A note on the expansion of the first order 
Melnikov function near a class of 3-polycycles

Yanqin Xiong\textsuperscript{1} and Maoan Han\textsuperscript{2,3,}\textsuperscript{1}

Abstract This paper comments that there exist some mistakes in the asymptotic expansion of the first order Melnikov function near a 3-polycycle given by Theorem 3.1 of \cite{2}. We present a correction to the theorem, and then use it to show that only one limit cycle can be found near a 3-polycycle for a class of quadratic systems.

Keywords Limit cycle, quadratic polynomial system, 3-polycycle.

MSC(2010) 34C05, 34C29, 34C25.

Consider a system of the form
\begin{align*}
\dot{x} &= F(x)y + \epsilon p(x, y), \\
\dot{y} &= G(x) + R(x)y^2 + \epsilon q(x, y)
\end{align*}

with
\begin{align*}
p(x, y) &= \sum_{j=0}^{n} \hat{p}_j(x)y^j, \\
q(x, y) &= \sum_{j=0}^{n} \hat{q}_j(x)y^j
\end{align*}

where \(\epsilon > 0\) is a small parameter, \(F(x), G(x), R(x), \hat{p}_j(x)\) and \(\hat{q}_j(x)\) are \(C^\infty\) functions in their variable \(x\).

Regarding system (1), we make two assumptions below as in \cite{2}:

(A1) \(F(0) = 0, F(x) = x F_1(x), F_1(0) > 0, G(0) > 0, R(0) < 0;\)

(A2) System (1)|\(\epsilon = 0\) has a 3-polycycle \(L_0\) passing through the saddle points \((0, \pm y_0)\) and \((x_1, 0)\), where \(y_0 > 0, x_1 > 0\). Please see Figure 1.

Then, by Lemma 2.1 of \cite{2}, one can see that system (1)|\(\epsilon = 0\) has an integrating factor of the form
\[
\mu(x) = x^\alpha \mu_0(x), \quad \alpha = \frac{-2R(0)}{F_1(0)} - 1,
\]

\textsuperscript{1}the corresponding author.
Email address: yqxiong@nuist.edu.cn(Y. Xiong), mahan@shnu.edu.cn(M. Han)
\textsuperscript{1}School of Mathematics and Statistics, Nanjing University of Information Science and Technology, Nanjing, 210044, China
\textsuperscript{2}Department of Mathematics, Zhejiang Normal University, Jinhua, Zhejiang, 321004, China
\textsuperscript{3}Department of Mathematics, Shanghai Normal University, Shanghai, 200234, China
\textsuperscript{*}The authors were supported by National Natural Science Foundations of China (11771296, 11931016, 11701289) and the Natural Science Foundation of Jiangsu Province(BK20170936).
where \( \mu_0(x) \in C^\infty, \mu_0(0) = \frac{1}{F_1(0)} > 0 \). Obviously, under the above assumptions, system (1)|\(t=0\) has a family of periodic orbits inside \( L_0 \) given by \( L_h, \ 0 < -h < h_0 \). Then, correspondingly we have the first order Melnikov function for system (1)

\[
M(h) = \int_{L_h} \mu(x)q(x,y)dx - \int_{L_h} \mu(x)q(x,y)dy, \quad 0 < -h < h_0.
\]  

(2)

Theorem 3.1 of [2] says that the above function has an asymptotic expansion of the form for \( 0 < -h \ll 1 \)

\[
M(h) = \sum_{i \geq 0, \beta_i \in \mathbb{Z}^+} B_i |h|^{\beta_i} + \left( \sum_{i \geq 0, \beta_i \in \mathbb{Z}^+} B_i |h|^{\beta_i} + \sum_{i \geq 0} C_{2i+1} |h|^{i+1} \right) \ln|h| + \sum_{i \geq 0} b_i h^i.
\]  

(3)

However, we recently found that the term \( C_{2i+1} |h|^{i+1} \) in (3) should be corrected as \( C_{2i+1} |h|^{i+1} \). The reason is that the authors used \( |h|^{i+1} \) for \( h^{i+1} \) in (3.8) of [2] by mistake. In fact, according to Theorem 3.2.9 of [1], the formula in (3.8) in page 374 of [2] should be corrected as

\[
I_2(h) = \sum_{i \geq 0} C_{2i+1} |h|^{i+1} \ln|h| + N_1(h).
\]

