Some New Exact Solutions for Time Fractional Thin-film Equation

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Abstract In this paper, the invariant subspace method is utilized to obtain some new exact solutions for the time fractional thin-film equation. The fractional derivative in the considered equation is given in Riemann-Liouville and Caputo senses. Some new invariant subspaces have been obtained that are not reported in the literature before.

Keywords Time-fractional thin-film equation, Riemann-Liouville fractional derivative, Caputo fractional derivative, Invariant subspace method, New exact solutions.


1. Introduction

Exact solutions of nonlinear evolution equations play a very important role in the study of nonlinear physical phenomena. Many methods can be utilized for obtaining exact solutions of nonlinear evolution equations such as Backlund transformation method [6, 9], Lie group method [1, 3, 19, 24], the tanh method [2, 5, 22, 29], the exp(−φ(z))—expansion method [12, 13, 15, 17], the exp function approach [16] and the invariant subspace method (ISM) [7, 21].

The importance of the invariant subspace method comes from it is not only used for solving nonlinear evolution equations but also it can be used for solving fractional nonlinear evolution equations. Very recently, it is widely utilized for investigating exact solutions of fractional nonlinear evolution equations (see for example [4, 8, 10, 11, 14, 18, 20, 26–28, 30]).

In this paper, we use the ISM to investigate some new solutions of the time-fractional thin-film equation [27]

$$\frac{\partial^\alpha u}{\partial t^\alpha} = -u \left( \frac{\partial^4 u}{\partial x^4} \right) + \beta \left( \frac{\partial u}{\partial x} \right) \left( \frac{\partial^3 u}{\partial x^3} \right) + \gamma \left( \frac{\partial^2 u}{\partial x^2} \right)^2 , \quad t > 0 , \quad 0 < \alpha \leq 1 . \quad (1.1)$$

Equation (1.1) can be used as a model for thin film flow on a substrate [7]. Here, $u(x,t)$ denotes the height of the fluid. The invariant subspaces and some exact solutions of the Eq. (1.1) (when $\alpha = 1$) have been obtained in [7, 30]. Some exact solution of Eq. (1.1) have been obtained in [27] using the ISM. The main aim of

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this paper is to obtain some new invariant subspaces and some new exact solutions of Eq. (1.1). The rest of the paper is organized as follows: The basics and definitions of the ISM are introduced in Section 2. The new invariant subspaces and exact solutions of Eq. (1.1) are discussed in Section 3. Section 4 discuss the results and conclusion of the paper.

2. ISM: Time Fractional Partial Differential Equations (PDEs)

Consider a time fractional PDE
\[ \frac{\partial^\alpha u}{\partial t^\alpha} = F[u] = F(x, t, u, u_x, u_{xx}, \ldots), \quad 0 < \alpha \leq 1, \tag{2.1} \]
where $F[u]$ is a nonlinear differential operator of order $k$. The $n$-dimensional invariant subspace $W_n = \mathcal{L}\{f_1(x), f_2(x), \ldots, f_k(x)\}$ (where $n \leq 2k+1$) can be obtained from the solution of the following $n$th-order linear ordinary differential equation:
\[ L[y] = y^{(n)}(x) + a_{n-1}y^{(n-1)}(x) + \cdots + a_0 y(x) = 0, \tag{2.2} \]
where, the constant coefficients $a_{n-1}, a_{n-2}, \ldots, a_0$ can be obtained from the condition [13]
\[ L[F[u]]|_{L[u]=0} = 0. \]
The time fractional PDE (2.1) can be converted into a system of time fractional ordinary differential equations (ODEs) through the following theorem:

**Theorem 1** [13]. Let $W_n$ be the linear space spanned by $n$ linearly independent functions $\{f_i(x), i = 1, 2, \ldots, n\}$ and suppose that $W_n$ is invariant under the operator $F[u]$, which means that
\[ F\left[ \sum_{i=1}^n c_i f_i(x) \right] = \sum_{i=1}^n F_i(c_1, c_2, \ldots, c_n) f_i(x), \tag{2.3} \]
for whatever constants $c_1, c_2, \ldots, c_n$. Then, the fractional PDE (2.1) has the solution of the form
\[ u(x, t) = \sum_{i=1}^n c_i(t) f_i(x), \tag{2.4} \]
where the coefficients $c_1(t), c_2(t), \ldots, c_n(t)$ satisfy the following system of fractional ODEs
\[ \frac{d^\alpha c_i(t)}{dt^\alpha} = \psi_i(c_1(t), \ldots, c_n(t)), \quad i = 1, 2, \ldots, n. \tag{2.5} \]
Here, the fractional order derivative $\frac{\partial^\alpha u}{\partial t^\alpha}$ will be considered in the Riemann-Liouville sense in some cases. In some other cases the fractional derivative $\frac{\partial^\alpha u}{\partial t^\alpha}$ will be in the Caputo sense according to the availability of exact solutions of the obtained system of fractional ODEs after using the invariant subspace method. Basic definitions and properties of the Riemann-Liouville and Caputo fractional derivatives are given in Appendix A.
3. Invariant subspaces and exact solutions of the time fractional thin-film Eq. (1.1)

In this section, the invariant subspaces and exact solutions of the time fractional thin-film Eq. (1.1) will be investigated. Here, we investigate the invariant subspaces with dimensions \( n = \{ 4, 5, 6, 7, 8, 9 \} \).

