Journal of Nonlinear Modeling and Analysis
Volume 2, Number 1, March 2020, 125–130
http://jnma.ca; http://jnma.ijournal.cn
DOI:10.12150/jnma.2020.125

# A note on the expansion of the first order Melnikov function near a class of 3-polycycles

Yanqin Xiong[1] and Maoan Han[2,3,†]

**Abstract** This paper comments that there exist some mistakes in the asymptotic expansion of the first order Menikov function near a 3-polycycle given by Theorem 3.1 of [2]. We present a correction to the theorem, and then use it to show that only one limit cycle can be found near a 3-polycycle for a class of quadratic systems.

**Keywords** Limit cycle, quadratic polynomial system, 3-polycycle.

**MSC(2010)** 34C05, 34C29, 34C25.

Consider a system of the form

$$\dot{x} = F(x)y + \epsilon p(x, y), \quad \dot{y} = G(x) + R(x)y^2 + \epsilon q(x, y) \tag{1}$$

with

$$p(x, y) = \sum_{j=0}^{n} \hat{p}_j(x)y^j, \quad q(x, y) = \sum_{j=0}^{n} \hat{q}_j(x)y^j$$

where $\epsilon > 0$ is a small parameter, $F(x)$, $G(x)$, $R(x)$, $\hat{p}_j(x)$ and $\hat{q}_j(x)$ are $C^\infty$ functions in their variable $x$.

Regarding system (1), we make two assumptions below as in [2]:

**(A1)** $F(0) = 0$, $F(x) = xF_1(x)$, $F_1(0) > 0$, $G(0) > 0$, $R(0) < 0$;

**(A2)** System $(1)|_{\epsilon=0}$ has a 3-polycycle $L_0$ passing through the saddle points $(0, \pm y_0)$ and $(x_1, 0)$, where $y_0 > 0$, $x_1 > 0$. Please see Figure 1.

Then, by Lemma 2.1 of [2], one can see that system $(1)|_{\epsilon=0}$ has an integrating factor of the form

$$\mu(x) = x^\alpha \mu_0(x), \ \alpha = -\frac{2R(0)}{F_1(0)} - 1,$$

† the corresponding author.
Email address: yqxiong@nuist.edu.cn(Y. Xiong), mahan@shnu.edu.cn(M. Han)
[1] School of Mathematics and Statistics, Nanjing University of Information Science and Technology, Nanjing, 210044, China
[2] Department of Mathematics, Zhejiang Normal University, Jinhua, Zhejiang, 321004, China
[3] Department of Mathematics, Shanghai Normal University, Shanghai, 200234, China
* The authors were supported by National Natural Science Foundations of China (11771296,11931016,11701289) and the Natural Science Foundation of Jiangsu Province(BK20170936).

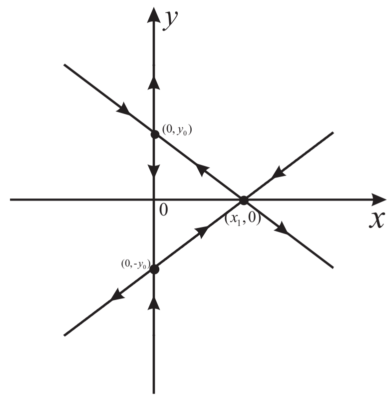


**Figure 1.** A 3-polycycle.

where $\mu_0(x) \in C^\infty$, $\mu_0(0) = \frac{1}{F_1(0)} > 0$. Obviously, under the above assumptions, system $(1)|_{\epsilon=0}$ has a family of periodic orbits inside $L_0$ given by $L_h$, $0 < -h < h_0$. Then, correspondingly we have the first order Melnikov function for system (1)

$$M(h) = \oint_{L_h} \mu(x)q(x,y)dx - \mu(x)q(x,y)dy, \quad 0 < -h < h_0. \tag{2}$$

Theorem 3.1 of [2] says that the above function has an asymptotic expansion of the form for $0 < -h \ll 1$

$$M(h) = \sum_{i\geq 0, \beta_i \notin \mathbb{Z}^+} B_i|h|^{\beta_i} + \Big(\sum_{i\geq 0, \beta_i \in \mathbb{Z}^+} B_i|h|^{\beta_i} + \sum_{i\geq 0} C_{2i+1}|h|^{i+1}\Big) \ln|h| + \sum_{i\geq 0} b_i h^i. \tag{3}$$

