem is an improvement to the first part of Theorem 2.2 given in [13].

Next, we consider a piecewise smooth near-Hamiltonian system of the following form:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{cases} \begin{pmatrix} H_y^+(x,y) + \varepsilon p^+(x,y,\varepsilon,\delta) \\ -H_x^+(x,y) + \varepsilon q^+(x,y,\varepsilon,\delta) \end{pmatrix}, & x > 0, \\ \begin{pmatrix} H_y^-(x,y) + \varepsilon p^-(x,y,\varepsilon,\delta) \\ -H_x^-(x,y) + \varepsilon q^-(x,y,\varepsilon,\delta) \end{pmatrix}, & x \leq 0, \end{cases} \tag{4.15}$$

where $H^\pm$, $p^\pm$ and $q^\pm$ are C^∞ functions, ε is a sufficiently small real parameter, and δ is a vector of bounded parameters. Suppose system (4.15) satisfies the following assumptions:

(I) There exist an interval $J = (\alpha, \beta)$ and two points $A(h) = (0, a(h))$ and $B(h) = (0, b(h))$ such that for $h \in J$,

$$\begin{aligned} &H^+(A(h)) = H^+(B(h)) = h, \\ &H^-(A(h)) = H^-(B(h)), \quad a(h) > b(h). \end{aligned}$$

(II) The equation $H^+(x,y) = h$, $x \geq 0$, defines an orbital arc L_h^+ starting from $A(h)$ and ending at $B(h)$; the equation $H^-(x,y) = H^-(A(h))$, $x \leq 0$, defines an orbital arc L_h^- starting from $B(h)$ and ending at $A(h)$, such that system $(4.15)|_{\varepsilon=0}$ has a family of clockwise oriented periodic orbits $L_h = L_h^+ \cup L_h^-$.

(III) The curves $L_h^\pm, h \in J$ are not tangent to the switch plane $x = 0$ at points $A(h)$ and $B(h)$. In other words, $H_y^\pm(A) \neq 0$ and $H_y^\pm(B) \neq 0$ for each $h \in J$.

Under the conditions (I)–(III), the closed orbits $\{L_h\}$ yield a crossing period annulus. The author [15] established a bifurcation function $F(h,\varepsilon)$ defined by

$$H^+(B_\varepsilon) - H^+(A) = \varepsilon F(h,\varepsilon,\delta),$$

where B_ε denotes the second intersection point of the positive orbit of (4.15) starting from A with $x = 0$ satisfying $\lim_{\varepsilon\to 0} B_\varepsilon = A$. Let $M(h,\delta) = F(h,0,\delta)$, which is called the first order Melnikov function of system (4.15). Then, from [9,15]

$$M(h,\delta) = \int_{\widehat{AB}} q^+ \mathrm{d}x - p^+ \mathrm{d}y|_{\varepsilon=0} + \frac{H_y^+(A)}{H_y^-(A)} \int_{\widehat{BA}} q^- \mathrm{d}x - p^- \mathrm{d}y|_{\varepsilon=0},\ h \in J. \quad (4.16)$$

The authors [9,15] gave some results on the number of limit cycles by using the zeros of the first order Melnikov function $M(h,\delta)$ bifurcating from the period annulus or a center of FF type. From (4.16), one can see that $M \in C^\infty(J)$. In general, we have for $h \in J$ and $|\varepsilon|$ sufficiently small

$$F(h,\varepsilon,\delta) = \sum_{j\geq 0} M_j(h,\delta)\varepsilon^j,$$

where $M_j \in C^\infty(J)$. Then, applying Theorem 2.2 to the function F, we obtain the following bifurcation theorem.

Theorem 4.4. *Consider the near-Hamiltonian system* (4.15) *with the conditions (I)–(III). Suppose that* $M_k \neq 0$ *for some* $k \geq 0$ *such that*

$$F(h,\varepsilon,\delta) = \sum_{j\geq k} M_j(h,\delta)\varepsilon^j.$$

If M_k has at most l zeros in $h \in J$ for all bounded δ, multiplicity taken into account, then for small $|\varepsilon| > 0$ and bounded δ system (4.15) *has at most l limit cycles bifurcating from the period annulus defined by L_h, multiplicity taken into account.*