3.1. Invariant subspaces when \( n = 4 \)

Applying the invariant subspace method to Eq. (1.1) taking into consideration the following auxiliary equation

\[ y^{(4)} + a_3 y^{(3)} + a_2 y'' + a_1 y' + a_0 y = 0, \]

we get the following cases:

**Case 1:** when \( \beta = 2, \gamma = -1 \). In this case, the invariant subspace is \( W_4 = L \{ \sin(a_3 x), \cos(a_3 x), e^{-(a_3 x)}, 1 \} \).

The exact solution of Eq. (1.1) can be written in the form

\[ u(x,t) = c_1(t) \sin(a_3 x) + C_2(t) \cos(a_3 x) + e^{-a_3 x} c_3(t) + C_4(t), \quad (3.1) \]

where \( a_3 \) is an arbitrary constant. Substitute Eq. (3.1) into Eq. (1.1) and compare the two sides of Eq. (1.1), to get

\[ \frac{d^\alpha C_1(t)}{dt^\alpha} = -a_3^4 C_1(t) C_4(t), \quad (3.2) \]
\[ \frac{d^\alpha C_2(t)}{dt^\alpha} = - a_3^4 C_2(t) C_4(t), \quad (3.3) \]
\[ \frac{d^\alpha C_3(t)}{dt^\alpha} = - a_3^4 C_3(t) C_4(t), \quad (3.4) \]
\[ \frac{d^\alpha C_4(t)}{dt^\alpha} = - 2 a_3^4 (C_1(t))^2 + C_2(t)^2. \quad (3.5) \]

Solving the system of equations (3.2)- (3.5) assuming that the fractional derivatives are in the Riemann–Liouville sense, we get

\[ C_4(t) = - \frac{1}{a_3^4} \frac{\Gamma(1 - \alpha)}{\Gamma(1 - 2\alpha)} t^{-\alpha}, \quad \alpha \neq \frac{1}{2}, \]
\[ C_3(t) = l_3 t^{-\alpha}, \]
\[ C_2(t) = \pm \sqrt{\frac{1}{2a_3^4} \left( \frac{\Gamma(1 - \alpha)}{\Gamma(1 - 2\alpha)} \right)^2 - l_1^2 t^{-\alpha}}, \]
\[ C_1(t) = l_1 t^{-\alpha}, \]

where \( l_1 \) and \( l_3 \) are arbitrary constants.

Hence, we obtain the following exact solution of the time fractional thin-film equation (1.1)
\[ u(x, t) = (l_1 \sin(a_3 x) \pm \sqrt{\frac{1}{2a_3^2} \left( \frac{\Gamma(1-\alpha)}{\Gamma(1-2\alpha)} \right)^2 - l_1^2 \cos(a_3 x) + l_3 e^{-a_3 x}} + \frac{-1}{a_3^3} \frac{\Gamma(1-\alpha)}{\Gamma(1-2\alpha)} t^{-\alpha}. \]  

(3.6)

The plot of solution (3.6) when \( l_1 = 1, l_3 = 1, a_3 = 1 \) and \( \alpha = 0.9 \) is given in Fig. 1. Also, the plot of solution (3.6) when \( l_1 = 1, l_3 = 0, a_3 = 1 \) and \( \alpha = 0.9 \) is given in Fig. 2.

Case 2: when \( \beta = 3 \) and \( \gamma = -2 \). In this case, the invariant subspace is

\[ W_4 = \mathcal{L} \{ 1, e^{-\frac{a_3}{2} x}, e^{-\frac{a_3}{2} x} \sin \left( \frac{\sqrt{\pi}}{4} x \right), e^{-\frac{a_3}{2} x} \cos \left( \frac{\sqrt{\pi}}{4} x \right) \} \]

and the exact solution of Eq. (1) can be given by
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\[ u(x, t) = C_1(t) e^{-\frac{a_3}{4} x} \cos \left( \frac{\sqrt{3}}{4} x \right) + C_2(t) e^{-\frac{a_3}{4} x} \sin \left( \frac{\sqrt{3}}{4} x \right) + C_3(t) e^{-\frac{a_3}{4} x} + C_4(t), \]  

(3.7)

where \( a_3 \) is an arbitrary constant. Substitute Eq. (3.7) into Eq. (1.1) and compare the two sides of Eq. (1.1), to get

\[ \frac{d^\alpha C_1(t)}{dt^\alpha} = \frac{1}{32} a_3^4 \left( C_1(t) - \sqrt{3} C_2(t) \right) C_4(t), \]  

(3.8)

\[ \frac{d^\alpha C_2(t)}{dt^\alpha} = \frac{1}{32} a_3^4 \left( \sqrt{3} C_1(t) + C_2(t) \right) C_4(t), \]  

(3.9)

\[ \frac{d^\alpha C_3(t)}{dt^\alpha} = \frac{1}{32} a_3^4 \left( 3 \left( C_1(t)^2 + C_2(t)^2 \right) + 2 C_3(t) C_4(t) \right), \]  

(3.10)

\[ \frac{d^\alpha C_4(t)}{dt^\alpha} = 0. \]  

(3.11)

Solving the system of equations (3.8)-(3.11) assuming that the fractional derivatives are in the Caputo sense, we get

\[ C_4(t) = 0, \]

\[ C_3(t) = -\frac{3a_3^4 (l_1^2 + l_2^2)}{32 \Gamma(1 + \alpha)} t^\alpha, \]

\[ C_2(t) = l_2, \]

\[ C_1(t) = l_1, \]

where \( l_1 \) and \( l_2 \) are arbitrary constants. Thus, the exact solution of the time fractional thin-film equation (1.1) is given by

\[ u(x, t) = l_1 e^{-\frac{a_3}{4} x} \cos \left( \frac{\sqrt{3}}{4} x \right) + l_2 e^{-\frac{a_3}{4} x} \sin \left( \frac{\sqrt{3}}{4} x \right) - \frac{3a_3^4 (l_1^2 + l_2^2)}{32 \Gamma(1 + \alpha)} t^\alpha e^{-\frac{a_3}{4} x}. \]  

(3.12)

The plot of solution (3.12) when \( l_1 = l_2 = 1, a_3 = 4 \) and \( \alpha = 0.9 \) is given in Fig.3.

