Journal of Nonlinear Modeling and Analysis http://jnma.ca; http://jnma.ijournal.cn
Volume 2, Number 3, September 2020, 447–465 DOI:10.12150/jnma.2020.447

Analysis of a Kind of Quitting Smoking Model with Beddington-DeAngelis Function*

Zhimin Li[1], Tailei Zhang[1,†], Jianzhong Gao[2] and Shu Fang[1]

Abstract In this paper, we discuss the dynamics of quitting smoking with Beddington-Deangelis function. Firstly, the basic reproduction number of the model is obtained by establishing the basic reproduction matrix. Then, by using the Routh-Hurwitz criterion and Lyapunov functionals and LaSalle's Invariant Principle and the second additive compound matrix, local and global dynamics of the model are analyzed. Based on the partial rank correlation coefficients (PRCCs), we discuss some biological implications and focus on the impact of some key model parameters. Finally, the numerical simulations show the theoretical analysis more intuitively, and we give some strategies to control the spread of smokers.

Keywords Beddington-DeAngelis function, Quitting smoking model, Local stability, Global stability.

MSC(2010) 34A30.

1. Introduction

Smoking is called "the fifth threat" by the World Health Organization (WHO) (The others are war, famine, plague and pollution). Among smoking-related deaths, chronic lung diseases accounted for 45%, lung cancer 15%, esophageal cancer, gastric cancer, liver cancer, stroke, heart disease and tuberculosis accounted for 40%. If the epidemic trend of smoking patterns is uncontrolled, it is expected that 3 million Chinese will die from tobacco-related diseases by 2050 [1, 2]. On October 27, 2017, World Health Organization International Agency for Research on Cancer lists smoking as a list of primary carcinogens. Epidemiological investigation showed that smoking is one of the important pathogenic factors of lung cancer. Smokers are 13 times more likely to develop lung cancer than non-smokers. About 85% of lung cancer deaths are caused by smoking. Smokers who are exposed to chemical carcinogens such as asbestos, nickel, uranium and arsenic at the same time are at higher risk of lung cancer. Smoking can reduce the activity of natural killer cells,

†the corresponding author.
Email address: t.l.zhang@126.com(T. Zhang), lizhimin_chd@126.com(Z. Li)
1School of Science, Chang'an University, Xi'an, Shaanxi 710064, China
2School of Ecological Environment, Northwestern Polytechnical University, Xi'an, Shaanxi 710072, China
*The authors were supported by Natural Science Basic Research Plan of Shaanxi Province (Grant No. 2018JM1011), the National Natural Science Foundation of China (Grant No. 11701041), the Fundamental Research Funds for the Central Universities, CHD (Grant No. 300102129202), and the Scientific Innovation Practice Project of Postgraduates of Chang'an University (Grant No. 300103002110).

thereby impairing the body's surveillance, killing and scavenging functions of tumor cell growth, which further explains that smoking is a high risk factor for multiple cancers. In China, the smoking rate of minors is on the rise, the age at which minors start to smoke is declining, and about 80,000 young people become long-term smokers every day. This situation not only affects the healthy growth of children, but also seriously affects the improvement of the physical fitness of our country as a whole. Therefore, the problem of smoking among minors has attracted more and more attention of the society [3].

Many scholars have done a lot of works on quitting smoking models [4–10]. In [9], Erturk et al. studied the dynamics of a quitting smoking model containing fractional derivatives. The unique positive solution for the fractional order model is presented. In [10], Zeb et al. studied a new quitting smoking model with square root of potential and occasional smokers of model. The local and global stability of the model and its general solution are discussed. Both non-negativity and conservative law for differential equations system are discussed.

Scholars have studied the predator-prey model in depth. Based on the predator-prey model, A.J.Lotka and V.Volterra proposed the famous Lotka-Volterra model [11] $x' = a_1x - b_1xy, y' = -a_2y + b_2xy$. This model is reasonable to some extent, but it ignores the factors of digestive saturation. Holling [12] proposed three functional response functions for different biological types in 1965:

(1) For filter predators, Holling I type functional response function with saturation is given.

$$\Phi(x) = \begin{cases} cx, & x \leqslant x_0, \\ cx_0, & x < x_0. \end{cases}$$

(2) For invertebrates, the Holling II type functional response function $\Phi(x) = \frac{\alpha x}{1+\omega x}$ is given. This functional response function reflects that when the amount of prey increases, the predator's prey will also increase until the number of predators reaches a saturated level.

(3) For vertebrates with complex behavior, the Holling III type functional response function $\Phi(x) = \frac{\alpha x^2}{1+\beta x^2}$ is given. When the number of prey is small, the predator learns to catch. When the number of prey increases, the predation rate increases accordingly. When the food is very full, the degree of hunger decreases and negative acceleration occurs to reach saturation. If the prey has defensive strategies, the predation behavior also belongs to this kind of functional response. The more general form of Holling III type functional response function is $\Phi(x) = \frac{\alpha x^2}{1+\omega x+\beta x^2}$.

In the study of biodynamics, when the amount of food increases to a certain extent, the population growth will be inhibited. In order to describe this inhibiting phenomenon, Andrews gave Holling IV type functional function: $\Phi(x) = \frac{\alpha x}{a+\omega x+\beta x^2}$. Later, Howell simplified it to $\Phi(x) = \frac{\alpha x}{a+x^2}$. It is unexpectedly found that the simplified function is more in line with the experimental data, and it also reduced the difficulty of research. Based on these practical factors and experimental results, some biologists and scholars realized that the predator's role should be added to the functional response function, so they established a new kind of functional response function, namely predator-dependent functional response. In 1975, the functional response function proposed by Beddington and DeAngelis is called Beddington-DeAngelis [13] functional response function, which is more in line with the actual situation. At present, the Beddington-DeAngelis function is used to study the dynamics model of infectious diseases [14, 15]. Based on population dynamics, this paper establishes a practical smoking cessation dynamics model. In the existing literature on smoking cessation models, the infection rate considered is relatively single. In this paper, smokers are divided into occasional smokers L and regular smokers S in detail during the modeling process, considering that both smokers have infectious effects on potential smokers P. Specifically, on the basis of common bilinear infection rates βPL and βPS, this paper adds a denominator $1 + a_iL + b_iS$ greater than 1, i.e. the inhibition of infection, which is only related to the number of smokers (including occasional smokers L and smokers S). This can be understood as non-smokers encounter people who smoke will produce rejection and avoidance behavior. In addition, the infection rates of regular smokers S and occasional smokers L were $\frac{\beta_1 PL}{1+a_1L+b_1S}$ and $\frac{\beta_2 PS}{1+a_2L+b_2S}$, respectively.

This paper is organized as follows: In Section 2, we will establish a quitting smoking model. In Section 3, the local stability and global asymptotic stability of the equilibria are proved. In Section 4, we will do numerical simulations to visually show the theoretical analysis in Section 3. Finally, we summarize the whole paper.

2. Model description

In order to describe the model, $P(t), L(t), S(t)$ and $Q(t)$ represent the number of potential smokers, occasional smokers, smokers and quit smokers at time t, respectively. The total population number at time t is expressed by $N(t) = P(t) + L(t) + S(t) + Q(t)$.

Main assumptions are as follows:

(i) λ is constant birth rate for the potential smokers. $\beta_i (i = 1, 2)$ is the contact rate.

(ii) μ is the natural death rate and d represents death rate for potential smokers, occasional smokers and quit smokers related to smoking disease. $d + \alpha$ represents death rate for smokers (high mortality due to heavy smoking). γ is the conversion rate from occasional smokers to smokers. δ is the rate of quitting smoking.

