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Global Stability of a Stochastic Lotka-Volterra Cooperative System with Two Feedback Controls*

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Abstract In this paper, a class of Lotka-Volterra cooperation system and corresponding stochastic system with two feedback controls which are affected by all species are considered. We obtain some sufficient criteria for local stability and global asymptotic stability of equilibria of the systems. Our study shows that these equilibria could be globally stable by adjusting coefficients of the feedback control variables and stochastic perturbation parameters. Numerical simulations are presented to demonstrate our main result.

Keywords Lotka-Volterra cooperative system, Feedback controls, Stochastic perturbation, Global stability.

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1. Introduction

The Lotka-Volterra system has been extensively investigated, see [1,2] and the references cited therein. During the last two decades, the study of dynamic behaviors of ecosystem with feedback controls has become one of important research topics, see [3,4] and the references cited therein. Some ecosystems have single feedback control strategy [5,6]. For example, [6] studied a cooperation system with one feedback control

$$\begin{cases} \frac{dx_1}{dt} = r_1 x_1(t)(\frac{K_1+\alpha_1 x_2(t)}{1+x_2(t)} - x_1(t) - d_1 u(t)), \\ \frac{dx_2}{dt} = r_2 x_2(t)(\frac{K_2+\alpha_2 x_1(t)}{1+x_1(t)} - x_2(t) - d_2 u(t)), \\ \frac{du}{dt} = -eu(t) + f_1 x_1(t) + f_2 x_2(t). \end{cases}$$

Another ecosystems have different feedback strategies to different species [4,7]. For instance, [7] considered a competitive system with two feedback controls

$$\begin{cases} \frac{dx_1}{dt} = x_1(t)(b_1 - a_{11}x_1(t) - a_{12}\int_0^{+\infty} K_1(s)x_2(t-s)ds - c_1 u_1(t)), \\ \frac{dx_2}{dt} = x_2(t)(b_2 - a_{21}\int_0^{+\infty} K_2(s)x_1(t-s)ds - a_{22}x_2(t) - c_2 u_2(t)), \\ \frac{du_1}{dt} = -e_1 u_1(t) + d_1 x_1(t), \\ \frac{du_2}{dt} = -e_2 u_2(t) + d_2 x_2(t). \end{cases}$$

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On the other hand, every feedback control variable in turn can be affected by one specie, such as the system in [7], or by more species, such as the system in [6]. These works showed that feedback control variables can usually influence the position of the positive equilibrium, and have no influence on the stability of the equilibrium under suitable conditions.

Meanwhile, the parameters involved in systems show random fluctuation due to environmental noise. Hence, many ecosystems with stochastic noise have been studied, see [8–12,14] and the references cited therein. For instance, [12] considered global stability of a stochastic SI epidemic model with feedback controls. They obtained that if the endemic equilibrium of the deterministic system is globally stable, then its corresponding stochastic system keeps the property provided the noise is sufficiently small.

Above phenomenons motivate us to propose and study the Lotka-Volterra type cooperative system with two feedback control variables as follows:

$$\begin{cases} dx_1(t) = x_1[b_1 - a_{11}x_1(t) + a_{12}x_2(t) - \alpha_1 u_1(t)]dt, \\ dx_2(t) = x_2[b_2 + a_{21}x_1(t) - a_{22}x_2(t) - \alpha_2 u_2(t)]dt, \\ du_1(t) = [-\eta_1 u_1(t) + a_1 x_1(t) + \beta_2 x_2(t)]dt, \\ du_2(t) = [-\eta_2 u_2(t) + \beta_1 x_1(t) + a_2 x_2(t)]dt, \end{cases} \tag{1.1}$$

where $a_{ij}, a_i, b_i, \eta_i, \alpha_i, \beta_i (i, j = 1, 2)$ are positive constants. $x_i(t)(i = 1, 2)$ is the density of population x_i at time t, and $u_i(t)(i = 1, 2)$ is feedback control variable. As for as we know there is no global stability result on the cooperate Lotka-Volterra system with two feedback controls which are affected by every specie.