**Case 3:** when \( \beta = \frac{9}{2} \) and \( \gamma = \frac{17}{2} \). The invariant subspace in this case is \( W_4 = \mathcal{L}\{1, \ e^{-\frac{a_3}{2} x}, \ e^{-a_3 x}, \ e^{\frac{a_3}{2} x} \} \) and the exact solution of Eq. (1) can be expressed as

\[ u(x, t) = e^{-\frac{a_3}{2} x} C_1(t) + e^{-a_3 x} C_2(t) + e^{\frac{a_3}{2} x} C_3(t) + C_4(t). \]  

(3.13)

where \( a_3 \) is an arbitrary constant. Substitute Eq. (3.13) into Eq. (1.1) and compare the two sides of Eq. (1.1), to obtain

\[ \frac{d^\alpha C_1(t)}{dt^\alpha} = -\frac{45}{8} a_3^4 C_2(t) C_3(t) - \frac{1}{16} a_3^4 C_1(t) C_4(t), \]  

(3.14)
Solving the system of equations (3.14)-(3.17) when the fractional derivatives are in the Riemann–Liouville sense, we get

\[ C_4(t) = \frac{-1}{a_3^4} \frac{\Gamma(1 + 2b_1 + \alpha)}{\Gamma(1 + 2b_1)} t^{-\alpha}, \quad b_1 \neq -\frac{1}{2}, \]

\[ C_3(t) = \frac{8}{9a_3^5 l_1} \frac{\Gamma(1 - \alpha)\Gamma(1 + \alpha + 2b_1)}{\Gamma(1 - 2\alpha)\Gamma(1 + 2b_1)} t^{-2\alpha - b_1}, \quad \alpha \neq \frac{1}{2}, \]

\[ C_2(t) = l_2 t^{2b_1 + \alpha}, \]

\[ C_1(t) = l_1 t^{b_1}, \]

where, \( l_1 \) is an arbitrary constant and

\[ l_2 = -\frac{a_3^4 l_1^2 \Gamma(1 - 2\alpha) (16 \Gamma(1 + b_1) \Gamma(1 + 2b_1) - l_3 \Gamma(1 + b_1 - \alpha))}{80l_3 \Gamma(1 - \alpha) \Gamma(1 + b_1 - \alpha)}, \]

\[ l_3 = \Gamma(1 + 2b_1 + \alpha). \]

The value of \( b_1 \) can be obtained numerically from the following relation

\[ \frac{\Gamma(1 - 2\alpha - b_1)}{\Gamma(1 - 3\alpha - b_1)} = \frac{1}{16} \frac{l_3}{\Gamma(1 + 2b_1)}. \]

Thus, the exact solution of the time fractional thin-film equation (1.1) is given by

\[ u(x, t) = l_1 t^{b_1} e^{-\frac{2\alpha}{a_3^4} x} + l_2 (t^{2b_1 + \alpha}) (e^{-a_3 x}) + \frac{8}{9a_3^5 l_1} \frac{\Gamma(1 - \alpha)\Gamma(1 + \alpha + 2b_1)}{\Gamma(1 - 2\alpha)\Gamma(1 + 2b_1)} (t^{-2\alpha - b_1}) (e^{\frac{a_3}{a_3^4} x}) - \frac{1}{a_3^4} \frac{\Gamma(1 + 2b_1 + \alpha)}{\Gamma(1 + 2b_1)} t^{-\alpha}. \]

(3.18)
In this case, the invariant subspace is \( W_4 = \mathcal{L}\{1,x,x^2,x^3\} \) and the exact solution of Eq. (1) can be expressed as

\[
 u (x, t) = C_1 (t) x^3 + C_2 (t) x^2 + C_3 (t) x + C_4 (t).
\]

(3.19)

Substitute Eq. (3.19) into Eq. (1.1) and compare the two sides of Eq. (1.1), to get

\[
 \frac{d^\alpha C_1 (t)}{dt^\alpha} = 0,
\]

(3.20)

\[
 \frac{d^\alpha C_2 (t)}{dt^\alpha} = 18 (\beta + 2\gamma) C_1 (t)^2,
\]

(3.21)

\[
 \frac{d^\alpha C_3 (t)}{dt^\alpha} = 12 (\beta + 2\gamma) C_1 (t) C_2 (t),
\]

(3.22)

\[
 \frac{d^\alpha C_4 (t)}{dt^\alpha} = 4\gamma C_2 (t)^2 + 6\beta C_1 (t) C_3 (t).
\]

(3.23)

The system of equations (3.20)- (3.23) can be solved when the fractional derivatives are in the Caputo sense in two subcases as follows:

**Case 4.1: when \( \beta \) and \( \gamma \) are arbitrary constants.**

This case is discussed in [27]. So, we will not discuss it here.