However, we recently found that the term $C_{2i+1}|h|^{i+1}$ in (3) should be corrected as $C_{2i+1}h^{i+1}$. The reason is that the authors used $|h|^{i+1}$ for $h^{i+1}$ in (3.8) of [2] by mistake. In fact, according to Theorem 3.2.9 of [1], the formula in (3.8) in page 374 of [2] should be corrected as

$$I_2(h) = \sum_{i\geq 0} C_{2i+1}h^{i+1} \ln|h| + N_1(h).$$

Therefore, about the expansion of (2) near $h = 0$, one has

**Theorem 1.** *Under* (A1) *and* (A2) *with* $\alpha \in (-1, 0) \cup (0, +\infty)$, *we have the asymptotic expansion below*

$$M(h) = \sum_{i\geq 0, \beta_i \notin \mathbb{Z}^+} B_i|h|^{\beta_i} + \Big(\sum_{i\geq 0, \beta_i \in \mathbb{Z}^+} B_i|h|^{\beta_i} + \sum_{i\geq 0} C_{2i+1}h^{i+1}\Big) \ln|h| + \sum_{i\geq 0} b_i h^i,$$

*where* $0 < -h \ll 1$, $\beta_i = \frac{\alpha+i}{\alpha+1}$ *and* $B_i$, $C_i$ *are constants defined in* [2].

Then, Corollary 3.2 of [2] should be corrected as the following

**Corollary 1.** *Under* (A1) *and* (A2) *with* $\alpha \in (-1, 0) \cup (0, +\infty)$, *then for* $0 < -h \ll 1$

**(1)** *if* $\alpha \in (-1, 0)$, *then*

$$M(h) = B_0|h|^{\frac{\alpha}{\alpha+1}} + b_0 + (B_1 - C_1)|h| \ln|h| + b_1 h + C_3 h^2 \ln|h| + O(h^2);$$

**(2)** *if* $\alpha \in (0,1)$, *then*

$$M(h) = b_0+B_0|h|^{\frac{\alpha}{\alpha+1}}+(B_1-C_1)|h|\ln|h|+b_1h+B_2|h|^{\frac{\alpha+2}{\alpha+1}}+C_3h^2\ln|h|+O(h^2);$$

**(3)** *if* $\alpha \in [1,+\infty) \setminus \{m-2,\ m\geq 3,\ m\in\mathbb{N}^+\}$, *then*

$$M(h) = b_0+B_0|h|^{\frac{\alpha}{\alpha+1}}+(B_1-C_1)|h|\ln|h|+b_1h+\sum_{i=2}^{[\alpha+2]} B_i|h|^{\frac{\alpha+i}{\alpha+1}}+C_3h^2\ln|h|+O(h^2);$$

**(4)** *if* $\alpha = m-2,\ m\geq 3,\ m\in\mathbb{N}^+$, *then*

$$M(h) = b_0+B_0|h|^{\frac{\alpha}{\alpha+1}}+(B_1-C_1)|h|\ln|h|+b_1h+\sum_{i=2}^{m-1} B_i|h|^{\frac{\alpha+i}{\alpha+1}}+(B_m+C_3)|h|^2\ln|h|+O(h^2).$$

Now, we present the formulas for $b_0, b_1, C_1, B_0, B_1$. Note that system (1) is a near-integrable system with the first integral of the form

$$H(x,y) = x^{\alpha+1}\Big[P_1(x)+\frac{1}{2}\mu_0(x)F_1(x)y^2\Big],$$

where

$$x^{\alpha+1}P_1(x) = -\int \mu(x)G(x)dx \in C^\infty.$$

Further, we suppose that the expansions of $\mu_0(x),\ F(x),\ G(x),\ P_1(x),\ p(x,y)$ and $q(x,y)$ at the origin are of the form respectively

$$\mu_0(x) = \sum_{j\geq 0} a_jx^j,\quad F(x) = \sum_{j\geq 1} f_jx^j,\quad G(x) = \sum_{j\geq 0} g_jx^j,\quad P_1(x) = \sum_{j\geq 0} p_jx^j,$$
$$p(x,y) = \sum_{j=0}^{n}\sum_{i\geq 0} p_{ij}x^iy^j,\quad q(x,y) = \sum_{j=0}^{n}\sum_{i\geq 0} q_{ij}x^iy^j.$$

Meanwhile, suppose that $H(x,y)$ at the point $(x_1,0)$ can be expanded as

$$H(x,y) = \frac{\lambda}{2}\big[y^2-(x-x_1)^2\big]+\sum_{i\geq 1} h_{i2}(x-x_1)^iy^2+\sum_{i\geq 3} h_{i0}(x-x_1)^i,\ \ \lambda = |\mu_1(x_1)\sqrt{F(x_1)G'(x_1)}|.$$