According to the above assumptions (i-ii), we can formulate a $PLSQ$ quitting smoking model as follows:

$$\begin{cases} \dfrac{\mathrm{d}P}{\mathrm{d}t} = \lambda - \dfrac{\beta_1 PL}{1 + a_1L + b_1S} - \dfrac{\beta_2 PS}{1 + a_2L + b_2S} - (\mu + d)P, \\ \dfrac{\mathrm{d}L}{\mathrm{d}t} = \dfrac{\beta_1 PL}{1 + a_1L + b_1S} + \dfrac{\beta_2 PS}{1 + a_2L + b_2S} - (\mu + d + \gamma)L, \\ \dfrac{\mathrm{d}S}{\mathrm{d}t} = \gamma L - (\mu + d + \alpha + \delta)S, \\ \dfrac{\mathrm{d}Q}{\mathrm{d}t} = \delta S - (\mu + d)Q. \end{cases} \tag{2.1}$$

The meaningful domain of the model (2.1) is

$$\Omega = \left\{ (P, L, S, Q) \in \mathbb{R}^4_+ : P + L + S + Q \leqslant \frac{\lambda}{\mu + d} \right\},$$

and it is easy to show that Ω is a positively invariant set.

One can easily verify that the non-smoker equilibrium is given by $E_0\left(\frac{\lambda}{\mu+d}, 0, 0, 0\right)$. According to the notation in [16], the Jacobian matrices F (of new infection terms) and V (of remaining transition terms) are given, respectively [17]. We have

$$F = \begin{bmatrix} \frac{\beta_1\lambda}{\mu+d} & \frac{\beta_2\lambda}{\mu+d} \\ 0 & 0 \end{bmatrix}, \quad V = \begin{bmatrix} \mu+d+\gamma & 0 \\ -\gamma & \mu+d+\alpha+\delta \end{bmatrix},$$

hence, the basic reproduction number R_0 of model (2.1) is given by

$$\mathcal{R}_0 = \rho(\mathrm{FV}^{-1}) = \frac{\beta_1\lambda}{(\mu+d)(\mu+d+\gamma)} + \frac{\beta_2\lambda\gamma}{(\mu+d)(\mu+d+\gamma)(\mu+d+\alpha+\delta)},$$

here ρ denote the spectral radius.

The transmission of smoking behavior is similar to the disease, so the T (type reproduction number [18]) can be used to describe the dynamic behavior of smokers. The type reproduction number defines the expected number of secondary infective cases of a particular population type caused by a typical primary case in completely potential smokers [16–18]. It is an extension of the basic reproduction number $\mathcal{R}_0$. Particularly, the type reproduction number T_1 for control of infection among humans is defined in the references [18–20] as

$$T_1 = e_1^T M(I - (I - P_1)M)^{-1} e_1,$$

provided the spectral radius of matrix $(I-P_1)M$ is less than one, i.e., $\rho((I-P_1)M) < 1$. Here I is the 2×2 identity matrix, vectors $e_1 = (1,\ 0)^T$, M is the next generation matrix, and P_1 is the 2×2 projection matrix with all zero entries expect that the $(1,1)$ entry is 1. Write $M = (m_{ij})$. The type reproduction number T_1 can be easily defined in terms of the elements m_{ij}:

$$T_1 = m_{11} + \frac{m_{12}m_{21}}{1 - m_{22}}, \tag{2.2}$$

T_1 exists provided $m_{22} < 1$. In view of $m_{21} = m_{22} = 0$, by (2.2), the type reproduction number associated with the infectious humans is given by

$$T_1 = \frac{\beta_1\lambda}{(\mu+d)(\mu+d+\gamma)} + \frac{\beta_2\lambda\gamma}{(\mu+d)(\mu+d+\gamma)(\mu+d+\alpha+\delta)} = \mathcal{R}_0.$$

It has been shown in [19] that $\mathcal{R}_0 < 1\ (=1, > 1) \Leftrightarrow T_1 < 1\ (=1, > 1)$.

3. Equilibria analysis

Through the simple calculation of model (2.1), we find that model (2.1) always has a non-smoker equilibrium E_0 and its smoker equilibrium E^* satisfies the following equation

$$\begin{cases} \lambda - \dfrac{\beta_1 PL}{1 + a_1L + b_1S} - \dfrac{\beta_2 PS}{1 + a_2L + b_2S} - (\mu+d)P = 0, \\ \dfrac{\beta_1 PL}{1 + a_1L + b_1S} + \dfrac{\beta_2 PS}{1 + a_2L + b_2S} - (\mu+d+\gamma)L = 0, \\ \gamma L - (\mu+d+\alpha+\delta)S = 0, \\ \delta S - (\mu+d)Q = 0. \end{cases} \tag{3.1}$$

After computing, we have

$$\begin{cases} P = \dfrac{\lambda}{\frac{\beta_1(\mu+d+\alpha+\delta)S}{\gamma\left(1+\left(\frac{a_1(\mu+d+\alpha+\delta)}{\gamma}+b_1\right)S\right)} + \frac{\beta_2 S}{1+\left(\frac{a_2(\mu+d+\alpha+\delta)}{\gamma}+b_2\right)S} + (\mu+d)}, \\ L = \dfrac{(\mu+d+\alpha+\delta)S}{\gamma}, \\ Q = \dfrac{\delta S}{\mu+d}, \\ P = \dfrac{(\mu+d+\gamma)(\mu+d+\alpha+\delta)}{\frac{\beta_1(\mu+d+\alpha+\delta)}{1+\left(\frac{a_1(\mu+d+\alpha+\delta)}{\gamma}+b_1\right)S} + \frac{\beta_2\gamma}{1+\left(\frac{a_2(\mu+d+\alpha+\delta)}{\gamma}+b_2\right)S}}. \end{cases} \tag{3.2}$$

Noting (3.2), the first and fourth equations are

$$\begin{cases} P = \Phi(S) = \dfrac{\lambda}{\frac{\beta_1(\mu+d+\alpha+\delta)S}{\gamma\left(1+\left(\frac{a_1(\mu+d+\alpha+\delta)}{\gamma}+b_1\right)S\right)} + \frac{\beta_2 S}{1+\left(\frac{a_2(\mu+d+\alpha+\delta)}{\gamma}+b_2\right)S} + (\mu+d)}, \\ P = \Psi(S) = \dfrac{(\mu+d+\gamma)(\mu+d+\alpha+\delta)}{\frac{\beta_1(\mu+d+\alpha+\delta)}{1+\left(\frac{a_1(\mu+d+\alpha+\delta)}{\gamma}+b_1\right)S} + \frac{\beta_2\gamma}{1+\left(\frac{a_2(\mu+d+\alpha+\delta)}{\gamma}+b_2\right)S}}. \end{cases} \tag{3.3}$$

Thus, the intersections of the curves $P = \Phi(S)$ and $P = \Psi(S)$ in $\left[0, \frac{\lambda}{\mu+d}\right]^2$ determine the nontrivial equilibria. The $P = \Psi(S)$ is a strictly increasing function. Meanwhile, it is clear that $\Phi(S)$ is a strictly decreasing function. From (3.3), we also have

$$\Phi(0) = \frac{\lambda}{\mu+d}, \ \Psi(0) = \frac{\mu+d+\gamma}{\beta_1 + \frac{\beta_2\gamma}{\mu+d+\alpha+\delta}}, \ \Phi(\frac{\lambda}{\mu+d}) < \Psi(\frac{\lambda}{\mu+d}). \tag{3.4}$$

From (3.1) and (3.4), we see that if $\mathcal{R}_0 > 1$, then $\Phi(0) > \Psi(0)$, which implies that there is a unique intersection in $\mathbb{R}^2_+$ between $\Phi(S)$ and $\Psi(S)$; if $\mathcal{R}_0 \leqslant 1$, then $\Phi(0) \leqslant \Psi(0)$, which indicates that there is no intersection between these curves in the interior of $\mathbb{R}^2_+$. Therefore we have the following existence, uniqueness and local stability theorem on E_0 and E^* of model (2.1).