**Case 4.2: when \( \beta = -2\gamma \)**

\[
 C_4 (t) = -\frac{12(-48 + l_1 l_3)\gamma}{\Gamma (1 + \alpha)} t^\alpha,
\]

\[
 C_3 (t) = l_3,
\]

\[
 C_2 (t) = l_2,
\]

\[
 C_1 (t) = l_1,
\]

where, \( l_1, l_2 \) and \( l_3 \) are arbitrary constants. Thus, the exact solution of the time fractional thin-film equation (1.1) is given by

\[
 u (x, t) = l_1 x^3 + l_2 x^2 + l_3 x - \frac{12(-48 + l_1 l_3)\gamma}{\Gamma (1 + \alpha)} t^\alpha.
\]

(3.24)

The plot of solution (3.24) when \( l_1 = l_2 = l_3 = 1 \) and \( \alpha = 0.9 \) is given in Fig. 4.

### 3.2. Invariant subspaces when \( n = 5 \)

Applying the ISM to Eq. (1.1) taking into consideration the auxiliary equation

\[
y^{(5)} + a_4 y^{(4)} + a_3 y^{(3)} + a_2 y'' + a_1 y' + a_0 y = 0,
\]

we get the following cases:
Figure 4. Plot of the solution (3.24) when $l_1 = l_2 = l_3 = 1$ and $\alpha = 0.9$.

**Case 5.1: when $\beta = \frac{9}{2}$, $\gamma = -\frac{7}{2}$.** The invariant subspace is given by

$$W_5 = \mathcal{L}\left\{1, \sin\left(\frac{\sqrt{a_3}x}{\sqrt{5}}\right), \cos\left(\frac{\sqrt{a_3}x}{\sqrt{5}}\right), \sin\left(2\frac{\sqrt{a_3}x}{\sqrt{5}}\right), \cos\left(2\frac{\sqrt{a_3}x}{\sqrt{5}}\right)\right\}.$$  

The exact solution of Eq. (1.1) can be written in the form

$$u(x,t) = C_5(t) + C_4(t) \cos\left(\frac{\sqrt{a_3}x}{\sqrt{5}}\right) + C_3(t) \sin\left(\sqrt{a_3}x\right) + C_2(t) \cos\left(2\frac{\sqrt{a_3}x}{\sqrt{5}}\right) + C_1(t) \sin\left(2\frac{\sqrt{a_3}x}{\sqrt{5}}\right),$$  

(3.25)

where $a_3$ is an arbitrary constants. Substitute Eq. (3.25) into Eq. (1.1) and compare the two sides of Eq. (1.1), to obtain

$$\frac{d^\alpha C_1(t)}{dt^\alpha} = -\frac{16}{25} a_3^2 C_1(t) C_5(t),$$  

(3.26)

$$\frac{d^\alpha C_2(t)}{dt^\alpha} = -\frac{16}{25} a_3^2 C_2(t) C_5(t),$$  

(3.27)

$$\frac{d^\alpha C_3(t)}{dt^\alpha} = \frac{1}{25} a_3^2 (45 C_2(t) C_3(t) - 45 C_1(t) C_4(t) - C_3(t) C_5(t)), $$  

(3.28)

$$\frac{d^\alpha C_4(t)}{dt^\alpha} = -\frac{1}{25} a_3^2 (45 C_1(t) C_3(t) + C_4(t) (45 C_2(t) + C_5(t))),$$  

(3.29)

$$\frac{d^\alpha C_5(t)}{dt^\alpha} = -\frac{9}{50} a_3^2 \left(16 C_1(t)^2 + 16 C_2(t)^2 + C_3(t)^2 + C_4(t)^2\right).$$  

(3.30)

Solving the system of equations (3.26)-(3.30) assuming that the fractional derivatives are in the Riemann–Liouville sense, we get

$$C_5(t) = -\frac{25 \Gamma(1 - \alpha)}{16 a_3^2 \Gamma(1 - 2\alpha)} t^{-\alpha}, \quad \alpha \neq \frac{1}{2},$$

where $\Gamma$ is the Gamma function.
\[ C_4(t) = l_4 \, t^{-\alpha}, \]
\[ C_3(t) = l_3 \, t^{-\alpha}, \]
\[ C_2(t) = l_2 \, t^{-\alpha}, \]
\[ C_1(t) = l_1 \, t^{-\alpha}, \]

where, \( l_1 \) is an arbitrary constant and

\[ l_2 = \pm \sqrt{-\frac{2304\alpha_3^3 l_1^2 (\Gamma(1-2\alpha))^2 + 625(\Gamma(1-\alpha))^2}{48\alpha_3^3 \Gamma(1-2\alpha)}}, \]

\[ l_3 = \pm \frac{5\sqrt{\pi}}{2^{1+4\alpha}} \frac{5 \sqrt{\Gamma(1-\alpha) + \sqrt{-2304\alpha_3^3 l_1^2 (\Gamma(1-\alpha))^2 + 625(\Gamma(1-\alpha))^2}}}{24 \sqrt{\Gamma(1-\alpha)}}, \]

\[ l_4 = \frac{l_3 (8l_1^2 + 16l_2^2 - l_3^2)}{8l_1 l_2}. \]

Therefore, we get the following exact solution of the time fractional thin-film equation (1.1)

\[ u(x,t) = \left( -\frac{25\Gamma(1-\alpha)}{16\alpha_3^3 \Gamma(1-2\alpha)} + l_4 \cos \left( \frac{\sqrt{\alpha_3} x}{\sqrt{5}} \right) + l_3 \sin \left( \frac{\sqrt{\alpha_3} x}{\sqrt{5}} \right) + l_2 \cos \left( \frac{2\sqrt{\alpha_3} x}{\sqrt{5}} \right) + l_1 \sin \left( \frac{2\sqrt{\alpha_3} x}{\sqrt{5}} \right) \right) t^{-\alpha}. \]

**Case 5.2: when \( \beta \) and \( \gamma \) are arbitrary constants.**

This case is discussed in [27]. So, we will not discuss it here.