Therefore, about the expansion of (2) near \( h = 0 \), one has

**Theorem 1.** Under (A1) and (A2) with \( \alpha \in (-1,0) \cup (0, +\infty) \), we have the asymptotic expansion below

\[
M(h) = \sum_{i \geq 0, \beta_i \in \mathbb{Z}^+} B_i |h|^{\beta_i} + \left( \sum_{i \geq 0, \beta_i \in \mathbb{Z}^+} B_i |h|^{\beta_i} + \sum_{i \geq 0} C_{2i+1} |h|^{i+1} \right) \ln|h| + \sum_{i \geq 0} b_i h^i,
\]

where \( 0 < -h \ll 1 \), \( \beta_i = \frac{\alpha+1}{4} \) and \( B_i, \ C_i \) are constants defined in [2].

Then, Corollary 3.2 of [2] should be corrected as the following

**Corollary 1.** Under (A1) and (A2) with \( \alpha \in (-1,0) \cup (0, +\infty) \), then for \( 0 < -h \ll 1 \)

1. if \( \alpha \in (-1,0) \), then

\[
M(h) = B_0 |h|^\frac{\alpha+1}{4} + b_0 + (B_1 - C_1)|h|\ln|h| + b_1 h + C_3 h^2 \ln|h| + O(h^2);
\]
(2) if $\alpha \in (0, 1)$, then

$$M(h) = b_0 + B_0|h|^\frac{\alpha+2}{\alpha+1} + (B_1 - C_1)|h|\ln |h| + b_1 h + B_2|h|^{\frac{\alpha+2}{\alpha+1}} + C_3 h^2 \ln |h| + O(h^2);$$

(3) if $\alpha \in [1, +\infty) \setminus \{m - 2, \ m \geq 3, \ m \in \mathbb{N}^+\}$, then

$$M(h) = b_0 + B_0|h|^\frac{\alpha+2}{\alpha+1} + (B_1 - C_1)|h|\ln |h| + b_1 h+ \sum_{i=2}^{[\alpha+2]} B_i|h|^{\frac{\alpha+2}{\alpha+1}} + C_3 h^2 \ln |h| + O(h^2);$$

(4) if $\alpha = m - 2, \ m \geq 3, \ m \in \mathbb{N}^+$, then

$$M(h) = b_0 + B_0|h|^\frac{\alpha+2}{\alpha+1} + (B_1 - C_1)|h|\ln |h| + b_1 h + \sum_{i=2}^{m-1} B_i|h|^{\frac{\alpha+2}{\alpha+1}} + (B_m + C_3)|h|^2 \ln |h| + O(h^2).$$

Now, we present the formulas for $b_0, b_1, C_1, B_0, B_1$. Note that system (1) is a near-integrable system with the first integral of the form

$$H(x, y) = x^{\alpha+1}\left(P_1(x) + \frac{1}{2} \mu_0(x)P_1(x)y^2\right),$$

where

$$x^{\alpha+1}P_1(x) = -\int \mu(x)G(x)dx \in C^\infty.$$ 

Further, we suppose that the expansions of $\mu_0(x), \ F(x), \ G(x), \ P_1(x), \ p(x, y)$ and $q(x, y)$ at the origin are of the form respectively

$$\mu_0(x) = \sum_{j \geq 0} a_j x^j, \ F(x) = \sum_{j \geq 1} f_j x^j, \ G(x) = \sum_{j \geq 0} g_j x^j, \ P_1(x) = \sum_{j \geq 0} p_j x^j,$$

$$p(x, y) = \sum_{j=0}^n \sum_{i \geq 0} p_{ij} x^i y^j, \ q(x, y) = \sum_{j=0}^n \sum_{i \geq 0} q_{ij} x^i y^j.$$

Meanwhile, suppose that $H(x, y)$ at the point $(x_1, 0)$ can be expanded as

$$H(x, y) = \frac{\lambda}{2} \left[y^2 - (x-x_1)^2\right] + \sum_{i \geq 1} h_{i2}(x-x_1)^i y^2 + \sum_{i \geq 3} h_{i0}(x-x_1)^i, \ \lambda = |\mu_1(x_1)|^{\frac{1}{2}} F(x_1)G'(x_1)|.$$