Theorem 3.1. *If $\mathcal{R}_0 \leqslant 1$, then system (2.1) has a unique equilibrium E_0. E_0 is locally asymptotically stable when $\mathcal{R}_0 < 1$ and Lyapunov stable when $\mathcal{R}_0 = 1$.*

Proof. We consider an equivalent system of model (2.1)

$$\begin{cases} \dfrac{\mathrm{d}P}{\mathrm{d}t} = \lambda - \dfrac{\beta_1 PL}{1 + a_1 L + b_1 S} - \dfrac{\beta_2 PS}{1 + a_2 L + b_2 S} - (\mu+d)P, \\ \dfrac{\mathrm{d}L}{\mathrm{d}t} = \dfrac{\beta_1 PL}{1 + a_1 L + b_1 S} + \dfrac{\beta_2 PS}{1 + a_2 L + b_2 S} - (\mu+d+\gamma)L, \\ \dfrac{\mathrm{d}S}{\mathrm{d}t} = \gamma L - (\mu+d+\alpha+\delta)S. \end{cases} \tag{3.5}$$

The Jacobian matrix of the vector field described by system (3.5) is

$$J=[J_{ij}]=\begin{bmatrix} -\frac{\beta_1 L}{1+a_1L+b_1S}-\frac{\beta_2 S}{1+a_2L+b_2S}-(\mu+d) & J_{12} & J_{13} \\ \frac{\beta_1 L}{1+a_1L+b_1S}+\frac{\beta_2 S}{1+a_2L+b_2S} & J_{22} & J_{23} \\ 0 & \gamma & -(\mu+d+\alpha+\delta) \end{bmatrix}, \tag{3.6}$$

where

$$\begin{aligned} J_{12} &= \frac{a_1\beta_1PL-\beta_2P(1+a_1L+b_1S)}{(1+a_1L+b_1S)^2}+\frac{a_2\beta_2PS}{(1+a_2L+b_2S)^2},\\ J_{13} &= \frac{b_2\beta_2PS-\beta_2P(1+a_2L+b_2S)}{(1+a_2L+b_2S)^2}+\frac{b_1\beta_1PL}{(1+a_1L+b_1S)^2},\\ J_{22} &= -J_{12}-(\mu+d+\gamma),\\ J_{23} &= -J_{13}. \end{aligned}$$

Computing the Jacobian matrix (3.6) at E_0, gives

$$J|_{E_0}=\begin{bmatrix} -(\mu+d) & -\frac{\beta_1\lambda}{\mu+d} & -\frac{\beta_2\lambda}{\mu+d} \\ 0 & \frac{\beta_1\lambda}{\mu+d}-(\mu+d+\gamma) & \frac{\beta_2\lambda}{\mu+d} \\ 0 & \gamma & -(\mu+d+\alpha+\delta) \end{bmatrix}.$$

Let λ_1, λ_2 and λ_3 denote the eigenvalues of $J|_{E_0}$, where $\lambda_1=-(\mu+d)<0$. It is easy to verify that λ_2 and λ_3 satisfying equation

$$\lambda^2-a\lambda+b=0,$$

where

$$\begin{aligned} a &= \frac{\beta_1\lambda}{\mu+d}-(\mu+d+\gamma)-(\mu+d+\alpha+\delta),\\ b &= \left((\mu+d+\gamma)-\frac{\beta_1\lambda}{\mu+d}\right)(\mu+d+\alpha+\delta)-\frac{\beta_2\lambda\gamma}{\mu+d}. \end{aligned}$$

According to the relation between root and coefficient, we get

$$\begin{aligned} \lambda_2+\lambda_3 &= \frac{\beta_1\lambda}{\mu+d}-(\mu+d+\gamma)-(\mu+d+\alpha+\delta),\\ \lambda_2\lambda_3 &= \left((\mu+d+\gamma)-\frac{\beta_1\lambda}{\mu+d}\right)(\mu+d+\alpha+\delta)-\frac{\beta_2\lambda\gamma}{\mu+d}. \end{aligned} \tag{3.7}$$

Since $\mathcal{R}_0\leqslant 1$, we can obtain $\lambda_2+\lambda_3<0$, $\lambda_2\lambda_3\geqslant 0$, that is, λ_2 and λ_3 have both negative real part. So, all of the eigenvalues of the characteristic equation are negative real part. Hence, the equilibrium E_0 is locally asymptotically stable in the interior of Ω. This completes the proof of Theorem 3.1. □

Theorem 3.2. *If $\mathcal{R}_0>1$, then system (2.1) has two equilibria E_0 and E^*. E_0 is unstable and E^* is locally asymptotically stable as $\mathcal{R}_0>1$.*

Proof. We know that E_0 is unstable when $\mathcal{R}_0>1$ from (3.7). Now, we will prove that the E^* is locally asymptotically stable. Evaluating the Jacobian matrix (3.6) at the E^*, we find the characteristic equation of $J|_{E^*}$ is given by

$$\lambda^3+c_1\lambda^2+c_2\lambda+c_3=0,$$

where

$$
\begin{aligned}
&c_1 = -(J_{11} + J_{22} + J_{33})|_{E^*},\\
&c_2 = (-J_{12}J_{21} + J_{11}J_{22} + J_{22}J_{33} + J_{11}J_{33} - J_{13}J_{31} - J_{23}J_{32})|_{E^*},\\
&c_3 = (-J_{22}J_{11}J_{33} + J_{22}J_{13}J_{31} + J_{12}J_{21}J_{33} + J_{11}J_{23}J_{32} - J_{12}J_{23}J_{31} - J_{13}J_{21}J_{32})|_{E^*}.
\end{aligned}
\tag{3.8}
$$

If $c_1 > 0, c_2 > 0, c_3 > 0$ and $c_1c_2 - c_3 > 0$ holds, according to the Routh-Hurwitz criterion, we know that E^* is locally asymptotically stable. In fact,

$$
\begin{aligned}
&J_{11} = -\frac{\beta_1 L}{1 + a_1L + b_1S} - \frac{\beta_2 S}{1 + a_2L + b_2S} - (\mu + d) < 0,\\
&J_{33} = -(\mu + d + \alpha + \delta) < 0,\\
&J_{22} = \frac{\beta_1 P(1 + a_1L + b_1S) - a_1\beta_1 PL}{(1 + a_1L + b_1S)^2} - \frac{a_2\beta_2 PS}{(1 + a_2L + b_2S)^2} - (\mu + d + \gamma),
\end{aligned}
\tag{3.9}
$$

From (2.1), we have

$$
\frac{\beta_1 PL}{1 + a_1L + b_1S} + \frac{\beta_2 PS}{1 + a_2L + b_2S} = (\mu + d + \gamma)L,
$$

so we obtain that

$$
\frac{\beta_1 PL}{1 + a_1L + b_1S} - (\mu + d + \gamma)L < 0,
$$

we can get that $J_{22} < 0$. Thus, $c_1 > 0$. From (3.6), we know $J_{21} > 0, J_{32} = \gamma > 0$, which implies

$$
\begin{aligned}
c_2 &= -J_{12}J_{21} + J_{11}J_{22} + J_{22}J_{33} + J_{11}J_{33} - J_{13}J_{31} - J_{23}J_{32}\\
&= -J_{12}J_{21} + J_{13}J_{32} + J_{11}J_{22} + J_{22}J_{33} + J_{11}J_{33}.
\end{aligned}
\tag{3.10}
$$

Since $-J_{12} = J_{22} + (\mu + d + \gamma)$we can obtain that

$$
\begin{aligned}
-J_{12}J_{21} + J_{11}J_{22} &= (J_{22} + (\mu + d + \gamma))\, J_{21} + J_{11}J_{22}\\
&= J_{22}J_{21} + J_{11}J_{22} + J_{21}(\mu + d + \gamma)\\
&= J_{22}(J_{21} + J_{11}) + J_{21}(\mu + d + \gamma)\\
&= -J_{22}(\mu + d) + J_{21}(\mu + d + \gamma) > 0.
\end{aligned}
\tag{3.11}
$$