If the system we consider has the following form

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{cases} \begin{pmatrix} f_1^+(x,y) + \varepsilon f_2^+(x,y,\varepsilon) \\ g_1^+(x,y) + \varepsilon g_2^+(x,y,\varepsilon) \end{pmatrix}, & x > 0, \\ \begin{pmatrix} f_1^-(x,y) + \varepsilon f_2^-(x,y,\varepsilon) \\ g_1^-(x,y,) + \varepsilon g_2^-(x,y,\varepsilon) \end{pmatrix}, & x \leq 0, \end{cases}$$

where the functions $f_1^\pm, f_2^\pm, g_1^\pm$ and $g_2^\pm$ are C^∞ functions such that the above unperturbed system has integrating factors μ_1 and μ_2 and first integrals H^+ and H^- respectively for $x > 0$ and $x \leq 0$, satisfying

$$\mu_1 f_1^+ = H_y^+,\quad \mu_1 g_1^+ = -H_x^+,$$
$$\mu_2 f_1^- = H_y^-,\quad \mu_2 g_1^- = -H_x^-,$$

then it is equivalent to a near-Hamiltonian system of the form (4.15), and the corresponding first order Melnikov function has the form

$$M(h) = \int_{\widehat{AB}} \mu_1(g_2^+ \mathrm{d}x - f_2^+ \mathrm{d}y)|_{\varepsilon=0} + \frac{H_y^+(A)}{H_y^-(A)} \int_{\widehat{BA}} \mu_2(g_2^- \mathrm{d}x - f_2^- \mathrm{d}y)|_{\varepsilon=0}.$$

Finally, in this section we consider a piecewise smooth near-Hamiltonian system with multiple parameters of the form:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{cases} \begin{pmatrix} H_y^+(x,y,\lambda) + \varepsilon p^+(x,y,\delta,\lambda) \\ -H_x^+(x,y,\lambda) + \varepsilon q^+(x,y,\delta,\lambda) \end{pmatrix}, & x > 0, \\ \begin{pmatrix} H_y^-(x,y,\lambda) + \varepsilon p^-(x,y,\delta,\lambda) \\ -H_x^-(x,y,\lambda) + \varepsilon q^-(x,y,\delta,\lambda) \end{pmatrix}, & x \leq 0, \end{cases} \quad (4.17)$$

where $H^\pm$, $p^\pm$ and $q^\pm$ are C^∞ functions, λ and ε are both sufficiently small real parameters with $0<\varepsilon\ll\lambda\ll 1$, and δ is a vector of bounded parameters. Suppose system (4.17) satisfies the following assumptions like (4.15):

(I^*) There exist an interval $J=(\alpha,\beta)$ and two points $A_\lambda(h)=(0,a(h,\lambda))$ and $B_\lambda(h)=(0,b(h,\lambda))$ such that for $h\in J$,

$$\begin{aligned}&H^+(A_\lambda(h),\lambda)=H^+(B_\lambda(h),\lambda)=h,\\&H^-(A_\lambda(h),\lambda)=H^-(B_\lambda(h),\lambda),\quad a(h,\lambda)>b(h,\lambda).\end{aligned}$$

(II^*) The equation $H^+(x,y,\lambda)=h$, $x\geq 0$, defines an orbital arc L_h^+ starting from $A_\lambda(h)$ and ending at $B_\lambda(h)$; the equation $H^-(x,y,\lambda)=H^-(A_\lambda(h),\lambda)$, $x\leq 0$, defines an orbital arc L_h^- starting from $B_\lambda(h)$ and ending at $A_\lambda(h)$, such that system $(4.17)|_{\varepsilon=0}$ has a family of clockwise oriented periodic orbits $L_h=L_h^+\cup L_h^-$.

(III^*) The curves $L_h^\pm, h\in J$ are not tangent to the switch plane $x=0$ at points $A_\lambda(h)$ and $B_\lambda(h)$. In other words, $H_y^\pm(A_\lambda,\lambda)\neq 0$ and $H_y^\pm(B_\lambda,\lambda)\neq 0$ for each $h\in J$.