Clearly, $\mu(x)p(x, y)$ and $\mu(x)q(x, y)$ can be expressed as at the point $(x_1, 0)$

$$\mu(x)p(x, y) = \sum_{j=0}^n \sum_{i \geq 0} a_{ij} (x-x_1)^i y^j, \ \mu(x)q(x, y) = \sum_{j=0}^n \sum_{i \geq 0} b_{ij} (x-x_1)^i y^j.$$ 

Then, by [2], we have

$$C_1 = -\frac{1}{\lambda} \left[(\mu p)_x + (\mu q)_y\right](x_1, 0),$$

$$C_3 = -\frac{1}{2\lambda^2} \left[(-3a_{30} - b_{21} + a_{12} + 3b_{03}) - \frac{1}{\lambda} (2a_{20} + b_{11})(3h_{30} - h_{12})\right],$$

$$b_0 = \begin{cases} \lim_{h \to 0^-} \int_{L_h} \mu p dx - \mu p dy + O(B_0) \ \alpha \in (-1, 0), \\ \lim_{h \to 0^-} \int_{L_h} \mu p dx - \mu p dy, \ \alpha \in (0, +\infty), \end{cases}$$
and the detailed proof. In fact, by using the corrected result, one can only get one limit cycle. Here, we give

\[
B_0 = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \frac{2^{j+\frac{3}{2}}}{j+1} p_0 \left( \frac{\alpha_j + 1}{\alpha + 1} \right)^{\alpha_j + 1} \alpha_0 p_{0,2j} B_0j,
\]

\[
B_1 = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \frac{2^{j+\frac{3}{2}}}{j+1} p_0 \left( \frac{\alpha_j + 2}{\alpha + 1} \right)^{\alpha_j + 2} \frac{1}{2} \left( (a_1 f_1 + a_0 f_2) p_0 \right) \left( \frac{\alpha_j + 1}{\alpha + 1} \right)^{\alpha_j + 1} \alpha_0 p_{0,2j} B_0j
\]

\[
+ \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \frac{2^{j+\frac{3}{2}}}{j+1} p_0 \left( \frac{\alpha_j + 1}{\alpha + 1} \right)^{\alpha_j + 2} \left[ a_0 q_{0,2j+1} (2j + 1) + (\alpha + 1) (a_0 p_{1,2j} + a_1 p_{0,2j}) \right] B_1j
\]

where \( B_{ij} \)'s are constants satisfying \( B_{0j} > 0 \) (resp. \( B_{0j} < 0 \)) for \( \alpha \in (-1, 0) \) (resp. \( \alpha \in (0, +\infty) \)), and \( \alpha_j = \frac{\alpha - 3}{2} - (\alpha + 1)j \).

Further, the authors of [2] using (3) (i.e incorrect results) studied a quadratic system of the form

\[
\dot{x} = a x y + \epsilon \sum_{i+j=0}^{2} p_{ij} x^i y^j, \quad \dot{y} = b + cx + dx^2 + ey^2 + \epsilon \sum_{i+j=0}^{2} q_{ij} x^i y^j, \quad (4)
\]

where \( \epsilon > 0 \) sufficiently small, \( a, b, d > 0, c, e < 0, \) and \( \frac{b}{c} = -\frac{2(a-2e)}{a} \). The unperturbed system (4)_{c=0} has a 3-polycycle. From Theorem 5.1 of [2], the authors [2] claimed to find at least 2 limit cycles near the 3-polycycle for system (4). In fact, by using the corrected result, one can only get one limit cycle. Here, we give the detailed proof.

One can easily see that system (4)_{c=0} is an integrable system having an integrating factor \( \mu(x) = \frac{1}{a} x^\alpha \), and it has a first integral given by

\[
H(x, y) = \frac{x^{\alpha+1}}{2} \left[ y^2 - \frac{d}{a - \epsilon} (x - \frac{b(a - 2e)}{ce})^2 \right], \quad \alpha = -1 - \frac{2e}{a}.
\]