Moreover, we study a corresponding stochastic system

$$\begin{cases} dx_1(t) = x_1[b_1 - a_{11}x_1(t) + a_{12}x_2(t) - \alpha_1 u_1(t)]dt + \sigma_1 x_1(t)(x_1(t) - x_1^*)dB_1(t), \\ dx_2(t) = x_2[b_2 + a_{21}x_1(t) - a_{22}x_2(t) - \alpha_2 u_2(t)]dt + \sigma_2 x_2(t)(x_2(t) - x_2^*)dB_2(t), \\ du_1(t) = [-\eta_1 u_1(t) + a_1 x_1(t) + \beta_2 x_2(t)]dt, \\ du_2(t) = [-\eta_2 u_2(t) + \beta_1 x_1(t) + a_2 x_2(t)]dt, \end{cases} \tag{1.2}$$

where $B_i(t)(i = 1, 2)$ is standard white noise, $\sigma_i(i = 1, 2)$ denotes the intensity of the noise, $P(x_1^*, x_2^*, u_1^*, u_2^*)$ is unique positive equilibrium of (1.1), see (2.1) in Section 2. Obviously, when $\sigma_1 = \sigma_2 = 0$, system (1.2) will reduce to (1.1). And systems (1.1), (1.2) have same positive equilibrium. As far as we know there is no global stability result on the stochastic cooperate Lotka-Volterra systems with two feedback controls which are affected by all species. One interesting issue is whether the system still admits a unique globally stable positive equilibrium or the system could have more complicate dynamic behaviors under the influence of more complicated feedback controls and standard white noise?

The rest of the paper is organized as follows. In the next section, we investigate the local stability of the equilibria of system (1.1). Section 3 is assigned to discuss the global stability property of system (1.1) and stochastic system (1.2). In Section 4, examples together with their numerical simulations are presented to illustrate the feasibility of our main result. Finally, we end this paper by a briefly discussion.

2. Local Stability

From the point of view of biology, we only discuss the positive solution of system (1.1). Assuming that the initial conditions of system (1.1) are

$$x_1(0) > 0, x_2(0) > 0, u_1(0) > 0, u_2(0) > 0.$$

Obviously, the solution of system (1.1) with the above initial value is well defined and positive for all $t > 0$. By calculation, (1.1) has three boundary equilibria:

$$P_1(x_{10}, 0, u_{10}, u_{20}), \quad P_2(0, x_{20}, u_{30}, u_{40}), \quad P_3(0, 0, 0, 0)$$

where

$$x_{10} = \frac{b_1\eta_1}{a_{11}\eta_1 + a_1\alpha_1}, u_{10} = \frac{a_1b_1}{a_{11}\eta_1 + a_1\alpha_1}, u_{20} = \frac{\beta_1b_1\eta_1}{\eta_2(a_{11}\eta_1 + a_1\alpha_1)},$$

$$x_{20} = \frac{b_2\eta_1}{a_{22}\eta_1 + a_2\alpha_2}, u_{30} = \frac{a_2b_2}{a_{22}\eta_1 + a_2\alpha_2}, u_{40} = \frac{\beta_2b_1\eta_1}{\eta_2(a_{22}\eta_1 + a_2\alpha_2)}.$$

Besides, if the following condition

$$(H1^*) \quad a_{11}a_{22} > a_{12}a_{21}, \quad a_{12}\eta_1 > \alpha_1\beta_2, \quad a_{21}\eta_2 > \alpha_2\beta_1, \quad a_1a_2 > \beta_1\beta_2$$

holds, then system (1.1) has a unique interior equilibrium $P(x_1^*, x_2^*, u_1^*, u_2^*)$:

$$\begin{aligned} x_1^* &= \frac{\eta_1b_1(\eta_2a_{22} + \alpha_2a_2) + \eta_2b_2(a_{12}\eta_1 - \alpha_1\beta_2)}{\Delta}, \\ x_2^* &= \frac{\eta_2b_2(\eta_1a_{11} + \alpha_1a_1) + \eta_1b_1(a_{21}\eta_2 - \alpha_2\beta_1)}{\Delta}, \\ u_1^* &= \frac{a_1x_1^* + \beta_2x_2^*}{\eta_1}, \quad u_2^* = \frac{\beta_1x_1^* + a_2x_2^*}{\eta_2} \end{aligned} \tag{2.1}$$

with

$$\begin{aligned} \Delta =& \eta_1\eta_2(a_{11}a_{22} - a_{12}a_{21}) + \alpha_1\alpha_2(a_1a_2 - \beta_1\beta_2) \\ &+ \alpha_1\eta_2(a_{11}\eta_2 + a_1a_{22}) + \alpha_2\eta_1(a_{11}a_2 + \beta_1a_{12}). \end{aligned}$$

Remark 2.1. In fact, the condition $(H1^*)$ can be replaced by weaker condition

$$(H1) \quad a_{11}a_{22} > a_{12}a_{21}, \quad a_{12}\eta_1 \geq \alpha_1\beta_2, \quad a_{21}\eta_2 \geq \alpha_2\beta_1, \quad \Delta > 0.$$

We will use $(H1)$ in the following discussion.

Now, we discuss the local stability of three nonzero equilibria of system (1.1).