Under the conditions (I^*)–(III^*), we have the first order Melnikov function of system (4.17)

$$M(h,\delta,\lambda)=\int_{\widehat{A_\lambda B_\lambda}}q^+\mathrm{d}x-p^+\mathrm{d}y+\frac{H_y^+(A_\lambda,\lambda)}{H_y^-(A_\lambda,\lambda)}\int_{\widehat{B_\lambda A_\lambda}}q^-\mathrm{d}x-p^-\mathrm{d}y. \tag{4.18}$$

Since λ is small, we have the expansion below

$$M(h,\delta,\lambda)=\sum_{j\geq 0}\widetilde{M}_j(h,\delta)\lambda^j.$$

The function $M(h,\delta,\lambda)$ can be used to study the number of limit cycles bifurcating from a period annulus, a center or a homoclinic and heteroclinic loop. Recently, Han and Liu [8] obtained formulas of $M_1(h,\delta)$ and $M_2(h,\delta)$ by using (4.18).

Similar to Theorem 3.4, we have

Theorem 4.5. *Let*

$$\widetilde{M}_j=0,\ j=0,\cdots,k-1,\quad \widetilde{M}_k\neq 0$$

for some $k\geq 1$. *If* $\widetilde{M}_k$ *has at most* l *zeros in* $h\in J$ *for all* δ *in a bounded set* V, *multiplicity taken into account, then for any compact set* $D\subset\Omega(=\bigcup_{h\in J}L_h)$ *there is* $\varepsilon_0=\varepsilon_0(D,V)>0$ *such that* (4.17) *has at most* l *limit cycles in* D *as* $\delta\in V$, $0<|\varepsilon|\ll|\lambda|<\varepsilon_0$, *multiplicity taken into account.*

5. One dimensional periodic differential equations

In this section, we consider the problem of finding the number of periodic solutions for one dimensional periodic differential equations.

Let J be an open interval of $\mathbf{R}$ and f be a C^k function on $\mathbf{R}\times J$ satisfying

$$f(t+T,x)=f(t,x),\ T>0,\ (t,x)\in\mathbf{R}\times J.$$

Then, the function f defines a periodic differential equation as follows

$$\dot{x}=f(t,x). \tag{5.1}$$

Suppose that (5.1) has a T periodic solution $x=\varphi(t), t\in\mathbf{R}$. Then, for all x_0 near $\varphi(0)$ (5.1) has a unique solution $x(t,x_0)$ satisfying $x(0,x_0)=x_0$ and defined for $t\in[0,T]$. Define $P(x_0)=x(T,x_0)$ which is called the Poincaré map of (5.1) near the solution $\varphi(t)$. Let $d(x_0)=P(x_0)-x_0$, which is called the displacement function of (5.1) near $\varphi(t)$. Obviously, we have $P,d\in C^k$ and

$$P(\bar{x}_0)=\bar{x}_0,\ d(\bar{x}_0)=0,$$

where $\bar{x}_0=\varphi(0)$.

Definition 5.1. If $d^{(l)}(\bar{x}_0)\neq 0, d^{(j)}(\bar{x}_0)=0$ for $j=0,\cdots,l-1$ for some $1\leq l\leq k$, then we say that the periodic solution φ has multiplicity l.

Now, consider a perturbed system of the form

$$\dot{x}=f(t,x)+F(t,x,\mu) \tag{5.2}$$

where $\mu\in\mathbf{R}^m$, F is C^k in $(t,x,\mu)\in\mathbf{R}\times J\times\mathbf{R}^m$, $F(t,x,0)=0$ and F is $T-$periodic in t. For $|\mu|$ small (5.1) has the Poincaré map $P(x_0,\mu)$ and the displacement function $d(x_0,\mu)=P(x_0,\mu)-x_0$ near φ. If the periodic solution φ has multiplicity l, then

$$\frac{\partial^l d}{\partial x_0^l}(\bar{x}_0,0)\neq 0,\ \frac{\partial^j d}{\partial x_0^j}(\bar{x}_0,0)=0,\ j=0,\cdots,l-1.$$

Thus, by Theorem 2.1 we have the following theorem immediately.