Clearly, $\mu(x)p(x,y)$ and $\mu(x)q(x,y)$ can be expressed as at the point $(x_1,0)$

$$\mu(x)p(x,y) = \sum_{j=0}^{n}\sum_{i\geq 0} a_{ij}(x-x_1)^iy^j,\quad \mu(x)q(x,y) = \sum_{j=0}^{n}\sum_{i\geq 0} b_{ij}(x-x_1)^iy^j.$$

Then, by [2], we have

$$C_1 = -\frac{1}{\lambda}\big[(\mu p)_x+(\mu q)_y\big](x_1,0),$$
$$C_3 = \frac{-1}{2\lambda^2}\Big[(-3a_{30}-b_{21}+a_{12}+3b_{03})-\frac{1}{\lambda}(2a_{20})+b_{11})(3h_{30}-h_{12})\Big],$$
$$b_0 = \begin{cases} \lim\limits_{h\to 0^-} \oint_{L_h} \mu q dx-\mu p dy+O(B_0) & \alpha\in(-1,0),\\ \lim\limits_{h\to 0^-} \oint_{L_h} \mu q dx-\mu p dy, & \alpha\in(0,+\infty),\end{cases}$$

$$b_1|_{B_0=B_1-C_1=0} = \lim_{h\to 0^-}\oint_{L_h}\big[(\mu p)_x+(\mu q)_y\big]dt,$$

$$B_0=\sum_{j=0}^{[\frac{n}{2}]}\frac{2^{j+\frac{3}{2}}}{2j+1}|p_0|^{-\frac{\alpha_j+1}{\alpha+1}}\alpha a_0 p_{0,2j}B_{0j},$$

$$B_1=\sum_{j=0}^{[\frac{n}{2}]}\frac{2^{j+\frac{3}{2}}}{2j+1}|p_0|^{-\frac{\alpha_j}{\alpha+1}}\Big[\frac{\alpha_j+2}{\alpha+1}|p_0|^{-\frac{\alpha+3}{\alpha+1}}p_1-\frac{1}{2}(2j+1)(a_1f_1+a_0f_2)|p_0|^{-\frac{2}{\alpha+1}}\Big]\alpha a_0 p_{0,2j}B_{1j}$$
$$+\sum_{j=0}^{[\frac{n}{2}]}\frac{2^{j+\frac{3}{2}}}{2j+1}|p_0|^{-\frac{\alpha_j+2}{\alpha+1}}\Big[a_0q_{0,2j+1}(2j+1)+(\alpha+1)(a_0p_{1,2j}+a_1p_{0,2j})\Big]B_{1j}$$

where $B_{ij}$'s are constants satisfying $B_{0j}>0$ (resp. $<0$) for $\alpha\in(-1,0)$ (resp. $\alpha\in(0,+\infty)$), and $\alpha_j=\frac{\alpha-3}{2}-(\alpha+1)j$.

Further, the authors of [2] using (3) (i.e incorrect results) studied a quadratic system of the from

$$\dot{x}=axy+\epsilon\sum_{i+j=0}^{2}p_{ij}x^iy^j,\quad \dot{y}=b+cx+dx^2+ey^2+\epsilon\sum_{i+j=0}^{2}q_{ij}x^iy^j,\qquad (4)$$

where $\epsilon>0$ sufficiently small, $a,\ b,\ d>0,\ c,\ e<0$, and $\frac{b}{e}=-\frac{c^2(a-e)}{d(a-2e)^2}$. The unperturbed system $(4)|_{\epsilon=0}$ has a 3-polycycle. From Theorem 5.1 of [2], the authors [2] claimed to find at least 2 limit cycles near the 3-polycycle for system (4). In fact, by using the corrected result, one can only get one limit cycle. Here, we give the detailed proof.

One can easily see that system $(4)|_{\epsilon=0}$ is an integrable system having an integrating factor $\mu(x)=\frac{1}{a}x^\alpha$, and it has a first integral given by

$$H(x,y)=\frac{x^{\alpha+1}}{2}\Big[y^2-\frac{d}{a-e}\Big(x-\frac{b(a-2e)}{ce}\Big)^2\Big],\quad \alpha=-1-\frac{2e}{a}.$$