Lemma 2.1. *Four roots of the following quartic equation*

$$\lambda^4 + c_1\lambda^3 + c_2\lambda^2 + c_3\lambda + c_4 = 0$$

have negative real parts if and only if

$$c_1 > 0, \ c_4 > 0, \ c_1c_2 - c_3 > 0, \ c_1c_2c_3 - c_1^2c_4 - {c_3}^2 > 0.$$

Proof. Let

$$\Delta_1 = c_1 > 0,\ \ \Delta_2 = \begin{vmatrix} c_1 & 1 \\ c_3 & c_2 \end{vmatrix} = c_1c_2 - c_3 > 0,$$

$$\Delta_3 = \begin{vmatrix} c_1 & 1 & 0 \\ c_3 & c_2 & c_1 \\ 0 & c_4 & c_3 \end{vmatrix} = c_1c_2c_3 - c_1^2c_4 - {c_3}^2 > 0,\ \ \Delta_4 = c_4\Delta_3 > 0.$$

By Hurwitz criterion ([13]), this equation has four roots with negative real parts since $\Delta_i > 0$ $(i = 1, 2, 3, 4)$. □

Theorem 2.1. *Assume that*

$$(H2) \qquad \frac{b_2\eta_2}{b_1\eta_1} < \frac{\alpha_2\beta_1 - a_{21}\eta_2}{a_{11}\eta_1 + a_1\alpha_1}$$

holds. Then the boundary equilibrium $P_1(x_{10}, 0, u_{10}, u_{20})$ *of system* (1.1) *is locally stable.*

Proof.
The Jacobian matrix of system (1.1) at $P_1(x_{10}, 0, u_{10}, u_{20})$ is

$$J_1(x_{10}, 0, u_{10}, u_{20}) = \begin{pmatrix} -a_{11}x_{10} & a_{12}x_{10} & -\alpha_1 x_{10} & 0 \\ 0 & b_2 + a_{21}x_{10} - \alpha_2 u_{20} & 0 & 0 \\ a_1 & \beta_2 & -\eta_1 & 0 \\ \beta_1 & a_2 & 0 & -\eta_2 \end{pmatrix}. \tag{2.2}$$

Let

$$\begin{aligned} &R := b_2 + a_{21}x_{10} - \alpha_2 u_{20}, && Q := \eta_1 + \eta_2 + a_{11}x_{10}, \\ &M := a_{11}\eta_1 x_{10} + a_{11}\eta_2 x_{10} + a_1\alpha_1 x_{10} + \eta_1\eta_2, && N := a_{11}x_{10}\eta_1\eta_2 + a_1\alpha_1 x_{10}\eta_2. \end{aligned} \tag{2.3}$$

The corresponding characteristic equation of matrix (2.2) is :

$$\lambda^4 + c_1\lambda^3 + c_2\lambda^2 + c_3\lambda + c_4 = 0,$$

where

$$c_1 = Q - R,\quad c_2 = M - RQ,\quad c_3 = N - RM,\quad c_4 = -RN.$$

It is obvious that $Q,\ M,\ N > 0$. Condition $(H2)$ implies

$$R < 0. \tag{2.4}$$

From (2.4), $c_1 > 0,\ c_4 > 0$. Moreover,

$$c_1c_2 - c_3 = QR^2 - Q^2R + QM - N,$$

and

$$QM - N = (a_{11}x_{10}\eta_1^2 + a_{11}x_{10}\eta_2^2 + a_{_{11}}^2 x_{_{10}}^2 \eta_1 + a_{_{11}}^2 x_{_{10}}^2 \eta_2 + \eta_1^2\eta_2 + \eta_2^2\eta_1 \\ + \alpha_1 a_1 x_{10}\eta_1 + 2a_{11}x_{10}\eta_1\eta_2 + a_{11}a_1\alpha_1 x_{10}^2) > 0. \tag{2.5}$$

Hence, $c_1c_2 - c_3 > 0$ from (2.4) and (2.5). Similarly, we have

$$\begin{aligned} c_1c_2c_3 - c_1^2c_4 - c_3^2 &= (c_1c_2 - c_3)c_3 - c_1^2c_4 \\ &= (N - QM)R^3 + Q(QM - N)R^2 \\ &\quad + M(N - QM)R + N(QM - N) \\ &> 0. \end{aligned} \tag{2.6}$$

According to Lemma 2.1, the boundary equilibrium $P_1(x_{10}, 0, u_{10}, u_{20})$ is locally stable. □

Similarly, we obtain the following Theorems.