Theorem 5.1. *Consider C^k periodic equation* (5.2)*. If for $\mu=0$ it has a periodic solution $x=\varphi(t)$ of multiplicity l with $1\leq l\leq k$, then for all $|\mu|$ small* (5.2) *has at most l periodic solutions near φ, multiplicity taken into account.*

Next, consider a periodic differential equation of the form

$$\dot{x}=F_0(t,x)+\varepsilon F(t,x,\varepsilon), \tag{5.3}$$

where $x\in\mathbf{R}$, $F_0\in C^{k+1}$, $F\in C^k$, F_0 and F are $T-$periodic in t, $k\geq 1$. Suppose that there exists an open interval $J\subset\mathbf{R}$ such that for $\varepsilon=0$ (5.3) has a family of $T-$periodic solutions $x=p(t,x_0)$ satisfying $p(0,x_0)=x_0$ with $x_0\in J$. Then, p is a C^k function in $(t,x_0)\in\mathbf{R}\times J$. Let $x(t,x_0,\varepsilon)$ denote the solution of (5.3) with $x(0,x_0,\varepsilon)=x_0\in J$. We can write

$$x(t,x_0,\varepsilon)=p(t,x_0)+\varepsilon\bar{x}(t,x_0,\varepsilon),$$

where $\bar{x}(0,x_0,\varepsilon)=0$. One can see easily that the function $\bar{x}$ satisfies a periodic differential equation of the form

$$\dot{\bar{x}}=\bar{F}(t,\bar{x},\varepsilon),\ \bar{F}\in C^k$$

with

$$\bar{F}(t,\bar{x},0)=\frac{\partial F_0}{\partial x}(t,p)x_1+F(t,p,0),$$

where $x_1=\bar{x}(t,x_0,0)$. This shows that $\bar{x}$ is C^k for $(t,x_0)\in\mathbf{R}\times J$ and $|\varepsilon|$ small. From the equation $\dot{x}_1=\bar{F}(t,x_1,0)$, we can solve that

$$x_1(t,x_0)=\mathrm{e}^{K(t,x_0)}\int_0^t \mathrm{e}^{-K(s,x_0)}F(s,p(s,x_0),0)\mathrm{d}s,$$

where

$$K(t, x_0) = \int_0^t \frac{\partial F_0}{\partial x}(s, p(s, x_0), 0)\mathrm{d}s.$$

Note that $p(t, x_0)$ satisfies

$$\dot{p} = F_0(t, p), \quad p(0, x_0) = x_0.$$

It follows that

$$\frac{\partial p}{\partial x_0}(t, x_0) = \mathrm{e}^{K(t,x_0)}, \ \frac{\partial p}{\partial x_0}(0, x_0) = 1.$$

Hence,

$$\mathrm{e}^{K(T,x_0)} = \frac{\partial p}{\partial x_0}(T, x_0) = \frac{\partial p}{\partial x_0}(0, x_0) = 1.$$

Therefore,

$$\bar{x}(T, x_0, 0) = x_1(T, x_0) = \int_0^T \left[\frac{\partial p}{\partial x_0}(t, x_0)\right]^{-1} F(t, p(t, x_0), 0)\mathrm{d}t.$$

Then, we have

$$d(x_0, \varepsilon) = P(x_0, \varepsilon) - x_0 = \varepsilon \bar{d}(x_0, \varepsilon), \tag{5.4}$$

where P and d denote the Poincaré map and displacement function of (5.3) respectively with $\bar{d} \in C^k$ and

$$\bar{d}(x_0, 0) = \int_0^T [\frac{\partial p}{\partial x_0}(t, x_0)]^{-1} F(t, p(t, x_0), 0)\mathrm{d}t \equiv d_0(x_0). \tag{5.5}$$

In general, we have for $0 \le l \le k$

$$\bar{d}(x_0, \varepsilon) = \sum_{j=0}^{l} d_j(x_0)\varepsilon^j + o(\varepsilon^l), \ x_0 \in J,$$

with

$$d_j(x_0) = \frac{1}{j!}\frac{\partial^j \bar{d}}{\partial x_0^j}(x_0, 0) \in C^{k-j}.$$

Now, applying Theorem 2.2 to the function d we have the following theorem.