Further it has a center at $\big(\frac{b(a-2e)}{c(e-a)},0\big)$ and three elementary saddle points at $\big(0,\pm\sqrt{\frac{-b}{e}}\big)$ and $\big(\frac{b(a-2e)}{ce},0\big)$, respectively. There exists a family of periodic orbits given by

$$L_h:\quad H(x,y)=h,\quad h\in(\eta,0),$$

around the center and bounded by a 3-polycycle through the three saddle points. Here, $\eta=-\frac{1}{2}\frac{d}{a-e}\frac{a^2}{e^2}\big[\frac{b(a-2e)}{c(e-a)}\big]^{\frac{2(a-e)}{a}}<0$. Corresponding to system (4), we have the following first order Melnikov function

$$M(h)=\oint_{L_h}\frac{1}{a}x^\alpha\sum_{i+j=0}^{2}q_{ij}x^iy^j\,dx-\frac{1}{a}x^\alpha\sum_{i+j=0}^{2}p_{ij}x^iy^j\,dy.$$

By using integration by parts, it follows that

$$M(h)=\frac{1}{a}\oint_{L_h}x^\alpha\Big\{q_{01}+(\alpha+1)p_{10}+\big[q_{11}+(\alpha+2)p_{20}\big]x+\frac{\alpha p_{00}}{x}\Big\}ydx+\frac{\alpha}{3a}p_{02}\oint_{L_h}x^{\alpha-1}y^3dx$$
$$=A_0\oint_{L_h}x^{\alpha-1}ydx+A_1\oint_{L_h}x^\alpha ydx+A_2\oint_{L_h}x^{\alpha+1}ydx,\qquad (5)$$

since $y^2 = 2hx^{-\alpha-1} + \frac{d}{a-e}\big[x - \frac{b(a-2e)}{xe}\big]^2$ along the curve $L_h$. Here,

$$A_0 = \frac{\alpha}{a}p_{00} + \frac{b\alpha}{a(a-e)}p_{02},$$
$$A_1 = \frac{1}{a}q_{01} + \frac{\alpha+1}{a}p_{10} + \frac{c\alpha}{a(a-e)}p_{02},$$
$$A_2 = \frac{1}{a}q_{11} + \frac{\alpha+2}{a}p_{20} + \frac{d\alpha}{a(a-e)}p_{02}.$$

About these integrals in (5), one finds that

**Lemma 1.** *For $h \in (\eta, 0)$, we have*

$$\oint_{L_h} x^\alpha y dx = \frac{c(e-a)}{b(a-2e)} \oint_{L_h} x^{\alpha+1} y dx.$$

**Proof.** Note that the curve $L_h$ can be rewritten as $y = \pm\sqrt{2hx^{-\alpha-1} + \frac{d}{a-e}\big[x - \frac{b(a-2e)}{ce}\big]^2}$ for $x_1(h) \le x \le x_2(h)$, where $x_1(h)$ and $x_2(h)$ are the solutions of the equation $-\frac{d}{a-e}\frac{x^{\alpha+1}}{2}\big[x - \frac{b(a-2e)}{ce}\big]^2 = h$ satisfying $0 < x_1(h) < \frac{b(a-2e)}{c(e-a)} < x_2(h)$. Then,

$$\begin{aligned}
&\frac{c(e-a)}{b(a-2e)} \oint_{L_h} x^{\alpha+1} y dx - \oint_{L_h} x^\alpha y dx \\
=&\oint_{L_h} x^\alpha \Big[\frac{c(e-a)}{b(a-2e)}x - 1\Big] y dx \\
=&2\int_{x_1(h)}^{x_2(h)} x^\alpha \Big[\frac{c(e-a)}{b(a-2e)}x - 1\Big] \sqrt{2hx^{-\alpha-1} + \frac{d}{a-e}\big[x - \frac{b(a-2e)}{ce}\big]^2} dx \\
=&2\int_{x_1(h)}^{x_2(h)} x^{\frac{\alpha-1}{2}} \Big[\frac{c(e-a)}{b(a-2e)}x - 1\Big] \sqrt{\frac{d}{a-e}x^{\alpha+1}\big[x - \frac{b(a-2e)}{ce}\big]^2 + 2h} dx. \qquad (6)
\end{aligned}$$

Let $u = x^{\frac{\alpha+1}{2}}\big[x - \frac{b(a-2e)}{ce}\big]$. Then, $du = \frac{\alpha+1}{2}\frac{b(a-2e)}{ce}x^{\frac{\alpha-1}{2}}\big[\frac{c(e-a)}{b(a-2e)}x - 1\big]dx$, and the function $u$ is monotonic for $x \in (x_1(h), \frac{b(a-2e)}{c(e-a)})$ (resp. $x \in (\frac{b(a-2e)}{c(e-a)}, x_2(h))$). Then, by the above discussion, one can derive that from (6)