### 3.3. Invariant subspaces when \( n = 6 \)

Applying the ISM to Eq. (1.1) taking into consideration the auxiliary equation
\[ y^{(6)} + a_5 y^{(5)} + a_4 y^{(4)} + a_3 y^{(3)} + a_2 y^{(2)} + a_1 y + a_0 y = 0, \]
we get the following case:

**Case 6: when \( \gamma = -\frac{3}{20} (-2 + 5\beta) \).** The invariant subspace is given by \( W_6 = \mathcal{L}\{x, x^2, x^3, x^4, x^5\} \). So, the solution of Eq. (1.1) can be expressed as

\[ u(x,t) = c_1(t) + c_2(t) x + c_3(t) x^2 + c_4(t) x^3 + c_5(t) x^4 + c_6(t) x^5. \]  

(3.31)

Substitute Eq. (3.31) into Eq. (1.1) and equate different powers of \( x \) to zero, to get

\[ \frac{d^\alpha C_1(t)}{dt^\alpha} = \left( \frac{6}{5} - 3\beta \right) C_3(t)^2 + 6\beta C_2(t) C_4(t) - 24 C_1(t) C_5(t), \]  

(3.32)

\[ \frac{d^\alpha C_2(t)}{dt^\alpha} = \frac{6}{5} (6 - 5\beta) C_3(t) C_4(t) + 24((-1 + \beta) C_2(t) C_5(t) - 5 C_1(t) C_6(t)), \]  

(3.33)

\[ \frac{d^\alpha C_3(t)}{dt^\alpha} = \left( \frac{54}{5} - 9\beta \right) C_4(t)^2 + \frac{12}{5} (-4 + 5\beta) C_3(t) C_5(t) + \ldots \]
The system of equations (3.32)-(3.37) can be solved in the following special case
when the fractional derivatives are in the Caputo sense

\[ C_6(t) = l_6, \quad C_3(t) = C_4(t) = C_5(t) = 0, \quad \beta = 2, \quad \gamma = -\frac{6}{5}, \]

\[ C_1(t) = l_1, \]

\[ C_2(t) = \frac{-120 l_1 l_6}{\Gamma(1 + \alpha)} t^\alpha, \]

where \( l_1 \) and \( l_6 \) are arbitrary constants. Therefore, we get the following exact solution of the time fractional thin-film equation (1.1)

\[ u(x, t) = l_1 - \frac{120 l_1 l_6}{\Gamma(1 + \alpha)} t^\alpha x + l_6 x^5. \]  

(3.38)

The plot of solution (3.38) when \( l_1 = l_6 = l \) and \( \alpha = 0.9 \) is given in Fig. 5.

3.4. Invariant subspaces when \( n = 7 \)

Applying the ISM to Eq. (1.1) taking into consideration the following auxiliary equation

\[ y^{(7)} + a_6 y^{(6)} + a_5 y^{(5)} + a_4 y^{(4)} + a_3 y^{(3)} + a_2 y'' + a_1 y' + a_0 y = 0, \]