Theorem 2.2. *Assume that*

$$(H3) \qquad \frac{b_1\eta_1}{b_2\eta_2} < \frac{\alpha_1\beta_2 - a_{12}\eta_1}{a_{22}\eta_2 + a_2\alpha_2}$$

holds. Then the boundary equilibrium $P_2(0, x_{20}, u_{30}, u_{40})$ *is locally stable.*

Theorem 2.3. *Assume that hypotheses* $(H1)$ *holds and* $\Delta_3 > 0$ *in Lemma 2.1, then the unique positive equilibrium* $P(x_1^*, x_2^*, u_1^*, u_2^*)$ *of system* (1.1) *is locally stable.*

Proof.

The Jacobian matrix of system (1.1) at $P(x_1^*, x_2^*, u_1^*, u_2^*)$ is

$$J_3(x_1^*, x_2^*, u_1^*, u_2^*) = \begin{pmatrix} -a_{11}x_{_1}^* & a_{12}x_{_1}^* & -\alpha_1 x_1^* & 0 \\ a_{21}x_2^* & -a_{22}x_2^* & 0 & -\alpha_2 x_2^* \\ a_1 & \beta_2 & -\eta_1 & 0 \\ \beta_1 & a_2 & 0 & -\eta_2 \end{pmatrix}. \tag{2.7}$$

The corresponding characteristic equation of matrix (2.7) is

$$\lambda^4 + c_1''\lambda^3 + c_2''\lambda^2 + c_3''\lambda + c_4'' = 0,$$

where

$$\begin{aligned} c''_1 =& a_{11}x_1^* + a_{22}x_2^* + \eta_1 + \eta_2, \\ c''_2 =& (a_{11}a_{22} - a_{12}a_{21})x_1^*x_2^* + (a_{11}\eta_1 + a_{11}\eta_2 + a_1\alpha_1)x_1^* \\ &+ (a_{22}\eta_1 a_{22}\eta_2 + a_2\alpha_2)x_2^* + \eta_1\eta_2, \\ c''_3 =& (a_{11}a_{22} - a_{12}a_{21})(\eta_1 + \eta_2)x_1^*x_2^* + (a_1\alpha_1 a_{22} + a_2\alpha_2 a_{11} + \beta_1\alpha_2 a_{12} \\ &+ \beta_2\alpha_1 a_{21})x_1^*x_2^* + (\eta_1\eta_2 a_{11} + a_1\alpha_1\eta_2)x_1^* + (\eta_1\eta_2 a_{22} + a_2\alpha_2\eta_1)x_2^*, \\ c''_4 =& (a_{11}a_{22} - a_{12}a_{21})\eta_1\eta_2 x_1^*x_2^* + (a_{12}\eta_1 - \alpha_1\beta_2)\alpha_2\beta_1 x_1^*x_2^* \end{aligned}$$

$$+ (\beta_2 a_{21} + a_1 a_{22})\alpha_1\eta_2 x_1^* x_2^* + a_1 a_2 \alpha_1 \alpha_2 x_1^* x_2^* + a_{11} a_2 \alpha_2 \eta_1 x_1^* x_2^*.$$

By virtue of condition $(H1)$,

$$a_{11}a_{22} - a_{12}a_{21} > 0, \quad a_{12}\eta_1 - \beta_2\alpha_1 \geq 0, \quad a_{21}\eta_2 - \beta_1\alpha_2 \geq 0.$$

Consequently, $c_1'' > 0$, $c_4'' > 0$. By calculation, we have

$$\begin{aligned} c''_1 c''_2 - c''_3 =&(a_{11}a_{22} - a_{12}a_{21})(a_{11}x_1^{*2}x_2^* + a_{22}x_1^* x_2^{*2} + x_1^* x_2^* \eta_1 + x_1^* x_2^* \eta_2) \\ &+ (a_{12}\eta_1 - \beta_2\alpha_1)a_{21}x_1^* x_2^* + (a_{21}\eta_2 - \beta_1\alpha_2)a_{12}x_1^* x_2^* \\ &+ a_{11}a_{22}(\eta_1 + \eta_2)x_1^* x_2^* + [a_{11}(\eta_1 + \eta_2)^2 + a_1\alpha_1\eta_1]x_1^* \\ &+ [a_{22}(\eta_1 + \eta_2)^2 + a_2\alpha_2\eta_2]x_2^* + (\eta_1 + \eta_2)\eta_1\eta_2 \\ >&0. \end{aligned}$$

It follows from Lemma 2.1 that positive equilibrium $P(x_1^*, x_2^*, u_1^*, u_2^*)$ is locally stable. This completes the proof of Theorem 2.3. □

Remark 2.2. We conjecture that condition $(H1)$ can ensure $\Delta_3 > 0$ in Lemma 2.1. However, we do not give a certain answer because its tedious calculation.