$$\begin{aligned}
&\frac{c(e-a)}{b(a-2e)} \oint_{L_h} x^{\alpha+1} y dx - \oint_{L_h} x^\alpha y dx \\
=&\frac{4}{\alpha+1}\frac{ce}{b(a-2e)}\Big[\int_{\left[\frac{2(e-a)h}{d}\right]^{\frac12}}^{\rho} \sqrt{\frac{d}{a-e}u^2 + 2h} dx + \int_{\rho}^{\left[\frac{2(e-a)h}{d}\right]^{\frac12}} \sqrt{\frac{d}{a-e}u^2 + 2h} dx\Big] \\
\equiv&0,
\end{aligned}$$

where $\rho = \frac{ab(a-2e)}{ec(e-a)}\big[\frac{b(a-2e)}{c(e-a)}\big]^{\frac{\alpha+1}{2}}$. This ends the proof. □

Thus, by Lemma 1, the function in (5) can be expressed as

$$M(h) = A_0 \oint_{L_h} x^{\alpha-1} y dx + \Big[A_1 + \frac{b(a-2e)}{c(e-a)}A_2\Big] \oint_{L_h} x^\alpha y dx. \qquad (7)$$

We note that $\alpha > -1$ and $\alpha \ne 0$. By Corollary 1, we should discuss the expansion of $M(h)$ for $\alpha \in (-1, 0)$ or $\alpha \in (0, +\infty)$, separately. Here, we only provide a proof for the case $\alpha \in (0, +\infty)$ since the proof for the case $\alpha \in (-1, 0)$ is similar. For $\alpha > 0$, from Corollary 1 again, the first order Melnikov function $M(h)$ in (7) near the 3-polycycle can be expanded as

$$M(h) = b_0 + B_0|h|^{\frac{\alpha}{\alpha+1}} + O(|h|\ln|h|), \quad 0 < -h \ll 1. \tag{8}$$

It is easy to see that

$$\begin{aligned} B_0 =& 2\sqrt{2}\Big(-\frac{b}{2e}\Big)^{\frac{1-\alpha}{2(1+\alpha)}} B_{00}\Big[\frac{\alpha}{a}p_{00} + \frac{b\alpha}{a(a-e)}p_{02}\Big] \\ =& 2\sqrt{2}\Big(-\frac{b}{2e}\Big)^{\frac{1-\alpha}{2(1+\alpha)}} B_{00}A_0. \end{aligned}$$

Then, we have from (8)

$$b_0 = \lim_{h\to 0^-} M(h) = \lim_{h\to 0^-} \oint_{L_h} \mu q dx - \mu p dy = \oint_{L_0} \mu q dx - \mu p dy,$$

which, implies that

$$\begin{aligned} b_0 =& -2A_0\sqrt{\frac{d}{a-e}}\int_0^{\frac{b(a-2e)}{ce}} x^{\alpha-1}\big[x - \frac{b(a-2e)}{ce}\big]dx - 2\big[A_1 + \frac{b(a-2e)}{c(e-a)}A_2\big] \\ & \times \sqrt{\frac{d}{a-e}}\int_0^{\frac{b(a-2e)}{ce}} x^{\alpha}\big[x - \frac{b(a-2e)}{ce}\big]dx \\ =& \frac{2A_0}{\alpha(\alpha+1)}\sqrt{\frac{d}{a-e}}\Big[\frac{b(a-2e)}{ce}\Big]^{\alpha+1} + \frac{2\big[A_1 + \frac{b(a-2e)}{c(e-a)}A_2\big]}{(\alpha+1)(\alpha+2)}\sqrt{\frac{d}{a-e}}\Big[\frac{b(a-2e)}{ce}\Big]^{\alpha+2}. \end{aligned}$$

Obviously, $B_0$ and $b_0$ can be taken as free parameters. We can choose them satisfying $0 < -b_0 \ll B_0 \ll 1$ such that the sign of the function $M(h)$ in (8) near $h = 0$ can be changed one time. This means that one can find only one limit cycle near the 3-polycycle. Further, as $b_0 = B_0 = 0$, we have

$$A_0 = 0, \quad A_1 + \frac{b(a-2e)}{c(e-a)}A_2 = 0.$$

Hence, when $b_0 = B_0 = 0$, we have $M(h) \equiv 0$ for $h \in (\eta, 0)$. Thus, one can find one limit cycle near the 3-polycycle by using the first order Melnikov function.

# References

[1] M. Han, *Bifurcation Theory of Limit Cycles*, Science Press, Beijing, 2013.

[2] Y. Wang and M. Han, *Limit cycle bifurcations by perturbing a class of integrable systems with a polycycle*, J. Math. Anal. Appl. 418 (2014) 357-386.