we get the following case:
Case 7: when $\beta = (\frac{1}{4} - \frac{3}{4}\gamma)$. The invariant subspace is given by $W_7 = \mathcal{L}\{1,\ x,\ x^2,\ x^3,\ x^4,\ x^5,\ x^6\}$. Therefore, the solution of Eq. (1.1) can be expressed as
\[
 u(x, t) = c_1(t) + c_2(t)x + c_3(t)x^2 + c_4(t)x^3 + c_5(t)x^4 + c_6(t)x^5 + c_7(t)x^6, \tag{3.39}
\]
Substitute Eq. (3.39) into Eq. (1.1) and equate different powers of $x$ to zero, to get
\[
\frac{d^\alpha C_1(t)}{dt^\alpha} = 4\gamma C_3(t)^2 - \frac{3}{2}(-2 + 5\gamma) C_2(t) C_4(t) - 24C_1(t)C_5(t), \tag{3.40}
\]
\[
\frac{d^\alpha C_2(t)}{dt^\alpha} = 3(2 + 3\gamma) C_5(t) C_4(t) - 6(2 + 5\gamma) C_2(t) C_5(t) - 120C_1(t)C_6(t), \tag{3.41}
\]
\[
\frac{d^\alpha C_3(t)}{dt^\alpha} = \frac{9}{2}(2 + 3\gamma) C_4(t)^2 - 12\gamma C_3(t) C_5(t) - 15(6 + 5\gamma) C_2(t) C_6(t) - 360C_1(t)C_7(t), \tag{3.42}
\]
\[
\frac{d^\alpha C_4(t)}{dt^\alpha} = 24(1 + \gamma) C_4(t) C_5(t) - 10(6 + 7\gamma) C_3(t) C_6(t) - 150(2 + \gamma) C_2(t) C_7(t), \tag{3.43}
\]
\[
\frac{d^\alpha C_5(t)}{dt^\alpha} = 24(1 + \gamma) C_5(t)^2 - \frac{15}{2}(2 + 3\gamma) C_4(t) C_6(t) - 60(4 + 3\gamma) C_3(t) C_7(t), \tag{3.44}
\]
\[
\frac{d^\alpha C_6(t)}{dt^\alpha} = 3(6 + 5\gamma)(2C_5(t) C_6(t) - 9C_4(t) C_7(t)), \tag{3.45}
\]
\[
\frac{d^\alpha C_7(t)}{dt^\alpha} = (6 + 5\gamma)(5C_6(t)^2 - 12C_5(t) C_7(t)). \tag{3.46}
\]
Solving the system (3.40)-(3.46) assuming that the fractional derivatives are in the Riemann–Liouville sense, we get
\[
\gamma = -\frac{2}{3};\quad C_1(t) = C_2(t) = C_3(t) = 0,
\]
\[
C_4(t) = \frac{\Gamma\left(\frac{1}{4} - \frac{3\alpha}{4} - \frac{b_1}{4}\right)}{72l_7} \cdot \frac{2\Gamma\left(\frac{1}{4} (4 - 3\alpha + b_1)\right)}{\Gamma\left(\frac{1}{4} (4 - 7\alpha + b_1)\right)} \cdot l^\frac{1}{2} (-3\alpha + b_1),
\]
\[
C_5(t) = \frac{\Gamma\left(\frac{1}{4} (4 - 3\alpha + b_1)\right)}{8\Gamma\left(\frac{1}{4} (4 - 7\alpha + b_1)\right)} \cdot l^{-\alpha},
\]
\[
C_6(t) = l_6 t^{-\alpha - b_1},
\]
\[
C_7(t) = l_7 t^{-\alpha - b_1},
\]
where $l_7$ is an arbitrary constant and
\[
l_6 = \pm \frac{\Gamma\left(d_1\right) \Gamma\left(d_2\right) + 4\Gamma\left(d_3\right) \Gamma\left(d_4\right)}{2\sqrt{\Gamma\left(d_3\right) \Gamma\left(d_2\right)}},
\]
\[
d_1 = \frac{1}{2} (2 - 3\alpha - b_1),\quad d_2 = \frac{1}{4} (4 - 7\alpha + b_1).\]
The value of $b_1$ can be obtained from the relation
\[ \frac{\Gamma (1 - \alpha) \Gamma \left( \frac{1}{4} (4 - 7\alpha + b_1) \right)}{\Gamma (1 - 2\alpha)} - \Gamma \left( \frac{1}{4} (4 - 3\alpha + b_1) \right) = 0. \]

Thus, the exact solution of Eq. (1.1) is
\[
u(x, t) = \frac{l_6 \left( \frac{\Gamma \left( \frac{1}{4} (4 - 3\alpha - b_1) \right) + 2\Gamma \left( \frac{1}{4} (4 - 3\alpha + b_1) \right) \Gamma \left( \frac{1}{4} (4 - 7\alpha + b_1) \right)}{\Gamma (1 - 2\alpha)} \right)}{72l_7 t^{\frac{1}{2}(3\alpha + b_1)} x^3} + \frac{\Gamma \left( \frac{1}{4} (4 - 3\alpha + b_1) \right)}{8\Gamma \left( \frac{1}{4} (4 - 7\alpha + b_1) \right)} t^{-\alpha} x^4 + l_6 t^{\frac{1}{2}(3\alpha - b_1)} x^5 + l_7 t^{\frac{1}{2}(3\alpha - b_1)} x^6.
\]

### 3.5. Invariant subspaces when $n = 8$

Applying the ISM to Eq. (1.1) taking into consideration the following auxiliary equation
\[ y^{(8)} + a_7 y^{(7)} + a_6 y^{(6)} + a_5 y^{(5)} + a_4 y^{(4)} + a_3 y^{(3)} + a_2 y'' + a_1 y' + a_0 y = 0, \]

we get the following case:

**Case 8: when $\beta = \frac{16}{7}$, $\gamma = -\frac{10}{7}$**. The invariant subspace is given by $W_8 = \mathcal{L}\{1, x, x^2, x^3, x^4, x^5, x^6, x^7\}$. Therefore, the solution of Eq. (1.1) can be expressed as
\[
u(x, t) = c_1(t) + c_2(t)x + c_3(t)x^2 + c_4(t)x^3 + c_5(t)x^4 + c_6(t)x^5 + c_7(t)x^6 + c_8(t)x^7.
\]

Substitute Eq. (3.47) into Eq. (3.11) and equate different powers of $x$ to zero, to get
\[ \frac{d^\alpha C_1(t)}{dt^\alpha} = -\frac{40}{7} C_3(t)^2 + \frac{96}{7} C_2(t) C_4(t) - 24 C_1(t) C_5(t), \]
\[ \frac{d^\alpha C_2(t)}{dt^\alpha} = -12 C_3(t) C_4(t) + 48 C_2(t) C_5(t) - 120 C_1(t) C_6(t), \]
\[ \frac{d^\alpha C_3(t)}{dt^\alpha} = -\frac{72}{7} C_4(t)^2 + \frac{120}{7} C_3(t) C_5(t) + \frac{120}{7} C_2(t) C_6(t) - 360 C_1(t) C_7(t), \]
\[ \frac{d^\alpha C_4(t)}{dt^\alpha} = -\frac{72}{7} C_4(t) C_5(t) + 40 C_3(t) C_6(t) - \frac{600}{7} C_2(t) C_7(t) - 840 C_1(t) C_8(t), \]
\[ \frac{d^\alpha C_5(t)}{dt^\alpha} = -\frac{72}{7} C_5(t)^2 + \frac{120}{7} C_6(t) C_4(t) + \frac{120}{7} C_3(t) C_7(t) - 360 C_2(t) C_8(t), \]
\[ \frac{d^\alpha C_6(t)}{dt^\alpha} = -\frac{48}{7} C_5(t) C_6(t) + \frac{216}{7} C_4(t) C_7(t) - 120 C_3(t) C_8(t), \]
\[ \frac{d^\alpha C_7(t)}{dt^\alpha} = -\frac{40}{7} C_6(t)^2 + \frac{96}{7} C_5(t) C_7(t) - 24 C_4(t) C_8(t), \]
\[ \frac{d^\alpha C_8(t)}{dt^\alpha} = -\frac{40}{7} C_7(t)^2 + \frac{96}{7} C_6(t) C_8(t) - 24 C_5(t) C_8(t). \]
\[ \frac{d^\alpha C_8 (t)}{dt^\alpha} = 0. \] (3.55)