We analyze the expression of c_2 from (3.10), we have

$$
\begin{aligned}
&J_{22}J_{33} + J_{11}J_{33} + J_{13}J_{32}\\
&= \gamma\left(\frac{b_2\beta_2 PS - \beta_2 P(1 + a_2L + b_2S)}{(1 + a_2L + b_2S)^2} + \frac{b_1\beta_1 PL}{(1 + a_1L + b_1S)^2}\right)\\
&\quad +(\mu + d + \alpha + \delta)\left((\mu + d + \gamma) + \frac{a_1\beta_1 PL - \beta_1 P(1 + a_1L + b_1S)}{(1 + a_1L + b_1S)^2}\right)\\
&\quad +(\mu + d + \alpha + \delta)\frac{a_2\beta_2 PS}{(1 + a_2L + b_2S)^2}\\
&\quad +(\mu + d + \alpha + \delta)\left(\frac{\beta_1 L}{1 + a_1L + b_1S} + \frac{\beta_2 S}{1 + a_2L + b_2S} + (\mu + d)\right),
\end{aligned}
$$

in that $\gamma L=(\mu+d+\alpha+\delta)S, (\mu+d+\gamma)L=\frac{\beta_1 PL}{1+a_1L+b_1S}+\frac{\beta_2 PS}{1+a_2L+b_2S}$, thus we have

$$(\mu+d+\alpha+\delta)=\frac{\gamma L}{S}, (\mu+d+\gamma)=\frac{\beta_1 P}{1+a_1L+b_1S}+\frac{\beta_2 PS}{(1+a_2L+b_2S)L},$$
$$\frac{\beta_2 PS}{(1+a_2L+b_2S)L}(\mu+d+\alpha+\delta)=\frac{\beta_2 PS}{(1+a_2L+b_2S)L}\frac{\gamma L}{S}=\frac{\gamma\beta_2 P}{1+a_2L+b_2S}.$$

Therefore,

$$\begin{aligned}
&J_{22}J_{33}+J_{11}J_{33}+J_{13}J_{32}\\
&=\gamma\left(\frac{b_2\beta_2 PS-\beta_2P(1+a_2L+b_2S)}{(1+a_2L+b_2S)^2}+\frac{b_1\beta_1PL}{(1+a_1L+b_1S)^2}\right)\\
&\quad+(\mu+d+\alpha+\delta)\left((\mu+d+\gamma)+\frac{a_1\beta_1PL-\beta_1P(1+a_1L+b_1S)}{(1+a_1L+b_1S)^2}\right)\\
&\quad+(\mu+d+\alpha+\delta)\frac{a_2\beta_2PS}{(1+a_2L+b_2S)^2}\\
&\quad+(\mu+d+\alpha+\delta)\left(\frac{\beta_1L}{1+a_1L+b_1S}+\frac{\beta_2S}{1+a_2L+b_2S}+(\mu+d)\right)\\
&=\gamma\left(\frac{b_2\beta_2PS}{(1+a_2L+b_2S)^2}+\frac{b_1\beta_1PL}{(1+a_1L+b_1S)^2}\right)-\gamma\frac{\beta_2P}{1+a_2L+b_2S}\\
&\quad+(\mu+d+\alpha+\delta)\left((\mu+d+\gamma)+\frac{a_1\beta_1PL}{(1+a_1L+b_1S)^2}\right)\\
&\quad+(\mu+d+\alpha+\delta)\left(-\frac{\beta_1P}{(1+a_1L+b_1S)}+\frac{a_2\beta_2PS}{(1+a_2L+b_2S)^2}\right)\\
&\quad+(\mu+d+\alpha+\delta)\left(\frac{\beta_1L}{1+a_1L+b_1S}+\frac{\beta_2S}{1+a_2L+b_2S}+(\mu+d)\right)\\
&=\gamma\left(\frac{b_2\beta_2PS}{(1+a_2L+b_2S)^2}+\frac{b_1\beta_1PL}{(1+a_1L+b_1S)^2}\right)-\gamma\frac{\beta_2P}{1+a_2L+b_2S}\\
&\quad+(\mu+d+\alpha+\delta)\left(\frac{\beta_1P}{1+a_1L+b_1S}+\frac{\beta_2PS}{(1+a_2L+b_2S)L}\right)\\
&\quad+(\mu+d+\alpha+\delta)\left(\frac{a_1\beta_1PL}{(1+a_1L+b_1S)^2}-\frac{\beta_1P}{1+a_1L+b_1S}+\frac{a_2\beta_2PS}{(1+a_2L+b_2S)^2}\right)\\
&\quad+(\mu+d+\alpha+\delta)\left(\frac{\beta_1L}{1+a_1L+b_1S}+\frac{\beta_2S}{1+a_2L+b_2S}+(\mu+d)\right)\\
&=\gamma\left(\frac{b_2\beta_2PS}{(1+a_2L+b_2S)^2}+\frac{b_1\beta_1PL}{(1+a_1L+b_1S)^2}\right)+(\mu+d+\alpha+\delta)\\
&\quad\times\left(\frac{a_1\beta_1PL}{(1+a_1L+b_1S)^2}+\frac{a_2\beta_2PS}{(1+a_2L+b_2S)^2}\right)-\gamma\frac{\beta_2P}{1+a_2L+b_2S}\\
&\quad+(\mu+d+\alpha+\delta)\left(\frac{\beta_1L}{1+a_1L+b_1S}+\frac{\beta_2S}{1+a_2L+b_2S}+(\mu+d)\right),
\end{aligned}$$

so we have

$$\begin{aligned}
&J_{22}J_{33}+J_{11}J_{33}+J_{13}J_{32}\\
&=\gamma\left(\frac{b_2\beta_2PS}{(1+a_2L+b_2S)^2}+\frac{b_1\beta_1PL}{(1+a_1L\quad+b_1S)^2}\right)+(\mu+d+\alpha+\delta)\\
&\quad\times\left(\frac{a_1\beta_1PL}{(1+a_1L+b_1S)^2}+\frac{a_2\beta_2PS}{(1+a_2L+b_2S)^2}\right)\\
&\quad+\gamma\frac{\beta_2P}{1+a_2L+b_2S}-\gamma\frac{\beta_2P}{1+a_2L+b_2S}\\
&\quad+(\mu+d+\alpha+\delta)\left(\frac{\beta_1L}{1+a_1L+b_1S}+\frac{\beta_2S}{1+a_2L\quad+b_2S}+(\mu+d)\right)\\
&>0,
\end{aligned}\tag{3.12}$$

From (3.10)-(3.12), we known that $c_2>0$.

Next, we will prove that c_3 is positive. Since $-J_{12}=J_{22}+(\mu+d+\gamma)$, so we can obtain

$$\begin{aligned}
c_3&=(-J_{22}J_{11}J_{33}+J_{12}J_{21}J_{33}+J_{22}J_{13}J_{31}+J_{11}J_{23}J_{32}-J_{12}J_{23}J_{31}-J_{13}J_{21}J_{32})|_{E^*}\\
&=(J_{12}+(\mu+d+\gamma))(-J_{21}-(\mu+d))J_{33}+J_{12}J_{21}J_{33}+J_{22}J_{13}J_{31}+J_{11}J_{23}J_{32}\\
&\quad-J_{12}J_{23}J_{31}-J_{13}J_{21}J_{32}\\
&=-J_{12}J_{21}J_{33}-(\mu+d)J_{12}J_{33}-(\mu+d+\gamma)J_{21}J_{33}-(\mu+d+\gamma)(\mu+d)J_{33}\\
&\quad+J_{12}J_{21}J_{33}+J_{22}J_{13}J_{31}+J_{11}J_{23}J_{32}-J_{12}J_{23}J_{31}-J_{13}J_{21}J_{32}\\
&=-(\mu+d)J_{12}J_{33}-(\mu+d+\gamma)J_{21}J_{33}-(\mu+d+\gamma)(\mu+d)J_{33}+J_{22}J_{13}J_{31}\\
&\quad+J_{11}J_{23}J_{32}-J_{12}J_{23}J_{31}-J_{13}J_{21}J_{32}\\
&=-(\mu+d)J_{12}J_{33}-(\mu+d+\gamma)J_{21}J_{33}\\
&\quad-(\mu+d+\gamma)(\mu+d)J_{33}+J_{11}J_{23}J_{32}-J_{13}J_{21}J_{32}\\
&=-(\mu+d)J_{12}J_{33}-(\mu+d+\gamma)J_{21}J_{33}\\
&\quad-(\mu+d+\gamma)(\mu+d)J_{33}+J_{13}J_{32}(-J_{11}-J_{21})\\
&=-(\mu+d)J_{12}J_{33}-(\mu+d+\gamma)J_{21}J_{33}-(\mu+d+\gamma)(\mu+d)J_{33}+J_{13}J_{32}(\mu+d).
\end{aligned}\tag{3.13}$$