3. Global Stability

In the section, we first discuss the global asymptotic stability of the nonzero boundary equilibria of (1.1). We will use the idea of [6,7] to prove the following theorems. Let

$$\begin{aligned} A &= \min\{\frac{2\eta_1}{\beta_2}, \quad \frac{2a_{11}a_1\alpha_2}{a_{12}a_1\alpha_2 + a_{21}a_2\alpha_1 + \beta_1\alpha_1\alpha_2}\}, \\ B &= \max\{\frac{\beta_1}{2\eta_2}, \quad \frac{a_{12}a_1\alpha_2 + a_{21}a_2\alpha_1 + \beta_2\alpha_1\alpha_2}{2a_{22}a_2\alpha_1}\}. \end{aligned}$$

Suppose

$$(H4) \qquad B < A.$$

Theorem 3.1. *Assume that* $(H2)$ *and* $(H4)$ *hold, then the boundary equilibrium* $P_1(x_{10}, 0, u_{10}, u_{20})$ *of system* (1.1) *is globally asymptotically stable.*

Proof. $P_1(x_{10}, 0, u_{10}, u_{20})$ satisfies

$$b_1 - a_{11}x_{10} - \alpha_1 u_{10} = 0, \quad -\eta_1 u_{10} + a_1 x_{10} = 0, \quad -\eta_2 u_{20} + \beta_1 x_{10} = 0. \tag{3.1}$$

Define a Lyapunov function:

$$V_1(t) = \omega_1(x_1 - x_{10} - x_{10}\ln\frac{x_1}{x_{10}}) + \omega_2 x_2 + \omega_3(u_1 - u_{10})^2 + \omega_4(u_2 - u_{20})^2, \tag{3.2}$$

where $\omega_i (i = 1, 2, 3, 4)$ is positive constant which will be determined. It is easy to see that $V_1 \geq 0$ when $x_i, u_i (i = 1, 2) \geq 0$. Calculating the derivative of $V_1(t)$ along the positive solution of system (1.1). With the help of (3.1),

$$V_1{}'(t) = \omega_1(x_1 - x_{10})[-a_{11}(x_1 - x_{10}) + a_{12}x_2 - \alpha_1(u_1 - u_{10})]$$

$$+\omega_2 x_2[b_2 + a_{21}x_1(t) - a_{22}x_2(t) - \alpha_2 u_2(t)]$$
$$+ 2\omega_3(u_1 - u_{10})[-\eta_1(u_1 - u_{10}) + a_1(x_1 - x_{10}) + \beta_2 x_2]$$
$$+ 2\omega_4(u_2 - u_{20})[-\eta_2(u_2 - u_{20}) + \beta_1(x_1 - x_{10}) + a_2 x_2].$$

Since $ab \leq \frac{\theta}{2}a^2 + \frac{1}{2\theta}b^2$ with some $\theta > 0$, we obtain

$$\begin{aligned} V_1{}'(t) =& - a_{11}\omega_1(x_1 - x_{10})^2 - a_{22}\omega_2 x_2^2 - 2\eta_1\omega_3(u_1 - u_{10})^2 - 2\eta_2\omega_4(u_2 - u_{20})^2 \\ &+ 2\beta_2 x_2(u_1 - u_{10}) + (a_{12}\omega_1 + a_{21}\omega_2)(x_1 - x_{10})x_2 \\ &+ (b_2 + a_{21}x_{10} - \alpha_2 u_{20})\omega_2 x_2 + 2\beta_1(x_1 - x_{10})(u_2 - u_{20}) \\ \leq& - [a_{11}\omega_1 - \frac{\theta}{2}(a_{12}\omega_1 + a_{21}\omega_2) - 2\beta_1 \cdot \frac{\theta}{2}](x_1 - x_{10})^2 \\ &- [a_{22}\omega_2 - \frac{1}{2\theta}(a_{12}\omega_1 + a_{21}\omega_2) - 2\beta_2 \cdot \frac{1}{2\theta}]x_2{}^2 \\ &- [2\omega_3\eta_1 - 2\beta_2 \cdot \frac{\theta}{2}](u_1 - u_{10})^2 - [2\omega_4\eta_2 - 2\beta_1 \cdot \frac{1}{2\theta}](u_2 - u_{20})^2 \\ &+ (b_2 + a_{21}x_{10} - \alpha_2 u_{20})\omega_2 x_2. \end{aligned}$$