Theorem 5.2. *Consider* (5.3)*. Suppose that there exists* $0 \le l \le k$ *such that*

$$d_l(x_0) \not\equiv 0, \ d_j(x_0) \equiv 0, \ j = 0, \cdots, l-1.$$

If d_l has at most n zeros in $x_0 \in J$ for some integer n with $0 \le n \le k-l$, multiplicity taken into account, then for any closed interval $I \subset J$ there exists a constant $\varepsilon_0 > 0$ such that for all $0 < |\varepsilon| < \varepsilon_0$ Equation (5.3) *has at most n periodic solutions whose ranges are in the set* $\bigcup\limits_{t\in \mathbf{R}, x_0 \in I} p(t, x_0)$*, multiplicity taken into account.*

Now, we consider a periodic differential equation with double small parameters of the form

$$\dot{x} = F_0(t, x, \lambda) + \varepsilon F(t, x, \lambda), \tag{5.6}$$

where F_0 and F are C^∞ functions, $T-$periodic in t and $0 < \varepsilon \ll \lambda \ll 1$ with $F_0(t, x, 0) = 0$. Suppose $\varepsilon = 0$ (5.6) has a family of periodic solutions $x = p(t, x_0, \lambda)$,

$x_0 \in J_\lambda$ with $p(0, x_0, \lambda) = x_0$. Then, by (5.4) and (5.5), Equation (5.6) has a displacement function of the form

$$d(x_0, \varepsilon, \lambda) = \varepsilon \bar{d}(x_0, \varepsilon, \lambda),$$
$$\bar{d}(x_0, 0, \lambda) = \int_0^T [\frac{\partial p}{\partial x_0}(t, x_0, \lambda)]^{-1} F(t, p(t, x_0, \lambda), \lambda) \mathrm{d}t.$$

Noting $\bar{d} \in C^\infty$, we can write

$$\bar{d}(x_0, 0, \lambda) = \sum_{j \geq 0} \widetilde{d}_j(x_0) \lambda^j.$$

Applying Theorem 2.2 to the function $\bar{d}$, we have the following theorem.

Theorem 5.3. *Consider* (5.6)*. If there exist integers $k \geq 0$, $l \geq 1$ such that*

$$\widetilde{d}_k(x_0) \not\equiv 0, \ \widetilde{d}_j(x_0) \equiv 0, \ 0 \leq j \leq k-1$$

and that $\widetilde{d}_k$ has at most l zeros in $x_0 \in \mathbf{R}$, multiplicity taken into account, then for any compact interval $I \subset \mathbf{R}$ there exists a constant $\delta > 0$ such that for $0 < \varepsilon \ll \lambda < \delta$ (5.6) *has at most l periodic solutions whose range is in I, multiplicity taken into account.*

The formulas of $\widetilde{d}_j(x_0)$ have been given in [17].

In the following, we consider a T-periodic differential equation of the form

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \varepsilon F(t, x, \varepsilon, \delta), \tag{5.7}$$

with $T > 0$ constant, F being given for $0 \leq t \leq T$ by

$$F(t, x, \varepsilon, \delta) = \begin{cases} F_1(t, x, \varepsilon, \delta), & (t, x) \in D_1, \\ F_2(t, x, \varepsilon, \delta), & (t, x) \in D_2, \\ \cdots & \\ F_k(t, x, \varepsilon, \delta), & (t, x) \in D_k, \end{cases}$$

where $x \in J \subset R$ with J an open interval, $|\varepsilon| < \varepsilon_0$, $\delta \in V \subset R^n$ with V a compact set, and k C^r functions $F_j(t, x, \varepsilon, \delta)$ are defined for all $(t, x) \in U(\bar{D}_j)$ with $U(\bar{D}_j)$ being an open set containing $\bar{D}_j$, $\bar{D}_j$ denoting the closure of the set D_j which has the following form

$$D_j = \{(t, x) | h_{j-1}(x) \leq t < h_j(x), x \in J\}, \quad j = 1, \cdots, k,$$

where $h_j(x)$ are C^r functions defined on J satisfying

$$h_0(x) = 0 < h_1(x) < \cdots < h_{k-1}(x) < T = h_k(x), \quad x \in J, \ k \geq 2, \ r \geq 1.$$

Define

$$l_j = \{(t, x) | t = h_j(x), x \in J\}, \quad j = 0, \cdots, k.$$

Then, equation (5.7) has the switch lines $l_1, \cdots, l_{k-1}$. We call it a k-piecewise C^r smooth periodic equation, as called in Han [6].