Since $\mu+d+\gamma=\frac{\beta_1P}{1+a_1L+b_1S}+\frac{\beta_2P}{1+a_2L+b_2S}\frac{S}{L}$, substituting in (3.13), we have

$$\begin{aligned}
c_3&=-(\mu+d)J_{12}J_{33}-(\mu+d+\gamma)J_{21}J_{33}-(\mu+d+\gamma)(\mu+d)J_{33}+J_{13}J_{32}(\mu+d)\\
&=(\mu+d)(\mu+d+\alpha+\delta)J_{12}+(\mu+d+\gamma)(\mu+d+\alpha+\delta)J_{21}+\gamma(\mu+d)J_{13}\\
&\quad+(\mu+d)(\mu+d+\alpha+\delta)\left(\frac{\beta_1P}{1+a_1L+b_1S}+\frac{\beta_2P}{1+a_2L+b_2S}\frac{S}{L}\right)\\
&=(\mu+d)(\mu+d+\alpha+\delta)\frac{\beta_1P}{1+a_1L+b_1S}+(\mu+d)(\mu+d+\alpha+\delta)J_{12}\\
&\quad+(\mu+d)(\mu+d+\alpha+\delta)\frac{\beta_2P}{1+a_2L+b_2S}\frac{S}{L}\\
&\quad+\gamma(\mu+d)J_{13}+(\mu+d+\gamma)(\mu+d+\alpha+\delta)J_{21},
\end{aligned}$$

so we obtain

$$\begin{aligned}c_3 &= (\mu+d)(\mu+d+\alpha+\delta)\frac{\beta_1 P}{1+a_1L+b_1S}+(\mu+d)(\mu+d+\alpha+\delta)J_{12}\\&\quad+(\mu+d)(\mu+d+\alpha+\delta)\frac{\beta_2 P}{1+a_2L+b_2S}\frac{S}{L}+(\mu+d)(\mu+d+\alpha+\delta)\frac{S}{L}J_{13}\\&\quad+(\mu+d+\gamma)(\mu+d+\alpha+\delta)J_{21}\\&> 0.\end{aligned}\tag{3.14}$$

By (3.14), we have

$$\begin{aligned}c_1c_2-c_3 &= (J_{11}+J_{22}+J_{33})(J_{12}J_{21}-J_{11}J_{22}-J_{22}J_{33}-J_{11}J_{33}+J_{23}J_{32})\\&\quad+J_{11}J_{22}J_{33}-J_{12}J_{21}J_{33}-J_{11}J_{23}J_{32}+J_{13}J_{21}J_{32}\\&= J_{11}J_{12}J_{21}+J_{22}J_{12}J_{21}+J_{33}J_{12}J_{21}-J_{11}^2J_{22}-J_{11}J_{22}^2\\&\quad-J_{11}J_{22}J_{33}-J_{11}J_{22}J_{33}-J_{22}^2J_{33}-J_{22}J_{33}^2-J_{11}^2J_{33}\\&\quad-J_{11}J_{22}J_{33}-J_{11}J_{33}^2+J_{11}J_{23}J_{32}+J_{22}J_{23}J_{32}+J_{33}J_{23}J_{32}\\&\quad+J_{11}J_{22}J_{33}-J_{12}J_{21}J_{33}-J_{11}J_{23}J_{32}+J_{13}J_{21}J_{32}.\end{aligned}\tag{3.15}$$

By simple calculation, we have

$$J_{23}=-J_{13}, J_{22}=-J_{12}-(\mu+d+\gamma), J_{11}=-J_{21}-(\mu+d).\tag{3.16}$$

According to (3.15) and (3.16), further we can obtain

$$\begin{aligned}c_1c_2-c_3 &= -J_{11}^2J_{22}-J_{11}J_{22}^2-J_{11}J_{22}J_{33}-J_{22}^2J_{33}-J_{22}J_{33}^2\\&\quad-J_{11}^2J_{33}-J_{11}J_{22}J_{33}-J_{11}J_{33}^2+J_{11}J_{12}J_{21}\\&\quad+J_{22}J_{12}J_{21}+J_{22}J_{23}J_{32}+J_{33}J_{23}J_{32}+J_{13}J_{21}J_{32},\end{aligned}$$

which the positive terms are

$$-J_{11}^2J_{22}, -J_{11}J_{22}^2, -J_{11}J_{22}J_{33}, -J_{22}^2J_{33}, -J_{22}J_{33}^2, -J_{11}^2J_{33}, -J_{11}J_{22}J_{33}, -J_{11}J_{33}^2,$$

and it contains negative terms, but the overall positive and negative uncertainties are

$$J_{11}J_{12}J_{21},\ J_{22}J_{12}J_{21},\ J_{22}J_{23}J_{32},\ J_{33}J_{23}J_{32},\ J_{13}J_{21}J_{32}.\tag{3.17}$$

Five items are analyzed in (3.17) as follows

(1) $J_{11}J_{12}J_{21}=-J_{11}(J_{22}+(\mu+d+\gamma))J_{21}=-J_{11}J_{22}J_{21}-J_{11}(\mu+d+\gamma)J_{21}$, here $-J_{11}J_{22}J_{21}$ is counteracted by the positive term $-J_{11}^2J_{22}$ in expression of $c_1c_2-c_3$.

(2) $J_{22}J_{12}J_{21}=-J_{22}(J_{22}+(\mu+d+\gamma))J_{21}=-J_{22}^2J_{21}-J_{22}(\mu+d+\gamma)J_{21}$, here $-J_{22}^2J_{21}$ is counteracted by the positive term $-J_{22}^2J_{11}$ in expression of $c_1c_2-c_3$.

(3) Since

$$\begin{aligned}\frac{\beta_2P}{1+a_2L+b_2S}\gamma &= \frac{\beta_2P}{1+a_2L+b_2S}\frac{(\mu+d+\alpha+\delta)S}{L}\\&= (\mu+d+\alpha+\delta)\frac{\beta_2P}{1+a_2L+b_2S}\frac{S}{L}\\&= (\mu+d+\alpha+\delta)(\mu+d+\gamma-\frac{\beta_1P}{1+a_1L+b_1S}).\end{aligned}$$

The negative term in expression of $J_{13}J_{21}J_{32}$ is

$$-\gamma J_{21}\frac{\beta_2 P}{1+a_2L+b_2S}=-J_{21}(\mu+d+\alpha+\delta)(\mu+d+\gamma-\frac{\beta_1 P}{1+a_1L+b_1S}). \quad (3.18)$$

So, (3.18) is counteracted by the positive term $-J_{11}J_{22}J_{33}$ in expression of $c_1c_2-c_3$.

(4) Similarly (3), the negative term $\gamma J_{33}\frac{\beta_2 P}{1+a_2L+b_2S}$ in the expression of $J_{33}J_{23}J_{32}=-\gamma J_{33}J_{13}$ can be counteracted by the positive term $-J_{22}J_{33}^2$ in the expression of $c_1c_2-c_3$.

(5) Similarly (3), the negative term $\gamma J_{22}\frac{\beta_2 P}{1+a_2L+b_2S}$ in the expression of $J_{22}J_{23}J_{32}=-\gamma J_{22}J_{13}$ can be counteracted by the positive term $-J_{22}^2J_{33}$ in the expression of $c_1c_2-c_3$.

Hence, by the above analysis, we have proved that $c_1c_2-c_3>0$.