Taking

$$\omega_1 = \frac{2a_1}{\alpha_1},\ \omega_2 = \frac{2a_2}{\alpha_2},\ \omega_3 = 1,\ \omega_4 = 1$$

and $\theta = \frac{A+B}{2}$, it follows from condition $(H4)$ that

$$\begin{aligned} &a_{11}\omega_1 - \frac{\theta}{2}(a_{12}\omega_1 + a_{21}\omega_2) - \beta_1\theta > 0, \\ &a_{22}\omega_2 - \frac{1}{2\theta}(a_{12}\omega_1 + a_{21}\omega_2) - \frac{\beta_2}{\theta} > 0, \\ &2\omega_3\eta_1 - \beta_2\theta > 0, \qquad 2\omega_4\eta_2 - \frac{\beta_1}{\theta} > 0. \end{aligned}$$

Condition $(H2)$ is equivalent to $b_2 + a_{21}x_{10} - \alpha_2 u_{20} < 0$. We have

$$\frac{dV_1}{dt} < 0, \quad \text{for all } \ x_1 > 0, x_2 > 0, u_1 > 0, u_2 > 0,$$

except the boundary equilibrium $P_1(x_{10}, 0, u_{10}, u_{20})$, and $\frac{dV_1}{dt}(P_1) = 0$. So the function $V_1(t)$ satisfies Lyapunov's asymptotic stability theorem. The boundary equilibrium $P_1(x_{10}, 0, u_{10}, u_{20})$ is globally asymptotically stable. □

Theorem 3.2. *Assume that* $(H3)$ *and* $(H4)$ *hold, then the boundary equilibrium* $P_2(0, x_{20}, u_{30}, u_{40})$ *of system* (1.1) *is globally asymptotically stable.*

Proof. Define a new Lyapunov function

$$V_2(t) = \omega_1 x_1 + \omega_2(x_2 - x_{20} - x_{20}\ln\frac{x_2}{x_{20}}) + \omega_3(u_1 - u_{30})^2 + \omega_4(u_2 - u_{40})^2.$$

The proof is similar to ones of Theorem 3.1, we omit it. □

Now, we discuss the global stability of the positive equilibrium $P(x_1^*, x_2^*, u_1^*, u_2^*)$ of system (1.1) and stochastic model (1.2). Obviously, system (1.2) reduces to system (1.1) as $\sigma_i = 0 (i = 1, 2)$. We investigate them together. Consider, for a start, the existence of global positive solution of system (1.2) with any positive initial value.

Lemma 3.1. *Assume that* $a_{11} > \frac{1}{2}\sigma_1^2$, $a_{22} > \frac{1}{2}\sigma_2^2$ *hold, then there is a unique solution* $(x_1(t), x_2(t), u_1(t), u_2(t))$ *of system* (1.2) *on* $t \geq 0$ *for any initial value* $(x_1(0), x_2(0), u_1(0), u_2(0)) \in \mathbb{R}_+^4$*, and the solution will remain in* $\mathbb{R}_+^4$ *with probability 1, namely,* $(x_1(t), x_2(t), u_1(t), u_2(t)) \in \mathbb{R}_+^4$ *for all* $t \geq 0$ *almost surely(a.s.), where* $\mathbb{R}_+^4 = \{(x_1, x_2, u_1, u_2) \in \mathbb{R}^4 | x_1 > 0, x_2 > 0, u_1 > 0, u_2 > 0\}$.

Proof. The proof of this Lemma is similar to the Lemma 1 in [12], we omit it. □

Let

$$\tilde{A} = \min\{\frac{2\eta_1}{\beta_2}, \quad \frac{2a_{11}a_1\alpha_2 - a_1\alpha_2\sigma_1^2 x_1^*}{a_{12}a_1\alpha_2 + a_{21}a_2\alpha_1 + \beta_1\alpha_1\alpha_2}\},$$
$$\tilde{B} = \max\{\frac{\beta_1}{2\eta_2}, \quad \frac{a_{12}a_1\alpha_2 + a_{21}a_2\alpha_1 + \beta_2\alpha_1\alpha_2}{2a_{22}a_2\alpha_1 - a_2\alpha_1\sigma_2^2 x_2^*}\}.$$

Suppose

$$(\tilde{H4}) \qquad \tilde{B} < \tilde{A}.$$

Theorem 3.3. *Assume that* $(H1), (\tilde{H4})$ *and* $a_{11} > \frac{1}{2}\sigma_1^2$, $a_{22} > \frac{1}{2}\sigma_2^2$ *hold, then the unique positive equilibrium* $P(x_1^*, x_2^*, u_1^*, u_2^*)$ *of system* (1.2) *is globally asymptotically stable with probability 1.*