Let

$$f(x,\delta)=\int_0^T F(t,x,0,\delta)\mathrm{d}t=\sum_{j=1}^{k}\int_{h_{j-1}(x)}^{h_j(x)} F(t,x,0,\delta)\mathrm{d}t.$$

For $x_0\in J$, denoted by $x(t,x_0,\varepsilon,\delta)$ the solution of equation (5.7) satisfying $x(0)=x_0$ for $t\in[0,T]$. As we did for smooth equations, the author [6] defined the Poincaré map of (5.7) as

$$P(x_0,\varepsilon,\delta)=x(T,x_0,\varepsilon,\delta)=x_0+\varepsilon\bar{g}_k(x_0,\varepsilon,\delta), \tag{5.8}$$

and obtained that $\bar{g}_k(x,0,\delta)=f(x,\delta)$. Also, the author [6] proved that for any given closed interval $I\subset J$, there exists $\varepsilon^*>0$ such that the function $\bar{g}_k(x_0,\varepsilon,\delta)$ is well defined and of C^r in (x_0,ε,δ) for all $x_0\in I$, $|\varepsilon|<\varepsilon^*$ and $\delta\in V$. Thus, by Lemma 2.1, we have the following expansion in ε

$$\bar{g}_k(x,\varepsilon,\delta)=\sum_{j=1}^{r} f_j(x,\delta)\varepsilon^{j-1}+o(\varepsilon^{r-1}) \tag{5.9}$$

for $x\in I$, $|\varepsilon|<\varepsilon^*$ and $\delta\in V$, where

$$f_j(x,\delta)=\frac{1}{(j-1)!}\frac{\partial^{j-1}\bar{g}_k}{\partial\varepsilon^{j-1}}(x,0,\delta)\in C^{r-j+1}.$$

By applying Theorem 2.2 to (5.9), we have the following theorem immediately.

Theorem 5.4. *Consider the periodic equation* (5.7). *If there exist integers* $l\geq 1, m\geq 1$ *satisfying* $l+m\leq r+1$ *such that*

$$f_l(x,\delta)\not\equiv 0,\ \ f_j(x,\delta)\equiv 0,\ \ 1\leq j\leq l-1,$$

and that the function f_l *has at most* m *zeros in* $x\in J$ *for all* $\delta\in V$, *multiplicity taken into account, then for any closed interval* $I\subset J$, *there exists* $\varepsilon_1=\varepsilon_1(I)>0$, *such that for* $0<|\varepsilon|<\varepsilon_1,\delta\in V$ *the periodic equation* (5.7) *has at most* m T-*periodic solutions, multiplicity taken into account with the property that the range of each of them is a subset of* I.

Remark 5.1. If $F(t,0,\varepsilon,\delta)=0, J=(0,+\infty)$, and the first non-zero function f_l has at most m zeros in $x\in J$ for all $\delta\in V$, multiplicity taken into account, then for any $N>0$, there exists $\varepsilon_1=\varepsilon_1(N)>0$ such that for $0<|\varepsilon|<\varepsilon_1$, $\delta\in V$, (5.7) has at most m positive periodic solutions, multiplicity taken into account, whose ranges are subsets of $(0,N]$.

Clearly, $f_1(x,\delta)=f(x,\delta)$. By Lemma 9 (the fundamental lemma) of [16], we have

$$f_2(x,\delta)=\int_0^T\left(D_x\tilde{F}_1(t,x,\delta)\int_0^t\tilde{F}_1(s,x,\delta)ds+\tilde{F}_2(t,x,\delta)\right)\mathrm{d}t,$$

where

$$\widetilde{F_1}(t,x,\delta)=F(t,x,0,\delta),$$

$$\widetilde{F}_2(t,x,\delta)=\begin{cases}\frac{\partial F_1(t,x,\varepsilon,\delta)}{\partial\varepsilon}|_{\varepsilon=0}, & (t,x)\in D_1,\\ \frac{\partial F_2(t,x,\varepsilon,\delta)}{\partial\varepsilon}|_{\varepsilon=0}, & (t,x)\in D_2,\\ \cdots & \\ \frac{\partial F_k(t,x,\varepsilon,\delta)}{\partial\varepsilon}|_{\varepsilon=0}, & (t,x)\in D_k,\end{cases} \equiv \frac{\partial F(t,x,0,\delta)}{\partial\varepsilon},$$

$$D_x\widetilde{F}_1(t,x,\delta)=\sum_{j=1}^{k}\chi_{D_j}D_xF_j(t,x,0,\delta),$$

with

$$\chi_{D_j}(t,x)=\begin{cases}1, & (t,x)\in D_j,\\ 0, & (t,x)\notin D_j.\end{cases}$$

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