According to the Routh-Hurwitz criterion, we know E^* is locally asymptotically stable. This completes the proof of Theorem 3.2. □

Theorem 3.3. *If $\mathcal{R}_0\leqslant 1$, E_0 is globally asymptotically stable in the region Ω.*

Proof. Let

$$F=\begin{bmatrix}\frac{\beta_1\lambda}{\mu+d} & \frac{\beta_2\lambda}{\mu+d}\\ 0 & 0\end{bmatrix},\quad V=\begin{bmatrix}\mu+d+\gamma & 0\\ -\gamma & \mu+d+\alpha+\delta\end{bmatrix}.$$

Write $y=(L,S)^T$, the system (2.1) satisfies

$$\frac{\mathrm{d}y}{\mathrm{d}t}\leqslant (F-V)y.$$

Let $\omega=\left(\frac{\beta_1\lambda}{\mu+d},\frac{\beta_2\lambda}{\mu+d}\right)$. In view of $\mathcal{R}_0=\rho(FV^{-1})=\rho(V^{-1}F)$, one can verify that $\omega V^{-1}F=\mathcal{R}_0\omega$. Motivated by [21], we define a Lyapunov function as follows:

$$L=\omega V^{-1}y.$$

Differentiating L along solutions of (2.1), we have

$$\frac{\mathrm{d}L}{\mathrm{d}t}=\omega V^{-1}\frac{\mathrm{d}y}{\mathrm{d}t}\leqslant \omega V^{-1}(F-V)y=(\mathcal{R}_0-1)\omega y.$$

If $\mathcal{R}_0<1$, then $\frac{\mathrm{d}L}{\mathrm{d}t}\leqslant 0$. Since $\frac{\mathrm{d}L}{\mathrm{d}t}=0$ implies that $\omega y=0$, then $L=S=0$. It follows from the equations of model (2.1) that $P=\frac{\lambda}{\mu+d}$ and $Q=0$. Hence, the only invariant set is the singleton $\left\{\left(\frac{\lambda}{\mu+d},0,0,0\right)\right\}$ as $\frac{\mathrm{d}L}{\mathrm{d}t}=0$.

In the case $\mathcal{R}_0=1$, then $\frac{\mathrm{d}L}{\mathrm{d}t}=0$ implies that $P=\frac{\lambda}{\mu+d}$ or $L=S=0$. Then, by a similar argument above, we find that the largest invariant set is the singleton $\left\{\left(\frac{\lambda}{\mu+d},0,0,0\right)\right\}$ as $\frac{\mathrm{d}L}{\mathrm{d}t}=0$.

By LaSalle's Invariant Principle [22], the E_0 is globally asymptotically stable in Ω when $\mathcal{R}_0\leqslant 1$. This completes the proof of Theorem 3.3. □

Next we give the proof of the global stability of E^*. Firstly, we give the corresponding preliminary and lemma. Suppose that $\mathcal{R}_0>1$. Hence, by Theorem 3.2, the system (2.1) has two equilibria E_0 and E^*. We now proceed to prove the global stability of the endemic equilibrium of (2.1) by using the geometric approach based on the second additive compound matrix [23]. The details on the geometric approach are as follows. Here we present the main result of the geometric approach for global stability, originally developed by Li and Muldowney [24]. We consider a dynamical system

$$\frac{\mathrm{d}X}{\mathrm{d}t} = f(X), \tag{3.19}$$

where $f : D \mapsto \mathbb{R}^n$ is a C^1 function and $D \subset \mathbb{R}^n$ is a simply connected open set. Let $P(X)$ be a $\begin{pmatrix} n \\ 2 \end{pmatrix} \times \begin{pmatrix} n \\ 2 \end{pmatrix}$ matrix-valued C^1 function in D, and set

$$Q = P_f P^{-1} + P J^{[2]} P^{-1},$$

where P_f is the derivative of P (entry-wise) along the direction of f, and $J^{[2]}$ is the second additive compound matrix of the Jacobian $J(X) = Df(X)$. Let $\mathcal{M}(Q)$ be the Lozinskii measure of Q with respect to a matrix norm i.e.,

$$\mathcal{M}(Q) = \lim_{h \to 0^+} \frac{|\mathrm{I} + hQ| - 1}{h},$$

where I represent the identity matrix. Define a quantity $\bar{q}_2$ as

$$\bar{q}_2 = \limsup_{t \to \infty} \sup_{X_0 \in K} \frac{1}{t} \int_0^t \mathcal{M}(Q(X(s, X_0)))\mathrm{d}s,$$

where K is a compact absorbing subset of D. Then the condition $\bar{q}_2 < 0$ provides a Bendixson criterion in D. As a result, the following lemma holds:

Lemma 3.1 (Theorem 3.1, [24])**.** *Assume that there exists a compact absorbing set $K \subset D$ and the system (3.19) has a unique equilibrium point X^* in D. Then X^* is globally asymptotically stable in D if $\bar{q}_2 < 0$.*

Theorem 3.4. *If $\mathcal{R}_0 > 1$, E^* is globally asymptotically stable in the Ω^0, the interior of Ω, provided that*

$$\sup\left\{\frac{\beta_1 P}{1 + a_1 L + b_1 S} - \frac{a_1 \beta_1 P L}{(1 + a_1 L + b_1 S)^2} - \frac{a_2 \beta_2 P S}{(1 + a_2 L + b_2 S)^2} : \left.\left.(P, L, S, Q) \in \mathbb{R}^4_+, P + L + S + Q \leqslant \frac{\lambda}{\mu + d}\right)\right\} \leqslant \frac{\gamma}{2}.$$

Proof. By Theorem3.2, E^* is unstable as $\mathcal{R}_0 > 1$, and it is on the boundary of the domain Ω. This implies that the number of smoking is uniformly persistent in Ω^0, namely, $\liminf_{t \to \infty} S(t) > c$ for some $c > 0$. It then follows from the compactness of Ω and the uniform persistence of system (2.1) that there exists a compact absorbing set in Ω. By Lemma 3.1, the following work is to construct a matrix-valued function such that $\bar{q}_2 < 0$.

For simplicity, we denote

$$\begin{aligned}
\theta_1 &= \frac{\beta_1 L}{1 + a_1 L + b_1 S} + \frac{\beta_2 S}{1 + a_2 L + b_2 S}, \\
\theta_2 &= \frac{\beta_1 P(1 + a_1 L + b_1 S) - a_1 \beta_1 P L}{(1 + a_1 L + b_1 S)^2} - \frac{a_2 \beta_2 P S}{(1 + a_2 L + b_2 S)^2}, \\
\theta_3 &= \frac{\beta_2 P(1 + a_2 L + b_2 S) - b_2 \beta_2 P S}{(1 + a_2 L + b_2 S)^2} - \frac{b_1 \beta_1 P L}{(1 + a_1 L + b_1 S)^2}.
\end{aligned}$$

The Jacobian matrix associated with the linearized system of (3.5) is

$$\tilde{J} = \begin{bmatrix} -\theta_1 - (\mu + d) & -\theta_2 & -\theta_3 \\ \theta_1 & \theta_2 - (\mu + d + \gamma) & \theta_3 \\ 0 & \gamma & -(\mu + d + \alpha + \delta) \end{bmatrix}$$

and its second additive compound matrix is

$$\tilde{J}^{[2]} = \begin{bmatrix} j_{11} & \theta_3 & \theta_3 \\ \gamma & j_{22} & -\theta_2 \\ 0 & \theta_1 & j_{33} \end{bmatrix},$$

where

$$j_{11} = -\theta_1 + \theta_2 - (2\mu + 2d + \gamma),\ j_{22} = -\theta_1 - (2\mu + 2d + \alpha + \delta),$$
$$j_{33} = \theta_2 - (2\mu + 2d + \gamma + \alpha + \delta).$$

We now take $Q = diag\left[1, \frac{L}{S}, \frac{L}{S}\right]$, then $Q_f Q^{-1} = diag\left[0, \frac{L'}{L} - \frac{S'}{S}, \frac{L'}{L} - \frac{S'}{S}\right]$, where f denote the vector field of (2.1). Thus, we have

$$Q\tilde{J}^{[2]}Q^{-1} = \begin{bmatrix} m_{11} & \theta_3 \frac{S}{L} & \theta_3 \frac{S}{L} \\ \gamma \frac{L}{S} & m_{22} & -\theta_2 \\ 0 & \theta_1 & m_{33} \end{bmatrix},$$

where

$$m_{11} = -\theta_1 + \theta_2 - (2\mu + 2d + \gamma),\ m_{22} = -\theta_1 - (2\mu + 2d + \alpha + \delta),$$
$$m_{33} = \theta_2 - (2\mu + 2d + \gamma + \alpha + \delta).$$