Proof. Define a Lyapunov function

$$\begin{aligned} V_3(t) =& \omega_1(x_1 - x_1{}^* - x_1{}^* \ln \frac{x_1}{x_1{}^*}) + \omega_2(x_2 - x_2{}^* - x_2{}^* \ln \frac{x_2}{x_2{}^*}) \\ &+ \omega_3(u_1 - u_1{}^*)^2 + \omega_4(u_2 - u_2{}^*)^2 \end{aligned}$$

with $\omega_1 = \frac{2a_1}{\alpha_1}, \omega_2 = \frac{2a_2}{\alpha_2}, \omega_3 = 1, \omega_4 = 1$. Obviously, $V_3(t) \geq 0$ provided with $x_i, u_i (i = 1, 2) \geq 0$. Calculating the derivative of $V(t)$ along the positive solution of system (1.2). Note that

$$\begin{aligned} &b_1 = a_{11}x_1{}^* - a_{12}x_2{}^* + \alpha_1 u_1{}^*, \quad b_2 = -a_{21}x_1{}^* + a_{22}x_2{}^* + \alpha_2 u_2{}^*, \\ &- \eta_1 u_1{}^* + a_1 x_1{}^* + \beta_2 x_2{}^* = 0, \quad -\eta_2 u_2{}^* + \beta_1 x_1{}^* + a_2 x_2{}^* = 0 \end{aligned}$$

and $ab \leq \frac{\theta}{2}a^2 + \frac{1}{2\theta}b^2$ with some $\theta > 0$. By using Itô's formula, we have

$$dV_3(t) = LV_3(t)dt + \omega_1\sigma_1(x_1 - x_1^*)^2 dB_1(t) + \omega_2\sigma_2(x_2 - x_2^*)^2 dB_2(t)$$

with

$$\begin{aligned} LV_3 =& -\omega_1(a_{11} - \frac{1}{2}\sigma_1^2 x_1{}^*)(x_1 - x_1{}^*)^2 - \omega_2(a_{22} - \frac{1}{2}\sigma_2^2 x_2{}^*)(x_2 - x_2{}^*)^2 \\ &- 2\eta_1\omega_3(u_1 - u_1{}^*)^2 - 2\eta_2\omega_4(u_2 - u_2{}^*)^2 + (a_{12}\omega_1 + a_{21}\omega_2)(x_1 - x_1{}^*)(x_2 - x_2^*) \\ &+ 2\beta_2\omega_3(x_2 - x_2^*)(u_1 - u_1{}^*) + 2\beta_1\omega_4(x_1 - x_1{}^*)(u_2 - u_2{}^*) \\ \leq& -[\omega_1(a_{11} - \frac{1}{2}\sigma_1^2 x_1{}^*) - \frac{\tilde{\theta}}{2}(a_{12}\omega_1 + a_{21}\omega_2) - 2\beta_1\frac{\tilde{\theta}}{2}](x_1 - x_1{}^*)^2 \\ &- [\omega_2(a_{22} - \frac{1}{2}\sigma_2^2 x_2{}^*) - \frac{1}{2\tilde{\theta}}(a_{12}\omega_1 + a_{21}\omega_2) - 2\beta_2\frac{1}{2\hat{\theta}}](x_2 - x_2{}^*)^2 \end{aligned}$$

$$-[2\omega_3\eta_1 - 2\beta_2\frac{\tilde{\theta}}{2}](u_1 - u_1{}^*)^2 - [2\omega_4\eta_2 - 2\beta_1\frac{1}{2\tilde{\theta}}](u_2 - u_2{}^*)^2.$$

Take $\theta = \frac{\tilde{A}+\tilde{B}}{2}$. From $(H1)$, $(\tilde{H}4)$, we obtain that

$$LV_3 < 0, \quad \text{for all} \;\; x_1 > 0, x_2 > 0, u_1 > 0, u_2 > 0$$

except the positive equilibrium $P(x_1^*, x_2^*, u_1^*, u_2^*)$. By the stability theory of stochastic differential equations([14]), the positive equilibrium $P(x_1^*, x_2^*, u_1^*, u_2^*)$ is globally asymptotically stable with probability 1, which completes the proof. □

4. Numerical Simulations

Here, we only give some numerical simulations about the globally asymptotically stable of positive equilibrium $P(x_1^*, x_2^*, u_1^*, u_2^*)$. The cases are similar for the boundary equilibria.