Solving the system (3.48)-(3.55) assuming that the fractional derivatives are in the Caputo sense, we get

\[ C_1 (t) = C_2 (t) = C_3 (t) = C_4 (t) = C_5 (t) = 0, \]
\[ C_6 (t) = l_6, \]
\[ C_7 (t) = - \frac{40l_6^2}{\Gamma (\alpha + 1)} t^\alpha, \]
\[ C_8 (t) = l_8, \]

where \( l_6 \) and \( l_8 \) are constants. Therefore, the exact solution of Eq. (1.1) is

\[ u (x, t) = l_6 x^5 - \frac{40l_6^2}{\Gamma (\alpha + 1)} t^\alpha x^6 + l_8 x^7. \]

### 3.6. Invariant subspaces when \( n = 9 \)

Applying the ISM to Eq. (1.1) taking into consideration the following auxiliary equation

\[ y^{(9)} + a_8 y^{(8)} + a_7 y^{(7)} + a_6 y^{(6)} + a_5 y^{(5)} + a_4 y^{(4)} + a_3 y^{(3)} + a_2 y'' + a_1 y' + a_0 y = 0, \]

we get the following case:

**Case 9: when \( \gamma = -\frac{45}{28}, \beta = \frac{5}{2} \).** The invariant subspace is given by \( W_9 = \mathcal{L}\{1, x, x^2, x^3, x^4, x^5, x^6, x^7, x^8\} \). Therefore, the solution of Eq. (1.1) can be expressed as

\[ u (x, t) = c_1 (t) + c_2 (t) x + c_3 (t) x^2 + c_4 (t) x^3 + c_5 (t) x^4 + c_6 (t) x^5 + c_7 (t) x^6 + c_8 (t) x^7 + c_9 (t) x^8, \] (3.56)

Substitute Eq. (3.56) into Eq. (3.11) and equate different powers of \( x \) to zero, to get

\[ \frac{d^\alpha C_1 (t)}{dt^\alpha} = - \frac{45}{7} C_3 (t)^2 + 15 C_2 (t) C_4 (t) - 24 C_1 (t) C_5 (t), \] (3.57)
\[ \frac{d^\alpha C_2 (t)}{dt^\alpha} = - \frac{60}{7} C_3 (t) C_4 (t) + 36 C_2 (t) C_5 (t) - 120 C_1 (t) C_6 (t), \] (3.58)
\[ \frac{d^\alpha C_3 (t)}{dt^\alpha} = - \frac{90}{7} C_4 (t)^2 + \frac{132}{7} C_3 (t) C_5 (t) + 30 C_2 (t) C_6 (t) - 360 C_1 (t) C_7 (t), \] (3.59)
\[ \frac{d^\alpha C_4 (t)}{dt^\alpha} = - \frac{108}{7} C_4 (t) C_5 (t) + \frac{360}{7} C_3 (t) C_6 (t) - 60 C_2 (t) C_7 (t) - 840 C_1 (t) C_8 (t), \] (3.60)
\[ \frac{d^\alpha C_5 (t)}{dt^\alpha} = - \frac{108}{7} C_5 (t)^2 + \frac{135}{7} C_6 (t) C_4 (t) + \frac{330}{7} C_3 (t) C_7 (t) - 315 C_2 (t) C_8 (t). \]
\[
\frac{d^\alpha C_6(t)}{dt^\alpha} = -\frac{108}{7} C_5(t) C_6(t) + \frac{360}{7} C_4(t) C_7(t) - 60 C_3(t) C_8(t)
- 840 C_2(t) C_9(t), \quad (3.61)
\]

\[
\frac{d^\alpha C_7(t)}{dt^\alpha} = -\frac{90}{7} C_6(t)^2 + \frac{132}{7} C_5(t) C_7(t) + 30 C_4(t) C_8(t) - 360 C_3(t) C_9(t),
\quad (3.62)
\]

\[
\frac{d^\alpha C_8(t)}{dt^\alpha} = -\frac{60}{7} C_6(t) C_7(t) + 36 C_5(t) C_8(t) - 120 C_4(t) C_9(t),
\quad (3.63)
\]

\[
\frac{d^\alpha C_9(t)}{dt^\alpha} = -\frac{45}{7} C_7(t)^2 + 15 C_6(t) C_8(t) - 24 C_5(t) C_9(t).
\quad (3.65)
\]