Thus, the matrix $\mathrm{H} = Q_f Q^{-1} + Q\tilde{J}^{[2]}Q^{-1}$ can be written in the following block form:

$$\mathrm{H} = \begin{bmatrix} \mathrm{H}_{11} & \mathrm{H}_{12} \\ \mathrm{H}_{21} & \mathrm{H}_{22} \end{bmatrix},$$

where

$$\mathrm{H}_{11} = -\frac{\beta_1 L}{1 + a_1 L + b_1 S} - \frac{\beta_2 S}{1 + a_2 L + b_2 S} + \frac{\beta_1 P(1 + a_1 L + b_1 S) - a_1 \beta_1 P L}{(1 + a_1 L + b_1 S)^2}$$
$$- \frac{a_2 \beta_2 P S}{(1 + a_2 L + b_2 S)^2} - (2\mu + 2d + \gamma),$$
$$\mathrm{H}_{12} = \left(\frac{\beta_2 P(1 + a_2 L + b_2 S) - b_2 \beta_2 P S}{(1 + a_2 L + b_2 S)^2} - \frac{b_1 \beta_1 P L}{(1 + a_1 L + b_1 S)^2} \right) \frac{S}{L} \begin{bmatrix} 1 & 1 \end{bmatrix},$$
$$\mathrm{H}_{21} = \begin{bmatrix} \gamma \frac{L}{S} \\ 0 \end{bmatrix},\ \mathrm{H}_{22} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix},$$

with

$$\begin{aligned}
h_{11} &= -\frac{\beta_1 L}{1+a_1L+b_1S} - \frac{\beta_2 S}{1+a_2L+b_2S} - (2\mu+2d+\alpha+\delta) + \frac{L'}{L} - \frac{S'}{S},\\
h_{12} &= \frac{a_1\beta_1 PL - \beta_1 P(1+a_1L+b_1S)}{(1+a_1L+b_1S)^2} + \frac{a_2\beta_2 PS}{(1+a_2L+b_2S)^2},\\
h_{21} &= \frac{\beta_1 L}{1+a_1L+b_1S} + \frac{\beta_2 S}{1+a_2L+b_2S},\\
h_{22} &= \frac{\beta_1 P(1+a_1L+b_1S) - a_1\beta_1 PL}{(1+a_1L+b_1S)^2} - \frac{a_2\beta_2 PS}{(1+a_2L+b_2S)^2}\\
&\quad -(2\mu+2d+\gamma+\alpha+\delta) + \frac{L'}{L} - \frac{S'}{S}.
\end{aligned}$$

The vector norm $|\cdot|$ in $\mathbb{R}^3$ is chosen as

$$|(x_1,x_2,x_3)| = \max\left\{|x_1|,|x_2|,|x_3|\right\}.$$

One can verify that the Lozinskii measure $\mathcal{M}\,(\mathrm{H})$ with respect to this norm can be estimated as

$$\mathcal{M}(\mathrm{H}) \leqslant \sup\left\{g_1, g_2\right\},$$

where

$$g_1 = \mathcal{M}_1(\mathrm{H}_{11}) + |\mathrm{H}_{12}|\,, g_2 = |\mathrm{H}_{21}| + \mathcal{M}_1(\mathrm{H}_{22}).$$

Here $|\mathrm{H}_{12}|$ and $|\mathrm{H}_{21}|$ are matrix norms induced by the l_1 vector norm, $\mathcal{M}_1$ denote the Lozinskii measure with respect to the l_1 norm. In detail,

$$\begin{aligned}
g_1 = \mathrm{H}_{11} + \mathrm{H}_{12} &= -\frac{\beta_1 L}{1+a_1L+b_1S} - \frac{\beta_2 S}{1+a_2L+b_2S} - \frac{a_2\beta_2 PS}{(1+a_2L+b_2S)^2}\\
&\quad + \frac{\beta_1 P(1+a_1L+b_1S) - a_1\beta_1 PL}{(1+a_1L+b_1S)^2} - (2\mu+2d+\gamma)\\
&\quad + \left(\frac{\beta_1 P(1+a_2L+b_2S) - a_2\beta_2 PS}{(1+a_2L+b_2S)^2} - \frac{a_1\beta_1 PL}{(1+a_1L+b_1S)^2}\right)\frac{S}{L},\\
g_2 &= |\mathrm{H}_{21}| + \max\left\{h_{11} + |h_{21}|, |h_{12}| + h_{22}\right\}\\
&= \gamma\frac{L}{S} + \max\Bigg\{-(2\mu+2d+\alpha+\delta) + \frac{L'}{L} - \frac{S'}{S},\\
&\quad \left|\frac{a_1\beta_1 PL - \beta_1 P(1+a_1L+b_1S)}{(1+a_1L+b_1S)^2} + \frac{a_2\beta_2 PS}{(1+a_2L+b_2S)^2}\right| - \frac{a_2\beta_2 PS}{(1+a_2L+b_2S)^2}\\
&\quad + \frac{\beta_1 P(1+a_1L+b_1S) - a_1\beta_1 PL}{(1+a_1L+b_1S)^2} - (2\mu+2d+\gamma+\alpha+\delta) + \frac{L'}{L} - \frac{S'}{S}\Bigg\}\\
&= \gamma\frac{L}{S} - (2\mu+2d+\alpha+\delta) + \frac{L'}{L} - \frac{S'}{S}\\
&\quad + \max\left\{0, 2\left(\frac{\beta_1 P(1+a_1L+b_1S) - a_1\beta_1 PL}{(1+a_1L+b_1S)^2} - \frac{a_2\beta_2 PS}{(1+a_2L+b_2S)^2}\right) - \gamma\right\}.
\end{aligned}$$

Since

$$L' = \frac{\beta_1 PL}{1+a_1L+b_1S} + \frac{\beta_2 PS}{1+a_2L+b_2S} - (\mu+d+\gamma)L,$$

we obtain

$$-(\mu+d+\gamma)=\frac{L'}{L}-\frac{\beta_1 P}{1+a_1L+b_1S}-\frac{S}{L}\frac{\beta_2 P}{1+a_2L+b_2S}.$$

This leads to

$$\begin{aligned} g_1 =& -\frac{\beta_1 L}{1+a_1L+b_1S}-\frac{\beta_2 S}{1+a_2L+b_2S}+\frac{\beta_1 P(1+a_1L+b_1S)-a_1\beta_1 PL}{(1+a_1L+b_1S)^2}\\ &-\frac{a_2\beta_2 PS}{(1+a_2L+b_2S)^2}-(\mu+d)+\frac{L'}{L}-\frac{\beta_1 P}{1+a_1L+b_1S}-\frac{S}{L}\frac{\beta_2 P}{1+a_2L+b_2S}\\ &+\left(\frac{\beta_1 P(1+a_2L+b_2S)-a_2\beta_2 PS}{(1+a_2L+b_2S)^2}-\frac{a_1\beta_1 PL}{(1+a_1L+b_1S)^2}\right)\frac{S}{L},\\ \leqslant& \frac{L'}{L}-(\mu+d). \end{aligned} \tag{3.20}$$

Since $S'=\gamma L-(\mu+d+\alpha+\delta)S$, we have

$$\frac{S'}{S}=\gamma\frac{L}{S}-(\mu+d+\alpha+\delta).$$

This leads to

$$\begin{aligned} g_2 =& \max\left\{0,2\left(\frac{\beta_1 P(1+a_1L+b_1S)-a_1\beta_1 PL}{(1+a_1L+b_1S)^2}-\frac{a_2\beta_2 PS}{(1+a_2L+b_2S)^2}\right)-\gamma\right\}\\ &-(\mu+d)+\frac{L'}{L}\\ \leqslant& \frac{L'}{L}-(\mu+d), \end{aligned} \tag{3.21}$$

if $\left(\frac{\beta_1 P(1+a_1L+b_1S)-a_1\beta_1 PL}{(1+a_1L+b_1S)^2}-\frac{a_2\beta_2 PS}{(1+a_2L+b_2S)^2}\right)\leqslant\frac{\gamma}{2}$. Thus, (3.20) and (3.21) yield

$$\mathcal{M}(\mathrm{H})\leqslant\frac{L'}{L}-(\mu+d).$$

It follows from $0\leqslant L\leqslant\frac{\lambda}{\mu+d}$ that

$$\frac{\ln(L(t))-\ln(L(0))}{t}\leqslant\frac{\mu+d}{2},$$

for t sufficiently large. We then obtain

$$\frac{1}{t}\int_0^t\mathcal{M}(\mathrm{H})\mathrm{d}s\leqslant\frac{1}{t}\int_0^t\left(\frac{L'}{L}-(\mu+d)\right)\mathrm{d}s=\frac{1}{t}\left(\ln\frac{L(t)}{L(0)}\right)-(\mu+d)\leqslant-\frac{\mu+d}{2}, \tag{3.22}$$

if t is large enough. This in turn implies that $\bar{q}_2\leqslant-\frac{\mu+d}{2}<0$. It completes the proof. □