Example 4.1. Consider system (1.2) with $\sigma_1 = 0, \sigma_2 = 0$ and $b_1 = 1, b_2 = 1, \alpha_1 = 0.5, \alpha_2 = 1.5, a_{11} = 2, a_{12} = 0.5, a_{21} = 2, a_{22} = 2, \eta_1 = 3, \eta_2 = 0.5, \beta_1 = 0.5, \beta_2 = 2, a_1 = 0.5, a_2 = 1$. By calculation, we have $\Delta = 15.5 > 0$, $\tilde{A} = 1.7143 > \tilde{B} = 1.4375$. Condition $(H1)$ and $(\tilde{H}4)$ hold. By Theorem 3.3, the positive equilibrium $P(0.50, 0.25, 0.25, 1.00)$ is globally stable, see Fig.1(a).

However, $(\tilde{H}4)$ is not necessary condition to ensure global stability of the positive equilibrium point even if $\sigma_1 = \sigma_2 = 0$.

Example 4.2. Consider system (1.2) with $\sigma_1 = 0, \sigma_2 = 0$ and $b_1 = 1, b_2 = 1.5, \eta_1 = 2, \eta_2 = 3, \beta_1 = 1, \beta_2 = 2, \alpha_1 = 0.5, \alpha_2 = 1, a_{11} = 2, a_{12} = 3, a_{21} = 0.75, a_{22} = 2, a_1 = 1, a_2 = 1$. By calculation, we have $\Delta = 29 > 0$, $\tilde{A} = 1.0323 < \tilde{B} = 2.1875$. Here, $(\tilde{H}4)$ does not hold. Numerical simulation illustrates that the positive equilibrium $P(x_1^*, x_2^*, u_1^*, u_2^*)$ is still globally stable, see Fig.1(b).

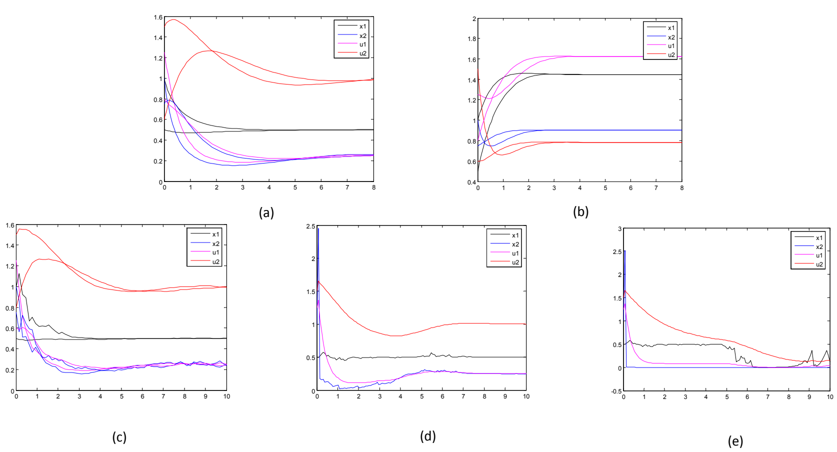

Figure 1. Numerical simulation of the solutions of Example 4.1-4.4.

Example 4.3. Consider stochastic system (1.2) corresponding to the Example 4.1. If we chose $\sigma_1 = 0.8, \sigma_2 = 0.9$, then $a_{11} > \frac{1}{2}\sigma_1^2,\ a_{22} > \frac{1}{2}\sigma_2^2$. By calculation, we have $\tilde{A} = 1.5771 > \tilde{B} = 1.4687$. By virtue of Theorem 3.3, the positive equilibrium $P(0.50, 0.25, 0.25, 1.00)$ of the stochastic system is globally asymptotically stable. The numerical simulation can be seen in Fig.1(c).

The following example shows that the conditions of Theorem 3.3 are not necessary. On the other hand, the property of positive equilibrium can be changed if noise is large.

Example 4.4. Consider stochastic perturbation of the system (1.2) corresponding to the Example 4.1. Choose $\sigma_1 = 5.16, \sigma_2 = 3.6$ in Example 4.1, then, $a_{11} < \frac{1}{2}\sigma_1^2,\ a_{22} < \frac{1}{2}\sigma_2^2,\ \ \tilde{A} = -3.9912 < \tilde{B} = 7.5658$. The solutions of (1.2) still tends to the positive equilibrium $P(0.50, 0.25, 0.25, 1.00)$, see Fig.1(d). If we change noise a little, such as $\sigma_1 = 5.16, \sigma_2 = 3.7$, However, x_2 tends to 0, which implies that the specie will die out, see Fig.1(e).

5. Conclusions

In this paper, we consider a class of Lotka-Volterra cooperation system with two feedback control variables which have been affected by all species and a corresponding stochastic perturbation system. The stabilization criteria are derived for local and global asymptotic stability of these systems. Finally, numerical simulation support these conclusions. The study shows that these systems can keep globally stable by adjusting the coefficients of the feedback variables and stochastic perturbation parameters.

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