Solve the system (3.57)-(3.65) assuming that the fractional derivatives are in the Riemann–Liouville sense, to get

\[
C_1(t) = C_2(t) = C_3(t) = C_4(t) = 0,
\]

\[
C_5(t) = l_5 t^{-\alpha},
\]

\[
C_6(t) = l_6 t^{\frac{1}{2}(5\alpha - b_1)},
\]

\[
C_7(t) = l_7 t^{\frac{1}{2}(3\alpha - b_1)},
\]

\[
C_8(t) = l_8 t^{\frac{1}{2}(7\alpha - b_1)},
\]

\[
C_9(t) = l_9 t^{-2\alpha - b_1},
\]

where \(l_6\) is an arbitrary constant and

\[
l_5 = -\frac{7\Gamma(m_5)}{108\Gamma(m_6)},
\]

\[
l_7 = \frac{16^2\Gamma(m_5)\Gamma(m_2)}{2(11\Gamma(m_5)\Gamma(m_2) + 9\Gamma(m_6)\Gamma(m_4))},
\]

\[
l_8 = \frac{90\Gamma^2(m_6)\Gamma(m_5)\Gamma(m_4)\Gamma(m_2)}{7(7\Gamma(m_5)\Gamma(m_1) + 3\Gamma(m_6)\Gamma(m_3))(11\Gamma(m_5)\Gamma(m_2) + 9\Gamma(m_6)\Gamma(m_4))},
\]

\[
l_9 = -\frac{15(3l_5^2 - 7l_6l_8)}{7(24l_5\Gamma(1 - 3\alpha - b_1) + \Gamma(1 - 2\alpha - b_1))},
\]

\[
m_1 = 1 - \frac{11\alpha}{4} - \frac{3b_1}{4},
\]

\[
m_2 = 1 - \frac{5\alpha}{2} - \frac{b_1}{2},
\]

\[
m_3 = 1 - \frac{7\alpha}{4} - \frac{3b_1}{4},
\]

\[
m_4 = 1 - \frac{3\alpha}{2} - \frac{b_1}{2},
\]

\[
m_5 = 1 - \alpha,
\]
The value of $b_1$ can be obtained from the relation

$$
\frac{\Gamma \left(1 - \frac{5a}{4} - \frac{b_1}{4}\right)}{\Gamma \left(1 - \frac{3a}{4} - \frac{b_1}{4}\right)} - \frac{\Gamma \left(1 - \alpha\right)}{\Gamma \left(1 - 2\alpha\right)} = 0.
$$

Thus, we get the following exact solution of Eq. (1.1)

$$
u(x, t) = l_5 t^{-\alpha} x^4 + l_6 t^{\frac{1}{2}(-5a-b_1)} x^5 + l_7 t^{\frac{1}{2}(-3a-b_1)} x^6 + l_8 t^{\frac{1}{2}(-7a-b_1)} x^7 + l_9 t^{-2a-b_1} x^8. \quad (3.66)
$$

4. Conclusion

In this paper, we have investigated the time-fractional thin-film equation (1.1) using the ISM. We have concentrated on higher dimensional invariant subspaces (when $n = 4, 5, 6, 7, 8, 9$). These higher dimensional invariant subspaces are usually ignored in research works due to the difficulty of solving the resulting system of fractional ODEs. Many new exact solutions of Eq. (1.1) have been obtained which have not been reported in [27]. Also, a new invariant subspace (namely, when $n = 5, \beta = \frac{9}{2}$ and $\gamma = -\frac{7}{2}$) have been derived.

Appendix A

In this Appendix, the definition of the Riemann-Liouville and Caputo fractional derivatives are introduced. Also, we introduce some basic properties of these fractional derivatives.

Definition 1 [23, 25]. The Riemann-Liouville fractional derivative is given by

$$
_{0}D_{x}^{\alpha} f(x) = \begin{cases}
\frac{1}{\Gamma(m-\alpha)} \left(\frac{d}{dx}\right)^{m} \int_{0}^{x} (x-t)^{m-\alpha-1} f(t) \, dt, & m-1 < \alpha < m
\end{cases},
$$

where $m$ is an integer number.

Definition 2 [23, 25]. The Caputo fractional derivative is given by

$$
_{0}C_{x}^{\alpha} f(x) = \begin{cases}
\frac{1}{\Gamma(m-\alpha)} \int_{0}^{x} (x-t)^{m-\alpha-1} f^{(m)}(t) \, dt, & m-1 < \alpha < m
\end{cases},
$$

where $m$ is an integer number.

The Riemann-Liouville and Caputo fractional derivatives satisfy the following properties [23, 25]:

$$
_{0}D_{x}^{\alpha} (f(x) + g(x)) = _{0}D_{x}^{\alpha} f(x) + _{0}D_{x}^{\alpha} g(x),
$$

$$
_{0}C_{x}^{\alpha} (f(x) + g(x)) = _{0}C_{x}^{\alpha} f(x) + _{0}C_{x}^{\alpha} g(x),
$$

$$
_{0}D_{x}^{\alpha} (x^\beta) = \begin{cases}
0, & \text{if } \beta > -1 \text{ and } \alpha - \beta \in \{0, 1, 2, ..., m-1\},
\frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} t^{\beta-\alpha}, & \text{if } \beta > -1 \text{ and } \alpha - \beta \notin \mathbb{N},
\end{cases}
$$

$$
_{0}C_{x}^{\alpha} (x^\beta) = \begin{cases}
0, & \text{if } \beta \in \{0, 1, 2, ..., m-1\},
\frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} t^{\beta-\alpha}, & \text{if } \beta \in \mathbb{N} \text{ and } \beta \geq m, \text{ or } \beta \notin \mathbb{N} \text{ and } \beta > m-1.
\end{cases}
$$
References