4. Numerical simulation

In this section, some numerical results of system (2.1) are presented for supporting the analytic results obtained above. All the parameters values are estimated. The model parameters are taken as:

(1) $\lambda = 0.4, \beta_1 = 0.35, \beta_2 = 0.4, a_1 = 0.3, a_2 = 0.23, b_1 = 0.4, b_2 = 0.3, \mu = 0.2, d = 0.21, \gamma = 0.02, \delta = 0.21, \alpha = 0.015$, then $\mathcal{R}_0 = 0.8227$. According to Theorem 3.3, we get the number of people who smoke tends to zero. Here we choose initial value $P(0) = 0.7, L(0) = 0.6, S(0) = 0.7, Q(0) = 0.4$, (see Figure 1).

(2) $\lambda = 0.4, \beta_1 = 0.35, \beta_2 = 0.4, a_1 = 0.3, a_2 = 0.23, b_1 = 0.4, b_2 = 0.3, \mu = 0.01, d = 0.21, \gamma = 0.02, \delta = 0.21, \alpha = 0.015$, then $\mathcal{R}_0 = 2.7877$. According to Theorem 3.4, we know the number of smokers will continue. Here we choose initial value $P(0) = 0.7, L(0) = 0.6, S(0) = 0.7, Q(0) = 0.4$, (see Figure 2).

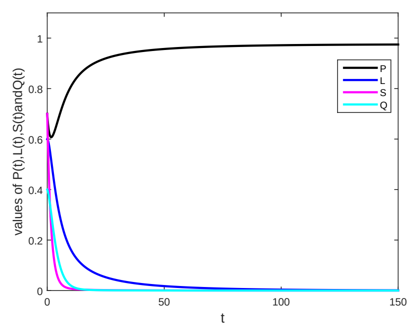

Figure 1. The global stability of the model (2.1).

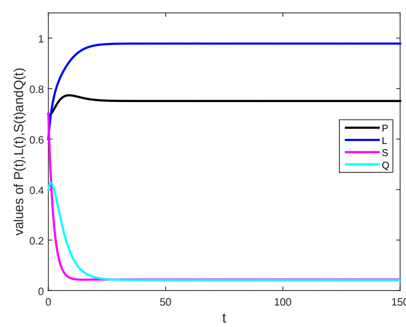

Figure 2. The permanence of the model (2.1).

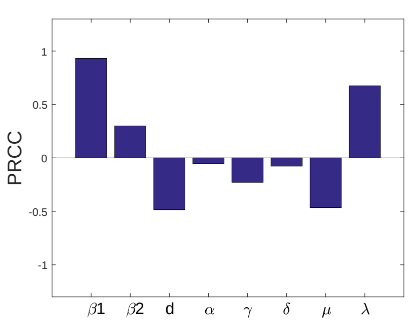

Figure 3. PRCCs for the aggregate $\mathcal{R}_0$ and each input parameter variable.

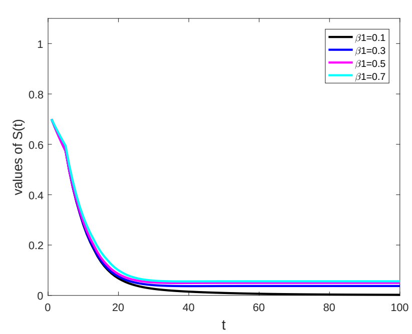

Figure 4. The influence of parameter β_1 on the value of $S(t)$.

(3) From Figure 3, it can be shown results for the partial rank correlation coefficients (PRCCs) [21] and these results illustrate the dependence of $\mathcal{R}_0$ on each parameter. From the biological significance of the established model, we only studied the sensitivity of contact rate $\beta_i (i = 1, 2)$, the rate of quitting smoking δ, the conversion rate from occasional smokers to smokers γ, the death rate for smokers $d + \alpha$ (high mortality due to heavy smoking) on $\mathcal{R}_0$.

In Figures 4-5, this changes value of parameter $\beta_i (i = 1, 2)$ and other parameters values are the same as Figure 2. In Figure 6, this changes value of parameter $d + \alpha$ and other parameters values are the same as Figure 2. In Figure 7, this changes value of parameter γ and other parameters values are the same as Figure 2. In Figure 8, this changes value of parameter δ and other parameters values are the same as Figure 2.

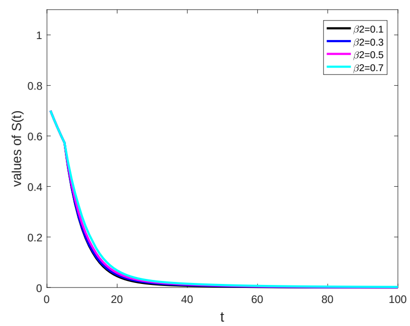

Figure 5. The influence of parameter β_2 on the value of $S(t)$.

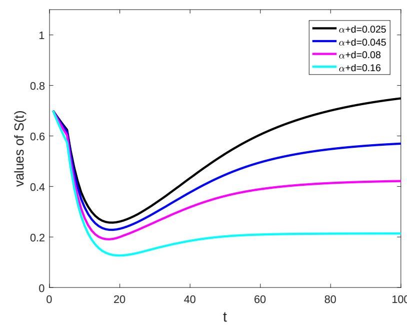

Figure 6. The influence of parameter $\alpha + d$ on the value of $S(t)$.

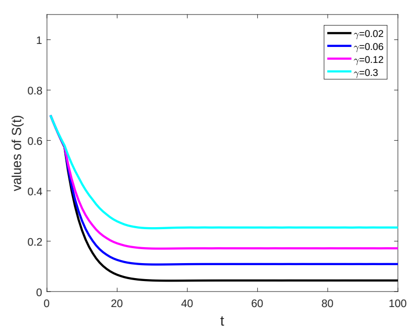

Figure 7. The influence of parameter γ on the value of $S(t)$.

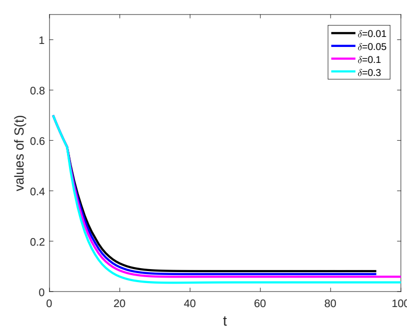

Figure 8. The influence of parameter δ on the value of $S(t)$.

5. Discussion

In this paper, considering the propagation of smoking behavior is similar to the predator-prey model, we introduce a more general Beddington-DeAngelis function into the quitting smoking model to establish a non-linear quitting smoking model. For ODE model, we have rigorously proved the reproduction number $\mathcal{R}_0$, and analyzed the local and global stability of the model from the perspective of the reproduction number. In particular, when $\mathcal{R}_0 \leqslant 1$, the equilibrium E_0 of non-smokers is globally asymptotically stable (see Figure 1); when $\mathcal{R}_0 > 1$, the equilibrium E^* of smokers is globally asymptotically stable (see Figure 2). From the partial rank correlation coefficient (PRCCs), the effects of different parameters on smokers' transmission are obtained (see Figure 3-8). Finally, the theoretical analysis is shown more intuitively by numerical simulations. Our results show that in order to reduce and control the number of smokers, we can increase the rate of quitting smoking δ and decrease contact rate $\beta_i (i = 1, 2)$ by scientific means from the above analysis (see Figures 4-5, 8).

Acknowledgements

We are grateful to the editors and the referees for their valuable comments and suggestions that greatly improved the presentation of this paper.

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