Further it has a center at \((\frac{b(a-2e)}{2(a-2e)}, 0)\) and three elementary saddle points at \((0, \pm \sqrt{\frac{b}{c}})\) and \((\frac{b(a-2e)}{ce}, 0)\), respectively. There exists a family of periodic orbits given by

\[
L_h : \quad H(x, y) = h, \quad h \in (\eta, 0),
\]

around the center and bounded by a 3-polycycle through the three saddle points. Here, \( \eta = -\frac{1}{2} \frac{d}{a - \epsilon} \frac{a^2}{c} \left[ \frac{b(a-2e)}{ce-a} \right]^{\frac{2a-\epsilon}{a}} < 0 \). Corresponding to system (4), we have the following first order Melnikov function

\[
M(h) = \frac{1}{a} \int_{L_h} x^\alpha \sum_{i+j=0}^{2} q_{ij} x^i y^j dx - \frac{1}{a} x^\alpha \sum_{i+j=0}^{2} p_{ij} x^i y^j dy.
\]

By using integration by parts, it follows that

\[
M(h) = \frac{1}{a} \int_{L_h} x^\alpha \left( \frac{\alpha_0 p_{01}}{x} + \frac{\alpha_0 p_{20}}{x} \right) y dx + \frac{\alpha_0 p_{02}}{3a} \int_{L_h} x^{\alpha-1} y^3 dx
\]

\[
= A_0 \int_{L_h} x^{\alpha-1} y dx + A_1 \int_{L_h} x^\alpha y dx + A_2 \int_{L_h} x^{\alpha+1} y dx,
\]

(5)
since \( y^2 = 2hx^{-\alpha-1} + \frac{d}{a-e} \left[ x - \frac{b(a-2e)}{ce} \right]^2 \) along the curve \( L_h \). Here,

\[
\begin{align*}
A_0 &= \frac{\alpha}{a} p_{00} + \frac{b\alpha}{a(a-e)} p_{02}, \\
A_1 &= \frac{1}{a} q_{01} + \frac{\alpha + 1}{a} p_{10} + \frac{c\alpha}{a(a-e)} p_{02}, \\
A_2 &= \frac{1}{a} q_{11} + \frac{\alpha + 2}{a} p_{20} + \frac{d\alpha}{a(a-e)} p_{02}.
\end{align*}
\]

About these integrals in (5), one finds that

**Lemma 1.** For \( h \in (\eta, 0) \), we have

\[
\int_{L_h} x^\alpha ydx = \frac{c(e-a)}{b(a-2e)} \int_{L_h} x^{\alpha+1} ydx.
\]

**Proof.** Note that the curve \( L_h \) can be rewritten as \( y = \pm \sqrt{2hx^{-\alpha-1} + \frac{d}{a-e} \left[ x - \frac{b(a-2e)}{ce} \right]^2} \) for \( x_1(h) \leq x \leq x_2(h) \), where \( x_1(h) \) and \( x_2(h) \) are the solutions of the equation

\[
- \frac{d}{a-e} \frac{x^{\alpha+1}}{2} \left[ x - \frac{b(a-2e)}{ce} \right]^2 = h \text{ satisfying } 0 < x_1(h) < \frac{b(a-2e)}{c(e-a)} < x_2(h).
\]

Then,

\[
\begin{align*}
&\int_{L_h} x^\alpha ydx - \int_{L_h} x^\alpha ydx \\
&= \int_{L_h} x^\alpha \left[ \frac{c(e-a)}{b(a-2e)} x - 1 \right] ydx \\
&= 2 \int_{x_1(h)}^{x_2(h)} x^\alpha \left[ \frac{c(e-a)}{b(a-2e)} x - 1 \right] \sqrt{2hx^{-\alpha-1} + \frac{d}{a-e} \left[ x - \frac{b(a-2e)}{ce} \right]^2} dx \\
&= 2 \int_{x_1(h)}^{x_2(h)} x^\alpha \left[ \frac{c(e-a)}{b(a-2e)} x - 1 \right] \sqrt{\frac{d}{a-e} x^{\alpha+1} \left[ x - \frac{b(a-2e)}{ce} \right]^2 + 2hdx}.
\end{align*}
\]

Let \( u = x^\alpha \left[ x - \frac{b(a-2e)}{ce} \right] \). Then, \( du = \frac{\alpha + 1}{b(a-2e)} x^\alpha \left[ \frac{c(e-a)}{b(a-2e)} x - 1 \right] dx \), and the function \( u \) is monotonic for \( x \in (x_1(h), \frac{b(a-2e)}{c(e-a)}) \) (resp. \( x \in (\frac{b(a-2e)}{c(e-a)}, x_2(h)) \)). Then, by the above discussion, one can derive that from (6)

\[
\begin{align*}
&\frac{c(e-a)}{b(a-2e)} \int_{L_h} x^{\alpha+1} ydx - \int_{L_h} x^\alpha ydx \\
&= \frac{4}{\alpha+1} \frac{ce}{b(a-2e)} \left[ \int_{\rho}^{\rho} \frac{d}{a-e} u^2 + 2hdx + \int_{\rho}^{\rho} \frac{z(e-a)h}{a-e} \right]^{\frac{1}{2}} \sqrt{\frac{d}{a-e} u^2 + 2hdx} \\
&= 0,
\end{align*}
\]

where \( \rho = \frac{a(b-2e) + b(a-2e)}{c(e-a)} \). This ends the proof. \( \blacksquare \)

Thus, by Lemma 1, the function in (5) can be expressed as

\[
M(h) = A_0 \int_{L_h} x^{\alpha-1} ydx + [A_1 + \frac{b(a-2e)}{c(e-a)} A_2] \int_{L_h} x^\alpha ydx.
\]

We note that \( \alpha > -1 \) and \( \alpha \neq 0 \). By Corollary 1, we should discuss the expansion of \( M(h) \) for \( \alpha \in (-1, 0) \) or \( \alpha \in (0, +\infty) \), separately. Here, we only provide a proof.
for the case $\alpha \in (0, +\infty)$ since the proof for the case $\alpha \in (-1, 0)$ is similar. For $\alpha > 0$, from Corollary 1 again, the first order Melnikov function $M(h)$ in (7) near the 3-polycycle can be expanded as

$$M(h) = b_0 + B_0|h|^{\frac{1}{\alpha + 2}} + O(|h| \ln |h|), \quad 0 < -h \ll 1. \quad (8)$$

It is easy to see that

$$B_0 = 2\sqrt{2} \left( -\frac{b}{2e} \right)^{\frac{1}{\alpha + 2}} B_{00} \left[ \frac{\alpha}{a} p_{00} + \frac{b_0}{a(a - e)} p_{02} \right]$$

$$= 2\sqrt{2} \left( -\frac{b}{2e} \right)^{\frac{1}{\alpha + 2}} B_{00} A_0.$$

Then, we have from (8)

$$b_0 = \lim_{h \to 0^-} M(h) = \lim_{h \to 0^-} \oint_{L_h} \mu q dx - \mu p dy = \oint_{L_0} \mu q dx - \mu p dy,$$

which, implies that

$$b_0 = -2A_0 \sqrt{\frac{d}{a - e}} \int_0^{b(a - 2e)/ce} x^\alpha \left[ x - \frac{b(a - 2e)}{ce} \right] dx - 2[A_1 + \frac{b(a - 2e)}{c(e - a)}] A_2$$

$$\times \sqrt{\frac{d}{a - e}} \int_0^{b(a - 2e)/ce} x^\alpha \left[ x - \frac{b(a - 2e)}{ce} \right] dx$$

$$= \frac{2A_0}{\alpha(\alpha + 1)} \sqrt{\frac{d}{a - e}} \left[ \frac{b(a - 2e)}{ce} \right]^{\alpha + 1} + 2 \left[ A_1 + \frac{b(a - 2e)}{c(e - a)} A_2 \right] \sqrt{\frac{d}{a - e}} \left[ \frac{b(a - 2e)}{ce} \right]^{\alpha + 2}.$$

Obviously, $B_0$ and $b_0$ can be taken as free parameters. We can choose them satisfying $0 < -b_0 \ll B_0 \ll 1$ such that the sign of the function $M(h)$ in (8) near $h = 0$ can be changed one time. This means that one can find only one limit cycle near the 3-polycycle. Further, as $b_0 = B_0 = 0$, we have

$$A_0 = 0, \quad A_1 + \frac{b(a - 2e)}{c(e - a)} A_2 = 0.$$

Hence, when $b_0 = B_0 = 0$, we have $M(h) \equiv 0$ for $h \in (\eta, 0)$. Thus, one can find one limit cycle near the 3-polycycle by using the first order Melnikov function.

**References**
